Solving real business cycle models by solving systems of first order conditions

John H. Cochrane[1]

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[1]Graduate School of Business, University of Chicago. john.cochrane@gsb.uchicago.edu.
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1 Introduction

These notes describe how to set up and solve real business cycle methods by the King Plosser Rebelo method of linearizing first order conditions. Much of the discussion follows Campbell (1992). However, Campbell shows you how to get analytic solutions with masses of algebra. Here I show you how to get numerical answers as quickly as possible. Analytic solutions are great, but the formulas are so complicated that you end up evaluating them numerically anyway in order to figure out what they say.

2 Fixed labor model, and method in detail

Start with an economy with fixed labor supply

$$\max E \sum_{j=0}^{\infty} \beta^j u(C_j) \ s.t.$$

$$\begin{aligned} Y_t = A_t^{\alpha} K_t^{1-\alpha} &= C_t + I_t \\ K_{t+1} &= (1-\delta) K_t + I_t \\ \ln A_{t+1} &= \rho \ln A_t + (1-\rho) gt + \epsilon_t. \end{aligned}$$

The technology shock is assumed to be an AR(1) around a trend growth path gt.

Step 1: Write first order conditions.

I substitute the constraints to obtain

$$\max E \sum_{j=0}^{\infty} \beta^j u(C_j) \ s.t.$$

$$K_{t+1} - (1-\delta) K_t - A_t^{\alpha} K_t^{1-\alpha} + C_t = 0.$$

Denote by $\beta^t \lambda^t$ the Lagrange multiplier on the constraint (one for each date t) Then the first order conditions are

$$\begin{aligned} \partial/\partial C_t : \ u'(C_t) &= \lambda_t \\ \partial/\partial K_{t+1} : \lambda_t &= \beta E_t \left[\lambda_{t+1} \left((1-\alpha) \left(\frac{A_{t+1}}{K_{t+1}} \right)^{\alpha} + (1-\delta) \right) \right] \\ \partial/\partial \lambda_t : \ K_{t+1} &= (1-\delta) K_t + A_t^{\alpha} K_t^{1-\alpha} - C_t. \end{aligned}$$

The first just interprets λ_t as the shadow value of wealth. The second is our old friend. The term multiplying λ_{t+1} in brackets is the one period return to capital. If you invest one dollar, you get the depreciated dollar back plus the marginal product of capital.)

At this point, it is convenient to eliminate either consumption or λ. KPR eliminate consumption, Campbell eliminates λ. I'll follow the latter approach. Furthermore, it's a good time to introduce a parametric form for the utility function. I'll use $u'(C) = C^{-\gamma}$. The algebra will be a little more transparent if we keep R separate for now. (We'll substitute back for R in the C equation later.) Thus, our system is

$$C_t^{-\gamma} = \beta E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right)$$

$$R_{t+1} = (1-\alpha) \left(\frac{A_{t+1}}{K_{t+1}} \right)^{\alpha} + (1-\delta)$$

$$K_{t+1} = (1-\delta) K_t + A_t^{\alpha} K_t^{1-\alpha} - C_t.$$

Finally, it is clearer if we express the first order conditions in terms of variables that do not grow,

$$1 = \beta E_t \left(\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} \right) \tag{1}$$

$$R_{t+1} = (1-\alpha) \left(\frac{A_{t+1}}{K_{t+1}} \right)^{\alpha} + (1-\delta) \tag{2}$$

$$\frac{K_{t+1}}{K_t} = (1-\delta) + \left(\frac{A_t}{K_t} \right)^{\alpha} - \frac{C_t}{K_t}. \tag{3}$$

We want to solve them for K_t and C_t in terms of the shocks $\{A_t\}$. This is an unpleasantly nonlinear system of difference equations. KPR suggests we linearize them near a nonstochastic steady state, and find a solution to the linearized system.

Step 2. Characterize nonstochastic steady state.

Guess that nonstochastic steady state exists in which Y, C, I, A, K all grow at a common rate $G = (1+g)$. If so, it must satisfy the first order conditions, so it must satisfy

$$1 = \beta \, G^{-\gamma} R \tag{4}$$

$$R = (1-\alpha) \left(\frac{A}{K} \right)^{\alpha} + (1-\delta) \tag{5}$$

$$G = (1-\delta) + \left(\frac{A}{K} \right)^{\alpha} - \frac{C}{K} \tag{6}$$

We'll use these later on to help get numbers for the nonstochastic steady state.

Step 3. Linearize first order conditions

It's better to end up with predictions for log variables. Denote logs and deviations from steady states as follows.

$$x = \ln(X); \; \tilde{x}_t = \ln(X_t); \; x_t = \ln(X_t) - \ln(X)$$

Let's start with the capital accumulation equation 3. Write it as

$$e^{\tilde{k}_{t+1}-\tilde{k}_t} = (1-\delta) + e^{\alpha\tilde{a}_t-\alpha\tilde{k}_t} - e^{\tilde{c}_t-\tilde{k}_t}$$

Taking a first order Taylor expansion, $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$,

$$G + \; G(k_{t+1} - k_t) = (1-\delta) + \left(\frac{A}{K}\right)^{\alpha} + \left(\frac{A}{K}\right)^{\alpha}(\alpha a_t - \alpha k_t) - \frac{C}{K} - \frac{C}{K}(c_t - k_t).$$

The constant terms cancel, by virtue of 6, leaving

$$G(k_{t+1} - k_t) = \left(\frac{A}{K}\right)^{\alpha}(\alpha a_t - \alpha k_t) - \frac{C}{K}(c_t - k_t).$$

Rearranging,

$$Gk_{t+1} = \left[G - \alpha\left(\frac{A}{K}\right)^{\alpha} + \frac{C}{K}\right]k_t + \left[\alpha\left(\frac{A}{K}\right)^{\alpha}\right]a_t - \left[\frac{C}{K}\right]c_t$$

We can make this simpler by substituting from 6 to obtain

$$G - \alpha\left(\frac{A}{K}\right)^{\alpha} + \frac{C}{K} = (1-\delta) + (1-\alpha)\left(\frac{A}{K}\right)^{\alpha} = R$$

so

$$Gk_{t+1} = [R]\,k_t + \left[\alpha\left(\frac{A}{K}\right)^{\alpha}\right]a_t - \left[\frac{C}{K}\right]c_t.$$

However, I won't spend a lot of time making the formulas look pretty in this way. Our objective is to get numerical solutions, so I want to get to the computer as fast as possible.

This equation is exactly what we're looking for. We now have a *linear* equation linking next period's capital to today's capital, technology and consumption. We will have *numbers* for the quantities in brackets.

The return equation proceeds similarly. Write 2 as

$$e^{\tilde{r}_{t+1}} = (1-\alpha)e^{\alpha\tilde{a}_{t+1}-\alpha\tilde{k}_{t+1}} + (1-\delta).$$

Taking the Taylor expansion (the constant term drops as before)

$$Rr_{t+1} = (1-\alpha)\left(\frac{A}{K}\right)^{\alpha}(\alpha a_{t+1} - \alpha k_{t+1}).$$

The consumption first order condition is a little trickier because of the expectation. We will linearize the term inside the expectation, and then take it. Write 1 as

$$1 = \beta E_t\left(e^{-\gamma c_{t+1}+\gamma c_t}e^{r_{t+1}}\right)$$

Taking the expansion,

$$1 = \beta E_t \left(G^{-\gamma} R + \ G^{-\gamma} R(-\gamma(c_{t+1} - c_t) + r_{t+1}) \right)$$

$$1 = \beta G^{-\gamma} R + \ \beta G^{-\gamma} R E_t \left(-\gamma(c_{t+1} - c_t) + r_{t+1} \right).$$

Again, as a result of the nonstochastic steady state condition 4 we can cancel terms leaving

$$0 = E_t \left(\gamma(c_{t+1} - c_t) - r_{t+1} \right).$$

This equation has a nice interpretation. If expected returns are expected to be high, so should consumption growth, since today's consumption level is low as people save more for the future. However, as risk aversion γ (really intertemporal substitution aversion in this case) rises, consumption growth is less and less sensitive to returns.

Summarizing, we have produced the following system of linearized first order conditions,

$$\gamma E_t c_{t+1} = \gamma c_t + E_t r_{t+1}$$

$$R r_{t+1} = (1 - \alpha) \left(\frac{A}{K} \right)^{\alpha} (\alpha a_{t+1} - \alpha k_{t+1}).$$

$$G k_{t+1} = [R]\, k_t + \left[\alpha \left(\frac{A}{K} \right)^{\alpha} \right] a_t - \left[\frac{C}{K} \right] c_t$$

Step 4. Use steady state restrictions to get numbers for linearized first order conditions

We have three steady state conditions, so we can derive three quantities from assumptions about the others. Which quantities one takes as given, i.e. estimated from data or imposed by the researcher's curiosity about a parameter's effect, and which parameters one infers from the steady state are a matter of choice.

Campbell advocates one nice calibration scheme. Assume or estimate

$$R = 1 + r = 10.6/4; \quad G \ = 1 + g = 1.02/4; \quad \alpha = 0.67; \quad \delta = 0.1/4$$

and a range of assumed values for γ and ρ. (The 1.4 terms adjust annual values to quarterly data.) Given these quantities, the steady state conditions imply

$$\beta = G^{\gamma}/R$$

$$\left(\frac{A}{K} \right)^{\alpha} = \frac{r + \delta}{1 - \alpha}$$

$$\frac{C}{K} = \left(\frac{A}{K} \right)^{\alpha} - (g + \delta) = \frac{r + \delta}{1 - \alpha} - (g + \delta).$$

Now we have numbers for all the terms in square brackets in the linearized first order conditions.

Other values for nonstochastic steady state quantities follow. For example,

$$\frac{Y}{K} = \left(\frac{A}{K}\right)^{\alpha}; \ \frac{I}{K} = \frac{Y}{K} - \frac{C}{K}.$$

Then, it's easy to find C/Y, I/Y or any other quantity you might want.

There are lots of other such schemes one can use. All that matters is that you respect the nonstochastic steady state conditions in picking numbers for the quantities in brackets.

This procedure is sometimes called "calibration". However, it's really estimation. We have a set of moments, things like $E(C_t/Y_t), E(R_t)$. We want the model's predictions for these things to be reasonably close to values found in the data. That's what this calibration procedure achieves. But there is no excuse for not making this a real estimation exercise and attaching standard errors to the results, to reflect sampling uncertainty about the values of the parameters so recovered. Burnside, Eichenbaum and Rebelo (1991) and Eichenbaum (1991) show how to do this.

Step 5. Substitute everything else into the c and k equations, put it in standard form.

In this case, all we have to get rid of is r leading to

$$E_t c_{t+1} = c_t + \frac{1}{\gamma} E_t \left(\left[\frac{\alpha(1-\alpha)\left(\frac{A}{K}\right)^{\alpha}}{R} \right] (a_{t+1} - \ k_{t+1}) \right)$$

$$G k_{t+1} = [R]\, k_t + \left[\alpha \left(\frac{A}{K}\right)^{\alpha} \right] a_t - \left[\frac{C}{K}\right] c_t.$$

We have numbers for all the things in square brackets, so this is a system of the form

$$\begin{aligned} E_t c_{t+1} &= \ b_{ca} E_t a_{t+1} + b_{ck} k_{t+1} + b_{cc} c_t \\ k_{t+1} &= b_{kk} k_t + b_{ka} a_t + b_{kc} c_t \\ a_{t+1} &= \rho a_t + \epsilon_{t+1} \end{aligned}$$

where I added back the equation for the technology shock.

Finally, it is convenient to express the system in standard form, in which each variable at t+1 is expressed in terms of time t values of the other variables. Here all we have to do is substitute from the k and a equations into the c equation, to get

$$E_t c_{t+1} = \ b_{ca} \rho a_t + b_{ck}(b_{kk} k_t + b_{ka} a_t + b_{kc} c_t) + b_{cc} c_t$$

$$E_t c_{t+1} = \ b_{ck} b_{kk} k_t + (b_{ck} b_{ka} + b_{ca}\rho) a_t + a_t + (b_{ck} b_{kc} + b_{cc}) c_t.$$

Using new letters to denote these combinations of $b's$,we have reduced the problem to solving the following *linear* system of difference equations

$$E_t c_{t+1} = \; d_{ck} k_t + d_{ca} a_t + d_{cc} c_t \tag{7}$$

$$k_{t+1} = b_{kk} k_t + b_{ka} a_t + b_{kc} c_t \tag{8}$$

$$a_{t+1} = \rho a_t + \epsilon_{t+1}. \tag{9}$$

Step 6. Solve the system

A. Campbell's method

There are several ways to solve this system for $\{c_t, k_t\}$ as a function of the $\{a_t\}$ sequence. Campbell suggests the *method of undetermined coefficients.* This solution method is conceptually easy, and in the hands of an algebra master like Campbell, can be used to delay going to the computer until you want to evaluate the answer.

The linear system of equations that we are staring at must be the first order conditions of some linear quadratic problem. Thus, it is a pretty good guess that the policy function will express consumption as a *linear* function of the state variables k_t and a_t,

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t.$$

We see if we can find numbers η_{ck} and η_{ca} to make this work. Substitute the guess in our system of linear equations,

$$E_t(\eta_{ck} k_{t+1} + \eta_{ca} a_{t+1}) = \; d_{ck} k_t + d_{ca} a_t + d_{cc}(\eta_{ck} k_t + \eta_{ca} a_t)$$

$$k_{t+1} = b_{kk} k_t + b_{ka} a_t + b_{kc}(\eta_{ck} k_t + \eta_{ca} a_t)$$

$$a_{t+1} = \rho a_t + \epsilon_{t+1}.$$

Simplifying, and eliminating $E_t a_{t+1}$ by the third equation,

$$\eta_{ck} k_{t+1} = \; (d_{ck} + d_{cc}\eta_{ck}) k_t + (d_{ca} + (d_{cc} - \rho)\eta_{ca}) a_t$$

$$k_{t+1} = (b_{kk} + b_{kc}\eta_{ck}) k_t + (b_{ka} + b_{kc}\eta_{ca}) a_t$$

Using the second equation to eliminate k from the first equation,

$$\eta_{ck}((b_{kk} + b_{kc}\eta_{ck}) k_t + (b_{ka} + b_{kc}\eta_{ca}) a_t) = \; (d_{ck} + d_{cc}\eta_{ck}) k_t + (d_{ca} + (d_{cc} - \rho)\eta_{ca}) a_t.$$

Simplifying,

$$[\eta_{ck}(b_{kk} + b_{kc}\eta_{ck}) - (d_{ck} + d_{cc}\eta_{ck})] k_t + \; [\eta_{ck}(b_{ka} + b_{kc}\eta_{ca}) - \; (d_{ca} + (d_{cc} - \rho)\eta_{ca})] a_t = 0.$$

This must hold for every value of k and $a,$ so each term must be separately zero. The first term is

$$b_{kc}\eta_{ck}^2 + (b_{kk} - d_{cc})\eta_{ck} - d_{ck} = 0$$

This gives a quadratic for η_{ck}. Hence,

$$\eta_{ck} = \frac{-(b_{kk} - d_{cc}) \pm \sqrt{(b_{kk} - d_{cc})^2 + 4b_{kc}d_{ck}}}{2b_{kc}}.$$

Pick the positive root, or the one that does not lead to an explosive solution.

Given η_{ck} we can find η_{ca} from the second term,

$$[b_{kc}\eta_{ck} + (\rho - d_{cc})]\eta_{ca} + (\eta_{ck}b_{ka} - \ d_{ca}) = 0$$

Hence,

$$\eta_{ca} = \frac{d_{ca} - \eta_{ck}b_{ka}}{b_{kc}\eta_{ck} + (\rho - d_{cc})} \ .$$

We're done! We can simulate the system

$$a_{t+1} = \rho a_t + \epsilon_{t+1}$$

$$k_{t+1} = (b_{kk} + b_{kc}\eta_{ck})k_t + (b_{ka} + b_{kc}\eta_{ca})a_t$$

and then find values for c,

$$c_t = \eta_{ck}k_t + \eta_{ca}a_t.$$

If we want other variables, we can find them as well. For example, the production function implies

$$y_t = \alpha a_t + (1 - \alpha)k_t$$

We can find a linear rule for investment from

$$I_t = Y_t - C_t \Rightarrow i_t = \frac{Y}{I}y_t - \frac{C}{I}c_t$$

B. KPR/Hansen Sargent method.

Option 1: eigenvector tricks.

Write (7)-9) in matrix form as

$$\begin{bmatrix} E_t c_{t+1} \\ k_{t+1} \\ E_t a_{t+1} \end{bmatrix} = \begin{bmatrix} d_{cc} & d_{ck} & d_{ca} \\ b_{kc} & b_{kk} & b_{ka} \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix} = W \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}.$$

One of the eigenvalues of W is greater than 1, so this generically leads to an explosive path. We expect this. Consumption should end up being the present value of *future* income, not of past income, so we expect to find a root that must be solved forward. If you don't set consumption equal to the right present value of future income, capital will explode up or down. However, if we pick c_t just right—equal to the present value of future income, the system is not explosive. Here's a neat way to do this.

First, find the eigenvalue decomposition $W = P\Lambda P^{-1}$where P has the eigenvectors of W as its columns, and Λ has eigenvalues of W down its diagonal[1]. To keep things straight, call the explosive eigenvalue λ_1,and put it in the top left corner of Λ. Now can write

$$\begin{bmatrix} E_t c_{t+1} \\ k_{t+1} \\ E_t a_{t+1} \end{bmatrix} = P\Lambda P^{-1} \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

and hence,

$$\begin{bmatrix} E_t c_{t+j} \\ k_{t+j} \\ E_t a_{t+j} \end{bmatrix} = P \begin{bmatrix} \lambda_1^j & & \\ & \lambda_2^j & \\ & & \lambda_3^j \end{bmatrix} P^{-1} \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}.$$

The only way to keep this from being explosive is if the term to the right of the Λ matrix has a zero multiplying λ_1, i.e. if

$$P^{-1} \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ d \end{bmatrix}.$$

Multiplying by P and eliminating the column of P corresponding to the zero, it must be true that

$$\begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix} = P[.,2:3] \begin{bmatrix} b \\ d \end{bmatrix}$$

Inverting the last two rows,

$$\begin{bmatrix} b \\ d \end{bmatrix} = P[2:3,2:3]^{-1} \begin{bmatrix} k_t \\ a_t \end{bmatrix}$$

Hence, we obtain the decision rule for consumption in terms of capital and the shock.

$$c_t = P[1,2:3]P[2:3,2:3]^{-1} \begin{bmatrix} k_t \\ a_t \end{bmatrix}. \tag{10}$$

As above, once we have this decision rule, we are done. We simulate a from its AR(1),

$$a_t = \rho a_{t-1} + \epsilon_t.$$

[1] If you haven't seen this before, start with the definition of eigenvalue and eigenvector: a number λ and a vector x such that $Wx = \lambda x$. There are typically as many such eigenvalues and vectors as there are columns of W. Stacking the $x's$ next to each other and calling the result P, we obtain $WP = P\Lambda$. Inverting, we obtain the diagonalization $W = P\Lambda P^{-1}$.

Then we find k from

$$k_{t+1} = b_{kk}k_t + b_{ka}a_t + b_{kc}c_t$$

$$k_{t+1} = \left(\left[\begin{array}{cc} b_{kk} & b_{ka} \end{array}\right] + P[1,2:3]P[2:3,2:3]^{-1}\right)\left[\begin{array}{c} k_t \\ a_t \end{array}\right].$$

We find c_t from10, and so forth.

Option 2: decouple dynamics, solve unstable root forward.

This method is particularly useful if you *don't* want to use an AR(1) or other Markovian shock—if you want to specify an arbitrary series for expected future shocks, such as a war, rather than specify how expected future shocks are functions of state variables. Write the first two equations of 7-9 in matrix form as

$$\left[\begin{array}{c} c_{t+1} \\ k_{t+1} \end{array}\right] = W\left[\begin{array}{c} c_t \\ k_t \end{array}\right] + Qa_t.$$

It's tempting to invert this AR(1) in standard form to get

$$\left[\begin{array}{c} c_t \\ k_t \end{array}\right] = \sum_{j=0}^{\infty} W^j Q a_{t-j}.$$

However, this isn't the answer. Again, W contains an unstable eigenvalue which has to be solved forward. This time, let's explicitly solve the unstable root forward.

The matrix W has the eigenvalue decomposition $W = P\Lambda P^{-1}$ Then, write our equations as

$$\left[\begin{array}{c} E_t c_{t+1} \\ k_{t+1} \end{array}\right] = P\Lambda P^{-1}\left[\begin{array}{c} c_t \\ k_t \end{array}\right] + Qa_t.$$

Multiply through by P^{-1}, we obtain

$$P^{-1}\left[\begin{array}{c} E_t c_{t+1} \\ k_{t+1} \end{array}\right] = \Lambda P^{-1}\left[\begin{array}{c} c_t \\ k_t \end{array}\right] + P^{-1}Qa_t.$$

Now, define new variables z_t and w_t by

$$\left[\begin{array}{c} z_t \\ w_t \end{array}\right] = P^{-1}\left[\begin{array}{c} c_t \\ k_t \end{array}\right].$$

These new variables follow

$$\left[\begin{array}{c} E_t z_{t+1} \\ E_t w_{t+1} \end{array}\right] = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right]\left[\begin{array}{c} z_t \\ w_t \end{array}\right] + P^{-1}Qa_t$$

where $\lambda_1 > 1$, and $\lambda_2 < 1$. These variables are uncoupled, so we can find each separately. z follows

$$E_t z_{t+1} = \lambda_1 z_t + (P^{-1}Q)_1 a_t.$$

Since $\lambda_1 > 1$, we solve this one forward[2],

$$z_t = -\sum_{j=0}^{\infty} \frac{1}{\lambda_1^j}(P^{-1}Q)_1 E_t a_{t+j}$$

We can use this for any process for a_t. (That's an advantage of this solution method.) Plugging in for our AR(1) process,

$$z_t = -\sum_{j=0}^{\infty} \frac{1}{\lambda_1^j}(P^{-1}Q)_1 \rho^j a_t = -\frac{\lambda_1}{\lambda_1 - \rho}(P^{-1}Q)_1 a_t$$

Since $\lambda_2 < 1$, we solve w_t backwards. The moving average representation isn't particularly useful, so it's easier to express our solution for w_t in recursive form.

We're done! We find z_t and w_t recursively by

$$a_t = \rho a_{t-1} + \epsilon_t$$

$$\begin{aligned} z_t &= -\frac{\lambda_1}{\lambda_1 - \rho}(P^{-1}Q)_1 a_t \\ w_t &= \lambda_2 w_{t-1} + (P^{-1}Q)_2 a_t \end{aligned}$$

Then, we find c and k by

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = P \begin{bmatrix} z_t \\ w_t \end{bmatrix}$$

and other variables as above.

3 Basic variable labor model

Now we add variable labor supply. This is a very important modification. Business cycle variations are all about variations in hours and employment.

$$\max E \sum_{j=0}^{\infty} \beta^j \left[\ln(C_j) + \theta \frac{(1-N_t)^{1-\gamma}}{1-\gamma} \right] \quad s.t.$$

$$Y_t = (A_t N_t)^{\alpha} K_t^{1-\alpha} = C_t + I_t$$

[2] In case you forgot, you can do this one by lag operators, or recursively,

$$z_t = \frac{1}{\lambda_1} E_t z_{t+1} - (P^{-1}Q)_1 a_t$$

$$z_t = \frac{1}{\lambda_1^2} E_t z_{t+2} - \frac{1}{\lambda_1}(P^{-1}Q)_1 E_t a_{t+1} - (P^{-1}Q)_1 a_t$$

and so forth.

$$K_{t+1} = (1-\delta)K_t + I_t$$

$$\ln A_{t+1} = \rho \ln A_t + (1-\rho)gt + \epsilon_t$$

We go through the same steps.

Step 1. Find first order conditions

Again, we get rid of some of the constraints, and obtain a Lagrangian

$$\max E \sum_{t=0}^{\infty} \beta^t \left(\ln(C_t) + \theta \frac{(1-N_t)^{1-\gamma}}{1-\gamma} \right) - \beta^t \lambda_t \left(K_{t+1} - (1-\delta)K_t - (A_t N_t)^\alpha K_t^{1-\alpha} + C_t \right).$$

The first order conditions are as before, with an additional condition for labor.

$$\partial/\partial C_t : \quad \frac{1}{C_t} = \lambda_t$$

I'll go ahead and eliminate λ right away from the remaining conditions,

$$\partial/\partial K_{t+1} : \frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} \left((1-\alpha) \left(\frac{A_{t+1} N_{t+1}}{K_{t+1}} \right)^\alpha + (1-\delta) \right) \right]$$

$$\partial/\partial N_t : \ \theta (1-N_t)^{-\gamma} = \frac{1}{C_t} \alpha \ N_t^{\alpha-1} \ A_t^\alpha K_t^{1-\alpha}$$

$$\partial/\partial \lambda_t : \ K_{t+1} = (1-\delta)K_t + (A_t N_t)^\alpha K_t^{1-\alpha} - C_t$$

The labor first order condition has a natural interpretation. Write it as

$$v'(N_t) = u'(C_t)/F_N$$

and it says that the wage or marginal product of labor equals the marginal rate of substitution between consumption and leisure.

Again, it's convenient to express the first order conditions in terms of variables that are stationary, and to separate out the definition of return

$$1 = \beta E_t \left[\frac{C_t}{C_{t+1}} R_{t+1} \right]$$

$$R_{t+1} = (1-\alpha) \left(\frac{A_{t+1} N_{t+1}}{K_{t+1}} \right)^\alpha + (1-\delta)$$

$$\theta (1-N_t)^{-\gamma} \ = \ \alpha \frac{A_t^\alpha}{C_t} \left(\frac{K_t}{N_t} \right)^{1-\alpha} \ = \ \alpha \left(\frac{A_t N_t}{K_t} \right)^\alpha \frac{K_t}{C_t} \frac{1}{N_t}$$

$$\frac{K_{t+1}}{K_t} = (1-\delta) + \left(\frac{A_t N_t}{K_t} \right)^\alpha - \frac{C_t}{K_t}$$

Characterize nonstochastic steady state

$$1 = \beta G^{-1} R$$

$$R = (1-\alpha)\left(\frac{AN}{K}\right)^{\alpha} + (1-\delta)$$

$$\theta(1-N)^{-\gamma} = \alpha \frac{A^{\alpha}}{C}\left(\frac{K}{N}\right)^{1-\alpha}$$

$$G = (1-\delta) + \left(\frac{AN}{K}\right)^{\alpha} - \frac{C}{K}$$

Not much news. One more condition, but another free parameter θ. In addition we need a value for N. 1/4-1/3 are typical values, since about that fraction of time is spent working.

Linearize around the steady state

Write each variable X as e^x, and take derivatives with respect to x, remembering that the constant terms will cancel.

$$1 = \beta E_t\left[\frac{C_t}{C_{t+1}} R_{t+1}\right] \Rightarrow \; E_t(c_{t+1} - c_t) = r_{t+1}).$$

Next,

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}N_{t+1}}{K_{t+1}}\right)^{\alpha} + (1-\delta) \Rightarrow$$

$$Rr_{t+1} = (1-\alpha)\left(\frac{AN}{K}\right)^{\alpha}(\alpha a_t + \alpha n_t - \alpha k_t)$$

and using the steady state conditions to simplify

$$Rr_{t+1} = (r+\delta)(\alpha a_t + \alpha n_t - \alpha k_t)$$

Next,

$$\theta(1-N_t)^{-\gamma} = \alpha \frac{A_t^{\alpha}}{C_t}\left(\frac{K_t}{N_t}\right)^{1-\alpha} \Rightarrow$$

$$\theta(1-N)^{-\gamma-1}\gamma N n_t = \alpha \frac{A^{\alpha}}{C}\left(\frac{K}{N}\right)^{1-\alpha}(\alpha a_t + (1-\alpha)k_t - c_t - (1-\alpha)n_t)$$

Using the steady state conditions to simplify,

$$\frac{\gamma N}{1-N} n_t = \alpha a_t + (1-\alpha)k_t - c_t - (1-\alpha)n_t$$

$$\left(1 + \gamma\frac{N}{1-N} - \alpha\right) n_t = \alpha a_t + (1-\alpha)k_t - c_t.$$

Next,

$$\frac{K_{t+1}}{K_t} = (1-\delta) + \left(\frac{A_t N_t}{K_t}\right)^{\alpha} - \frac{C_t}{K_t} \Rightarrow$$

$$G(k_{t+1} - k_t) = \left(\frac{AN}{K}\right)^{\alpha} (\alpha a_t + \alpha n_t - \alpha k_t) - \frac{C}{K}(c_t - k_t)$$

$$Gk_{t+1} = \left(\frac{AN}{K}\right)^{\alpha} (\alpha a_t + \alpha n_t) - \frac{C}{K} c_t + \left(\frac{C}{K} + G - \alpha \left(\frac{AN}{K}\right)^{\alpha}\right) k_t$$

Using the steady state conditions to simplify,

$$\left(\frac{C}{K} + G - \alpha \left(\frac{AN}{K}\right)^{\alpha}\right) = (1-\delta) + (1-\alpha)\alpha \left(\frac{AN}{K}\right)^{\alpha} = R$$

so

$$Gk_{t+1} = \left(\frac{AN}{K}\right)^{\alpha} (\alpha a_t + \alpha n_t) - \frac{C}{K} c_t + \ Rk_t$$

Substitute everything into k, c equations, write in standard form with t+1 on left, t on right.

Our system is

$$E_t c_{t+1} = c_t + E_t r_{t+1}$$

$$Rr_{t+1} = (r+\delta)(\alpha a_{t+1} + \alpha n_{t+1} - \alpha k_{t+1})$$

$$\left(1 - \alpha + \gamma \frac{N}{1-N}\right) n_t = \ \alpha a_t \ + \ (1-\alpha) k_t - c_t$$

$$Gk_{t+1} = \left(\frac{AN}{K}\right)^{\alpha} (\alpha a_t + \alpha n_t) - \frac{C}{K} c_t + \ Rk_t$$

Plugging the R equation in the C equation

$$E_t c_{t+1} = c_t + \ \frac{(r+\delta)}{R} (\alpha E_t a_{t+1} + \alpha E_t n_{t+1} - \alpha k_{t+1})$$

Give the constants new names,

$$E_t c_{t+1} = b_{cc} c_t + b_{ck} k_{t+1} + b_{ca} E_t a_{t+1} + b_{cn} E_t n_{t+1}$$

$$n_t = b_{nc} c_t + b_{nk} k_t + b_{na} a_t$$

$$k_{t+1} = b_{kc} c_t + b_{kk} k_t + b_{ka} a_t \ + b_{kn} n_t$$

At this point, you can substitute the n equation into the c and k equation, and get a system in c, k, a, in the standard form in which $t+1$ variables are functions of the t variables. Then use either Campbell's solution or the eigenvalue trick given above to find c as a function of k and a, and finally n and other variables.

To organize this algebra (especially in larger systems) it is useful to use matrix notation. Our system of equations is

$$\begin{bmatrix} {}_tc_{t+1} \\ k_{t+1} \\ {}_ta_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & b_{ck} & b_{ca} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} {}_tc_{t+1} \\ k_{t+1} \\ {}_ta_{t+1} \end{bmatrix} + \begin{bmatrix} b_{cc} & 0 & 0 \\ b_{kc} & b_{kk} & b_{ka} \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix} +$$

$$+ \begin{bmatrix} b_{cn} \\ 0 \\ 0 \end{bmatrix} [{}_tn_{t+1}] + \begin{bmatrix} 0 \\ b_{kn} \\ 0 \end{bmatrix} [n_t]$$

and

$$[n_t] = \begin{bmatrix} b_{nc} & b_{nk} & b_{na} \end{bmatrix} \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

Substituting the n equation in the first equation,

$$\left(I - \begin{bmatrix} 0 & b_{ck} & b_{ca} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} b_{cn} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} b_{nc} & b_{nk} & b_{na} \end{bmatrix} \right) \begin{bmatrix} {}_tc_{t+1} \\ k_{t+1} \\ {}_ta_{t+1} \end{bmatrix} =$$

$$= \left(\begin{bmatrix} b_{cc} & 0 & 0 \\ b_{kc} & b_{kk} & b_{ka} \\ 0 & 0 & \rho \end{bmatrix} + \begin{bmatrix} 0 \\ b_{kn} \\ 0 \end{bmatrix} \begin{bmatrix} b_{nc} & b_{nk} & b_{na} \end{bmatrix} \right) \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

$$A \begin{bmatrix} {}_tc_{t+1} \\ k_{t+1} \\ {}_ta_{t+1} \end{bmatrix} = B \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

$$\begin{bmatrix} {}_tc_{t+1} \\ k_{t+1} \\ {}_ta_{t+1} \end{bmatrix} = A^{-1}B \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$$

Now we've got it in standard form, and we can solve it just as before.

4 A general approach to solving KPR models.

The point here is to mechanize the steps of substituting for variables such as n, i, etc. and to allow you to go straight from first order conditions to numerical solution. Stacking up first order conditions, constraints, definitions, etc. we have a system of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_{t+1} \\ x_t \\ z \end{bmatrix} = 0.$$

Here, x denotes the state and control variables (usually c, k, a, or sometimes λ, k, a). z contains definitions and other first order conditions, for example, it will usually contain the definition of r_{t+1}, first order conditions for n_{t+1} and n_t that allow you to substitute out n from the other equations, and possibly equations for $i_t, i_{t+1}, c_t, c_{t+1}, ...$

Step 1: substitute out other variables, find transition equation for state.

The first step is to derive a transition equation for the state and control with all the other variables substituted out.

$$C\begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} + Dz \ = 0 \ \Rightarrow z = -D^{-1}C\begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix}$$

This is also used at the end to find values for other variables of interest.

Then,

$$A\begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} + Bz_t = 0 \Rightarrow (A - BD^{-1}C)\begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} = 0$$

or

$$\begin{bmatrix} E & F \end{bmatrix}\begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} = 0$$

Then, finally,

$$x_{t+1} = -E^{-1}Fx_t = Wx_t$$

Step 2: Find value of control for stable solution.

I assume that w has one explosive eigenvalue and the rest stable. I also assume that one element, the first is the control (usually c or λ).

First, find the eigenvalue decomposition $W = P\Lambda P^{-1}$where P has the eigenvectors of W as its columns, and Λ has eigenvalues of W down its diagonal. To keep things straight, call the explosive eigenvalue λ_1,and put it in the top left corner of Λ. Now can write

$$x_{t+1} = P\Lambda P^{-1}x_t.$$

For example,

$$\begin{bmatrix} E_t c_{t+j} \\ k_{t+j} \\ E_t a_{t+j} \end{bmatrix} = P\begin{bmatrix} \lambda_1^j & & \\ & \lambda_2^j & \\ & & \lambda_3^j \end{bmatrix} P^{-1}\begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}.$$

The only way to keep this from being explosive is if the term to the right of the Λ matrix has a zero multiplying λ_1, i.e. if

$$P^{-1}\begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ d \end{bmatrix}.$$

Multiplying by P and eliminating the column of P corresponding to the zero, it must be true that

$$\begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix} = P[.,2:3]\begin{bmatrix} b \\ d \end{bmatrix}$$

Inverting the last two rows,

$$\begin{bmatrix} b \\ d \end{bmatrix} = P[2:3, 2:3]^{-1} \begin{bmatrix} k_t \\ a_t \end{bmatrix}$$

Hence, we obtain the decision rule for consumption in terms of capital and the shock.

$$c_t = P[1, 2:3]P[2:3, 2:3]^{-1} \begin{bmatrix} k_t \\ a_t \end{bmatrix} = \eta_{c,ka} \begin{bmatrix} k_t \\ a_t \end{bmatrix}.$$

Then, the state transition matrix is

$$\begin{bmatrix} k_{t+1} \\ a_{t+1} \end{bmatrix} = W[2:3,1]c_t + W[2:3,2:3] \begin{bmatrix} k_t \\ a_t \end{bmatrix} = \Big(W[2:3,1]\eta_{c,ka} + W[2:3,2:3]\Big) \begin{bmatrix} k_t \\ a_t \end{bmatrix}$$

$$\begin{bmatrix} k_{t+1} \\ a_{t+1} \end{bmatrix} = T \begin{bmatrix} k_t \\ a_t \end{bmatrix}$$

5 Balanced growth restrictions

$$\max \sum \beta^t u(C_t, N_t) \ s.t.$$

$$Y_t = A_t^\alpha N_t^\alpha K_t^{1-\alpha}$$

$$K_{t+1} = (1-\delta)K_t + I_t$$

First order conditions

$$u_c(t) = \beta u_c(t+1) R_{t+1}$$

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}N_{t+1}}{K_{t+1}}\right)^\alpha + (1-\delta)$$

$$u_n(t)/F_N = u_c(t)$$

$$u_n(t) = u_c(t)\alpha A_t^a \left(\frac{K_t}{N_t}\right)^{1-\alpha}$$

$$u_n(t) = u_c(t)\alpha \left(\frac{A_t}{K_t}\right)^\alpha K_t N_t^{\alpha-1}$$

We're looking for a balanced growth path in which "great ratios" $A/K, Y/K, C/Y, I/Y, ...$ are constant, i.e. K, A, C, I, Y all grow at the same rate. and N does not grow. Hence, we need preferences such that

$$\frac{u_c(t+1)}{u_c(t)} = \text{const.}$$

$$\frac{u_c(t)K_t}{u_n(t)} = \text{const.}$$

along the growth path.

Example 1:

$$u(C,N) = \ln(C) + \ v(N).$$
$$\frac{u_c(t+1)}{u_c(t)} = \frac{C_t}{C_{t+1}} = \text{const.}$$
$$\frac{u_c(t)K_t}{u_n(t)} = \frac{K_t}{C_t}\frac{1}{v_n(N_t)} = \text{const.}$$

Example 2:

$$u(C,N) = \frac{C^{1-\gamma}}{1-\gamma} + \ v(N).$$
$$\frac{u_c(t+1)}{u_c(t)} = \left(\frac{C_t}{C_{t+1}}\right)^{\gamma} = \text{const.}$$
$$\frac{u_c(t)K_t}{u_n(t)} = \frac{K_t}{C_t^{\gamma}}\frac{1}{v_n(N_t)}.$$

This is *not* constant unless $\gamma = 1$. Balanced growth and separable utility (working more doesn't change your marginal utility of consumption) *requires* log utility over consumption. To get higher curvature, we need to add nonseparability.

Example 3:

$$U(C,N) = \frac{(C^{\rho}(1-N)^{1-\rho})^{1-\gamma}}{1-\gamma}$$
$$\frac{u_c(t+1)}{u_c(t)} = \frac{(C_{t+1}^{\rho}(1-N_{t+1})^{1-\rho})^{-\gamma}(1-N_{t+1})^{1-\rho}\rho C_{t+1}^{\rho-1}}{(C_t^{\rho}(1-N_t)^{1-\rho})^{-\gamma}(1-N_t)^{1-\rho}\rho C_t^{\rho-1}} = \left(\frac{C_{t+1}}{C_t}\right)^{-\rho\gamma+\rho-1} = \text{const.}$$
$$\frac{u_c(t)K_t}{u_n(t)} = -\frac{(C_t^{\rho}(1-N_t)^{1-\rho})^{-\gamma}(1-N_t)^{1-\rho}\rho C_t^{\rho-1}K_t}{(C_t^{\rho}(1-N_t)^{1-\rho})^{-\gamma}C_t^{\rho}(1-\rho)(1-N_t)^{-\rho}} = \text{const.} \times \frac{K_t}{C_t} = \text{const.}$$

In this case, getting more consumption makes you work less.

$$u_n = (1-\rho)(C_t^{\rho}(1-N_t)^{1-\rho})^{-\gamma}C_t^{\rho}(1-N_t)^{-\rho} = (1-\rho)C_t^{\rho(1-\gamma)}(1-N_t)^{-\gamma(1-\rho)-\rho}$$
$$u_{nc} = (1-\rho)(\rho(1-\gamma))C_t^{\rho(1-\gamma)-1}(1-N_t)^{-\gamma(1-\rho)-\rho}$$

If $\gamma > 1$ (more curved than log) $u_{nc} < 0$: raising consumption lowers the marginal utility of work and hence work. If $\gamma < 1, u_{nc} > 0$ and raising consumption raises the marginal utility of work.

Example 4:

$$u(C,N) = (C^{\rho} + \theta(1-N)^{\rho})^{\frac{1}{\rho}(1-\gamma)}/(1-\gamma)$$
$$u_c = (C^{\rho} + \theta(1-N)^{\rho})$$