*Econometrica*, Vol. 69, No. 1 (January, 2001), 69–116

# LONG-TERM DEBT AND OPTIMAL POLICY IN THE FISCAL THEORY OF THE PRICE LEVEL

BY JOHN H. COCHRANE[1]

The fiscal theory says that the price level is determined by the ratio of nominal debt to the present value of real primary surpluses. I analyze long-term debt and optimal policy in the fiscal theory. I find that the maturity structure of the debt matters. For example, it determines whether news of future deficits implies current inflation or future inflation. When long-term debt is present, the government can trade current inflation for future inflation by debt operations; this tradeoff is not present if the government rolls over short-term debt. The maturity structure of outstanding debt acts as a "budget constraint" determining which periods' price levels the government can affect by debt variation alone. In addition, debt policy—the expected pattern of future state-contingent debt sales, repurchases and redemptions—matters crucially for the effects of a debt operation. I solve for optimal debt policies to minimize the variance of inflation. I find cases in which long-term debt helps to stabilize inflation. I also find that the optimal policy produces time series that are similar to U.S. surplus and debt time series. To understand the data, I must assume that debt policy offsets the inflationary impact of cyclical surplus shocks, rather than causing price level disturbances by policy-induced shocks. Shifting the objective from price level variance to inflation variance, the optimal policy produces much less volatile inflation at the cost of a unit root in the price level; this is consistent with the stabilization of U.S. inflation after the gold standard was abandoned.

KEYWORDS: Fiscal theory of the price level, government debt, price level, inflation.

## 1. INTRODUCTION

THE FISCAL THEORY STATES that the price level is determined by the ratio of nominal debt to the present value of real primary surpluses,

$$\frac{\text{nominal debt}}{\text{price level}} = \text{present value of real surpluses}. \tag{1}$$

The fiscal theory is developed by Leeper (1991), Sims (1994, 1997), Woodford (1995, 1997, 1998a, 1998b) and Dupor (2000) with one-period debt, building on Sargent and Wallace (1981). Cochrane (1999, 2000) reviews the fiscal theory, argues for its plausibility, and addresses many theoretical disputes.

In this paper, I extend the fiscal theory to include long-term debt. With long-term debt, the nominal value of the debt on the left-hand side of (1) is not fixed; it depends on nominal bond prices which in turn depend on expected future price levels. To see why this fact might matter, suppose that there is bad

[1]I thank the CRSP, Graduate School of Business, and the National Science Foundation for research support, and I thank Andrea Eisfeldt for research assistance. I thank Angel Serrat, Michael Woodford, and an anonymous referee for unusually helpful comments. An early draft of this paper circulated under the title "Maturity Matters: Long Term Debt in the Fiscal Theory of the Price Level."

news about future surpluses so the right-hand side of (1) declines. If there is no long-term debt, the nominal value of government debt is predetermined, so the price level must rise to re-equilibrate (1). However, if long-term bonds are outstanding, their *relative* price and thus the *numerator* of the left-hand side might fall instead, leaving today's price level unchanged. Lower bond prices today correspond to expectations of higher price levels in the future, so long-term debt means that bad news about future surpluses can result in future rather than current inflation.

To analyze issues of this sort, I solve equations like (1) for the price level, with current and expected future surpluses and debt on the right-hand side. I present an exact solution, but it is algebraically complex. I also present two approximate solutions which are more convenient for many applications.

### *Comparative Statics*

I use the solutions to understand the obvious comparative statics exercises: (i) How does the price level react to current and future surpluses, holding debt constant? (ii) How does the price level react to current and future debt, holding surpluses constant? Answers to the first question are particularly useful in thinking about events such as currency crashes or the ends of hyperinflations. Answers to the second question suggest ways in which government choices about the quantity and maturity structure of nominal debt can cause inflation or offset the inflationary impact of surplus shocks. They also allow us to think about open market operations, deliberate "twists" in the maturity structure, and other debt-management issues.

In answer to the first question, I find that the effects of surpluses on the price level depend on debt *policy*: Current and expectations of future state-contingent debt sales and redemptions matter as well as the maturity structure of outstanding debt. The effects are often surprisingly different than those in the short-term debt case. For example, if the government pays off outstanding perpetuities rather than roll over short-term debt, the price level at each date is determined by the surplus at that date rather than by the present value of surpluses.

In answer to the second question, I find that the effects of debt on the price level also depend on the maturity structure and on expectations of future debt policy. For example, I find that the government can trade inflation today for inflation in the future, with no change in surpluses, if and only if some long-term debt is outstanding. Suppose that the government sells some additional debt, holding surpluses constant. If no long-term debt is outstanding, the government faces a unit-elastic demand curve. Bonds are nominal claims to the same real resources, so bond prices fall one-for-one with the number sold; real revenue from bond sales and the price level today are unaffected by the number sold. However, if there are long-term bonds outstanding, selling extra debt dilutes the existing long-term bonds as claims to the fixed stream of future real resources. In this case, unexpected debt sales can raise revenue today and lower today's price level, with no change in current or future surpluses, or in the total market value of debt. Of course, selling more debt today with constant surpluses always raises the price level later, as fixed real resources must pay off a larger nominal debt.

This limited control of the timing of inflation is a different mechanism than that studied by Sargent and Wallace (1981). In that paper, there is a monetary friction, debt is real, and the monetary authority determines when seignorage revenue will be earned. The mechanism works with short-term debt. Here, there is no monetary friction, debt is nominal, the treasury determines the price level path, all revenues are held fixed, and the mechanism only works if long-term debt is present.

For most of the comparative statics, state-contingent debt policy—when the debt is expected to be repurchased, redeemed or rolled over—is crucially important to the resulting price level and nominal interest rate path. Thus, questions such as "what is the effect of an open market operation?" or "what is the effect of a change in the maturity structure" cannot be answered without specifying the full date- and state-contingent change in debt policy, as well as any implicit changes in current and expected future surpluses. As always in dynamic intertemporal models, one must think about policy rules or state-contingent sequences, rather than think about decisions taken in isolation.

### *Optimal Policy*

After studying the comparative statics of debt and surplus movements, I ask what debt and surplus policies optimally smooth inflation, paying particular attention to motivations for long-term debt. The three elements of the government's policy choice are the average maturity structure, the choice of state-contingent debt sales and redemptions in response to fiscal shocks, and a limited control of the surplus. I add each element in turn and analyze the results in terms of the above comparative statics.

I start by analyzing optimal *fixed-debt* policy, in which the government determines only the steady state level of debt and its maturity structure; it does not adjust debt in response to surplus shocks, and it cannot control the surplus. I find that short maturity structures are preferred when the present value of the surplus varies by less than the surplus itself; while long maturity structures are preferred when surpluses build up following a shock so that the present value varies by more than the surplus itself. This finding is a natural result of the comparative statics: the price level responds to the present value of surpluses with a short maturity structure, while the price level responds to the surplus at each date with a long maturity structure.

I then analyze optimal *active* policy, in which the government can also change the amount of debt and its maturity structure each period in response to surplus shocks. Now there is a second motivation for long-term debt. If long-term debt is outstanding, the government can smooth inflation by occasionally and unexpectedly devaluing long-term bonds, trading a lower price level today for a higher price level in the future. This action can smooth inflation after a shock has hit. I study a quantitative example in which the optimal fixed-debt policy consists of short-term debt, but the optimal active policy includes long-term debt so that the government can smooth inflation by such ex-post devaluations.

Finally, I add a limited control over the long-term surplus in order to model better the situation faced by the U.S. government and the fact that debt sales do seem to come with promises of increased long-run surpluses. This optimal policy analysis solves some empirical puzzles. A simpleminded application of (1) and its comparative-static predictions for the effects of surplus and debt shocks seems disastrous for the fiscal theory in U.S. data. However, if we regard the U.S. government as solving such an optimal policy problem, *adapting* debt and fiscal policy to defend price level stability in the face of cyclical surplus shocks rather than *causing* price level disturbances by exogenous surplus and debt movements, we can explain many of the initially puzzling features of the data.

For example, equation (1) suggests that the price level should move together with total nominal debt. On the reasonable assumption that the present value of the surplus is high when the surplus itself is high, it also suggests that the price level should move inversely with the surplus and that the real value of the debt should move together with the surplus. But none of these patterns is an even vaguely plausible description of U.S. data. Figure 1 presents the primary Federal surplus/consumption ratio and CPI inflation.[2] If anything there is a slight positive correlation between surplus and inflation at business cycle frequencies. Figure 2 presents the surplus/consumption ratio together with the level and

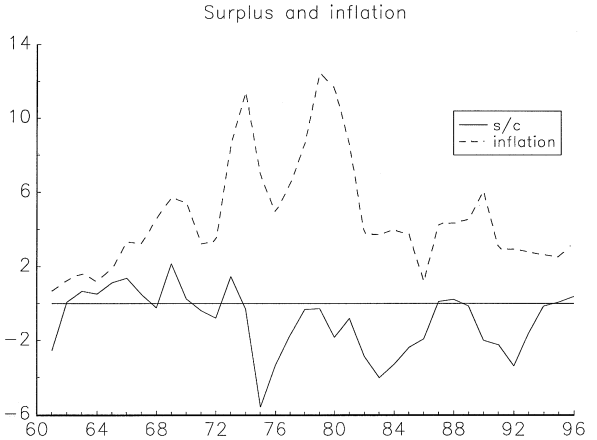


FIGURE 1.—Federal primary surplus/nondurable + services consumption, and CPI inflation. Both series are expressed as percentages.

[2]For all the empirical work in this article, I use data from and presented in more detail in Cochrane (1999). I constructed the value of the debt as the market value of all outstanding treasury securities, and inferred the surplus from the rate of return on government debt and the quantities outstanding. Dividing by consumption gives a more plausibly stationary series, and the theory adapts easily to this transformation by adding consumption growth to the "rate of return" in the formulas.

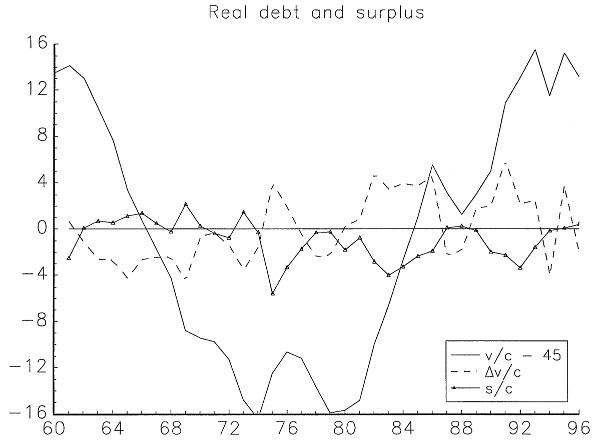


FIGURE 2.—Real value of the debt/consumption; difference of real value/consumption, and surplus/consumption ratio. All series are expressed as percentages. $V/c$ is shifted down by 45 percentage points to fit on the same graph.

difference in total real value of the debt. Comparing the two figures, we can see that there is little correlation between the level of debt and the price level, inflation, or the surplus, as debt moves much more slowly than any of the other series. The surplus is nicely *negatively* correlated with *changes* in debt. Unsurprisingly with a constant price level, but surprisingly in terms of (1), high surpluses pay down the debt.

By contrast, I find that the optimal policies that smooth inflation in the face of cyclical surplus shocks produce time series that are similar to these U.S. time series in many dimensions. For example, the optimal policies generate a negative correlation between surpluses and debt growth, as in the data.

### *A Few Comments on the Fiscal Theory*

At heart, the fiscal theory recognizes that even apparently unbacked fiat money is, together with nominal debt, a residual claim on government surpluses, and values them as such. For example, stock is valued by

$$\frac{\text{number of shares}}{\text{price per share}} = \text{present value of future earnings}. \tag{2}$$

If Microsoft stock became numeraire, unit of account, and medium of exchange, we would try to understand price level determination—the rate of exchange between goods and one share of Microsoft—via this equation. The fiscal theory values government-issued nominal debt in exactly the same way. (Cochrane (2000) pursues the stock analogy in depth.)

As this analogy makes clear, the fiscal theory needs no frictions—no money demand or theory of money—to determine the price level. The fiscal theory can describe a well-determined price level for apparently unbacked fiat money in a completely cashless economy, one in which just-maturing government bonds are units of account but not media of exchange. The stock analogy also suggests that the fiscal theory's predictions for the price level will not be much affected by the presence of monetary frictions—if some categories of debt help to facilitate transactions. The only potential effects are the small fiscal consequences of seignorage or interest rate spreads on transactions-facilitating assets. The analogy also shows that fiscal price level determination is immune to financial innovation and to private note issue. An agent can issue a claim to a share of stock, payable from *his* holdings, with no "dilution" effect on the value of the underlying shares, even if the agent's claim trades at a discount due to the risk that he may default. In the same way, agents can create and trade claims to government debt or banknotes with no effect on a fiscally-determined price level.

The basic fiscal theory equation (1) is, like the stock example equation (2), an equilibrium valuation equation, not a constraint. There is nothing that *forces* Microsoft (or Amazon.com!) to adjust future earnings to match current valuations, any more than calling (1) a "government budget constraint" *forces* the government to raise future taxes in response to an "off-equilibrium" deflation. Since the equations apply just as well to an economy that uses Microsoft stock as numeraire and medium of exchange, the fiscal theory does not require that one assume anything different about government and private budget constraints.

Initially, the idea that nominal debt and surpluses are policy instruments may seem strange. Most of the above-cited fiscal theory analyses include a monetary friction, and a monetary policy (control of an interest rate or monetary aggregate) thus implicitly determines the evolution of nominal debt. With no monetary friction, however, nominal debt does become the nominal policy tool directly.

It is also unusual that nominal debt and surpluses are *separate* policy instruments. We are used to thinking of debt as evolving from a surplus decision. For example, with perfect foresight, the real value of one-period nominal debt $B_{t-1}$ that matures at $t$ evolves as

$$\frac{B_{t-1}}{p_t} = s_t + \frac{1}{r}\frac{B_t}{p_{t+1}},$$

where $p_t$ = price level, $s_t$ = primary surplus, and $r$ = gross real interest rate. Thus, next period's debt seems to be determined from last period's debt and this period's surplus. This analysis is correct for real debt, or if prices are determined elsewhere (e.g. by $M_t v = p_t y$). In a fiscal equilibrium, however, the sequences $\{B_t, s_t\}$ are chosen first, and prices follow; the government does not take the price sequence $\{p_t\}$ as fixed when deciding on $\{B_t, s_t\}$. For example, if the government contemplates doubling $B_{t-1}$, it knows that $p_t$ will also double, just as Microsoft knows that its share price will halve if it does a split. Thus, the government can happily contemplate a change in debt with no change in surpluses. The government can choose debt and surplus as separate policy instruments, even in a completely cashless economy, and not just in a limit as in Woodford (1998a).

Except for occasional currency reforms, changes in nominal debt with no change in surpluses are unfamiliar policy paths. Most extra sales of nominal debt increase the real value of total debt, and thus *must* come with an increase in expected future surpluses, since the total real value of debt always equals the present value of future surpluses. (A simultaneous decrease in the real discount rate is theoretically possible, but unlikely in this context.) Thus, our *experience* is largely composed of increases in debt that accompany a decreased current surplus and increased future surpluses, and, as we shall see, for good reasons: changes in debt with no accompanying change in surpluses have dramatic effects on the price level, and most governments do not want to cause sharp fluctuations in the price level. However, the fact that most policy actions consist of simultaneous changes in two levers should not cloud the fact that the two policy levers *are* nominal debt and real surpluses. We can analyze what happens if each is moved without moving the other, and then we can better understand why optimal policy typically consists of coincident movements in both levers.

Since the models here are frictionless, standard Modigliani-Miller theorems by which the maturity structure of the debt is irrelevant for *real* quantities still apply. I study the effects of the maturity structure on the *nominal* price level; such effects can occur even in a frictionless economy and desired nominal results (such as smoothing inflation) can determine optimal maturity structures.

The issues in this paper are different than those studied by most of the literature on the maturity structure of government debt. Lucas and Stokey (1983), Blanchard and Missale (1994), and many others analyze time-consistency and precommitment issues. I ignore these important issues; I describe government policy by a sequence of state-contingent choices of debt and of the surplus, and I presume that the government can commit to carrying out such a policy once chosen. Taxes are lump sum, so this analysis is different from Missale's (1997) objective of smoothing real government revenues over the cycle with distortionary taxation, or Calvo and Guidotti's (1992) mixture of distorting taxes and time-consistency issues. Both issues are important considerations for future research.

## 2. FISCAL THEORY WITH LONG-TERM DEBT

### 2.1. *The Basic Equations*

Let $B_t(j)$ denote the face value of zero-coupon nominal bonds outstanding at the end of period $t$ that come due in period $j$. Let $Q_t(j)$ denote the nominal price at time $t$ of a bond that matures at time $j$. Of course, $Q_t(t) = 1$ and $B_t(j) = 0$ for $j \leq t$. Let $p_t$ denote the price level and let $s_t$ denote the real primary surplus, i.e. tax collections less government purchases. The appendix summarizes notation.

I model a frictionless economy in which no cash is held overnight. The economy need not be "cashless;" transactions may be facilitated by money—claims to just-maturing government bonds—created each morning and retired each night via repurchase agreements rather than by direct exchange of maturing bonds, and any amount of private money, bonds, banknotes, checking accounts etc. may be created with no effect on the formulas that determine the price level. Ignoring monetary frictions simplifies the algebra a great deal without altering the first-order predictions of the fiscal theory. I assume a risk-neutral economy with constant gross real interest rate $1/\beta$; this assumption simplifies the formulas with no great loss of generality.

The entire analysis flows from two equivalent equilibrium conditions, derived below. The *flow condition* says that the real primary surplus $s_t$ must equal bond redemptions plus net repurchases,

$$(3) \qquad \frac{B_{t-1}(t)}{p_t} - \sum_{j=1}^{\infty} \beta^j E_t\left(\frac{1}{p_{t+j}}\right)[B_t(t+j) - B_{t-1}(t+j)] = s_t,$$

while the *present value condition* says that the real value of outstanding debt equals the present value of real surpluses,

$$(4) \qquad \frac{B_{t-1}(t)}{p_t} + \sum_{j=1}^{\infty} \beta^j E_t\left(\frac{1}{p_{t+j}}\right) B_{t-1}(t+j) = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}.$$

As discussed below, the terms $\beta^j E_t(1/p_{t+j})$ give real bond prices in terms of expected future price levels. I use whichever form is more convenient for a given application. I use discrete time for clarity, but the model works just as well in continuous time.

An *equilibrium* is a sequence of prices $\{p_t\}$, of surpluses $\{s_t\}$, and of debt of all maturities $\{B_t(t+j),\ j = 1, 2, \ldots \infty\}$ such that equation (3) or (4) holds at each date and state.

We are interested in finding the price level for various specifications of the debt and surplus policy choices. A *solution* is the equilibrium price sequence for given debt and surplus sequences, i.e. an equation with $p_t$ on the left and other quantities on the right.[3] Because prices multiply quantities in (3)–(4), solutions are not trivial to find.

I describe government policy by the state-contingent sequences of prices and debt, $\{s_t, B_{t-1}(t+j)\}$. I assume that the government can commit to such a sequence once chosen.

[3] Prices, surpluses, and debt are each random variables, so $\{p_t\}$ denotes a sequence of random variables, with $p_t$ in the time-$t$ information set. Thus, the qualification "each date *and state*." I limit attention to positive and finite values of the surplus and debt, $0 < E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} < \infty$ and $0 < \sum_{j=0}^{\infty} Q_t(t+j) B_t(t+j) < \infty$, and to rule out $0/0$, $0 < B_t(t+j) < \infty$.

### 2.2. *Derivation*

To derive (3)–(4), start with the accounting identity that the primary surplus equals purchases less sales of bonds,

$$B_{t-1}(t) - \sum_{j=1}^{\infty} Q_t(t+j)[B_t(t+j) - B_{t-1}(t+j)] = p_t s_t. \tag{5}$$

To express bond prices in terms of future price levels, denote equilibrium marginal utility by $\rho^t u'(c_t)$, and conditional expectation by $E_t^*$ so

$$Q_t(t+j) = E_t^*\left(\rho^j \frac{u'(c_{t+j})}{u'(c_t)} \frac{p_t}{p_{t+j}}\right) = \beta^j E_t\left(\frac{p_t}{p_{t+j}}\right). \tag{6}$$

The right-hand equality simplifies notation with the assumption of a constant real discount factor $\beta = E_t[\rho u'(c_{t+1})/u'(c_t)]$ and by denoting expectation $E_t$ with respect to a risk-neutral set of probabilities. The latter step just simplifies notation, avoiding a marginal utility in every formula. The model is frictionless, so changes in the price level sequence do not affect equilibrium consumption or the real interest rate.

Substituting the one-period bond price (6) in (5) and dividing by $p_t$, we obtain (3). To derive (4) note that (3) can be written as

$$E_t(1 - \beta L^{-1}) v_t = s_t,$$

where

$$v_t \equiv \sum_{j=0}^{\infty} \beta^j \left(\frac{1}{p_{t+j}}\right) B_{t-1}(t+j).$$

Iterating forward on $v_t$, or applying $E_t(1-\beta L^{-1})^{-1}$ to both sides, together with the equilibrium condition $\lim_{T\to\infty} E_t \beta^T v_T = 0$, we obtain (4) and vice versa.

## 3. SOLUTIONS IN SPECIAL CASES, AND SURPLUS COMPARATIVE STATICS

For several specifications of debt policy—the path of $\{B_t(t+j)\}$—we can easily derive solutions. These solutions also allow us to address the comparative statics question, how does the price level react to changes in current and expected future surpluses, holding debt constant?

### 3.1. *One Period Debt*

Suppose that the government only issues one period debt, rolled over every period. This is the standard case analyzed in the fiscal theory, for example Woodford (1995). All terms $B_{t-1}(t+j)$ other than $B_{t-1}(t)$ are zero. Then, the present value condition, (4), specializes to a solution directly,

$$p_t = \frac{B_{t-1}(t)}{E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}}. \tag{7}$$

With one period debt, future surpluses affect the price level today. The price level today responds only to the present value of surpluses.

While this case is familiar to fiscal-theory readers, it is *not* generally true that the present value condition is also a solution, as we see in the remaining cases.

### 3.2. *No New Debt*

Suppose instead that a full maturity structure is outstanding at time 0, and the government neither issues new debt nor repurchases outstanding debt before it matures. For example, the government could pay off a perpetuity. In this case, debt due at $t$ is constant over time, $B_{t-1}(t) = B_{t-2}(t) = B_0(t)$. The *flow* condition (3) is now also a solution,

$$p_t = \frac{B_{t-1}(t)}{s_t}. \tag{8}$$

Now, prices are determined by bonds that fall due at each date divided by that date's surplus. Shocks to future deficits have no influence at all on the current price level. Instead, long-term bond prices, reflecting future inflation, entirely absorb the shocks to the present value of surpluses. To see this fact, apply (8) at $t+j$; a shock to expected $s_{t+j}$ changes expected $1/p_{t+j}$ and thus changes bond prices $Q_t(t+j) = \beta^j E_t(p_t/p_{t+j})$. Since it is so much simpler, this maturity structure should prove more useful than rolled over short-term debt in many theoretical applications of the fiscal theory.

### 3.3. *k-period Debt*

As an intermediate example, suppose that each period the government issues $B_t(t+k)$ $k$-period discount bonds each period, and then lets them mature. With this debt policy, $B_t(t+k) = B_{t+1}(t+k) = \cdots = B_{t+k-1}(t+k)$. The flow condition (3) then becomes

$$\frac{B_{t-k}(t)}{p_t} - \beta^k E_t\left(\frac{1}{p_{t+k}}\right) B_t(t+k) = s_t.$$

This is a $k$-period difference equation, with solution

$$p_t = \frac{B_{t-k}(t)}{E_t \sum_{j=0}^{\infty} \beta^{jk} s_{t+jk}} = \frac{B_{t-1}(t)}{E_t \sum_{j=0}^{\infty} \beta^{jk} s_{t+jk}}.$$

The price level is still determined by a sort of present value, but only every $k$th term matters! For example, if the government issues 5 year debt, then expectations of surpluses in years 5, 10, 15, etc. matter to today's (0) price level, but surpluses in years 4, 6 etc. do not matter. As $k \to 1$ we recover the one period debt solution (7) in which all future deficits matter. As $k \to \infty$, we recover the case (8) in which only today's surplus matters to today's price level.

### 3.4. *Geometric Maturity Structure*

A geometric pattern gives a tractable way to analyze a rich maturity structure. Suppose that the amount of debt outstanding at the beginning of $t$ (end of $t-1$) that will mature at $t+j$ declines at a rate $\phi^j$:

$$B_{t-1}(t+j) = B_{t+j-1}(t+j)\phi^j. \tag{9}$$

Equivalently, the fraction of debt that matures at date $t$, sold at date $t-j$, follows a geometric pattern,

$$A_t(t+j) \equiv \frac{B_t(t+j) - B_{t-1}(t+j)}{B_{t+j-1}(t+j)} = \phi^{j-1}(1-\phi); \quad j \geq 1. \tag{10}$$

If the level of debt grows at a constant rate $B_{t-1}(t) = \theta_B^t$, then this specification also implies that debt declines geometrically with maturity at any given date, $B_{t-1}(t+j) = B_{t-1}(t)(\theta_B \phi)^j$. However, the latter conclusion is not the case for arbitrary movements in debt over time. A specification in which debt always falls geometrically with maturity does not lead to a simple price solution, since the government must do a lot of buying and selling of debt at all maturities to maintain it.

To derive a solution for this debt policy, plug (9) into the present value condition (4), and plug (10) into the flow condition (3). Adding the first and $\phi/(1-\phi)$ times the second equations and solving for $p_t$ we obtain the solution,

$$p_t = \frac{B_{t-1}(t)}{\phi s_t + (1-\phi)E_t\sum_{j=0}^{\infty}\beta^j s_{t+j}}. \tag{11}$$

This example also nests the one-period debt case and the no-change-in-debt case as $\phi$ varies from 0 to 1.

## 4. SOLUTIONS

The above analysis gave some special cases of solutions—price on the left and other variables on the right—but leaves one hungry for more general solutions, that apply for arbitrary debt policies. Here, I present an exact solution, and then two approximate solutions that are convenient in some situations.

### 4.1. *Exact Solution*

To find a solution for prices in terms of debt and surplus, I start with either the flow (3) or present value (4) conditions and recursively substitute the same equations for future values of prices $p_{t+j}$. After some ugly algebra that I relegate to the Appendix, the result can be expressed as

$$p_t = \frac{B_{t-1}(t)}{E_t\left[\sum_{j=0}^{\infty} \beta^j W_{t,j} s_{t+j}\right]}. \tag{12}$$

To define the $W$ weights, first denote the fraction of maturity $j$ debt issued at time $t$ by

$$A_t(t+j) \equiv \frac{B_t(t+j) - B_{t-1}(t+j)}{B_{t+j-1}(t+j)}; \quad j = 1, 2, \ldots. \tag{13}$$

Then, the $W$ are defined recursively by

$$\begin{aligned} W_{t,0} &= 1, \\ W_{t,1} &= A_t(t+1), \\ W_{t,2} &= A_{t+1}(t+2) W_{t,1} + A_t(t+2), \\ W_{t,3} &= A_{t+2}(t+3) W_{t,2} + A_{t+1}(t+3) W_{t,1} + A_t(t+3), \\ W_{t,j} &= \sum_{k=0}^{j-1} A_{t+k}(t+j) W_{t,k}. \end{aligned} \tag{14}$$

To get some sense of what this means, write out the first two terms of the general solution,

$$\begin{aligned} \frac{B_{t-1}(t)}{p_t} = E_t \Bigg[ s_t &+ \beta \left(1 - \frac{B_{t-1}(t+1)}{B_t(t+1)}\right) s_{t+1} \\ &+ \beta^2 \left\{ 1 - \left[ \frac{B_{t-1}(t+1)}{B_t(t+1)} \left(1 - \frac{B_t(t+2)}{B_{t+1}(t+2)}\right) \right.\right. \\ &\left.\left. + \frac{B_{t-1}(t+2)}{B_{t+1}(t+2)} \right] \right\} s_{t+2} + \cdots \Bigg]. \end{aligned} \tag{15}$$

The weights $W_{t,j}$ capture the effects of debt policy—the current *and future* maturity structure of the debt—on the relation between the price level and the sequence of surpluses.

### 4.2. *Approximate Solution with a Geometric Baseline*

Future surpluses enter (12) simply, though with complex coefficients. Thus, we can easily characterize the effects of surpluses on the price level for special cases of debt policy. Debt enters in a more complex and nonlinear manner, as seen in (15). Thus, to calculate the effects of debt policy on the price level, as well as for the optimal policy questions, I use an approximate solution which is much easier to manipulate. The approximate solution is based on a first-order Taylor expansion of the general solution about a simple baseline path.

The approximate solution takes derivatives around a baseline path $\{s_t^*, B_{t-1}^*(t+j), p_t^*\}$ with geometrically growing surplus and a geometric maturity structure,

$$s_t^* = s\theta^t, \tag{16}$$

$$B_{t-1}^*(t+j) = B_{t+j-1}^*(t+j)\phi^j, \tag{17}$$

$$\frac{B_{t-1}^*(t)}{p_t^*} = \frac{1-\delta\phi}{1-\delta} s_t^*, \tag{18}$$

where

$$\delta \equiv \beta\theta.$$

I denote by $\tilde{x}_t$ the proportional deviation of each variable $x_t$ from the baseline path,

$$\tilde{p}_t = \frac{p_t - p_t^*}{p_t^*}; \quad \tilde{s}_t = \frac{s_t - s_t^*}{s_t^*}; \quad \tilde{B}_{t-1}(t+j) = \frac{B_{t-1}(t+j) - B_{t-1}^*(t+j)}{B_{t-1}^*(t+j)}.$$

With this notation, two expressions for the approximate solution are convenient,

$$\tilde{p}_t = \tilde{B}_{t-1} - \delta\phi\tilde{B}_t - \left(\frac{1-\delta}{1-\delta\phi}\right)\left(\phi\tilde{s}_t + (1-\phi)\sum_{j=0}^{\infty}\delta^j E_t\tilde{s}_{t+j}\right), \tag{19}$$

where

$$\tilde{B}_{t-1} \equiv \sum_{j=0}^{\infty}(\delta\phi)^j \tilde{B}_{t-1}(t+j); \tag{20}$$

and, in lag operator notation,

$$\tilde{p}_t = \frac{(1-\delta\phi L^{-1})}{(1-\delta\phi M^{-1})}\tilde{B}_{t-1}(t) - \frac{1-\delta}{1-\delta\phi}E_t\frac{(1-\delta\phi L^{-1})}{(1-\delta L^{-1})}\tilde{s}_t, \tag{21}$$

where $M$ operates on maturity as $L$ operates on dates, $M^{-1}B_t(t+j) = B_t(t+j+1)$. $\tilde{B}_{t-1}$ is a nominal debt aggregate that I will use below. Keep in mind that it is an aggregate of *nominal*, face values of the debt, not an aggregate of *market values* of debt, since it is unaffected by variation in the price level and hence bond prices.

The approximation uses the baseline price level to value outstanding debt rather than the actual price levels, and it uses the baseline maturity structure rather than the actual maturity structure to capture the trade-off between current and future price levels. It also linearizes the product $B_{t-1}(t+j)/p_{t+j}$. As usual, linearizing a product gives the baseline value of each term times the deviation of the other and ignores terms in which deviations are multiplied by each other.

The surplus terms in equation (19) are comfortingly similar to those I derived above in equation (11) for a geometric maturity structure. The version in equation (21) shows that the price level is proportional to $1-\delta\phi L^{-1}$ of the present value of the surplus. For $\phi=0$, we recover the present value, but as $\phi \to 1/\delta$, price becomes proportional to *growth* in the present value of surpluses.

### *Derivation*

Taking derivatives of the present value condition (4) about the baseline path $p_t^*, B_{t-1}^*(t+j)$, $s_t^*$, we obtain an approximate version of the present value condition,

$$\sum_{j=0}^{\infty} \beta^j \frac{B_{t-1}^*(t+j)}{p_{t+j}^*}\left(\tilde{B}_{t-1}(t+j)-\tilde{p}_{t+j}\right) = \sum_{j=0}^{\infty} \beta^j s_{t+j}^* \tilde{s}_{t+j}. \tag{22}$$

Formula (22) will obviously lead to a convenient representation if the baseline path is geometric. To that end, I specify that the baseline path has a geometrically growing surplus, a geometric maturity structure as in (16)–(17), and that the ratio of debt to price grows geometrically,

$$\frac{B_{t-1}^*(t)}{p_t^*} = A\theta_A^t.$$

The baseline path must satisfy (4), which restricts its parameters,

$$\theta_A = \theta, \qquad \frac{s}{A} = \frac{1-\delta}{1-\delta\phi}. \tag{23}$$

The first equation says that the real value of the debt must grow at the same rate as the surplus. The second equation says that the level of real debt must equal the level of the present value of future surpluses. With these restrictions, we have (18).

The simplest such path features geometric growth in $p_t^*$ and $B_{t-1}^*(t)$,

$$p_t^* = p\theta_p^t, \quad B_{t-1}^*(t) = B\theta_B^t. \tag{24}$$

However, the individual terms $B_{t-1}^*(t)$ and $p_t^*$ need not grow geometrically, so long as their ratio does so. They may even be stochastic, and they may share a common unit root.

The baseline path must satisfy $\delta \le 1$ to keep the present value of surpluses finite, and $\delta\phi < 1$ to keep the present value of the debt finite. It is not necessary that $\phi \le 1$, but such maturity structures are unusual enough that we may want to impose $\phi \leq 1$ in practice. First, with $\phi < 1$, the government sells some debt of every maturity each period, and then redeems it all when it matures. At $\phi = 1$, the government sells or purchases no debt, simply redeeming a stock outstanding at the initial period. With $1 < \phi < 1/\delta$, the government *repurchases* a little bit of every maturity debt each period, from an initially outstanding stock. To see this, write debt sales each period $B_t^*(t+j) - B_{t-1}^*(t+j) = B_{t+j-1}^*(t)\phi^{j-1}(1-\phi)$. To say the least, such a path requires fundamentally new institutions. The most likely implementation are consols that promise an increasing coupon. The limit $\phi = 1/\delta$ corresponds to the limit that the coupons grow at the nominal interest rate.

Second, $1 < \phi < 1/\delta$ and debt and price level that grow over time imply that the face value increases with maturity, and therefore the total face value is infinite. At a minimum, this will pose a strain for current face-value based accounting practices. The *market value* still declines with maturity, which is why such parameters are allowed. To see this, note that with geometric debt growth as in (24), the face value of debt outstanding at each date is

$$B_{t-1}^*(t+j) = (\theta_B \phi)^j B_{t-1}^*(t),$$

while the *market* value of maturity $j$ debt is

$$\beta^j p_t^* / p_{t+j}^* B_{t-1}^*(t+j) = (\delta\phi)^j B_{t-1}^*(t).$$

Using the baseline path (16)–(18), with parameters (23), in (22), and taking conditional expectations, we obtain a linearized present value condition

$$(25) \qquad (1-\delta\phi) \sum_{j=0}^{\infty} (\delta\phi)^j \left( \tilde{B}_{t-1}(t+j) - E_t \tilde{p}_{t+j} \right) = (1-\delta) \sum_{j=0}^{\infty} \delta^j E_t \tilde{s}_{t+j}.$$

Since the weights are geometric, iterating (25) forward to solve for $\tilde{p}_t$ is easy, and gives the approximate price solution, (19)–(21).

### 4.3. *Approximate Solution with a Nongeometric Baseline Path*

A linearization about a nongeometric baseline path captures some effects that are ignored by the linearization about a geometric path, but without the full complexity of the general solution. In this case, the coefficients in the linearization are similar to the general solution, and thus algebraically complex. However, these coefficients only need to be evaluated once, and the approximate solution is then a convenient linear function of surplus and debt policy.

Generalize (17) to an arbitrary baseline maturity structure,

$$(26) \qquad B_{t-1}^*(t+j) = B_{t+j-1}^*(t+j)\phi_j.$$

The counterpart to (25) no longer has a geometric structure, so finding a solution requires more algebra. The solution, derived in the Appendix, is

$$\tilde{p}_t = \sum_{j=0}^{\infty} \beta^j D_j E_t \tilde{B}_{t-1+j} - \xi \sum_{j=0}^{\infty} \delta^j W_j E_t \tilde{s}_{t+j}. \tag{27}$$

Here,

$$\tilde{B}_{t-1} \equiv \sum_{j=0}^{\infty} \beta^j \phi_j \tilde{B}_{t-1}(t+j)$$

and

$$\xi \equiv (1-\delta) \sum_{j=0}^{\infty} \delta^j \phi_j$$

are natural generalizations of their counterparts with a geometric maturity structure. The $W_j$ are given by

$$\begin{aligned} W_0 &= 1, \\ W_1 &= A_1, \\ W_2 &= A_1 W_1 + A_2, \\ W_j &= \sum_{k=0}^{j-1} A_{j-k} W_k; \end{aligned}$$

these terms are the steady state level of the general-solution weights $W_{t,j}$. The $A_j$ are given by

$$A_j \equiv \phi_{j-1} - \phi_j; \tag{28}$$

these terms are the steady state level of the terms $A_t(t+j)$ in the general solution. The $D$ coefficients are recursively generated by

$$\begin{aligned} D_0 &= 1, \\ D_1 &= -\phi_1, \\ D_2 &= A_1 D_1 + A_2 D_0, \\ D_3 &= A_1 D_2 + A_2 D_1 + A_3 D_0, \\ D_{k+1} &= \sum_{i=0}^{k} A_{i+1} D_{k-i}. \end{aligned} \tag{29}$$

## 5. THE EFFECTS OF DEBT POLICY

I use the approximate solutions to answer, what are the effects of *debt* changes on the price level, holding *surpluses* constant? With no change in surpluses, the approximate solution about a geometric baseline path, (19), simplifies to

$$\tilde{p}_t = \tilde{B}_{t-1} - \delta\phi\tilde{B}_t. \tag{30}$$

The first $\tilde{B}$ term in (30) means that an increase in debt at date $t-1$, $\tilde{B}_{t-1}$, that is repurchased at $t$ (so that $\tilde{B}_t$ does not also change) moves the price level $\tilde{p}_t$ one for one. With one-period debt this effect is simple: more debt as a claim to the same fixed resources must result in a higher price level. The solution shows that more long-term debt at time $t-1$ also raises the price level at time $t$, even though the debt does not come due until later. The price level rises when the debt is repurchased, not when it matures. Working through the definition of the debt aggregate $\tilde{B}_t$ in (20), if maturity $j$ debt $B_{t-1}(t+j)$ increases 1% and is then repurchased at time $t$, the price level rises by $(\delta\phi)^j$ percentage points. Thus, the effect of increased debt on the price level is attenuated for longer term debt and as the maturity structure shortens.

The second $\tilde{B}$ term in (30) means that an increase in debt at date $t$, $\tilde{B}_t$, can decrease the price level at time $t$, but *only* if some long-term debt is outstanding, i.e. if $\phi > 0$. If the government just rolls over short-term debt, this effect does not exist. New long-term debt dilutes outstanding long-term debt as a claim to fixed future resources. The more long-term debt is currently outstanding, the less the dilution, and hence the more revenue the government can raise for each dollar of extra long-term bond sales. In turn, the more real revenue raised, and used to redeem currently maturing bonds, the greater the impact on the price level.

Only the aggregates $\tilde{B}_t$ enter this approximate solution, so analysis using the approximate solution will not distinguish changes in the debt aggregate $\tilde{B}_{t-1}$ brought about by changes in debt of different maturities. The approximation values changes in debt at the steady state price level, as any first-order approximation must. Thus, analysis using this approximation will be silent about the effects of state-contingent maturity rearrangements. Study of such policy will require a second-order approximation or the exact solution, and will not allow us to use simple linear time-series tools.

In most cases the government does not sell long-term debt and then repurchase it one period later. Rather, it sells additional long-term debt and then lets it mature. To calculate the effects of such a policy, suppose that at time 0 the government sells an additional 10 year bond and then lets that bond mature. Normalizing to $\tilde{B}_9(10) = 1$, we have $\tilde{B}_8(10) = 1/\phi, \tilde{B}_7(10) = 1/\phi^2, \ldots, \tilde{B}_0(10) = 1/\phi^9$. (Since the approximation takes proportional deviations from steady states, a \$1 increase in the quantity outstanding is a larger proportional increase for longer maturity bonds.) Using the definition of $\tilde{B}_t$, (20) and (30), the resulting price path is

$$\tilde{p}_0 = -\phi\delta^{10},$$

$$\tilde{p}_t = (1-\phi)\delta^{10-t}, \quad t = 1, 2, \ldots 9,$$

$$\tilde{p}_{10} = 1.$$

Figure 3 plots this price path. At date 0, we only have the second, negative debt term in (19); the price level is reduced if there is long-term debt outstand-

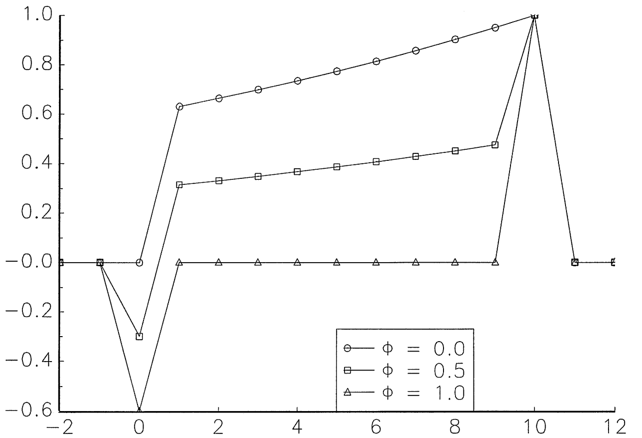


FIGURE 3.—Effect on the price level of an increase in 10 period debt at time 0 that is allowed to mature, starting in a steady state with a geometric maturity structure.

ing. At time 10, we only have the first, positive term in (19), so the price level rises by 1.0 for any maturity structure. One more bond must be redeemed from the same set of resources. In the intermediate dates, both terms in (19) are present. With long term debt, they cancel so there is no intermediate effect on the price level. With shorter-term debt, the price level increases all the way out to period 10.

The crucial question for the effects of a debt sale is the pattern of expected future sales and repurchases. For example, the price level path reported in Figure 3 requires only $\tilde{B}_9 = 1, \tilde{B}_8 = \delta, \tilde{B}_7 = \delta^2, \ldots, \tilde{B}_0 = \delta^9$. This pattern can be achieved just as well by selling an additional *one* period bond and then rolling over that debt 10 periods before repaying it. All that matters to the price path is when the debt is expected to be repaid.

The most important real-world debt operation is an open market operation. In this model, an open market operation is exactly the same thing as a debt sale or repurchase. For example, to repurchase a bond, the government issues additional just-maturing bonds, or equivalently, money. The comparative statics show that the effects of such an operation on the price level and hence nominal interest rates depend crucially on the maturity structure of outstanding debt, on simultaneous surplus movements (whether the government spends additional cash), and on expectations of when and how the debt will be retired—whether by raising future surpluses, or by competing with debt that would be retired on a given day. A wide variety of results is possible by different specifications of these components of the policy change.

Similarly, a revenue-neutral shortening or lengthening of the maturity structure of the debt, as practiced by the Kennedy administration and discussed in the early Clinton years (see Hall and Sargent (1997)) will have effects on the price level and nominal interest rates that depend crucially on the pattern of expected repayments. If the government simply raises debt of maturity $j$ and lowers that of maturity $k$, in a way that the aggregate $\tilde{B}_t$ is unaffected (revenue-neutral at baseline prices), and then restores this pattern every period as the debt matures (i.e., sells some $j-1$ maturity debt next period, buys some $j$ maturity debt, etc.) then there is no effect whatsoever. However, if the government lets the twist mature, then the price level will rise when the $j$ debt matures and decline when the $k$ debt matures; this expectation will show up in interest rates at the moment of the initial twist.

### 5.1. *Additional Effects with a Nongeometric Steady State*

The $D$ coefficients in (27) measure the effect on the price path of an expected bond sale at date $-1$, which will be repurchased at date 0,

$$D_k = \frac{1}{\delta^{j+k}\phi_j} \frac{\partial \tilde{p}_{-k}}{\partial \tilde{B}_{-1}(j)}.$$

Note that $D_{-1}, D_{-2\ldots} = 0$. Thus, despite the fact that long-term debt may be sold, there is no effect on prices past period 0 when the debt is repurchased.

$D_0 = 1$, so selling a little more debt at period $-1$ and then buying it back at period 0 raises the period 0 price level. Since $D_1 = -\phi_1 \le 0$, selling a little more debt at time $-1$ can lower time $-1$ prices, but only if there is some long term debt outstanding—if $\phi_1 \neq 0$. Interestingly, whether selling a little extra $j$ period debt affects prices immediately depends on the presence of outstanding time 1 debt ($\phi_1$), not time $j$ debt ($\phi_j$). The maturity of the debt *that is sold* does not matter; what matters is when that nominal debt will be repurchased, and compete with other debt for the fixed pool of resources.

In general, the terms $D_2, D_{3\ldots}$ are present, so prices at $t$ *can* be affected by all future expected debt changes. These terms all specialize to zero with the geometric steady state, in which case the price level at $t$ is only affected by $\tilde{B}_{t-1}$ and $\tilde{B}_t$. To see the force of this effect, we need an example in which the maturity structure is far from geometric. Suppose that the steady state maturity structure is $\phi_1 = 1$, $\phi_2 = \phi_3 = \cdots = 0.5$. The government combines some short-term debt with some extremely long term debt, for example a perpetuity. Figure 4 plots the response of prices to an anticipated debt sale at time 0, which is then repurchased at time 1, for this case. All the interesting dynamics before time 0 would be absent with a geometric steady state.

### 5.2. *Postponing Inflation—The Limits of Debt Policy*

As we have seen, additional sales of long-term debt can lower the price level today while raising it in the future, when some long-term debt is outstanding, even with no change in surpluses. To what extent can the government affect the price level today through unexpected bond sales? For example, can it completely offset surplus shocks?

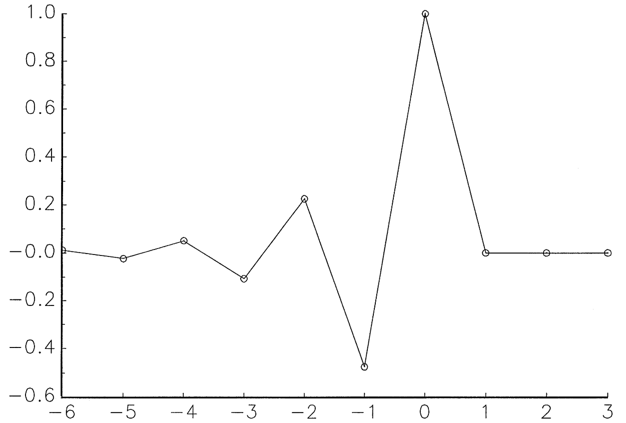


FIGURE 4.—Price path in response to an anticipated debt sale at time −1, which is then repurchased at time 0. The steady state maturity structure is $\phi_1 = 1$, $\phi_2 = \phi_3 = \cdots = 0.5$, and the discount factor is $\delta = 0.95$.

The present value condition (4) answers these questions directly and exactly. Rewriting the condition slightly,

$$\sum_{j=0}^{\infty} \beta^j E_t\left(\frac{1}{p_{t+j}}\right) B_{t-1}(t+j) = \sum_{j=0}^{\infty} \beta^j E_t(s_{t+j}). \tag{31}$$

We can read this equation as "budget constraint" for achievable expected inverse price levels. *The maturity structure of outstanding debt $B_{t-1}(t+j)$ gives the rates at which the government can trade off the price level today for (expected inverse) price levels in the future.*

The government can always raise future prices by selling more debt; the issue is whether such sales affect today's prices. With outstanding long-maturity debt, terms $B_{t-1}(t+j) > 0$, $j \geq 1$ in (31) are present, so that raising future price levels (by selling more long-term debt) can lower today's price level. If only one-period bonds are outstanding, these terms are absent so there is nothing the government can do with debt policy to affect prices today.

Furthermore, *there is a debt policy—a choice of $\{B_t(t+i), B_{t+1}(t+i)\ldots; i = 1,2,\ldots\infty\}$ that achieves any set of (expected inverse) price paths consistent with the constraint* (31). To verify this fact, we can construct a policy that works for a given price path. It is not unique. Let the government adjust its maturity structure once, determining $B_t(t+j)$, and then let the debt mature with no further purchases or sales. Future price levels are then given by the solution (8), and taking expectations at time $t$,

$$E_t\left(\frac{s_{t+j}}{B_{t+j-1}(t+j)}\right) = E_t\left(\frac{1}{p_{t+j}}\right).$$

Therefore, if the government sets

$$B_t(t+j) = \frac{E_t(s_{t+j})}{E_t\left(\frac{1}{p_{t+j}}\right)}$$

and lets debt mature so that $B_t(t+j) = B_{t+j-1}(t+j)$, the desired path of future price levels $\{E_t(1/p_{t+j})\}$ results. Equation (31) produces the price level at time $t$.

The converse statement is also true. *If there is no j period debt outstanding at time t, then there is no debt policy—no choice of* $(B_{t+1}(t+i), B_{t+2}(t+i)\ldots; i = 1,2,\ldots\infty\}$*—by which the government can lower the price level at time t in exchange for raising the price level at time* $t+j$.

Can the government go so far as to attain a *constant* price level in the face of surplus shocks by appropriately buying and selling bonds? The constraint (31) shows that this much is not possible, because debt at time $t$ must be in the time $t$ information set. Take innovations of equation (31), resulting in

$$\sum_{j=0}^{\infty} \beta^j (E_t - E_{t-1})(s_{t+j}) = \sum_{j=0}^{\infty} \beta^j B_{t-1}(t+j)(E_t - E_{t-1})\left(\frac{1}{p_{t+j}}\right).$$

A constant price level implies $(E_t - E_{t-1})(1/p_{t+j}) = 0$ for all $j$. The right side is zero and the left side is not, so this cannot be a solution. This conclusion holds in continuous time versions of the model as well.

With one period debt, we had

$$\frac{B_{t-1}(t)}{p_t} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}.$$

Since debt was predetermined, the price level had to absorb any shocks to the present value of future surpluses. Now we have equation (31). Debt of each maturity is still predetermined, so revisions in the (expected inverse) price level *sequence* must absorb any surplus shocks.

The government could attain a constant price level via debt policy alone if it issued *state-contingent* nominal debt. For example, suppose that the government issued state-contingent debt at time 0 and engaged in no further debt sales or repurchases. Let $B(\sigma^t)$ denote the amount of nominal debt that comes due at date $t$ in state $\sigma^t$. Similarly, let $s(\sigma^t)$ denote the real surplus at time $t$ in state $\sigma^t$. The budget identity at each date is then simply

$$p(\sigma^t)s(\sigma^t) = B(\sigma^t).$$

In this case, the government can attain any stochastic process for prices, including a constant price level, by choosing the appropriate state-contingent debt structure. Though dynamic trading of long-term debt allows a greater array of state-contingencies than does short term debt, it does not attain this complete-markets or state-contingent limit. In this paper, I focus on non-state-contingent nominal debt because that is the nearly universal structure of nominal government debt.

## 6. OPTIMAL DEBT POLICY

We have seen that debt policy can affect the price level. Now, I search for policies that *optimally* smooth inflation. I proceed in three stages: First, I find an optimal *fixed-debt* policy, i.e. an optimal steady state maturity structure, given that the government does not adjust debt ex-post in response to shocks. Then, I allow the government also to pursue *active* debt policy, adjusting the level of debt of various maturities in order to offset surplus shocks. Finally, I allow the government to control part of the surplus as well.

We can anticipate some of the qualitative results. As we have seen, with fixed debt, shorter maturity structures relate today's price to many leads of the surplus, while long maturity structures relate today's price to fewer leads of the surplus. Therefore, a short maturity structure smooths inflation if surpluses have a large transitory component, while a long maturity structure will smooth inflation when surpluses build following a shock. Long maturity structures also make active debt policy possible, so that the government can smooth a surplus shock as it happens by selling more long-term debt. This fact weighs in favor of a long maturity structure, even when short-term debt is the optimal fixed-debt policy.

### 6.1. *Statement of the Problem*

Given a stochastic process for the surplus $\{s_t\}$, the government picks the parameters governing the steady state maturity structure $\phi$ and a debt policy $\{\tilde{B}_t(t+j)\}$ to minimize the variance of inflation,

$$\min[\text{var}(\tilde{p}_t - \tilde{p}_{t-1})], \tag{32}$$

given that prices are generated by the approximate solution (19). I state the objective and constraints in terms of steady states and deviations about the steady state, since I use the approximate price solution to solve the problems. In order to use the approximate solution, I constrain the government's choice to a geometric steady state. A natural constraint set for the steady state maturity structure is $0 < \phi \leq 1$. However, as discussed above, solutions with $1 < \phi < \delta^{-1}$ are possible though unusual given today's institutions. Thus, when the objectives point to high values of $\phi$, I will study solutions limited by $\phi \leq 1$ as well as solutions limited only by $\phi < 1/\delta$. Debt $\tilde{B}_t(t+j)$ must be in the time-$t$ information set and must obey $\lim_{T\to\infty}(\delta\phi)^T \tilde{B}_T = 0$.

Smoothing the volatility of inflation is a reasonable characterization of postwar central bank objectives. In this model, the level of inflation is arbitrary and so it is not interesting to add it to the objective. Modeling "inflation" as the difference of proportional deviations from the steady state as in (32) rather than the ratio of price levels is an analytically convenient simplification. I also consider the objective of minimizing variance of the price level, $\min \text{var}(\tilde{p}_t)$, which is a plausible characterization of monetary policy objectives in the prewar, gold-standard regime. The methods adapt easily to other objectives. For example, one can minimize the variance of unexpected inflation $\min \operatorname{var}(\tilde{p}_t - E_{t-1}\tilde{p}_t)$, motivated by the Lucas (1972, 1973) world in which only unexpected money has real effects.

Following a long tradition in monetary economics, for example Sargent and Wallace (1975), I do not delay or complicate the analysis by justifying the price-smoothing objective from welfare maximization in an economy with specific frictions.

### 6.2. *Fixed-debt Policy*

I start by analyzing *fixed-debt* policies: The government chooses only a geometric steady state maturity structure, governed by the parameter $\phi$, in order to minimize the variance of inflation given that prices are generated by the approximate solution (19). I calculate results for an AR(2) surplus process,

$$\tilde{s}_t = (\lambda_1 + \lambda_2)\tilde{s}_{t-1} - (\lambda_1 \lambda_2)\tilde{s}_{t-2} + \varepsilon_t.$$

Figure 5 presents the optimal steady state maturity parameter $\phi$ as a function of the two roots $\lambda_1$ and $\lambda_2$. The calculation is detailed in the Appendix. For every stationary AR(1) (one root equal to zero, the other strictly less than one; this region is not shown in Figure 5 for clarity) the optimal maturity is short, $\phi = 0$. In these cases the variance of the present value of the surplus is smaller than the variance of the surplus, so short-term debt smooths inflation by making the price level equal to the smoother series. For the same reason, $\phi = 0$ is optimal for two relatively small AR(2) roots, as can be seen in the lower left-hand corner of Figure 5.

Two large positive roots $\lambda$ produce hump-shaped impulse response functions that continue to rise after an initial shock, and for which the present value varies by more than the series itself. In this case, the longest possible maturity

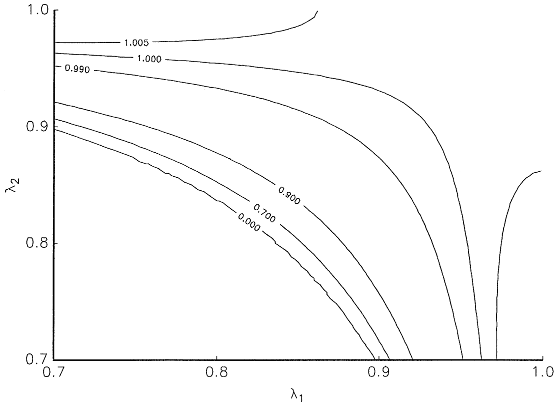


FIGURE 5.—Optimal geometric maturity $\phi$ of passive debt policies that minimizes the variance of inflation, as a function of the two roots of the AR(2) surplus process. $\delta = 0.95$.

debt $\phi = 1/\delta$ minimizes the variance of the *price level*. Long maturities are also useful in this case to minimize the variance of *inflation*, but as Figure 5 shows, the optimal maturity is interior $0 < \phi < 1/\delta$, and interestingly is never much above $\phi = 1$. This case is not implausible, as many macroeconomic time series have hump-shaped impulse-response patterns with roots roughly those of this region.

### 6.3. *Active Debt Policy*

Next, I allow the government to adjust debt of all maturities, still keeping the surplus process exogenous. As we have seen, this option gives another motivation for long-term debt, since state-contingent debt sales can postpone a shock to the price level if long-term debt is outstanding.

The problem now is to minimize $\text{var}(\tilde{p}_t)$ or $\text{var}(\tilde{p}_t - \tilde{p}_{t-1})$ by choice of $\phi$ and $\tilde{B}_t(t+j)$ at each date, given the surplus process, which I denote

$$\tilde{s}_t = \sum_{j=0}^{\infty} \eta_j \varepsilon_{t-j} = \eta(L)\varepsilon_t.$$

I solve this problem[4] by first finding the optimal price process, for a given steady state maturity structure $\phi$. Write the price process as a function of surplus shocks as

$$\tilde{p}_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} = \psi(L)\varepsilon_t.$$

I choose the coefficients $\psi_j$, subject to a constraint that the price process must be achievable by some debt policy (choice of $\{\tilde{B}_{t-1}\}$). We can express that constraint conveniently as follows. Write the linearized version of the present value condition (25) as

$$E_t \sum_{j=0}^{\infty} (\delta\phi)^j \tilde{p}_{t+j} = \tilde{B}_{t-1} - \frac{1-\delta}{1-\delta\phi} E_t \sum_{j=0}^{\infty} \delta^j \tilde{s}_{t+j}. \tag{33}$$

Since $\tilde{B}_{t-1}$ is in the $t-1$ information set, taking innovations of (33) yields a relation between the responses of price and surplus to shocks that does not involve debt,

$$(1-\delta\phi)\psi(\delta\phi) = -(1-\delta)\eta(\delta). \tag{34}$$

Thus, I choose the weights $\{\psi_j\}$ to minimize the variance of the price level or inflation rate subject to the constraint (34). This operation is enough to fully characterize the optimal price process for given $\phi$. Then, taking the variance of price level or inflation, I find the optimal maturity structure $\phi$. Finally, I solve (33) for debt $\tilde{B}_{t-1}$ to characterize the debt policy that supports the optimal price process.

[4] I thank Mike Woodford for suggesting this solution strategy.

*Minimize the Variance of the Price Level*

The objective is

$$\min \operatorname{var}(\tilde{p}_t) = \min_{\{\psi_j\}} \sum_{j=0}^{\infty} \psi_j^2 \sigma_\varepsilon^2$$

subject to (34). A straightforward Lagrangian minimization gives the optimum price level process,

$$\tilde{p}_t = -\frac{\left(1-(\delta\phi)^2\right)(1-\delta)\eta(\delta)}{(1-\delta\phi)} \frac{1}{(1-\delta\phi L)} \varepsilon_t,$$

with variance

$$\sigma^2(\tilde{p}_t) = -\frac{(1-\delta)^2 \eta(\delta)^2}{(1-\delta\phi)^2} \sigma^2(\varepsilon).$$

The minimum variance occurs with $\phi = 0$, and the resulting optimal price level process is

$$\tilde{p}_t = -(1-\delta)\eta(\delta)\varepsilon_t. \tag{35}$$

*The minimal-variance price level follows an i.i.d. process*. Interestingly, this is true *for any surplus process*. Price variance is greater, the greater the response of the present value of the surplus to its shocks, measured by $\eta(\delta)$.

Solving (33) for $\tilde{B}_{t-1}$, the debt policy supporting the optimal price process is[5]

$$\tilde{B}_{t-1} = (1-\delta)\left[E_{t-1} \sum_{j=0}^{\infty} \delta^j \tilde{s}_{t+j}\right].$$

The debt process offsets all the time $t$ present value of the surplus that is known as of $t-1$. The price level then absorbs the *shock* to the present value of surpluses only. With $\phi = 0$ debt policy can only affect the expected price level but cannot offset shocks as they come. Since $\operatorname{var}(\tilde{p}_t) = \operatorname{var}(E_{t-1}(\tilde{p}_t)) + \operatorname{var}(\tilde{p}_t - E_{t-1}\tilde{p}_t)$, debt policy adjusts to set $\operatorname{var}(E_{t-1}(\tilde{p}_t)) = 0$, by making the price level an i.i.d. process.

[5]Solving (33), for $\tilde{B}_{t-1}$, using $\phi = 0$ and substituting (35), we have

$$\begin{aligned}\tilde{B}_{t-1} &= -(1-\delta)\eta(\delta)\varepsilon_t + (1-\delta)\left(E_{t-1}\sum_{j=0}^{\infty}\delta^j\tilde{s}_{t+j} + (E_t - E_{t-1})\sum_{j=0}^{\infty}\delta^j\tilde{s}_{t+j}\right)\\ &= -(1-\delta)\eta(\delta)\varepsilon_t + (1-\delta)\left(E_{t-1}\sum_{j=0}^{\infty}\delta^j\tilde{s}_{t+j} + \eta(\delta)\varepsilon_t\right)\end{aligned}$$

and hence the result.

*Minimize Variance of Inflation*

The algebra is a bit more complex in this case, so I present it in the Appendix. The objective is

$$\min_{\{\psi_j\}} [\text{var}(\tilde{p}_t - \tilde{p}_{t-1})]$$

subject to (34). For a given $\phi$, the minimum-variance inflation process is

$$(36) \qquad (1-L)(1-\delta\phi L)\tilde{p}_t = -\left(1-(\delta\phi)^2\right)(1-\delta)\eta(\delta)\varepsilon_t.$$

Now, *for any surplus process, inflation follows an AR*(1). Notice that, in order to minimize the variance of *inflation*, the price *level* becomes nonstationary. The active debt policy fundamentally transforms the price level process. With a fixed-debt policy, the price level would be stationary, following stationary fluctuations in the present value of surpluses. By making the price *level* nonstationary, *inflation* can be smoothed.

The minimum variance of inflation for given $\phi$ is

$$(37) \qquad \text{var}[(1-L)\tilde{p}_t] = \left(1-(\delta\phi)^2\right)(1-\delta)^2\eta(\delta)^2\sigma_\varepsilon^2.$$

This function declines monotonically in $\phi$. Therefore, *long-term debt lowers the variance of inflation*, for any surplus process. The advantages of active debt policy are important. For example, we found that short-term debt minimized the variance of inflation with fixed-debt policies and an AR(1) surplus. Equation (37) shows that we get exactly the opposite conclusion with active debt policy. Long-term debt makes active debt policy possible, and the ability to offset shocks as they come by diluting and devaluing outstanding long-term debt dominates the fixed-debt inflation-smoothing properties of a short maturity structure.

The solution (35) that minimized the variance of the price *level* gives much more volatile *inflation* than the solution (36) that minimizes the variance of *inflation*. Evaluating the variance of inflation from the minimized price level variance solution (35), it is

$$\text{var}(1-L)\tilde{p}_t = (1-\delta)^2\eta(\delta)^2\sigma^2(\varepsilon).$$

The minimized variance of inflation given by (37) is lower by a factor $(1-(\delta\phi)^2)$. At $\phi = 1$, $\delta = 0.95$, for example, this means that the variance of inflation is only about 10 percent of what it would be under a policy that minimized the variance of the price level. On the other hand, the solution (36) that minimizes the variance of inflation gives a unit root and hence an *infinite* variance of the price level.

The contrast between (35) and (36) thus conforms broadly with experience: under a gold standard, the price *level* was stationary, and inflation was quite volatile. Now, the variance of inflation is much lower, but the price *level* wanders slowly and seems to have no long-run mean. Thus, the shift in the character of U.S. inflation from the prewar to the postwar period can be understood as a shift from a price-level targeting objective to an inflation smoothing objective subject to the constraints imposed by the fiscal theory of the price level.

*An AR*(1) *Example*

To give a better sense of the optimal policies, I report calculations based on an $AR(1)$ surplus

$$\tilde{s}_t = \rho \tilde{s}_{t-1} + \varepsilon_t.$$

With $\phi = 1$, the price level and debt policy that minimize inflation then simplify to

$$(38) \qquad (1-L)(1-\delta L)\tilde{p}_t = -\frac{(1-\delta^2)(1-\delta)}{(1-\delta\rho)}\varepsilon_t,$$

$$(39) \qquad (1-L)(1-\delta L)\tilde{B}_{t-1} = -\left(\frac{(1-\rho)+\delta(1-\delta)}{(1-\delta\rho)} - \delta L\right)\tilde{s}_{t-1}.$$

(I derive the debt policy in the Appendix.)

Debt $\tilde{B}_t$ depends on the whole history of surpluses despite the AR(1) surplus structure. In order to produce a unit root in the price level, nominal debt policy $\tilde{B}_{t-1}$ also has a unit root. The sign of the first term on the right-hand side of (39) is positive, so the government sells additional debt when there is a negative surplus shock. This action lowers the price level at the moment of the shock, but raises the price level in the future. The result is a smoother path of *inflation* at the cost of a more volatile—a unit root in fact—price *level*. Since the approximate solution values changes in debt by the steady state bond prices, the solution does not prescribe which maturities should be changed in the active debt policy.

Figure 6 presents artificial time series for debt growth, surplus and price level for this model. As in the data, but in contrast to an AR(1) surplus, fixed-debt model, there is no visible correlation between debt or the surplus and the price level and little correlation with inflation (not shown for clarity). As in the data, nominal debt growth is negatively correlated with the surplus. However, the surplus is still positively correlated with the *real* value of the debt in this model, as it must be in any AR(1) surplus model. To match the fact in the data that both real and nominal debt growth are negatively correlated with the surplus, I consider policy that affects the surplus below.

To emphasize how important active debt policy is to this case, Figure 7 contrasts inflation from the optimal active policy with (i) the inflation that results from a fixed-debt policy with long term debt ($\phi = 1$), and (ii) the inflation that results from the optimal fixed-debt policy, which uses short term debt ($\phi = 0$). With either fixed-debt policy, the price level is perfectly positively

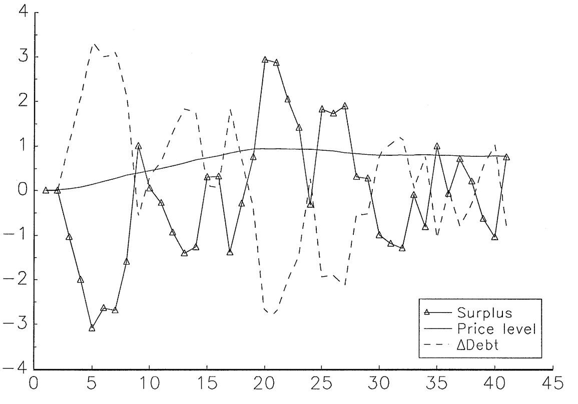


FIGURE 6.—Artificial data from optimal active debt policy with an AR(1) surplus. Parameters are $\rho = 0.6$, $\delta = 0.95$, $\phi = 1$.

correlated with the surplus, and so inflation is perfectly correlated with surplus growth. We see that active policy dramatically smooths inflation relative to the long-maturity fixed-debt policy, and also smooths inflation more than the optimal, short-maturity, fixed-debt policy.

*The Limit* $\phi \to 1/\delta$

The variance of inflation in (37) continues to decline in $\phi$ all the way to the limit $\phi = 1/\delta$. Thus, as in Woodford (1998b), we find a motive for this technically possible but unusual maturity structure. As $\phi \to 1/\delta$, the market value of debt approaches a constant at all maturities, and hence the market value of debt at any maturity approaches zero. As a result, proportional deviations from this steady state explode to infinity. Specifically, the limit of the debt policy (derived

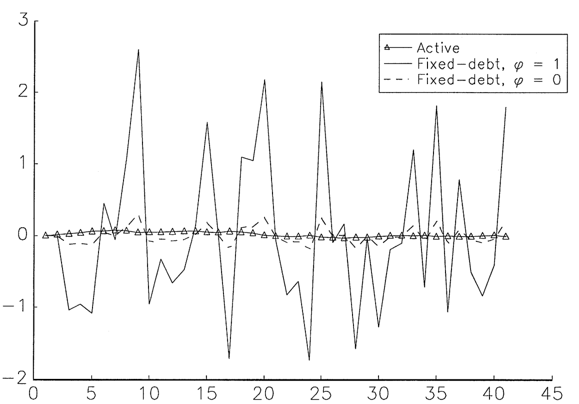


FIGURE 7.—Artificial inflation data from three debt policies with an AR(1) surplus. Parameters are $\rho = 0.6$, $\delta = 0.95$.

in the Appendix) as $\phi \to 1/\delta$ is

$$\lim_{\delta\phi \to 1} (1 - \delta\phi)(1 - L)\tilde{B}_{t-1} = -\frac{(1-\delta)(1-\rho)}{(1-\delta\rho)}\tilde{s}_{t-1}.$$

Finally, note that although the limit $\phi \to 1/\delta$ produces a zero variance of inflation, it does not produce a zero variance of the price level. The price level has a unit root, and thus infinite variance all the way to the limit. As above, $\phi = 0$ minimizes the variance of the price level.

## 7. OPTIMAL SURPLUS AND DEBT POLICY

Last, I add a limited control over the surplus. Governments have at least some control over the surplus as well as nominal debt, and a realistic policy optimization exercise should recognize this fact. Most importantly, the vast majority of debt sales come together with an implicit or explicit promise to increase future surpluses.

A second and related issue is that the AR(1) or AR(2) surplus processes investigated above, though they are natural examples and plausible descriptions of the *univariate* behavior of the U.S. real primary surplus, lead to a completely counterfactual description of the *joint* behavior of surplus and real debt. Simple AR surplus processes imply that the real value of the debt should be positively correlated with surpluses. In the data, as shown in Figure 2, high surpluses are associated with *declining* real debt. Canzoneri, Cumby, and Diba (1998) use these counterfactual predictions to reject the fiscal theory with an AR(1) surplus process.

To make this point precisely, denote the real value of the debt

$$v_t \equiv \frac{1}{p_t}\sum_{j=0}^{\infty} Q_t(t+j)B_{t-1}(t+j).$$

The present value condition (4) says that the real value of the debt—of any maturity structure—is equal to the present value of real surpluses:

$$v_t = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}. \tag{40}$$

With an AR(1) surplus, $s_t = \rho s_{t-1} + \varepsilon_t$, the surplus and real value of debt are perfectly positively correlated:

$$v_t = \frac{1}{1-\beta\rho}s_t. \tag{41}$$

This result holds for *any* debt policy, including the active debt policy analyzed above. More generally, any time series process in which the present value on the right-hand side of (40) moves positively with the series itself predicts that surpluses should be positively correlated with debt.

Therefore, to describe plausibly the joint behavior of the surplus and real debt—the fact that real debt declines when surpluses are high—the surplus must follow a process whose level is negatively correlated with its long-run and present values. This statement has nothing to do with the fiscal theory, since equation (40) is entirely in real terms and holds in all models, fiscal or not.

On first glance processes with negative long-run responses seem strange. On second glance they suggest that surpluses respond to real debt values, the "Ricardian regime" or "passive" special case that invalidates the fiscal theory. But on third glance such processes are a natural outcome of a debt policy run to smooth inflation in the face of transitory surplus shocks.

In a recession, the government must finance a deficit—a negative shock to $s_t$. It can do one of three things:

1. It can inflate away existing debt. For example, with one-period debt we have

$$\frac{B_{t-1}(t)}{p_t} + E_t\left(\frac{1}{p_{t+1}}\right)\beta B_t(t+1) = s_t.$$

If the government does not change nominal debt $B_t(t+1)$ and future surpluses $s_{t+j}$, a negative $s_t$ shock will be met by a rise in $p_t$, i.e. by inflating away the real value of outstanding debt.

2. As discussed above, if long-term debt is outstanding, the government can sell additional long-term debt with no change in future surpluses; this action devalues outstanding long-term debt, causing future rather than current inflation.

3. The government can sell additional debt, *while promising to increase future surpluses*. For example, with one-period debt, an increase in debt sales $B_t(t+1)$ while holding future surpluses $s_{t+1}$ constant results in an equiproportionate increase in the future price level $p_{t+1}$ and hence does not raise any revenue or affect prices at time $t$. But if the government can promise to raise future surpluses, then it can sell more debt $B_t(t+1)$ with no effect on $p_{t+1}$; hence it can raise more revenue without inflating away existing debt. In this last example a negative surplus shock today is followed by an increased surplus in the future.

The first two options lead to large swings in inflation. The third strategy leads to much less volatile inflation. Hence, we expect a government that wishes to smooth inflation to follow something like the third strategy. And in fact we routinely think of governments offsetting current fiscal stringency by borrowing, and implicitly or explicitly promising to raise future taxes or cut future spending to pay off the resulting debt. If they did not do so, the total real value of the debt would not rise when governments issue extra nominal debt. Thus, we routinely think of surplus processes, which, under partial government control, have response functions that reverse sign after a shock.

The first two options also lead to real values of the debt that are positively correlated with the surplus. The fact that high surpluses seem to pay down the real value of the debt is not an accounting identity; it results from the government's *choice* to do so rather than to finance deficits by inflating away the value of outstanding debt.

### 7.1. *A Model of Optimal Fiscal Policy*

Here, I pursue a model that captures the intuition of the last few paragraphs. First, we must describe the surplus process. There is a cyclical component to the surplus that is by and large beyond the government's control. In a recession, lower income means less tax revenue, and entitlement and other program-based spending automatically rises. Denote this cyclical portion of the surplus

$$c_t = \rho c_{t-1} + \varepsilon_t. \tag{42}$$

The government does control a long-term component of the surplus. By changing tax rates and the terms of government programs, it alters the overall level of the surplus. For good optimal-taxation reasons it does not change tax rates and spending policies to offset the transitory, cyclically-induced component of the surplus, for example raising tax rates in recessions and lowering them in booms. Let the controllable component of the surplus follow a random walk,

$$z_t = z_{t-1} + \zeta_t. \tag{43}$$

The actual surplus is the sum of the two components,

$$\tilde{s}_t = c_t + z_t.$$

(The random walk is a convenient simplification. The model works in much the same way if $z_t$ follows any process $z_t = \eta z_{t-1} + \zeta_t$ that is more persistent than $c_t$, $\eta \gg \rho$ so that $z_t$ controls the long-run surplus.)

Next, we must state the government's problem. The government picks the change in the controllable component of the surplus $\zeta_t$ at each date. $\zeta_t$ must be in the time-$t$ information set, and it must not be predictable from time $t-1$ information. The government also picks nominal debt $\tilde{B}_t$ in the time $t$ information set, and the steady state maturity structure $\phi$. The government picks $\phi, \{\zeta_t\}, \{\tilde{B}_t\}$ to minimize the variance of inflation, given that the price level is determined by (19), which specializes given this surplus structure to

$$(1-L)\tilde{p}_t = -\frac{(1-\delta)(1-\delta\phi\rho)}{(1-\delta\rho)(1-\delta\phi)}(1-L)c_t - \zeta_t + (1-L)(L-\delta\phi)\tilde{B}_t. \tag{44}$$

Now we can study solutions to this problem. There are policies that set the variance of inflation to zero. The government may choose $\phi$ arbitrarily (the optimal policy is not unique) and then chooses debt and the long-run component of the surplus according to

$$(1-L)\tilde{B}_t = -\frac{1-\delta}{1-\delta\phi}\frac{1-\rho}{1-\delta\rho}c_t, \tag{45}$$

$$\zeta_t = -\frac{1-\delta}{1-\delta\rho}\varepsilon_t. \tag{46}$$

To check this solution, plug these choices into (44) and verify that each power of $L$ on the right-hand side is equal to zero.

### 7.2. *Character of the Solution*

I compare the time-series process predicted by the inflation-minimization problem with actual time series in two ways, by comparing graphs of artificial with real data, and by comparing the predicted time-series processes with estimates of actual time-series processes.

Analytically, we can see from (46) that shocks to the long-run surplus are negatively correlated with shocks to the transitory component of the surplus. As expected, the government meets a short-run negative surplus shock by raising surpluses in the long-run.

#### *A Graph of Artificial Data*

Figures 8 and 9 plot simulated time series from the optimal policy system. The parameters are $\rho = 0.6$ and $\delta = 0.95$. The pictures are identical for any value of $\phi \in [0, 1/\delta)$. The random number draw is the same across the two pictures.

In Figure 8 we see how the surplus is generated from its permanent and transitory components. There are periodic recessions, in which the transitory component of the surplus declines, and booms in which it rises. The government slightly raises the permanent component of the surplus in the recessions and lowers it in the booms. This change has little effect on the short-run properties of the surplus, since the actual surplus tracks the transitory component closely. But it has a dramatic effect on the long-run or present value properties of the surplus. The long-run surplus *rises* in recessions so the government can raise revenue by selling debt, and it *falls* in booms as the government pays off debt.

Figure 9 presents the joint properties of the total surplus, debt and debt growth. Comparing Figure 9 to actual data in Figure 2 we notice the similarity.

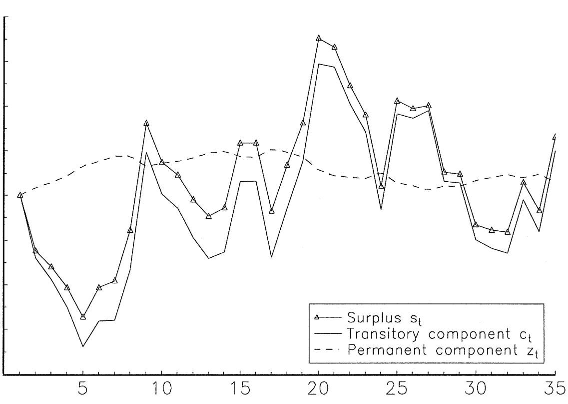


FIGURE 8.—Simulated surplus, and its permanent and transitory components.

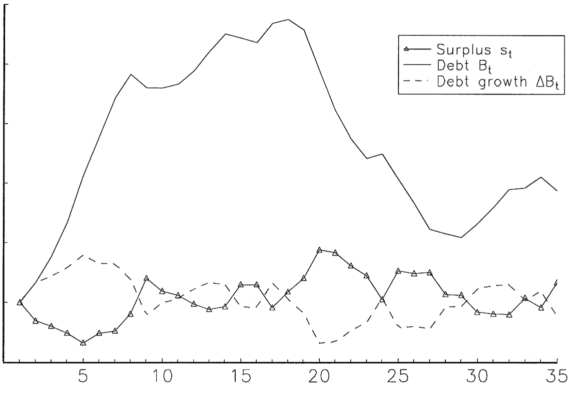


FIGURE 9.—Simulated surplus, debt and debt growth.

Debt is not well correlated with the surplus, and it wanders at much lower frequency than the surplus; *growth* in debt is nicely negatively correlated with the surplus. The simple model thus accounts for the initially puzzling time-series behavior of debt and surplus, and shows why despite a simple AR(1) input, the result is far from the perfect positive correlation of debt and surplus that a pure AR(1) surplus process predicts.

(Since the quantities $\tilde{B}, \tilde{s}$ denote proportional deviations from steady state, Figure 9 presents

$$\hat{B}_t \equiv \frac{1-\delta\phi}{1-\delta}\tilde{B}_t.$$

This transformation converts the debt series to the same units—real and relative to the surplus steady state—as the surplus series. This transformation also completely removes $\phi$ from the time-series properties of $\hat{B}_t, \tilde{s}_t$ in this example. Since the price level is constant, there is no distinction between real and nominal debt in the simulated data.)

*Time Series Processes*

For a slightly more formal comparison of model and data, we can compare the time-series process of debt and surplus predicted by the simple model to those we can estimate in the data. Debt and surplus in the model follow the joint time series process

$$(1-\rho L)(1-L)\hat{B}_t = -\frac{(1-\rho)}{(1-\delta\rho)}\varepsilon_t, \tag{47}$$

$$(1-\rho L)(1-L)\tilde{s}_t = \frac{(1-\rho)}{(1-\delta\rho)}(\delta - L)\varepsilon_t, \tag{48}$$

and hence the two series are related by

$$\tilde{s}_t = -(\delta - L)\hat{B}_t. \tag{49}$$

These relations hold for any value of the steady state maturity structure $\phi$.

Of course, inflation is not constant in actual data, and there is no linear function linking debt and surplus with no error term. Therefore, a formal test of (47)–(49) rejects the model. Nonetheless, we can see to what extent this model captures features of the data, as the above graphs suggest it does.

*Debt Process*

Table I presents regression estimates of the total debt process (47), using data described in the Introduction. (Since the model has no growth and no inflation, the table runs the regression using the ratio of total real Federal debt to consumption.) The table verifies that an AR(2) with one root near unity and one root around 0.5 is an excellent fit to this process.

*Debt-surplus Relation*

Equation (49) is consistent with the finding in the data that the surplus is strongly negatively correlated with *changes* in the total value of the debt, and given the debt process (47), poorly correlated with the level of the total value of the debt. To quantify this relation, Table II presents a regression of surplus on debt.

The relative values of the coefficients on current and lagged debt conform to the prediction of (49). The absolute values are about a half too small. There is of

TABLE I

DEBT AUTOREGRESSIONS

| | $\hat{B}_{t-1}$ | $\hat{B}_{t-2}$ | $\hat{B}_{t-3}$ | $\bar{R}^2$ | DW |
|---|---|---|---|---|---|
| $\hat{B}_t =$ | 1.42 | −0.49 | | 0.93 | 2.16 |
| std. error | (0.16) | (0.16) | | | |
| $\hat{B}_t =$ | 1.31 | −0.16 | −0.22 | 0.93 | 1.94 |
| std. error | (0.18) | (0.31) | (0.18) | | |
| | $\Delta\hat{B}_{t-1}$ | $\Delta\hat{B}_{t-2}$ | $\bar{R}^2$ | DW | |
| $\Delta\hat{B}_t =$ | 0.45 | | 0.18 | 2.07 | |
| std. error | (0.16) | | | | |
| $\Delta\hat{B}_t =$ | 0.36 | 0.17 | 0.17 | 1.91 | |
| std. error | (0.18) | (0.19) | | | |

*Note*: $\hat{B}_t$ is the total real market value of Federal debt divided by nondurable plus services consumption. Sample 1960–1996. Regressions include a constant.

TABLE II

REGRESSION OF SURPLUS ON DEBT

| | $\hat{B}_t$ | $\hat{B}_{t-1}$ | $\bar{R}^2$ | DW |
|---|---|---|---|---|
| $s_t =$ | $-0.44$ | 0.48 | 0.66 | 2.12 |
| std. error | (0.06) | (0.06) | | |

*Note*: $s_t$ denotes the Federal primary surplus divided by nondurable plus services consumption. $\hat{B}_t$ is the total real market value of Federal debt divided by nondurable plus services consumption. Sample 1960–1996. The regression includes a constant.

course no error in (49), while there is an error in the actual data. The data for Table II obey the identity

$$s_t = r_t \hat{B}_{t-1} - \hat{B}_t,$$

where $r_t$ is the gross ex-post real return on the government bond portfolio less the consumption growth rate. Therefore, the error term in the regression is largely the real return on government bonds. That return was low in the first half of the sample, when the surplus and right-hand side of (49) was high, and high in the latter part of the sample when the surplus and right-hand side of (49) was low. There is a decade-long movement in the error term, correlated with the right-hand variable. This fact lowers both coefficients but does not affect their relative values.

*Surplus Process*

The surplus/consumption ratio is well-modeled as an AR(1), or at most an AR(2). Table III presents autoregressions. The autocorrelation function also has a classic AR(1) shape, with at most a small secondary hump with $t$ statistics around 1.5.

To digest this estimate, we need the univariate surplus process predicted by the model. Equation (48) represents the evolution of $s_t$ from shocks to the

TABLE III

SURPLUS AUTOREGRESSIONS

| | $s_{t-1}$ | $s_{t-2}$ | $\bar{R}^2$ | DW |
|---|---|---|---|---|
| $s_t =$ | 0.56 | | 0.30 | 1.62 |
| std. error | (0.14) | | | |
| $s_t =$ | 0.72 | $-0.23$ | 0.34 | 1.99 |
| std. error | (0.17) | (0.17) | | |

*Note*: $s_t$ denotes the Federal primary surplus divided by nondurable plus services consumption. Sample 1960–1996. Regressions include a constant.

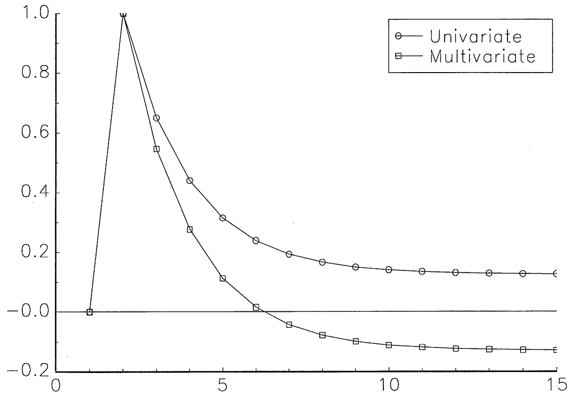


FIGURE 10.—Response of the surplus to univariate Wold representation shocks $\eta_t$ and to fundamental, multivariate shocks $\varepsilon_t$. Parameters are $\delta = 0.965$, $\rho = 0.6$.

bivariate $\{s_t, B_t\}$ system. However, since $\delta < 1$ the moving average term is not invertible. Hence, equation (48) is not the univariate Wold representation as is recovered by autoregressions or univariate ARMA estimation. The univariate Wold representation predicted by the model is[6]

$$(50) \qquad (1-L)\tilde{s}_t = \left(\frac{1-\delta L}{1-\rho L}\right)\eta_t; \quad \eta_t = \tilde{s}_t - \text{Proj}\big(\tilde{s}_t | \tilde{s}_{t-1}, \tilde{s}_{t-2} \ldots\big).$$

Figure 10 contrasts the univariate (response to $\eta$) and multivariate (response to $\varepsilon$) response functions predicted by the model. The univariate response function is very close to an AR(1): I use $\delta = 0.95$ so the unit root on the left-hand side nearly cancels the moving average root on the right-hand side, leaving only the autoregressive root $(1-\rho L)$. At long horizons, the univariate response function stops decaying at a positive value $(1-\delta)/(1-\rho) = 0.125$ so it is in fact even more persistent than an AR(1). A researcher examining the univariate properties of $s_t$ from this model would undoubtedly stop at an AR(1); most diagnostics are not capable of noticing the long-run divergence from an AR(1) implied by the near-canceling of roots. Thus, the univariate surplus process is broadly consistent with the data.

[6]To find the univariate representation, write the spectral density of (48)

$$S_{(1-L)\tilde{s}_t}(z) = \left(\frac{1-\rho}{1-\delta\rho}\right)^2 \delta^2 \frac{(1-\frac{1}{\delta}z)(1-\frac{1}{\delta}z^{-1})}{(1-\rho z)(1-\rho z^{-1})}\sigma_\varepsilon^2$$

$$= \left(\frac{1-\rho}{1-\delta\rho}\right)^2 \frac{(1-\delta z)(1-\delta z^{-1})}{(1-\rho z)(1-\rho z^{-1})}\sigma_\varepsilon^2.$$

$\delta < 1$ and $\rho < 1$, so this corresponds to the Wold representation.

*A Subtle Trap for Empiricists*

Figure 10 reminds us of a subtle trap for empiricists. What could be more natural in evaluating the fiscal theory than to fit a surplus process, take its expected present value, and then test whether the real value of the debt does indeed correspond to the estimated present value of the surplus? A reader of Hansen, Roberds, and Sargent (1991) already knows that one cannot follow this procedure; present values in such a test must be calculated from the *joint* debt-surplus process, because the univariate surplus model cannot reveal agents' information sets. Furthermore, we have already seen in (50) that the shock to agents' information sets cannot be recovered from current and past surpluses. Figure 10 shows what will go wrong if we try to take present values using the univariate process: The univariate response is always positive, while the true response function to shocks to agents' information is eventually negative. Thus "present values" calculated from responses to the univariate shock move positively with the surplus itself, while the true present value moves negatively with surpluses.

To give a better feel for this problem, Figure 11 plots simulated surplus time series together with the true simulated value of the debt, the value predicted by an AR(1) and the value predicted from the correct univariate process. The true value of the debt is equal to the true present value of the surplus, $v_t = E_t \sum_{j=0}^{\infty} \beta^j \tilde{s}_{t+j}$ as in previous plots. The AR(1) debt prediction uses the AR(1) model $\tilde{s}_t = \rho \tilde{s}_{t-1} + \varepsilon_t$ to calculate the present value

$$v_t^{\text{AR}(1)} = E_t \sum_{j=0}^{\infty} \delta^j \tilde{s}_{t+j} = \frac{1}{1-\delta\rho} \tilde{s}_t .$$

As the graph shows, this calculation predicts a value of the debt that is perfectly correlated with the surplus, and nothing at all like the true value of the debt.

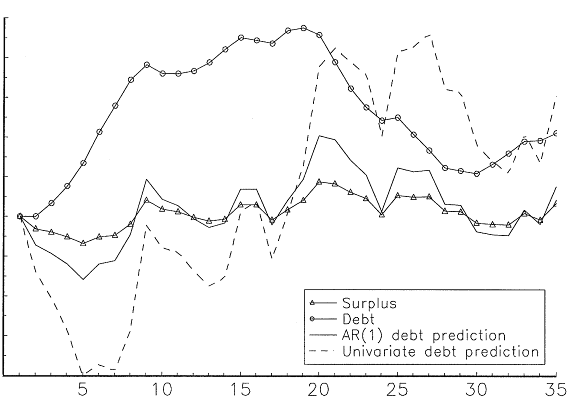


FIGURE 11.—Artificial time series of surplus, real debt, and real debt predicted from the present values of an AR(1) surplus process and the univariate (Wold) surplus process.

The univariate debt prediction uses the true univariate surplus process (50) rather than the AR(1) approximation to calculate the present value of the surplus:[7]

$$v_t^{\text{univariate}} \equiv E\left( \sum_{j=0}^{\infty} \delta^j \tilde{s}_{t+j} \,\middle|\, \tilde{s}_t, \tilde{s}_{t-1}, \tilde{s}_{t-2}, \ldots \right) = \frac{(1+\delta)(1-\delta\frac{1+\rho}{1+\delta}L)}{(1-\delta\rho)(1-\delta L)} \tilde{s}_t. \tag{51}$$

This prediction for the value of the debt is again positively correlated with the surplus and has no resemblance to the true debt process.

In sum, a researcher who fit a univariate surplus model and compared its present value to the value of the debt, using data from this artificial economy, would reject the present value condition. He would most likely fit an AR(1) surplus process, coming to the dramatically counterfactual prediction that debt and surplus should be perfectly correlated. With a lot of data and memories of the unit root debates he might fit the correct univariate process, but he would still come to a dramatically counterfactual prediction for debt. As in the analysis of Hansen, Roberds, and Sargent (1991), the only way to fit correctly the debt-surplus process in such a way that the value of debt equals the present value of surpluses is to estimate the *joint* debt-surplus process. (And even this procedure does not test the fiscal theory, since the present value condition holds in both "Ricardian" and "non-Ricardian" regimes, but that's a separate point.)

## 8. CONCLUSION

I started by analyzing the comparative statics of the fiscal theory—the effect of changing surpluses with the debt held constant, and the effect of changing debt with the surplus held constant—while allowing for long-term debt. These comparative statics are quite different from the standard case with only short-

[7] To derive this formula, express the surplus as a sum of two AR(1) components, driven by the shocks $\eta$,

$$s_t = c_t^* + z_t^*, \quad (1-\rho L)c_t^* = \frac{\delta-\rho}{1-\delta\rho}\eta_t, \quad (1-L)z_t^* = \frac{1-\delta}{1-\delta\rho}\eta_t. \tag{63}$$

Equation (63) gives the same univariate representation for $s_t$ as (50). Then,

$$v_t = \left( \frac{1}{1-\delta\rho} c_t^* + \frac{1}{1-\delta} z_t^* \right).$$

Equation (51) results by substituting back for $s_t$ from $c_t^*$ and $z_t^*$.

term debt. Depending on the maturity structure and debt policy—expectations of future debt sales and repurchases—today's price level can be determined by the present value of all future surpluses, by today's surplus alone, or by a rich variety of intermediate cases. If and only if long-term debt is outstanding, a debt sale can depress the price level today by devaluing outstanding debt. Debt and surplus *policy*—expectations of future state-contingent sales, repurchases and expenditures—matter crucially to the results; one-period changes in debt and surplus cannot be studied in isolation.

Then, I considered the question of *optimal* debt and surplus policy in pursuit of stable inflation. I found that long-term debt can be useful when the present value of surpluses varies by more than surpluses themselves. Perhaps more importantly, long-term debt allows the government to offset surplus shocks as they come. In this case, and especially when the government can choose the long-term surplus as well, the optimal policy produces artificial time series that display many initially puzzling properties of actual time series.

The optimal policies that I study here do not perfectly describe U.S. time series. Their primary failing is that they are too successful: they produce less variation of inflation than we observe. In addition, the nature of the optimization problems and the approximate solution conspired so I could not say much about the optimal maturity structure and especially about optimal state-contingent variation in the optimal maturity structure.

One can follow two paths in response to this criticism, both with long histories in the optimal monetary policy literature. Either the problem is harder than the model specifies, or inflation was simply a mistake.

The first path suggests that we add further complications to the models, so that optimal policy produces greater variation in inflation and the maturity choice is not degenerate. Most obviously, one could add price stickiness or some other friction. Such frictions would revive the inflation-output trade-offs that were a central part of classical monetary policy analyses such as Sargent and Wallace (1975), and they would generate an explicit welfare maximization problem in the modern general equilibrium tradition. Woodford (1998a) has analyzed fiscal models with such frictions, and the optimal policy exercises are waiting to be solved. In addition, one could use the exact solution rather than the approximate solutions; this path could generate more interesting results at a large cost in computational complexity. For example, in the approximate solutions only the sums $\tilde{B}_{t-1} = \sum_{j=0}^{\infty}(\delta\phi)^j \tilde{B}_{t-1}(t+j)$ matter to the price level. In a general solution, deliberate state-contingent lengthening and shortening of the maturity structure can affect the time-series process of inflation.

Most importantly, the component of the surplus under the government's control should be modeled as the result of distorting taxes, following the theory of dynamic optimal distortionary taxation (for example, Chari, Christiano, and Kehoe (1994), and Lucas and Stokey (1983)). Inflation is a state-contingent default, and perhaps this theory can shed light on why it is chosen. While such a state-contingent default in response to the low productivity and low surplus growth of the 70's may be fairly easy to generate, it will be harder to generate the cyclical state-contingent default in booms, when inflation is greatest, rather than busts, when it is lowest. Also, since ex-post devaluations are so useful in smoothing inflation, the time-consistency issues mentioned in the introduction will be important.

Alternatively, perhaps inflation was simply a mistake and we should advocate better policy, as monetarists charged for years that fluctuating inflation was due to mistakes by the Fed and $k$-percent or other rules would produce less volatile inflation. However, to make sense of the data, I had to assume that the government already does a great deal of inflation smoothing, aggressively using active fiscal policy to offset cyclical surplus shocks. Therefore, a $k$-percent debt growth rule would result in much *more* inflation volatility than we observe.[8] Improvements will involve more subtle changes in the dynamic, state-contingent path of surpluses and debt.

*Graduate School of Business, University of Chicago, 1101 E. 58th St., Chicago, IL 60637 U.S.A.; Federal Reserve Bank of Chicago; and NBER; john.cochrane@gsb.uchicago.edu; http://www-gsb.uchicago.edu/fac/john.cochrane/research/Papers/.*

*Manuscript received October, 1998; final revision received January, 2000.*

## APPENDIX

### Summary of Important Notation

$s_t$ = primary (net of interest) surplus.
$p_t$ = price level.
$B_t(j)$ = debt due at $j$ outstanding at the end of period $t$.
$Q_t(j)$ = nominal price of \$1 face value due at $j$, as of time $t$.
$\beta$ = discount factor; $1/\beta$ = gross real interest rate.
$v_t$ = real value of the debt.
$p_t^*, s_t^*, B_t^*$ = baseline path for approximation.
$\phi$ steady state debt parameter.
$\tilde{p}_t = p_t/p_t^* - 1$; proportional deviation from steady state.
$\tilde{s}_t = s_t/s_t^* - 1$; proportional deviation from steady state.
$\tilde{B}_{t-1}(t+j) = B_{t-1}(t+j)/B_{t-1}^*(t+j) - 1$; proportional deviation from steady state.
$\tilde{B}_{t-1} = \sum_{j=0}^{\infty}(\delta\phi)^j \tilde{B}_{t-1}(t+j)$ debt aggregate.
$\theta$ = baseline surplus growth rate, $s_t^* = s\theta^t$.
$\delta = \beta\theta$.

[8]The growth rate in the real value of the debt, which directly measures the present value of surpluses is extremely variable. (Figure 12 of Cochrane (1999) plots real debt growth; one can also see its volatility in Figure 2.) For example, real debt growth rose from $-11\%$ in 1973 to $+13\%$ in 1975, then fell to $-8\%$ in 1979, rising to $+15\%$ in 1982. If debt policy had consisted of $k$-percent growth in one-period debt, this real debt growth would have translated one-for-one to inflation. Inflation did vary in this period, but not by 20 percentage points over each cycle.

## Optimal Fixed-debt Policy

With fixed debt, $\tilde{B}=0$, so (21) becomes

$$(1-L)\tilde{p}_t = -\frac{1-\delta}{1-\delta\phi}(1-L)E_t\frac{(1-\delta\phi L^{-1})}{(1-\delta L^{-1})}\tilde{s}_t .$$

Defining

$$v_t \equiv E_t \frac{1}{(1-\delta L^{-1})}\tilde{s}_t ,$$

and using $\Delta \equiv (1-L)$, we have

$$\text{var}(1-L)\tilde{p}_t = \left(\frac{1-\delta}{1-\delta\phi}\right)^2 \text{var}[\Delta v_t - \delta\phi \Delta E_t v_{t+1}]. \tag{52}$$

The first order condition with respect to $\phi$ gives

$$\delta\phi = \frac{\text{cov}(\Delta v_t, \Delta v_t - \Delta E_t v_{t+1})}{\text{cov}(\Delta E_t v_{t+1}, \Delta v_t - \Delta E_t v_{t+1})}. \tag{53}$$

The objective (52) rises to infinity at $\phi = 1/\delta$. Therefore, if there is no solution to (53) in $0 \le \phi < 1/\delta$, the solution is $\phi = 0$.

The terms on the right-hand side of (53) are easy to calculate if we model the surplus as an element of a vector AR(1),

$$\tilde{s}_t = e' x_t , \tag{54}$$

where

$$x_t = A x_{t-1} + J\varepsilon_t ; \quad E(\varepsilon\varepsilon') = I. \tag{55}$$

With this structure, (53) results in

$$\delta\phi = \frac{e'(I-A)Ve}{e'(I-A)VA'e}, \tag{56}$$

where

$$V \equiv \text{var}(\Delta v_t) = (I-\delta A)^{-1}\,\Sigma_{\Delta x}(I-\delta A)^{-1\prime},$$

$$\Sigma_{\Delta x} = E(\Delta x_t \Delta x_t') = (A-I)\left(\sum_{j=0}^{\infty} A^j JJ' A^{j\prime}\right)(A-I)' + JJ'.$$

Despite the simple appearance of (56), substituting in the formulas for an AR(2) does not yield a simple expression, so I use this form for the calculation.

## Active Debt Policy with Exogenous Surplus

*Deriving $\psi(L)$*

Section 6.3 describes the setup and notation. Our objective is to minimize

$$\min_{\{\psi_j\}} \text{var}(\tilde{p}_t - \tilde{p}_{t-1}) = \psi_0^2 + \sum_{j=1}^{\infty}(\psi_j - \psi_{j-1})^2,$$

subject to the constraint

$$(1-\delta\phi)\psi(\delta\phi) = -(1-\delta)\eta(\delta) \tag{57}$$

and that the sequence $\{\psi_j\}$ must not explode. I proceed by a straightforward Lagrangian maximization. Taking the derivatives with respect to $\psi_j$ leads to

$$\psi_0 - (\psi_1 - \psi_0) = \lambda,$$

$$\psi_j - \psi_{j-1} - (\psi_{j+1} - \psi_j) = \lambda(\delta\phi)^j, \quad j \geq 1,$$

where $\lambda$ is the Lagrange multiplier on the constraint (57). Iterating forward from $j=0$, we can express

$$\psi_j = -\frac{\lambda}{1-\delta\phi}\left(j - \frac{\delta\phi(1-(\delta\phi)^j)}{1-\delta\phi}\right) + (j+1)\psi_0.$$

To keep $\lambda_j$ from growing without bound, the $j$ terms must cancel, so we must have

$$\frac{\lambda}{1-\delta\phi} = \psi_0.$$

Substituting, the $\psi_j$ must follow

$$\psi_j = \psi_0\left(\frac{1-(\delta\phi)^{j+1}}{1-\delta\phi}\right).$$

We can express this result in lag operator notation as

$$\psi(L) = \frac{\psi_0}{1-\delta\phi}\left(\frac{1}{1-L} - \frac{\delta\phi}{1-\delta\phi L}\right)$$

$$= \frac{\psi_0}{(1-L)(1-\delta\phi L)}.$$

I determine the remaining free parameter $\psi_0$ to satisfy the constraint (57).

$$\psi_0 = -(1-(\delta\phi)^2)(1-\delta)\eta(\delta),$$

thus

$$\psi(L) = -\frac{(1-(\delta\phi)^2)(1-\delta)\eta(\delta)}{(1-L)(1-\delta\phi L)}. \tag{58}$$

*Debt Policy*

Next, we need to characterize the debt policy that supports the desired price level. We can simply state the policy by solving (33) for debt,

$$\tilde{B}_{t-1} = E_t\frac{1}{1-\delta\phi L^{-1}}\tilde{p}_t + \frac{1-\delta}{1-\delta\phi}E_t\frac{1}{1-\delta L^{-1}}\tilde{s}_t.$$

It is useful to also express $\tilde{B}_{t-1}$ in terms of the history of $\tilde{p}$, $\tilde{s}$, or $\varepsilon$. Using

$$E_t\frac{1}{1-\delta\phi L^{-1}}\tilde{p}_t = \frac{\psi(L) - \delta\phi L^{-1}\psi(\delta\phi)}{1-\delta\phi L^{-1}}\varepsilon_t,$$

we can express the first term as

$$E_t\frac{1}{1-\delta\phi L^{-1}}\tilde{p}_t = -\frac{(1-\delta)\eta(\delta)}{(1-\delta\phi)}\frac{(1-\delta^2\phi^2 L)}{(1-L)(1-\delta\phi L)}\varepsilon_t.$$

We can write the second term as

$$E_t \frac{1}{1-\delta L^{-1}} \tilde{s}_t = E_{t-1} \frac{1}{(1-\delta L^{-1})} \tilde{s}_t + \eta(\delta)\varepsilon_t.$$

Pulling both terms together,

$$\tilde{B}_{t-1} = \frac{1-\delta}{1-\delta\phi}\left[E_{t-1}\frac{1}{1-\delta L^{-1}}\tilde{s}_t + \eta(\delta)\left(1-\frac{(1-\delta^2\phi^2 L)}{(1-L)(1-\delta\phi L)}\right)\varepsilon_t\right] \tag{59}$$

$$= \frac{1-\delta}{1-\delta\phi}\left[E_{t-1}\frac{1}{1-\delta L^{-1}}\tilde{s}_t - \eta(\delta)\left(\frac{1-\delta^2\phi^2}{(1-L)} + \delta\phi\right)\frac{1}{(1-\delta\phi L)}\varepsilon_{t-1}\right].$$

With an AR(1) surplus, $\eta(L) = (1-\rho L)^{-1}$, equation (59) reduces to (39) presented in the text. Using (58) to eliminate $\varepsilon$ in favor of $p$, and collecting terms, we have a characterization in terms of the present value of $s$ and past $p$:

$$\tilde{B}_{t-1} = \frac{1-\delta}{1-\delta\phi}E_{t-1}\sum_{j=0}^{\infty}\delta^j\tilde{s}_{t+j} + \frac{1}{1-\delta\phi}\left(\tilde{p}_{t-1} + \frac{\delta\phi}{(1-(\delta\phi)^2)}(1-L)\tilde{p}_{t-1}\right). \tag{60}$$

Note that $\tilde{B}$ moves one for one with $\tilde{p}$; the other terms are all stationary.

## General Price Solution

This section derives equation (12). To simplify notation, let $t = 0$. Define

$$v_t = \sum_{j=0}^{\infty}\beta^j s_{t+j},$$

and define a sequence $\{X_j\}$ by

$$X_0 = 1,$$

$$X_1 = -\frac{B_{-1}(1)}{B_0(1)},$$

$$X_2 = -\frac{B_{-1}(2) + X_1 B_0(2)}{B_1(2)},$$

$$X_3 = -\frac{B_{-1}(3) + X_1 B_0(3) + X_2 B_1(3)}{B_2(3)},$$

$$X_j = -\sum_{k=0}^{j-1}\frac{B_{k-1}(j)}{B_{j-1}(j)}X_k.$$

I start with the present value condition (4), which implies

$$\frac{B_{-1}(0)}{p_0} = E_0\left\{v_0 - \beta\left(\frac{1}{p_1}\right)B_{-1}(1) - \beta^2\left(\frac{1}{p_2}\right)B_{-1}(2) - \cdots\right\}$$

at time 0 and

$$\frac{1}{p_1} = \frac{1}{B_0(1)} E_1 \left\{ v_1 - \beta \left( \frac{1}{p_2} \right) B_0(2) - \beta^2 \left( \frac{1}{p_3} \right) B_0(3) - \cdots \right\}$$

at time 1. Use time 1 to substitute in time 0,

$$\frac{B_{-1}(0)}{p_0} = E_0 \left\{ v_0 - \beta \frac{B_{-1}(1)}{B_0(1)} \left[ v_1 - \beta \frac{1}{p_2} B_0(2) - \ldots \right] - \beta^2 \left( \frac{1}{p_2} \right) B_{-1}(2) - \ldots \right\}.$$

Recognizing the definition of $X_1$

$$\frac{B_{-1}(0)}{p_0} = E_0 \left\{ v_0 + \beta X_1 v_1 + \beta^2 \left[ -(B_{-1}(2) + X_1 B_0(2)) \frac{1}{p_2} \right] + \beta^3 \left[ -(B_{-1}(3) + X_1 B_0(3)) \frac{1}{p_3} \right] + \ldots \right\}.$$

Substitute now for $1/p_2$:

$$\frac{1}{p_2} = \frac{1}{B_1(2)} E_2 \left[ v_2 - \beta \frac{1}{p_3} B_1(3) - \beta^2 \frac{1}{p_4} B_1(4) \ldots \right],$$

$$\frac{B_{-1}(0)}{p_0} = E_0 \left\{ v_0 + \beta X_1 v_1 + \beta^2 \left[ - \frac{B_{-1}(2) + X_1 B_0(2)}{B_1(2)} \left[ v_2 - \beta \frac{1}{p_3} B_1(3) - \ldots \right] \right] + \beta^3 \left[ -(B_{-1}(3) + X_1 B_0(3)) \frac{1}{p_3} \right] + \ldots \right\}.$$

Recognizing the definition of $X_2$,

$$\frac{B_{-1}(0)}{p_0} = E_0 \left\{ v_0 + \beta X_1 v_1 + \beta^2 X_2 v_2 - \beta^3 \left( [B_{-1}(3) + X_1 B_0(3) + X_2 B_1(3)] \frac{1}{p_3} \right) + \ldots \right\}.$$

Continuing in this way, we have

$$\frac{B_{-1}(0)}{p_0} = E_0 \sum_{j=0}^{\infty} \beta^j X_j v_j.$$

This expression is already a solution. However, it is more elegant to collect terms in $s_j$ on the right-hand side, resulting in

$$\frac{B_{-1}(0)}{p_0} = E_0 \{ 1 + (1 + X_1) \beta s_1 + (1 + X_1 + X_2) \beta^2 s_2 + \ldots \},$$

$$\frac{B_{-1}(0)}{p_0} = E_0 \sum_{j=0}^{\infty} \beta^j \left( \sum_{k=0}^{j} X_k \right) s_j = \sum_{j=0}^{\infty} \beta^j W_j s_j.$$

This is the price solution (12). The last equality defines $W_j$. We can find a more direct definition rather than via $X_j$. Proceeding through time,

$$W_0 = X_0 = 1,$$

$$W_1 = 1 + X_1 = \frac{B_0(1) - B_{-1}(1)}{B_0(1)} = A_0(1),$$

$$W_2 = 1 + X_1 + X_2 = W_1 + X_2 = \frac{W_1 B_1(2) - X_1 B_0(2) - B_{-1}(2)}{B_1(2)}$$

$$= \frac{W_1 B_1(2) - (W_1 - 1) B_0(2) - B_{-1}(2)}{B_1(2)} = W_1 A_1(2) + A_0(2),$$

$$W_3 = W_2 + X_3 = \frac{W_2 B_3(3) - X_2 B_2(3) - X_1 B_1(3) - B_0(3)}{B_3(3)}$$

$$= \frac{W_2 B_2(3) + (W_1 - W_2) B_1(3) + (1 - W_1) B_0(3) - B_{-1}(3)}{B_2(3)}$$

$$= W_2 A_2(3) + W_1 A_1(3) + A_0(3),$$

and so forth.

## Approximate Solution with a Nongeometric Steady State

We can differentiate the general solution (12) with respect to debt, evaluated at the baseline path, to find that the approximation for the surplus terms is

$$\tilde{p}_t = -\xi \sum_{j=0}^{\infty} \delta^j W_j E_t \tilde{s}_{t+j}.$$

The hard part is unraveling the $W_{t,j}$ terms to find the effects on $p_t$ of a change in debt $dB_{t+j}(t+k)$. We could proceed directly by differentiating $W_{t,j}$. We could also substitute the nongeometric steady state into the differentiated present value condition (22), yielding

$$\sum_{j=0}^{\infty} \delta^j \phi_j E_t \tilde{p}_{t+j} = \sum_{j=0}^{\infty} \delta^j \phi_j \tilde{B}_{t-1}(t+j) - \xi \sum_{j=0}^{\infty} \delta^j E_t \tilde{s}_{t+j}. \tag{61}$$

However, the terms in expected future prices no longer have a geometric pattern representable by the easily invertible operator $(1 - \delta\phi L^{-1})^{-1}$. Therefore, one must iterate (61) forward manually, by substituting the same equation at $t+1, t+2$, etc. It turns out to be easiest to track the effects on $\{p_t\}$ of a single debt operation and then add up the results to give the approximate solution. All three approaches give the same result, of course.

Start by a direct Taylor expansion,

$$\tilde{p}_t \approx \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\partial(\tilde{p}_t)}{\partial \tilde{B}_{t-1+k}(t+k+j)} \tilde{B}_{t-1+k}(t+k+j). \tag{62}$$

To evaluate the partial derivatives, it is easiest to fix the date of the intervention at $t = -1$, repurchased at $t = 0$, and then evaluate the effect on the price sequence $\{p_t\}$ of a small change in $B_{-1}(j)$. I start with the real time $t$ flow condition, (3), which I repeat here for convenience:

$$s_t + \sum_{j=1}^{\infty} \beta^j E_t \left( \frac{1}{p_{t+j}} \right) [B_t(t+j) - B_{t-1}(t+j)] = \frac{B_{t-1}(t)}{p_t}.$$

At time $t = 1,2,3,\ldots, B_{-1}(j)$ does not enter this condition, or the general solution. Therefore prices at and past date 1 are not affected.

At time $t = 0$, the condition is

$$s_0 + \beta E_0\left(\frac{1}{p_1}\right)[B_0(1) - B_{-1}(1)] + \cdots + \beta^j E_0\left(\frac{1}{p_j}\right)[B_0(j) - B_{-1}(j)] + \cdots = \frac{B_{-1}(0)}{p_0}.$$

Take the derivative of this condition with respect to $B_{-1}(j)$, evaluated at the baseline path. Using $d\tilde{p} = dp/p^*$, etc., the result is

$$\frac{\partial \tilde{p}_0}{\partial \tilde{B}_{-1}(j)} = \delta^j \phi_j.$$

It is convenient to capture this expression with a standardized derivative $D_0$,

$$D_0 \equiv \frac{1}{\delta^j \phi_j} \frac{\partial \tilde{p}_0}{\partial \tilde{B}_{-1}(j)} = 1.$$

At time $t = -1$, the condition is

$$s_{-1} + \beta E_{-1}\left(\frac{1}{p0}\right)[B_{-1}(0) - B_{-2}(0)] + \cdots$$
$$+ \beta^{j+1} E_{-1}\left(\frac{1}{p_j}\right)[B_{-1}(j) - B_{-2}(j)] + \cdots = \frac{B_{-2}(-1)}{p_{-1}}.$$

Taking the derivative with respect to $B_{-1}(j)$ again, we now have direct terms ($B_{-1}(j)$ varies) and indirect terms, since $p_0$ varies as well. The result is

$$\frac{\partial \tilde{p}_{-1}}{\partial \tilde{B}_{-1}(j)} = \delta[1 - \phi_1]\frac{\partial \tilde{p}_0}{\partial \tilde{B}_{-1}(j)} - \delta^{j+1}\phi_j = -\phi_1 \delta^{j+1}\phi_j.$$

It is convenient to capture this result with

$$D_1 \equiv \frac{1}{\delta^{j+1}\phi_j} \frac{\partial \tilde{p}_{-1}}{\partial \tilde{B}_{-1}(j)} = -\phi_1.$$

At a generic time $t = -k$, the condition is

$$s_{-k} + \sum_{i=1}^{\infty} \beta^i E_{-k}\left(\frac{1}{p_{-k+i}}\right)[B_{-k}(-k+i) - B_{-k-1}(-k+i)] = \frac{1}{p_{-k}}[B_{-k-1}(-k)].$$

Again taking derivatives with respect to $B_{-1}(j)$,

$$\frac{\partial \tilde{p}_{-k}}{\partial \tilde{B}_{-1}(j)} = \sum_{i=1}^{k+1} \delta^i (\phi_{i-1} - \phi_i) \frac{\partial \tilde{p}_{-k+i}}{\partial \tilde{B}_{-1}(j)}.$$

Expressing this result in terms of standardized derivatives $D_j$,

$$D_k = \frac{\partial \tilde{p}_{-k}}{\delta^{j+k}\phi_j \partial \tilde{B}_{-1}(j)} = \sum_{i=0}^{k} A_{i+1} D_{k-i}.$$

Now we can use these derivative expressions in the Taylor expansion (62):

$$\tilde{p}_t = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\partial(\tilde{p}_t)}{\partial \tilde{B}_{t-1+k}(t+k+j)} \tilde{B}_{t-1+k}(t+k+j)$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta^{j+k} \phi_j D_k \tilde{B}_{t-1+k}(t+k+j)$$

$$= \sum_{k=0}^{\infty} \delta^k D_k E_t \sum_{j=0}^{\infty} \delta^j \phi_j \tilde{B}_{t-1+k}(t+k+j),$$

$$\tilde{p}_t = \sum_{k=0}^{\infty} \delta^k D_k E_t \tilde{B}_{t-1+k}.$$

## REFERENCES


Blanchard, Olivier, and A. Missale (1994): "The Debt Burden and Debt Maturity," *American Economic Review*, 84, 309–319.

Calvo, Guillermo A., and Pablo E. Guidotti (1992): "Optimal Maturity of Nominal Government Debt: An Infinite-Horizon Model," *International Economic Review*, 33, 895–919.

Canzoneri, Matthew B., Robert E. Cumby, and Behzad T. Diba (1998): "Is the Price Level Determined by the Needs of Fiscal Solvency?" Working Paper, Georgetown University.

Chari, V. V., Lawrence J. Christiano, and Patrick J. Kehoe (1994): "Optimal Fiscal Policy in a Business Cycle Model," *Journal of Political Economy*, 102, 617–652.

Cochrane, John H. (1999): "A Frictionless View of U.S. Inflation," in *NBER Macroeconomics Annual*, ed. by Ben Bernanke and Julio J. Rotemberg. Cambridge MA: MIT Press, 323–384.

——— (2000): "Money as Stock: Price Level Determination with no Money Demand," NBER Working Paper 7498.

Dupor, William (2000): "Exchange Rates and the Fiscal Theory of the Price Level," *Journal of Monetary Economics*, 45, 613–630.

Hall, George, and Thomas J. Sargent (1997): "Accounting Properly for the Government's Interest Costs," *Federal Reserve Bank of Chicago Economic Perspectives*, 21, 18–28.

Hansen, Lars Peter, William Roberds, and Thomas J. Sargent (1991): "Time Series Implications of Present-Value Budget Balance and of Martingale Models of Consumption and Taxes," in *Rational Expectations Econometrics*, ed. by Lars Peter Hansen and Thomas J. Sargent. Boulder: Westview Press, 121–161.

Leeper, Eric (1991): "Equilibria Under 'Active' and 'Passive' Monetary Policies," *Journal of Monetary Economics*, 27, 129–147.

Lucas, Robert E., Jr. (1972): "Expectations and the Neutrality of Money," *Journal of Economic Theory*, 4, 103–124.

——— (1973): "Some International Evidence on Output-Inflation Trade-Offs," *American Economic Review*, 63, 326–334.

Lucas, Robert E., Jr., and Nancy L. Stokey (1983): "Optimal Fiscal and Monetary Policy in an Economy without Capital," *Journal of Monetary Economics*, 12, 55–93.

Missale, Alessandro (1997): "Managing the Public Debt: The Optimal Taxation Approach," *Journal of Economic Surveys*, 11, 235–265.

Sargent, Thomas J. (1986): *Rational Expectations and Inflation*. New York: Harper & Row.

Sargent, Thomas J., and Neil Wallace (1975): "'Rational' Expectations, the Optimal Monetary Instrument, and the Optimal Money Supply Rule," *Journal of Political Economy*, 83, 241–254.

——— (1981): "Some Unpleasant Monetarist Arithmetic," *Federal Reserve Bank of Minneapolis Quarterly Review*, 5, 1–17. (Reprinted in Sargent (1986).)

SIMS, CHRISTOPHER A. (1994): "A Simple Model for the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy," *Economic Theory*, 4, 381–399.

——— (1997): "Fiscal Foundations of Price Stability in Open Economies," Working Paper, Yale University.

WOODFORD, MICHAEL (1995): "Price Level Determinacy Without Control of a Monetary Aggregate," *Carnegie-Rochester Conference Series on Public Policy*, 43, 1–46.

——— (1997): "Control of the Public Debt: A Requirement for Price Stability?" in *The Debt Burden and Monetary Policy*, ed. by G. Calvo and M. King. London: Macmillan.

——— (1998a): "Doing Without Money: Controlling Inflation in a Post-Monetary World," *Review of Economic Dynamics*, 1, 173–219.

——— (1998b): "Public Debt and the Price Level," Manuscript, Princeton University.