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Rethinking Production Under Uncertainty

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1 Introduction

Production possibilities in uncertain environments are usually modeled by standard (certainty) production functions augmented with shocks, for example,

$$y = \epsilon f(k) \tag{1}$$

where $y =$ output (tomorrow), $k =$ input (today), and $\epsilon =$ the shock (revealed tomorrow). The firm's choice variable is the input k.

This paper explores a different representation for technology under uncertainty. There are two equivalent ways to think about this representation. First, the firm is also given choice of the distribution of the shock variable ϵ, which is constrained to lie in a convex set. For example, the constraint could be that the second moment of ϵ must be below some value.

Second, consider the production possibility set induced by a production function of the form (1). The firm can only transform inputs today to outputs tomorrow in fixed proportions across states. The firm cannot transform output across states of nature, so the production possibility set has a kink, as illustrated in Figure 1. I study production sets whose borders are instead smooth (differentiable), as illustrated in Figure 2.

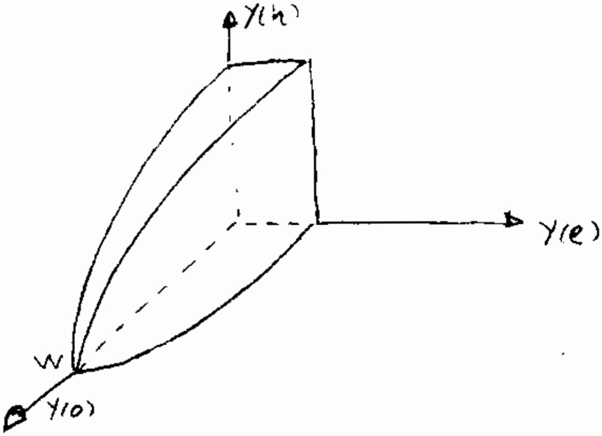


Figure 1: Standard production possibility set in a two state world. The technology is $y(s) = \epsilon(s)k^{1/2}$, $y(0) = W - k$.

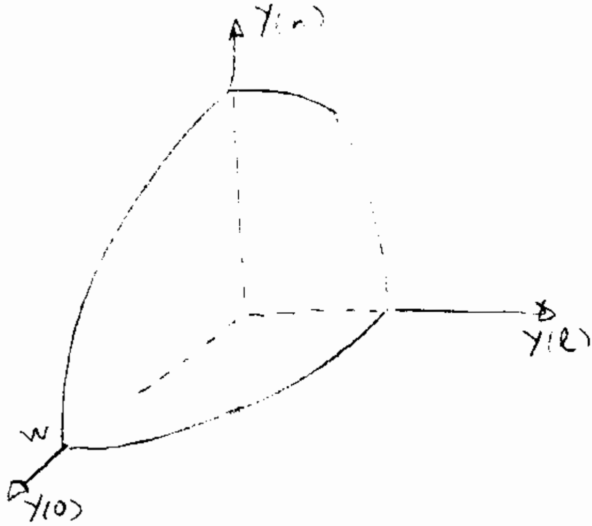


Figure 2: Smooth production possibility set. The firm can to change the distribution of output – the pattern of $y(1)$ vs. $y(2)$.

Why is this representation of production possibilities *reasonable?* First, one may simply argue that the *lack* of kinks is the most natural production set, and ask what empirical evidence there is *for* them, as we do when studying nonstochastic production functions. It seems natural to start with the presumption that the firm has at least *some* control over the distribution of outputs conditional on inputs, and ask for compelling evidence that it has absolutely none. Historically, aggregate production functions with kinks are not the result of such evidence. Instead, shocks were simply tacked on to deterministic intertemporal production functions familiar from growth theory.

Second, smooth production sets can occur when one aggregates standard production functions. Section 2 explores a model in which a firm has access to several different technologies or processes, each of which has a different distribution of shocks. By varying its input across the different processes, the firm can change the distribution of the shock in the *aggregate* production function that relates the firm's total output to its total inputs.

This approach is analogous to the standard result that an aggregate of Leontief production functions can produce a smooth function such as Cobb-Douglas. In fact, the latter result is a standard justification for smooth *input* requirement sets given that individual machines or production processes are generally fixed-coefficient. I apply the same logic to the multiple *outputs* (across states of nature) of a firm that operates in an uncertain environment.

One response to this observation is to inveigh against the use of aggregated data. However, this is not practical advice. National, category and industry level aggregates are a useful and informative source of data. Fluctuations of these aggregates and their correlation with asset returns *define* the empirical phenomena that we want to explain. And perfectly disaggregated data are unlikely to ever be available. Even at the plant level, the firm can choose what kinds of machines to install, the nature and durability of construction materials, etc. *All* data are somewhat aggregated.

Why is this representation of technology *useful* or *interesting?* My direct interest is in the construction of production-based asset pricing models, with the following motivation.

A large empirical literature has uncovered a tantalizing list of correlations between macroeconomic data and asset returns. For example, many of the same variables forecast stock and bond returns as well as GNP, such as the term premium and default premium. Stock returns forecast GNP. Regressions of stock returns on leads and lags of GNP can yield R^2 more than 50%. Ad-hoc macroeconomic factors can do a good job of explaining the cross-section of stock returns.

To make sense of this empirical work, we need economic models that tie asset returns to macroeconomic variables. Yet economic modeling of the link between aggregate consumption and asset returns has not been a great empirical success. Current research in this area features transactions costs, liquidity constraints, lumpy goods purchases, uninsured individual income risk and other heterogeneities. This research seems likely to produce a solid understanding of why there is *no* useful link between aggregate consumption and asset returns, rather than to produce a successful specification relating available consumption data to asset returns. Since general equilibrium models with production require a correctly specified consumption side, they do not avoid this empirical difficulty.

In this context, production based models try to exploit relations between production variables and asset returns derived from the maximizing behavior of firms. The hope is that they will allow us to describe the links between asset returns and macroeconomic aggregates, even while the consumption side of the problem is poorly understood. (Cochrane (1991, 1992) contain detailed motivations for this approach with reference to puzzles in the empirical asset pricing literature.)

When studying the consumption side of the problem, the condition that a contingent claim hyperplane is tangent to an indifference set leads to the familiar relation

$$\text{asset price} = E\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}\text{payoff}\right). \tag{2}$$

Figure 1 makes clear why a pure production-based asset pricing model is not possible using standard representations of technology. Since there is a kink in the production set, many different contingent claim price hyperplanes are consistent with any point the firm might choose.

Thus, my motivation for studying production sets that are smooth across states as well as dates is that they allow one to describe the link between production variables and asset returns with *no* information about preferences, just as the consumption model describes a link between asset returns and consumption that is independent of the production technology. However, this representation of technology should also be useful in many other applications.

1.1 Overview of the paper

Section 2 explores a simple model in which a smooth production set is derived from underlying fixed-shock technologies. Section 3 motivates the form of the production functions I use throughout this paper. I add choice of the shock ϵ and the constraint

$$E[(\frac{\epsilon}{\theta})^{\alpha}] \leq 1.$$

to standard intertemporal technologies such as (1). (θ is an underlying productivity shock that may make it easier to produce in some states than in others.) Then, section 3 derives the first order conditions of firms with these production functions. The conditions say two things. First, the firm removes arbitrage opportunities between the one period return on investment, denoted R^I, and asset returns. Second, they imply formulas for the marginal rate of *transformation* as a function of output, capital, etc.

The paper includes three sets of calculations. Section 4 calculates the effects of smooth technologies on simple standard stochastic growth models. Section 5 constructs Mehra-Prescott style economies in which *consumption* is a fixed stochastic process and we read asset returns from marginal rates of *transformation*. This model is a laboratory for thinking about what features of production technology will be useful for matching stylized facts about asset returns and macroeconomic variables. Section 6 estimates and tests a production - based asset pricing model, i.e., $1 = E(mR)$ where m is the marginal rate of transformation implied by the model (a function of output, investment, labor, etc. data) and R is a set of asset and investment returns.

2 A simple aggregation model.

The approach in this paper is to model the aggregated (smooth) production possibility set directly, rather than derive the structure of these sets from (unobservable) primitives. However, it is useful as motivation to sketch a model in which a smooth aggregated production set is derived from underlying traditional technologies.

Consider a two-state world, in which the firm has *two* technologies. For example, think of a farmer who can plant in two fields. One field does well in wet weather, the other in dry weather. The farmer can then shape the risk exposure of his *total* output to weather by varying the amount planted in each of the the two fields.

To make this story precise in a parametric example, let the technologies of field i be

$$y_i(s) \le \epsilon_i(s)k_i^\eta; \; s = h \text{ or } l; \; i = 1 \text{ or } 2.$$

Total output is

$$y(s) = y_1(s) + y_2(s)$$

and total inputs are constrained by initial capital less initial sales,

$$k_1 + k_2 = k = W - y(0).$$

We only observe the aggregates: $k, W, y(0)$ and $y(s)$. Thus, we ask what this structure implies for the *aggregate* production possibilities set, $\{y(h),\ y(l),\ -k\}$. The answer is straightforward given this functional form. First, write the technology in matrix form,

$$\begin{bmatrix} y(h) \\ y(l) \end{bmatrix} = \begin{bmatrix} \epsilon_1(h) & \epsilon_2(h) \\ \epsilon_1(l) & \epsilon_2(l) \end{bmatrix} \begin{bmatrix} k_1^\eta \\ k_2^\eta \end{bmatrix}$$

$$\mathbf{y} = E\mathbf{k}^\eta.$$

Next, Iassume that the matrix of shocks E is invertible. Otherwise, there really is only one technology at work. Then, we can find the pattern of inputs required to get a given state-pattern of outputs,

$$\mathbf{k} = (E^{-1}\mathbf{y})^{1/\eta}.$$

Finally, the first period resource constraint implies

$$k = \mathbf{1}'\mathbf{k} \geq \mathbf{1}'(E^{-1}\mathbf{y})^{1/\eta}.$$

Thus, we have the production set

$$\{(\mathbf{y}, -k) \text{ such that } 0 \geq \mathbf{1}'(E^{-1}\mathbf{y})^2 - k\}.$$

Figure 3 graphs this production set. As the figure shows, though the *individual* production technologies have kinks or fixed shocks, the *aggregate* technology is smooth.[1]

[1]The derivation ignored the possibility of creating a {y(h), y(l)} by disposal from a larger output. The latter possibility expands the production set to fill the solid lines in figure 3.

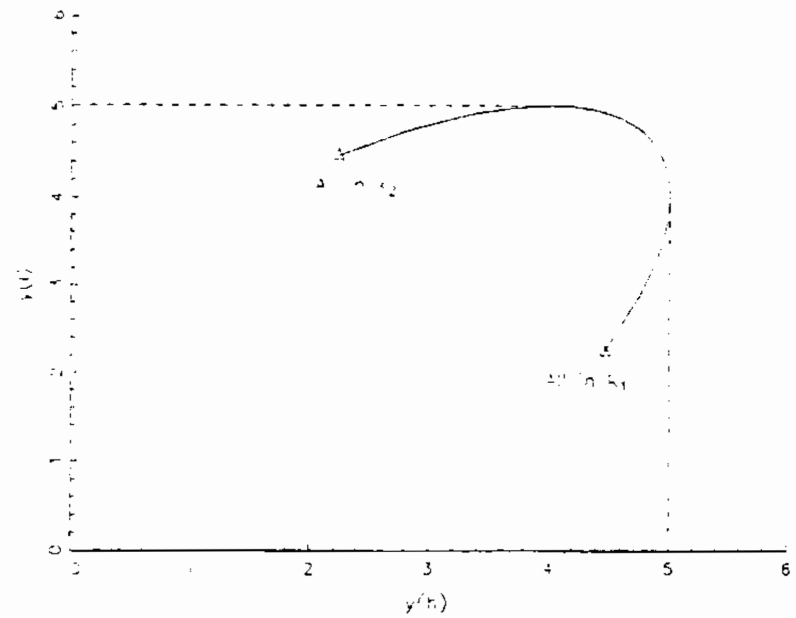


Figure 3: Aggregate production set $\{y(h), y(l), -k\}$ induced by two standard technologies, $y_i(s) \le \epsilon_i(s)k_i^{\gamma}$, $i = 1, 2$.

For continuous-state economies, we subdivide technology into finer units of analysis; each square foot of land may have a slightly different sensitivity to weather. Thus, consider individual technologies indexed by z (co-ordinates, in our farm example). Indexing states of nature by ω, we write aggregate output as

$$y(\omega) = \int dz\ \epsilon(\omega, z) f(k(z)).$$

If we can invert this relation to

$$f(k(z)) = \int dPr(\omega)\ \lambda(\omega, z) y(\omega)$$

then the firm can rearrange its output to attain any random variable.[2]

If the number of states is greater than the number of technologies, or if the shock distribution is not invertible, the firm may still be able to rearrange its output among a restricted set of random variables, and thus we may be able to price interesting subsets of all contingent claims. The appendix takes this issue up in detail.

[2]Alternatively, we can derive smooth production sets by allowing the firm to continuously vary its investment in a few technologies, as in the dynamic spanning literature (Duffie and Huang (198x)).

It is likely that marginal rates of transformation do not in fact exist to price every random variable, for example pure preference shocks or sunspots. Rather than model this fact directly, however, it is easier to write production functions that are *completely* smooth, and make sure we price assets whose payoffs are plausibly related to technology.

3 Production functions

The individual technologies $f(k(z))$ are not observable to economists, who must always study some level of aggregation. Therefore, instead of building up a lot of machinery about production functions $f(k(z))$ and variation in the distribution of shocks $\epsilon(\omega, z)$, I simply posit aggregate, smooth, production sets with convenient functional forms.

3.1 Functional forms in two-period models.

Consider a two period world with S states of nature at the second date. The firm's output is $y(s)$, its input is k. At a general level, we seek functions.

$$g(y(s), -k) : \Re^{S+1} \rightarrow \Re$$

that are monotone and convex, $g(\cdot) \geq 0$ for $\{y, -x\}$ feasible, $g(\cdot) < 0$ for $\{y, -x\}$ not feasible and g differentiable. The appendix describes features of production sets that imply such a function.

Separability across time and states makes the functions much more tractable. Perhaps the most natural approach, given the analogy to consumption-based models, is then to write the production function as

$$\beta \sum_s \rho(s) g(c(s)) + g(c(0)) \leq W, \ g(\cdot) \text{ monotone, convex}$$

where $c(s)$ represents net outputs sold by the firm in state s, and $c(0)$ denotes net outputs at time zero. (c is not necessarily nondurable consumption goods.) $\rho(s)$ is a probability measure.

However, it seems useful instead to incorporate standard production theory as far as possible. To that end, I specify a traditional-looking production function to describe the firm's ability to transform goods over *time*, but add the ability to substitute across *states*.

An obvious such choice is

$$\sum_s (a(s)y(s)^\alpha)^{1/\alpha} \leq f(k) \tag{3}$$

$$\alpha > 1, f(\cdot) \text{ monotone, concave}$$

.

In a continuous state space, this choice is

$$(\int dM(\omega)y(\omega)^\alpha)^{1/\alpha} \leq f(k).$$

Here, a or dM is *a* measure, not necessarily a (or the) probability measure. The expected utility axioms do not apply to firms, so probabilities do not enter into technology as they do into preferences. There is no obvious reason why the firm's technical ability to transform output into a state of nature should have anything to do with the probability that the state occurs.[3]

However, since it is very convenient to use a probability measure when integrating across a measure space, we can introduce the $R-N$ derivative and express the same production function as

$$(\int dPr(\omega)(y(\omega)/\theta(\omega))^\alpha)^{1/\alpha} \leq f(k)$$

or

$$E((y/\theta)^\alpha)^{1/\alpha} \leq f(k). \tag{4}$$

Finally, it will be convenient to think of the technology as a standard intertemporal technology augmented with choice of the shock. Thus, we can

[3]One could think of dynamic models in which the firm expends *R&D* efforts to make it easier to transform output to states with high probabilities, reflected in high contingent claims prices.

write (4) as

$$y = \epsilon f(k)$$
$$E((\epsilon/\theta)^{\alpha}) \leq 1.$$

α controls the firm's ability to transform output across states. As $\alpha \to \infty$, the chosen shock ϵ converges to θ. The choice $\epsilon = \theta$ is always feasible, and as α decreases, it becomes easier for the firm to transform output across states. Thus, we can think of θ as the underlying productivity shock, which the firm distorts to some new shock ϵ. As an example, consider $\alpha = 2$ and $\theta =$ constant. Then the firm can choose any technology shock whose second moment is less than θ^2, including $\epsilon = \theta$.

3.2 First order conditions in a two-period model.

The firm's objective is to maximize contingent claim value,

$$\max E(my) - k$$

subject to

$$y = \epsilon f(k)$$
$$E[(\epsilon/\theta)^{\alpha}] \leq 1.$$

m is the stochastic discount factor, or contingent claim price divided by probability.[4] The firm can choose k and the value of ϵ in each state of nature.

The first-order conditions are

$$\partial/\partial k: \qquad E(m\epsilon f'(k)) = 1 \tag{5}$$
$$\partial/\partial \epsilon: \qquad mf(k) = \lambda\alpha\frac{\epsilon^{\alpha-1}}{\theta^{\alpha}} \tag{6}$$

[4]This way of writing the objective does not require the existence of complete markets. A discount factor exists assuming only absence of arbitrage. Thus, this objective expresses the notion that firms remove arbitrage opportunities between physical investment and what assets are available, by synthesizing assets through marginal changes in output across states of nature.

and the constraints. Equation (5) is the familiar condition that the discounted value of the produce accruing to an additional unit of investment should equal its marginal cost. It can be written $1 = E(mR^I)$ where R^I is the investment return, $R^I \equiv \epsilon f'(k)$.

To remove the Lagrange multiplier λ in (6), multiply (6) by ϵ and take expectations to obtain

$$E(m\epsilon f(k)) = \lambda\alpha.$$

Using (5),

$$\lambda\alpha = \frac{f(k)}{f'(k)}.$$

Then, (6) yields

$$m = \frac{\epsilon^{\alpha-1}}{\theta^{\alpha} f'(k)} \tag{7}$$

This second condition describes the firm's choice of shock. As expected, the firm chooses a higher shock ϵ in states with higher contingent claim price or discount factor. For example, consider $\alpha = 2$ and $\theta =$ constant. The firm will not necessarily choose $\epsilon = \theta$ with no variance. Contingent claims to some states may be so attractive that the firm is willing to shift some output to those states, even though this lowers mean output and raises its variance. A higher θ makes it easier to produce in a given state, and so corresponds to a lower contingent claim price.

As with the consumption first order condition, we can use (7) to infer what the discount factor must be, given observations of the productivity shock ϵ. Since any asset or claim to payoff x is a bundle of contingent claims, we can write its price $p = E(mx)$. Thus, we may write the asset pricing implications of this model as

$$p = E\left((\frac{\epsilon}{\theta})^{\alpha} \frac{1}{\epsilon f'(k)} x \right)$$

Note that in general, the discount factor is *not* the inverse investment return $\epsilon f'(k)$.

3.3 Multiperiod production functions and first-order conditions.

Again, the most natural approach might be to write a production function that represents a general ability to transform across all states and dates, such as

$$\sum_t \beta^t \sum_s \rho(s^t) g(c(s^t)) \le W$$

However, it again seems more fruitful to maintain as much of the structure of standard production functions as possible. To this end, I add the above specification for the choice of shocks to standard intertemporal technologies.

I start with a simple technology, which I will use in studying stochastic growth models in section 4 and a Mehra-Prescott style model in section 5. The following section adds an adjustment cost. This feature is required to get the variation in investment returns to be anything like the variation in stock returns. I use the model with adjustment costs to specify and test an asset pricing model in section 6.

3.3.1 Multiple period model without adjustment costs.

The production function is

$$\begin{aligned} y_t &= \epsilon_t f(k_t) \\ k_{t+1} &= (1-\delta)k_t + i_t \\ E_t\left((\epsilon_{t+1}/\theta_{t+1})^\alpha\right) &\le 1. \end{aligned}$$

The firm's objective is

$$\max E \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} m_{t+k} \right) (y_{t+j} - i_{t+j})$$

Analogously to (5) and (7), the first order conditions can be summarized by

$$1 = E_t[m_{t+1}(\epsilon_{t+1} f'(k_{t+1}) + (1-\delta))] \tag{8}$$

and

$$m_{t+1} = \frac{\epsilon_{t+1}^{(\alpha-1)}/\theta^{\alpha}}{f'(k_{t+1}) + (1-\delta)E_t(\epsilon_{t+1}^{(\alpha-1)}/\theta^{\alpha})} \tag{9}$$

Equation (8) is the usual condition that the investment return should be correctly priced,

$$1 = E_t(mR_{t+1}^I);\ R_{t+1}^I \equiv \epsilon_{t+1}f'(k_{t+1}) + (1-\delta).$$

This condition occurs whether or not the firm can choose the shock ϵ. Equation (9) describes the choice of shock ϵ from the equality of marginal rates of transformation and discount factors.

3.3.2 Derivation of the first-order conditions

To derive first order conditions, it is useful to state the problem recursively:

$$V(k,\epsilon) = \max_{\{i,\epsilon'\}} \epsilon f(k) - i + E_t(mV(k',\epsilon'))$$

s.t.

$$k' = (1-\delta)k + i,$$
$$E_t((\epsilon'/\theta)^{\alpha}) = 1;$$

The first order conditions are

$$\begin{aligned}
\partial/\partial i: &\quad 1 = \lambda_1 \\
\partial/\partial k': &\quad E_t(mV_k(k',\epsilon')) = \lambda_1 \\
\partial/\partial \epsilon': &\quad mV_{\epsilon}(k',\epsilon') = \lambda_2 \alpha \epsilon'^{(\alpha-1)}/\theta^{\alpha} \\
\text{Envelope}: &\quad V_k = \epsilon f'(k) + \lambda_1(1-\delta) \\
&\quad V_{\epsilon} = f(k)
\end{aligned}$$

and constraints. Substituting for λ_1 and V_k, we obtain

$$\partial/\partial k' : 1 = E_t[m(\epsilon' f'(k') + (1-\delta))].$$

Substituting for $V_{\epsilon'}$ multiplying $\partial/\partial\epsilon'$ by ϵ' and taking expectations,

$$E_t(m\epsilon')f(k') = \lambda_2 \alpha E_t(\epsilon'^{\alpha}/\theta^{\alpha}) = \lambda_2\alpha.$$

Substituting for λ_2 in $\partial/\partial\epsilon'$ and canceling,

$$m = E_t(m\epsilon')\epsilon'^{(\alpha-1)}/\theta^{\alpha}.$$

To eliminate $E_t(m\epsilon')$, use the $\partial/\partial k'$ condition,

$$1 = E_t(m\epsilon')f'(k') + (1-\delta)E_t(m)$$

$$E_t(m\epsilon') = \frac{1-(1-\delta)E_t(m)}{f'(k')}$$

Thus,

$$m = \frac{1-(1-\delta)E_t(m)}{f'(k')}\frac{\epsilon'^{(\alpha-1)}}{\theta^{\alpha}} \tag{10}$$

Taking expectations and solving,

$$E_t(m) = \frac{E_t(\epsilon'^{(\alpha-1)}/\theta^{\alpha})}{f'(k') + (1-\delta)E_t(\epsilon'^{(\alpha-1)}/\theta^{\alpha})}.$$

Finally, (10) yields

$$m = \frac{\epsilon'^{(\alpha-1)}/\theta^{\alpha}}{f'(k') + (1-\delta)E_t(\epsilon'^{(\alpha-1)}/\theta^{\alpha})}. \tag{11}$$

3.3.3 First-order conditions with adjustment costs and variable labor

The firm's problem is now

$$\max E \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} m_{t+k} \right) (y_{t+j} - i_{t+j} - w_{t+j} l_{t+j})$$

s.t.

$$y_t = \epsilon_t k_t^{\eta} l_t^{1-\eta} - \gamma(i_t, k_t) \tag{12}$$

$$k_{t+1} = (1-\delta)(i_t + k_t)$$

$$E_t[(\frac{\epsilon_{t+1}}{\theta_{t+1}})^\alpha] = 1.$$

$\gamma(i,k)$ is the adjustment cost.

The first order conditions, derived in the appendix, now imply

$$1 = E_t(m_{t+1} R^I_{t+1}),$$

where

$$R^I_{t+1} \equiv (1-\delta)\,\frac{1 + \epsilon_{t+1}\eta k_{t+1}^{\eta-1} l_{t+1}^{1-\eta} + \gamma_i(i_{t+1}, k_{t+1}) - \gamma_k(i_{t+1}, k_{t+1})}{1 + \gamma_i(i_t, k_t)}. \tag{13}$$

A discount factor $m*$ for excess returns satisfies

$$0 = E_t(m * R^e)$$

This discount factor is simple to derive from the first order conditions, since time t variables including the Lagrange multiplier on the technology shock constraint can be divided into the 0 on the left hand side. One such discount factor is[5]

$$m* = \frac{\epsilon^\alpha}{y\theta^\alpha} \tag{14}$$

The actual discount factor is somewhat more complicated, since one has to keep the time t variables straight. It is

$$m = \frac{1}{(1-\delta)}\,\frac{\left(\frac{k_{t+1}}{l_{t+1}}\right)^{1-\eta}\frac{\epsilon_{t+1}^{\alpha-1}}{\theta_{t+1}^{\alpha}}}{\eta + E_t\left[\left(\frac{k_{t+1}}{l_{t+1}}\right)^{1-\eta}\left(\frac{\epsilon_{t+1}^{\alpha-1}}{\theta_{t+1}^{\alpha}}\right)\left(\frac{1+\gamma_i(i_{t+1},k_{t+1})-\gamma_k(i_{t+1},k_{t+1})}{1+\gamma_i(i_t,k_t)}\right)\right]} \tag{15}$$

[5] Formula (14) also works if the shock ϵ multiplies the adjustment cost as well, i.e. if

$$y_t = \epsilon_t\left(k_t^\eta l_t^{1-\eta} - \gamma(i_t, k_t)\right).$$

However, in this case m itself cannot be derived in closed form, since we cannot solve for the Lagrange multiplier on the shock choice constraint.

As with habit-persistence models, a conditional expectation appears non-linearly in the formula for the stochastic discount factor. To find moment conditions that do not depend on agents' information sets, we can start with $p_t = E_t(m_{t+1}x_{t+1})$, multiply both sides by the denominator of m_{t+1} and take unconditional expectations, to obtain

$$E\left((1-\delta)\left[\eta + (\frac{k_{t+1}}{l_{t+1}})^{1-\eta}\frac{\epsilon_{t+1}^{\alpha-1}}{\theta_{t+1}^{\alpha}}(\frac{1+\gamma_i(i_{t+1},k_{t+1})-\gamma_k(i_{t+1},k_{t+1})}{1+\gamma_i(i_t,k_t)})\right]p_t\right) =$$

$$= E\left((\frac{k_{t+1}}{l_{t+1}})^{1-\eta}\frac{\epsilon_{t+1}^{\alpha-1}}{\theta_{t+1}^{\alpha}}x_{t+1}\right).$$

I use the following functional form for the adjustment cost function:

$$\gamma(i,k) = \frac{\beta}{2}(\frac{i}{k})i \tag{16}$$

With this functional form, the production function (12) implies

$$\epsilon = [(\frac{k}{y})^{\eta}(\frac{l}{y})^{1-\eta} - \frac{\beta}{2}(\frac{i}{k})\frac{i}{y}]^{-1} \tag{17}$$

Which now defines $m*$ in terms of observables, up to the shock θ, which is discussed below. The investment return (13) becomes

$$R_{t+1}^{I} \equiv (1-\delta)\frac{1+\epsilon_{t+1}[\eta(\frac{k_{t+1}}{l_{t+1}})^{\eta-1} + \beta(\frac{i_{t+1}}{k_{t+1}}) - \frac{\beta}{2}(\frac{i_{t+1}}{k_{t+1}})^2]}{1+\epsilon\beta(\frac{i_t}{k_t}))} \tag{18}$$

4 Effect of technology shock choice on a stochastic growth model

In this section, I examine the effect of allowing choice of technology shock on a standard stochastic growth model.

Giving firms choice over technology shocks *per se* results in no extra implications. We can always specify the shock choice set so that firms would

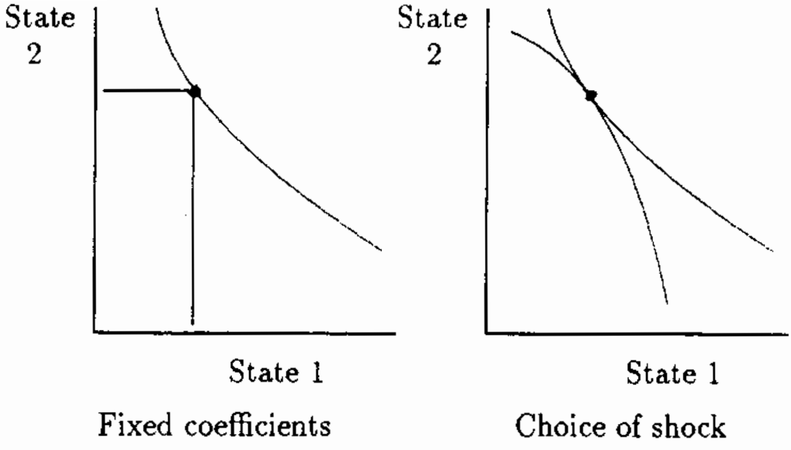


Figure 4: Firms can be hypothesized to choose any given distribution of shocks.

have chosen a given distribution of shocks. Figure 4 illustrates. Analogously, there is a production economy that gives the same consumption-asset price dynamics as any endowment economy and vice-versa.

However, we can ask the following questions. How does giving a firm choice over the actual technology shock alter the equilibrium of a model, holding other aspects of the model constant? In particular, can a firm choose a production shock that is *more* volatile than the underlying shock? If so, we might be able to understand the puzzling volatility of technology shocks required by real business cycle models as a *choice* given a set that does not favor one state much more than another. When will the firm choose a pro- or counter-cyclical technology shock? Choosing counter-cyclical shocks, in a sense described below, will be very useful in constructing successful asset pricing models, and it would be comforting if the result could occur in a standard general equilibrium model.

4.1 Set-up of the model

The planning problem is

$$\max E \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} + \psi \frac{(L_t)^{1-\gamma_n}}{1-\gamma_n} \right)$$

s.t.

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t$$

$$Y_t = A_t^{\xi} N_t^{\xi} K_t^{1-\xi}$$

$$E_t \left[\left(A_{t+1}^{\xi} / \Theta_{t+1}^{\xi} \right)^{\alpha} \right] \leq 1 \tag{19}$$

$$\theta_{t+1} = \phi\theta_t + \nu_{t+1}, \quad \nu_{t+1} \text{ i.i.d. } \mathcal{N}(0, \sigma_\nu^2) \tag{20}$$

$$N + L = 1$$

The shock is expressed as A^{ξ} to give a nonstochastic balanced growth path. I consider four special cases of increasing generality, and concomitant algebraic complexity. All the algebra is relegated to an appendix.

4.2 Two-period model with fixed labor supply

Restricting the model to two periods and ignoring the labor terms, we can solve the first-order conditions analytically. It will be convenient in what follows to examine logs of variables, denoted by lowercase letters.

The constraint that consumption exhaust second period product implies that consumption follows

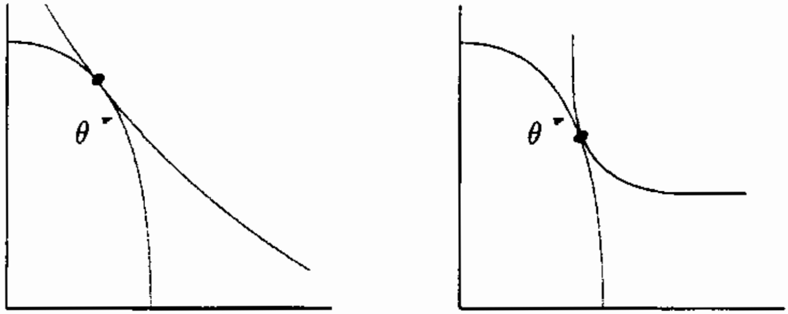
$$c_{t+1} = \xi a_{t+1} + (1-\xi) k_{t+1}$$

The shock a follows

$$a_{t+1} = \eta_{a\theta} \theta_{t+1} - \text{constant} \tag{21}$$

where

$$\eta_{a\theta} \equiv \frac{1}{1 + \frac{\gamma - 1}{\alpha}} \tag{22}$$



$\gamma < 1$ $\qquad\qquad$ $\gamma > 1$

Figure 5: How choice of shock ϵ depends on risk aversion γ.

When $\gamma = 1$, equation (21) reduces to $a_{t+1} = \theta_{t+1}$. If $\gamma > (<)\, 1$, then the elasticity $\eta_{a\theta}$ is $< (>)\, 1$. Thus, if consumers are less risk-averse than log, the firm *chooses* a pattern of technology shocks a that is *more* volatile than the underlying shock process θ, and vice-versa. Figure 5 illustrates. Finally, as $\alpha \to \infty$, the elasticity $\eta_{a\theta}$ approaches one, and $a = \theta$, for any value of γ. As $\alpha \to \infty$, the firm loses its ability to affect the production shock.

4.3 Two period model with variable labor supply

Next, I allow for varying labor supply within the two period model. Unfortunately, closed-form solutions of this model are not available. (Since labor is 1-leisure, the equilibrium quantities solve equations of the form $x = a(b-x)^c$.) Instead, I log-linearize the first-order conditions and constraints, to obtain "small deviation" solutions. Letting small case letters denote deviations of logs from means (i.e. constants suppressed), and with $\psi = 1$, the result is

$$a_{t+1} = \frac{1}{1 + \frac{\gamma-1}{\alpha\tau}}\, \theta_{t+1}. \tag{23}$$

$$c_{t+1} = \frac{\xi}{\tau}\, a_{t+1} = \frac{\xi}{\tau + \frac{\gamma-1}{\alpha}}\, \theta_{t+1} \tag{24}$$

$$l_{t+1} = \frac{\tau-1}{\tau}\frac{\phi}{\xi}\, a_{t+1} = \phi \frac{(\tau-1)}{\left(\tau+\frac{\gamma-1}{\alpha}\right)}\, \theta_{t+1} \tag{25}$$

where

$$\tau \equiv 1+\frac{\xi(\gamma-1)}{1+\gamma_n\phi},\ \phi=\frac{N}{L}$$

Implications

When $\xi = 0$ so production is unaffected by labor, or when $\gamma_n = \infty$ so consumers are unwilling to vary leisure, $\tau = 1$, the response of a to θ reduces to the value $\eta_{a\theta}$ (22) found in the model with no labor. Leisure (25) does not respond to shocks.

The direction of the response of a to a shock θ is still governed by risk aversion γ. From (23), the response of a is greater or less than one as γ is less than or greater than one. Labor has an effect on this relationship through the parameter τ which multiplies α in (23). When $\gamma > 1$, $\tau > 1$ as well.

Thus, when the firm chooses to attenuate production shocks, ($\gamma > 1$) the presence of labor causes the firm to attenuate less. However, when $\gamma < 1$, and the firm chooses to exacerbate production shocks, τ can take any value $\tau < 1$, including zero and negative values. Thus, the presence of labor can dramatically increase the sensitivity of shocks a to θ, and even result in a *negative* elasticity, or countercyclical choice of shock.

4.4 Infinite period model with fixed labor supply

Next, consider the infinite period model, but hold labor supply fixed.

An analytic solution exists for the special case of log utility and full depreciation, $\gamma = 1$, $\delta = 1$, $f(K) = K^\eta$, $\eta = 1-\xi$. The consumption and capital decision rules are well known[6]

$$C_t = (1-\beta\eta)A_t^\xi K_t^\eta,\ K_{t+1} = \beta\eta A_t^\xi K_t^\eta$$

[6] Try a value function, $V =$ const. $+ \frac{1}{1-\beta\eta}\ln(A_t K_t^\eta)$ and verify that it works.

In this case, we find

$$a = \theta$$

so the unit elasticity with log utility extends to this infinite-horizon case.

I follow Campbell (1992) in constructing an approximate analytical solution for other parameter values. Campbell's solution method log-linearizes the first-order conditions and constraints around values in a nonstochastic balanced growth path. The resulting linear system of equations can be solved analytically.

The first-order conditions and constraints are

$$C_t^{-\gamma} = E_t(\beta C_{t+1}^{-\gamma} R_{t+1}) \tag{26}$$

$$\beta C_{t+1}^{-\gamma} K_{t+1}^{1-\xi} = \Lambda_t \alpha A_{t+1}^{\xi(\alpha-1)} / \Theta_{t+1}^{\xi\alpha} \tag{27}$$

$$K_{t+1} = (1-\delta) K_t + A_t^{\xi} K_t^{1-\xi} - C_t \tag{28}$$

$$E_t\left[\left(A_{t+1}^{\xi}/\Theta_{t+1}^{\xi}\right)^{\alpha}\right]$$

where

$$R_{t+1} \equiv A_{t+1}^{\xi} K_{t+1}^{-\xi} + (1-\delta)$$

To obtain a log-linear version of the shock-choice constraint, hypothesize a solution in which $A_{t+1} = \lambda_t \Theta_{t+1}^{\eta_{a\theta}}$ as before. If so, A_{t+1} is also lognormal. Then, the shock choice constraint (20) is, in log form

$$E_t(a_{t+1}) = E_t(\theta_{t+1}). \tag{29}$$

Log-linearizing the other first order conditions and constraints, we obtain

$$E_t(\Delta c_{t+1}) = \sigma\lambda_3(E_t(a_{t+1}) - k_{t+1}) \tag{30}$$

$$-c_{t+1} + \sigma(1-\xi)k_{t+1} = \sigma\lambda_t + \sigma\xi(\alpha-1)a_{t+1} - \sigma\xi\alpha\theta_{t+1} \tag{31}$$

$$k_{t+1} = \lambda_1 k_t + \lambda_2 a_t + \lambda_4 c_t \tag{32}$$

$$E_t(a_{t+1}) = E_t(\theta_{t+1}) \cdot \tag{33}$$

$$\theta_{t+1} = \phi\theta_t + \nu_{t+1} \tag{34}$$

where

$$\lambda_1 = \frac{1+r}{1-g},\ \lambda_2 = \frac{\xi(r+\delta)}{(1-\xi)(1+g)},\ \lambda_3 = \frac{\xi(r+\delta)}{1+r},\ \lambda_4 = 1-\lambda_1-\lambda_2,$$

$$r,\ g = \text{steady state return to capital, growth rate },\ \sigma = 1/\gamma.$$

I guess a solution of the form

$$\begin{aligned} c_t &= \eta_{ck}k_t + \eta_{ca}a_t + \eta_{c\theta}\theta_t & (35) \\ a_{t+1} &= \phi\theta_t + \eta_{a\theta}(\theta_{t+1} - \phi\theta_t). & (36) \end{aligned}$$

Substituting this guess into the log-linearized first order conditions and constraints, and after some unpleasant algebra, we obtain

$$\begin{aligned} \eta_{a\theta} &= \frac{\sigma\xi\alpha - \eta_{c\theta}}{\eta_{ca} + \sigma\xi(\alpha-1)} \\ \eta_{ck} &= \frac{1}{2\lambda_4}\left(-Q_1 - \sqrt{Q_1^2 - 4\sigma\lambda_3\lambda_1\lambda_4}\right) \\ \eta_{ca} &= \frac{\lambda_2(\eta_{ck} + \sigma\lambda_3)}{1 - \lambda_4(\eta_{ck} + \sigma\lambda_3)} \\ \eta_{c\theta} &= \frac{-\phi(\eta_{ca} - \sigma\lambda_3)}{(\eta_{ck} + \sigma\lambda_3)\lambda_4 + \phi - 1} \end{aligned}$$

where

$$Q_1 \equiv \lambda_1 - 1 + \sigma\lambda_3\lambda_4,\ \sigma \equiv 1/\gamma.$$

To interpret the results, note that the consumption decision rule can also be written

$$c_t = \eta_{ck}k_t + (\eta_{ca} + \eta_{c\theta})a_t + \eta_{c\theta}(\theta_t - a_t).$$

or, substituting the decision rule for a,

$$c_t = \eta_{ck}k_t + (\eta_{ca} + \eta_{c\theta})a_t + \eta_{c\theta}(1 - \eta_{a\theta})(\theta_t - E_{t-1}\theta_t).$$

One can verify that $\eta_{ca} + \eta_{c\theta}$ gives the same expression as Campbell's expression for η_{ca}, derived in a model with $a = \theta$. Rewriting the decision rule this way, the first two terms are unaffected by the choice of shock parameter. Consumption responds to the difference between θ and a, or to the innovation in θ, since θ rather than a controls expectations of subsequent shocks.

It is also interesting to examine the innovation in consumption corresponding to an innovation in the underlying shock θ. Using $a_t - E_{t-1}a_t = \eta_{a\theta}(\theta_t - E_{t-1}\theta_t)$, the last equation implies

$$c_t - E_{t-1}c_t = (\eta_{ca}\eta_{a\theta} + \eta_{c\theta})(\theta_t - E_{t-1}\theta_t). \tag{37}$$

When there is no shock choice, $\eta_{a\theta} = 1$ and so disappears from the last expression.

What The Formulas Say

Four limits of the elasticities are easy to evaluate analytically. First, as $\alpha \to \infty$, $\eta_{a\theta} \to 1$. As expected, as the firm loses its ability to choose the shock a, the choice of a converges to θ.

Second, as $\phi \to 0$, $\eta_{c\theta} \to 0$. The only reason consumption responds to θ is that θ gives information about the choice set from which future technology shocks a will be drawn. When there is no such information (as in the two-period model) consumption doesn't respond to θ at all.

Third, as $\gamma \to 0$, $\eta_{a\theta} \to \alpha/(\alpha - 1)$. This is the same value as in the two-period model.

Fourth, varying the shock choice parameter α has no effect on the consumption elasticities η_{ck}, η_{ca} and $\eta_{a\theta}$. Its only effect is on the shock choice elasticity $\eta_{a\theta}$. As α increases, $\eta_{a\theta} \to 1$. As α decreases to one, $|\eta_{a\theta}| \to \infty$

Table 1 evaluates the elasticities $\eta_{a\theta}$ and $\eta_{c\theta}$ for the full depreciation case, $\delta = 1.0$. I use $g = 0.005$, $r = 0.015$ $\xi = 2/3$ as in Campbell (1992).

As in the two-period model, and as expected from the analytical solution for log utility, $\gamma = 1$ is a dividing line: for $\gamma > (<)1$, $\eta_{a\theta} > (<)1$ and $\eta_{c\theta} < (>)0$. However, in the two period model, as $\gamma \to \infty$, $\eta_{a\theta} \to 0$; the chosen shock approached a constant, and there were no negative values. This

$\eta_{c\theta}$

	γ					
ϕ	0	1/2	1	2	10	∞
0	0	0	0	0	0	0
0.9	-0.44	-0.14	0	0.15	0.4	0.08
0.95	-0.47	-0.15	0	0.16	0.46	0.16
1	-0.49	-0.16	0	0.18	0.54	0.99

$\eta_{a\theta}$

	γ					
ϕ	0	1/2	1	2	10	∞
0	2	1.3	1	0.75	0.36	0
0.9	2	1.3	1	0.59	-0.7	-8.2
0.95	2	1.3	1	0.57	-0.87	-16
1	2	1.3	1	0.56	-1.1	-100

Table 1: Elasticities with $\delta = 1$, $\alpha = 2$.

is no longer true; as $\gamma \to \infty$, $\eta_{a\theta}$ becomes a large negative number. The firm chooses *countercyclical* shocks. As $\gamma \to \infty$, the consumer desires a steady consumption stream. (As emphasized by Campbell, the model with $\gamma = \infty$ inherits some of the properties of the permanent income model.) A positive innovation in θ means that subsequent shocks will also be higher. To offset this, the firm chooses a very *low* realization in a for such states. I verify this intuition below.

Table 2 presents elasticities for a more realistic depreciation rate $\delta = 0.025$ (quarterly). The first thing to notice is that $\gamma = 1$ is no longer a special value. The shock elasticity $\eta_{a\theta}$ is greater than one for all values of ϕ, and the consumption elasticity $\eta_{c\theta}$ is zero at a value of risk aversion γ much less than one.

As in the $\delta = 1$ case, the shock elasticity $\eta_{a\theta}$ ranges from 2 to -100. Again, the negative elasticity allows the plan to give the consumer an even smoother stream than is possible with a fixed technology shock. To investigate this effect, table 3 presents the response of a consumption innovation to a θ innovation, as defined by equation (37). In the $\gamma = \infty$ column, the consumer

$\eta_{c\theta}$

	γ						
ϕ	0	0.1	0.5	1	2	10	∞
0	0	0	0	0	0	0	0
0.9	-10	-0.44	0.053	0.12	0.14	0.12	0.072
0.95	-11	-0.57	0.079	0.18	0.22	0.21	0.14
1	-11	-0.78	0.14	0.37	0.53	0.75	0.88

$\eta_{a\theta}$

	γ						
ϕ	0	0.1	0.5	1	2	10	∞
0	2	2	1.9	1.9	1.8	1.6	0
0.9	2	2	1.9	1.7	1.4	0.18	-8.2
0.95	2	2	1.9	1.6	1.2	-0.92	-16
1	2	2.1	1.8	1.4	0.38	-7.2	-100

Table 2: Elasticities with $\delta = 0.025$, $\alpha = 2$.

desires the smoothest possible consumption stream. In the model with no shock choice, one obtains permanent income-like results. For a transitory shock (low ϕ) consumption has a small positive response. (One can show that log consumption follows a random walk in this case.) However, when choice of shock is allowed, the increasingly negative (with ϕ) elasticity $\eta_{a\theta}$ allows the consumer to completely offset the shock, resulting in *no* consumption response!

Characterizing the shock process

An equivalent way of characterizing the effect of shock choice is to characterize the process for the chosen shock a_t. The model with choice is not observationally distinguishable from a model with no choice whose shock happened to follow the chosen shock process. From (36), we can write the a process as

$$a_t = \theta_t + (\eta_{a\theta} - 1)\nu_t \tag{38}$$

$$\theta_t = \phi\theta_{t-1} + \nu_t$$

Shock choice, $\eta_{ca}\eta_{a\theta} + \eta_{c\theta}$

	γ						
ϕ	0	0.1	0.5	1	2	10	∞
0	1.8	0.25	0.12	0.087	0.062	0.029	0
0.9	-8.3	-0.18	0.17	0.2	0.19	0.12	0
0.95	-8.8	-0.31	0.2	0.26	0.27	0.19	0
1	-9.4	-0.51	0.25	0.43	0.54	0.61	0

No shock choice, $\eta_{ca} + \eta_{c\theta}$

	γ						
ϕ	0	0.1	0.5	1	2	10	∞
0	0.89	0.13	0.063	0.046	0.034	0.018	0.0088
0.9	-9.2	-0.31	0.12	0.16	0.17	0.14	0.08
0.95	-9.7	-0.44	0.14	0.23	0.26	0.23	0.15
1	-10	-0.65	0.2	0.41	0.56	0.77	0.89

Table 3: Elasticities of consumption innovation w.r.t. θ innovation.

or

$$a_t = \left((1 - \phi L)^{-1} + (\eta_{a\theta} - 1)\right) \nu_t$$

Looking at equation (38), the effect of adding shock choice is to change the first element of the impulse-response function for the shocks by $(\eta_{a\theta} - 1)$.

For example, the strong negative values of $(\eta_{a\theta} - 1)$ found for high γ in table 2 result in a negative instantaneous response followed by the highly persistent positive AR(1) response. Roughly speaking, the negative instantaneous response makes the present value of the shocks zero, so no consumption change need take place.

4.5 Infinite-Period model with labor supply

Finally, I add labor supply to the infinite period model. I restrict attention to log utility, both to keep down the number of parameters and since separable labor and leisure and a stationary consumption/wage ratio require log utility (Campbell (1992), Ogaki and Cooley (1991), King Plosser and Rebelo

(1988)).

Again, I guess decision rules of the form

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t + \eta_{c\theta}\theta_t,$$

$$a_{t+1} = E_t(\theta_{t+1}) + \eta_{a\theta}(\theta_{t+1} - E_t(\theta_{t+1})).$$

Plugging these guesses into the linearized first order conditions and constraints, and after much tedious algebra, we find

$$\eta_{ck} = \frac{1}{2Q_2}\left(-Q_1 - \sqrt{Q_1^2 - 4Q_0 Q_2}\right)$$

$$\eta_{ca} = \frac{-\lambda_2(1+\nu\xi)[(1+\lambda_3\nu)\eta_{ck} + \lambda_3(1-\nu(1-\xi))]}{(\lambda_4 - \lambda_2\nu)[(1+\lambda_3\nu)\eta_{ck} + \lambda_3(1-\nu(1-\xi))] - 1}$$

$$\eta_{c\theta} = -\phi\frac{(1+\lambda_3\nu)\eta_{ca} - \lambda_3(1+\xi\nu)}{[(1+\lambda_3\nu)\eta_{ck} + \lambda_3(1-\nu(1-\xi))](\lambda_4 - \lambda_2\nu) + \theta(1+\lambda_3\nu) - 1}$$

$$\eta_{a\theta} = \frac{(1+\xi\nu)\eta_{c\theta} - \xi\alpha}{\xi^2\nu - \xi(\alpha-1) - (1-\xi\nu)\eta_{ca}}.$$

where

$$Q_2 = (1+\lambda_3\nu)(\lambda_4 - \lambda_2\nu)$$

$$Q_1 = (\lambda_4 - \lambda_2\nu)(\lambda_3(1-\nu(1-\xi))) + (1+\lambda_3\nu)(\lambda_1 + \lambda_2\nu(1-\xi)) - 1$$

$$Q_0 = \lambda_3(1-\nu(1-\xi))(\lambda_1 + \lambda_2\nu(1-\xi)).$$

λ_1, λ_2, λ_3, λ_4 are as given above, $\sigma_n \equiv 1/\gamma_n$ and

$$\nu \equiv \frac{(1-N)\sigma_n}{N + (1-\xi)(1-N)\sigma_n}.$$

.

What the formulas mean:

The elasticities reduce to the values given above for the model with fixed labor supply when $\gamma_n = 0$ and hence $\nu = 0$,

Tables 4 -6 evaluate some of these elasticities for a range of values of the leisure curvature parameter γ_n and the persistence parameter ϕ.

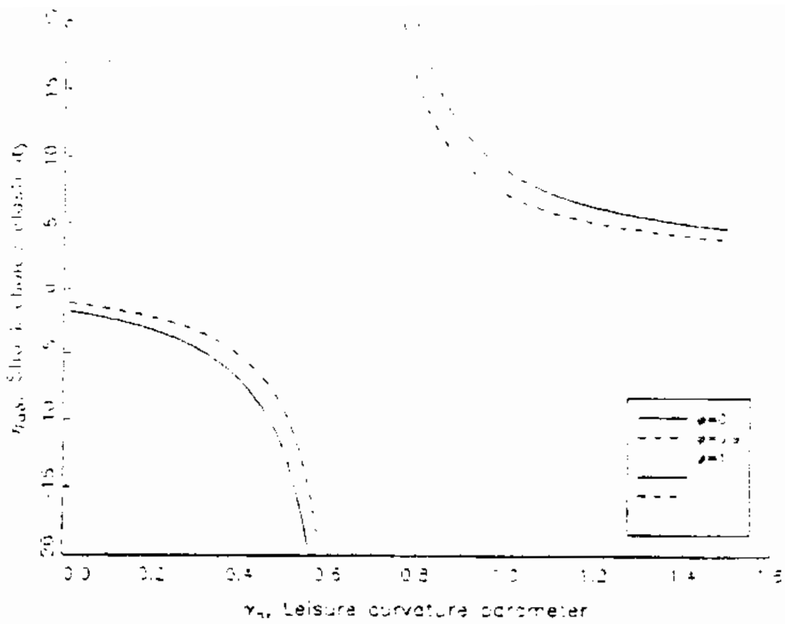


Figure 6: Shock elasticity $\eta_{a\theta}$ as a function of labor curvature parameter γ_n.

	$\eta_{a\theta}$						
	γ_n						
ϕ	0	0.1	0.5	1	2	10	∞
0	-1.7	-2.3	-12	9	3.6	2.1	1.9
0.9	-1.1	-1.6	-9.3	7.2	3.1	1.9	1.7
0.95	-0.79	-1.2	-7.9	6.4	2.8	1.8	1.6
1	-0.19	-0.44	-4.6	4.3	2.1	1.5	1.4

Table 4: Shock choice elasticity for variable labor model.

As shown in table 4, the shock elasticity $\eta_{a\theta}$ now can take on a wide range of positive and negative values. As in the two-period model, there is an interior singular point. It occurs at $\gamma_n \approx 0.53$. Figure 6 graphs the shock elasticity as a function of γ_n to make the behavior clearer. The shock elasticity is larger for smaller values of persistence ϕ.

Tables 5 present consumption elasticities and 6 presents labor supply elasticities. As with consumption, the addition of a choice of shock allows labor supply to be constant when $\gamma_n = \infty$.

Shock choice, $\eta_{ca}\eta_{a\theta} + \eta_{c\theta}$

	γ_n						
ϕ	0	0.1	0.5	1	2	10	∞
0	-0.19	-0.24	-1	0.66	0.23	0.11	0.087
0.9	0.053	0.0075	-0.63	0.68	0.33	0.22	0.2
0.95	0.16	0.12	-0.44	0.69	0.39	0.28	0.26
1	0.38	0.35	0.0098	0.7	0.52	0.45	0.43

No shock choice, $\eta_{ca} + \eta_{c\theta}$

	γ_n						
ϕ	0	0.1	0.5	1	2	10	∞
0	0.11	0.1	0.084	0.074	0.064	0.051	0.046
0.9	0.28	0.27	0.24	0.22	0.2	0.17	0.16
0.95	0.35	0.34	0.31	0.29	0.27	0.24	0.23
1	0.5	0.5	0.48	0.46	0.45	0.42	0.41

Table 5: Elasticities of consumption innovation w.r.t. θ innovation. Variable Labor Model.

Shock choice, $\eta_{na}\eta_{a\theta} + \eta_{n\theta}$

	γ_n						
ϕ	0	0.1	0.5	1	2	10	∞
0	-2.9	-3.5	-12	6.4	1.6	0.25	0
0.9	-2.3	-2.7	-9.6	5	1.3	0.2	0
0.95	-2	-2.4	-8.3	4.3	1.1	0.17	0
1	-1.5	-1.7	-5.2	2.6	0.65	0.099	0

No shock choice, $\eta_{na} + \eta_{n\theta}$

	γ_n						
ϕ	0	0.1	0.5	1	2	10	∞
0	1.7	1.5	1	0.71	0.45	0.12	0
0.9	1.2	1	0.73	0.54	0.35	0.092	0
0.95	0.95	0.85	0.61	0.45	0.3	0.08	0
1	0.49	0.44	0.32	0.24	0.16	0.046	0

Table 6: Elasticities of labor innovation w.r.t. θ innovation.

5 Considerations in specifying a technology for asset pricing models

It is useful in constructing general equilibrium models or specifying a discount factor model for asset pricing to consider some qualitative facts that the model should reproduce.

First, the discount factor must be volatile and correlated with asset returns. From the basic pricing equation for excess returns

$$0 = E(mR^e) = E(m)E(R^e) + \rho(m, R^e)\sigma(m)\sigma(R^e),$$

we have

$$\sigma(m) = -\frac{E(m)}{\rho(m, R^e)}\frac{E(R^e)}{\sigma(R^e)}.$$

The slope of the mean-standard deviation frontier $E(R^e)/\sigma(R^e)$ is about 0.2 in postwar quarterly data or about 0.4 annually. Thus, even if the discount factor and returns are perfectly correlated, we need $\sigma(m) = 0.2$ or 20% (40 % annually). This is a large value compared to the consumption growth, GNP growth, or even the market return. This requirement can lead to the implausibly high estimates of risk aversion coefficients. If m and R^e are less than perfectly correlated, even higher variances are required.[7]

Second, nearly risk-free rates are about $1 - 2\%$, so the mean discount factor should be around 0.98 annual or 0.995 quarterly. In the consumption-based model, risk aversion coefficients that generate adequate $\sigma(m)$ together with subjective discount factors $\beta < 1$ imply $E(m)$ around 0.85 and thus risk free rates around 15% per quarter. This is the "risk-free rate" puzzle.

Third, expected excess returns on stocks are positive. The mean excess return on stocks is about 7% per year. This fact implies that stock returns

[7]The requirement for high $\sigma(m)$ is the essence of the Mehra-Prescott (1985) equity premium puzzle. See Hansen and Jagannathan (1991), Gallant Hansen and Tauchen (1990) for a more sophisticated version of the calculation. Cochrane and Hansen (1992) emphasize the puzzle posed by the low correlation of asset returns with consumption growth.

must be negatively correlated with the discount factor. From the pricing equation $0 = E(mR^e)$ we have

$$E(R^e) = -\frac{\text{cov}(m, R^e)}{E(m)}.$$

Since $E(m) > 0$, and $E(R^e) > 0$, $\text{cov}(m, R^e) < 0$.

Fourth, excess returns are procyclical–they are positively correlated with growth rates of macroeconomic series such as consumption, output, and investment.

Facts three and four mean that the discount factor is likely to be *negatively* correlated with macroeconomic series. For example, the consumption-based model generates $m_{t+1} = \beta(c_{t+1}/c_t)^{-\gamma}, \gamma > 0$, so the discount factor m is negatively correlated with consumption growth. (Of course, $\text{Cov}(m, R) < 0$, $\text{cov}(R, y) > 0$ does not imply $\text{cov}(m, y) < 0$. However, it is convenient to start by thinking in terms of a one-shock model, in which case the sign of correlation is transitive.)

On the other hand, mean returns on long-term bonds are about the same as treasury bill returns, even though their standard deviation is almost that of stock portfolios. Thus, long term bond returns should show very little correlation with discount factors.

The negative correlation of discount factors and returns, and the negative correlation of discount factors and macro series are consistent with the standard real business cycle model model with stable preferences and procyclical technology shock. Fig. 7 illustrates in a two-state two-date endowment economy. It shows a higher discount factor (contingent claim price) in the "bust" state, and hence a negative correlation of the discount factor with consumption. The discount factor will also be negatively correlated with the returns of assets (such as claims to output or consumption) that pay off well in the "boom" state.

The pattern of correlations speaks against a view of the world with stable (state-invariant) *technology*, in which fluctuations are driven by *preferences*.

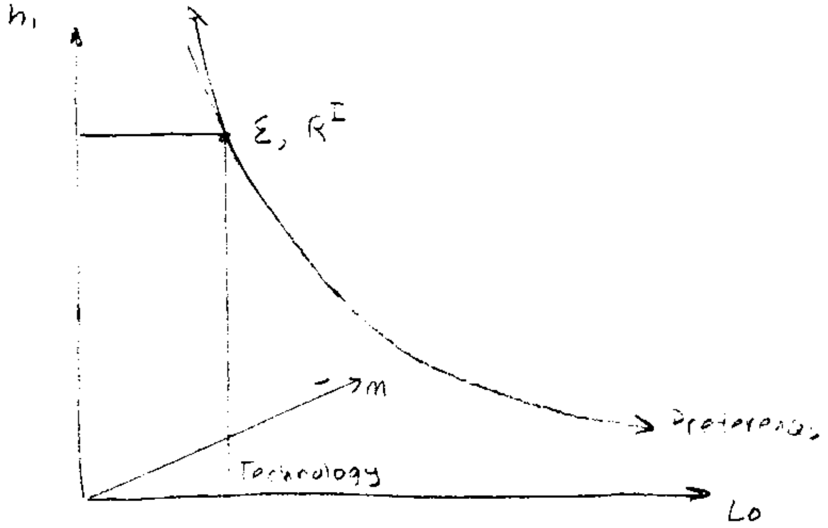


Figure 7: Negative correlation of returns and discount factor in standard model.

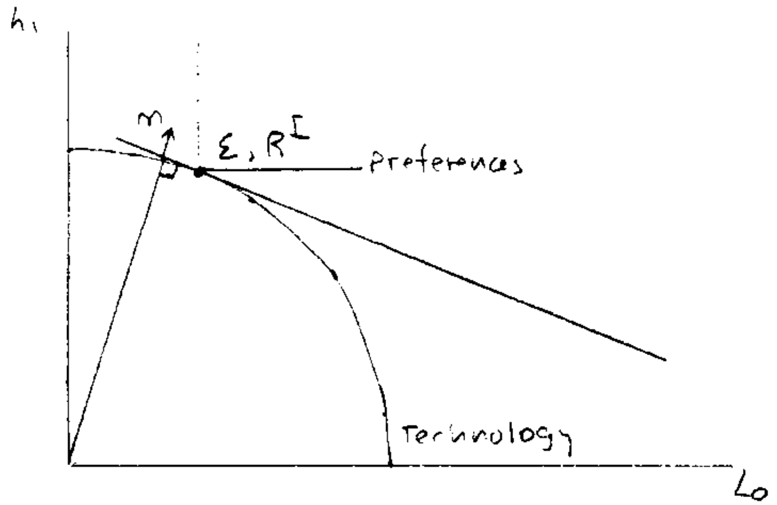


Figure 8: Positive correlation of returns and discount factor in simple production-based model with preference shocks.

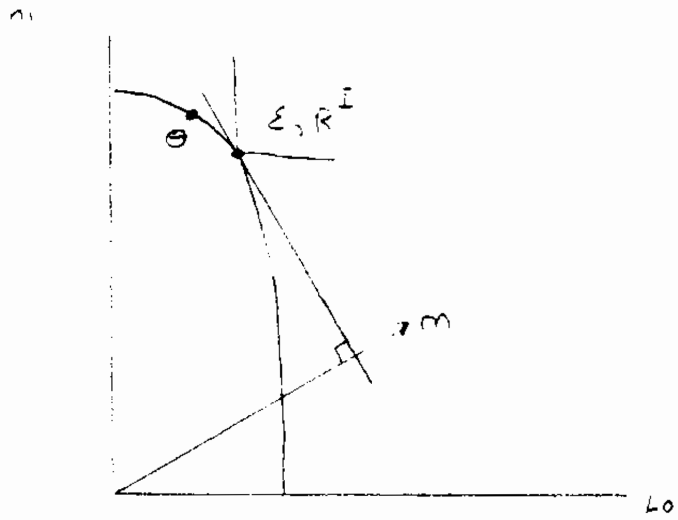


Figure 9: Negative correlation of returns and discount factor in a model with procyclical technology shocks.

(Stable does not mean "no shocks", it means that the choice set for shocks does not favor one state or the other.) For example, suppose the firm can choose the distribution of the endowment shock as in Fig. 8. In this world, output is higher in a boom *because* contingent claims prices are higher in a boom, not *despite* the fact that prices are lower in a boom. This model delivers a discount factor *positively* correlated with returns, and thus mean returns *lower* than the risk-free rate.

The models below include a variety of features to this end, but we can illustrate two simple possibilities right away. First, suppose there is an underlying productivity shock θ that is positively correlated with the "preference shock". In this case, the discount factor is again negatively correlated with returns. Fig. 9 illustrates.

Adding labor to the technology also helps to overcome the problem. Procyclical labor can take the place of a procyclical shock θ. For example, consider a technology defined by

$$y_t = \epsilon_t f(l_t),\ E((\epsilon_t/\theta)^\alpha) \leq 1.$$

The firm's production set for output is

$$E[(\frac{y_t}{\theta f(l_t)})^\alpha] \leq 1.$$

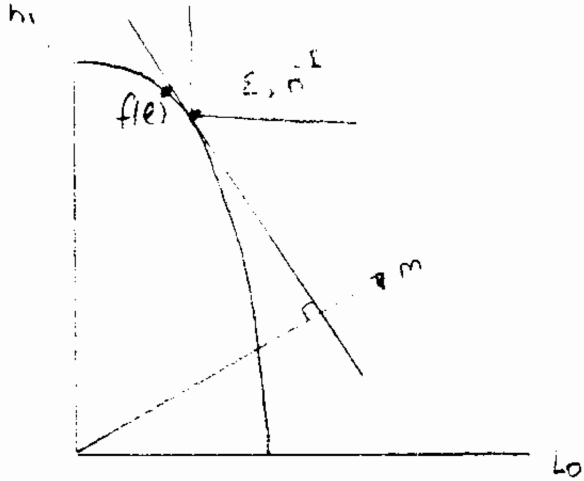


Figure 10: Negative correlation of returns and discount factor in a model with procyclical labor shocks.

Then, if labor l_t is higher in a boom, we obtain a production-possibility set such as illustrated in fig. 10.

6 A Mehra-Prescott Style Model

6.1 Specification

The model is specified as follows:

Technology:

$$\tilde{y}_t = \tilde{\epsilon}_t f(\tilde{k}_t) = \tilde{a}_t^{1-\eta} \tilde{k}_t^{\eta} \tag{39}$$

$$\tilde{y}_t = \tilde{c}_t + \tilde{i}_t \tag{40}$$

$$\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{i}_t \tag{41}$$

$$E_t \left[\left(\frac{\tilde{a}_{t+1}^{1-\eta}}{\tilde{\theta}_{t+1}^{1-\eta}} \right)^{\alpha} \right] \leq 1 \tag{42}$$

Preferences:

$$\min\left(\frac{\tilde{c}_t}{\tilde{\chi}_t}, \frac{\tilde{c}_{t+1}}{\tilde{\chi}_{t+1}}, \frac{\tilde{c}_{t+2}}{\tilde{\chi}_{t+2}}, \ldots\right) \tag{43}$$

Uncertainty:

$$\tilde{\chi}_t = G^t \chi(s_t, s_{t-1}), \tilde{\theta}_t = G^t \theta(s_t, s_{t-1}).$$

$$\Pr(s' \mid s) = \text{Prob}(s_{t+1} = s' \mid s_t = s).$$

The tilde notation distinguishes between levels and detrended values of any variable,

$$x_t \equiv \tilde{x}_t / G^t.$$

The technology is a standard fixed-labor technology, with the addition of the shock-choice possibility. In analogy to the Lucas (1978) - Mehra-Prescott endowment specification of technology, I regard consumption data as an expression of Leontief *preferences.* As with the production side of endowment economies, this specification is not restrictive. The asset pricing relations are valid for any preferences, given the technology and equilibrium consumption process. Using an endowment for production or Lenotief preferences is simply a convenience when one does not want to derive the equilibrium consumption process from more fundamental assumptions. Also, c need not represent nondurable / services consumption. c can represent durable goods, or whatever object the firm sells to consumers.

The model is set up to deliver a trend G in output, capital, and consumption. A Markov process for growth in the shocks rather than their deviations from trend would be more palatable. However, that assumption leads to solutions in which capital is a state variable, where here I am able to find solutions in which the current shock s is the only state variable. For the same reason, I do not include adjustment costs in the production function.

I allow both the consumption shock χ and the underlying production shock θ to be functions of the current as well as lagged state. This gives greater flexibility in describing variation in conditional distributions of both shocks. Of course, models in which one or the other is only a function of the current state are a straightforward special case.

6.2 Solving the model, finding asset prices

To solve the model, I follow the usual procedure. I find quantities by solving a planning problem; then I read asset prices from marginal rates of *transformation.*

6.2.1 Find Quantities

The planning problem is trivial in endowment economies. Equlibrium quantities are simply given by endowments. Things are not so simple in this case, for two reasons. First, consumers have fixed-coefficient preferences across dates and states, but would still welcome additional consumption goods in all dates and states. Thus, we have to solve for the level of consumption. Second, given that firms must deliver a sequence of consumptions $\{\tilde{c}_1, \tilde{c}_2, \ldots\}$, there are potentially many different combinations of output and investment choices that can deliver (at least) the given consumption sequence. Thus, we have to find the sequence of capital, investment and output that delivers the optimal consumption stream.

Since the technology allows one to transform output between all states and dates, consumption will be proportional to the consumption shock.

$$\tilde{c}_t = \gamma \tilde{\chi}_t,$$

or, detrending

$$c_t = c(s_t, s_{t-1}) = \gamma \chi(s_t, s_{t-1}).$$

Thus, the planning problem is to choose $\{\gamma, \tilde{y}_t, \tilde{c}_t, \tilde{x}_t, \tilde{a}_t\}$ to maximize γ subject to the technology (39) - (43) and $\tilde{c}_t = \gamma \tilde{\chi}(s_t, s_{t-1})$

Using (39) - (41) to substitute output and investment in terms of capital, and expressing the result with detrended variables, we can pose the problem with only the shock constraint remaining as

$$\max_{\{k_t, \gamma\}} \gamma \text{ s.t. } E_t\left[\left(\frac{Gk_{t+2} - (1-\delta)k_{t+1} + \gamma\chi(s_{t+1}, s_t)}{\theta(s_{t+1}, s_t)^{1-\eta} k_{t+1}^{\eta}}\right)^{\alpha}\right] \leq 1 \; \forall s_t \qquad (44)$$

I search for *stationary* solutions in which capital depends only on the current state.[8] Thus, the problem is

$$\max_{\{k(s),\gamma\}} \gamma \text{ s.t. } \sum_{s'} \Pr(s' \mid s)\left(\frac{Gk(s') - k(s)(1-\delta) + \gamma\chi(s,s')}{\theta(s,s')^{1-\eta}k(s)^{\eta}}\right)^{\alpha} \leq 1 \quad (45)$$

Given the stationary solution for γ and $k(s)$, the remaining variables can be found by

$$y(s,s') = Gk(s') - (1-\delta)k(s) + \gamma\chi(s,s')$$

$$c(s,s') = \gamma\chi(s,s')$$

$$i(s,s') = y(s,s') - c(s,s')$$

$$\epsilon(s,s') = \frac{y(s,s')}{k(s)^{\eta}}$$

Outuput y_t and the shock ϵ_t will be functions of both s_{t-1} and s_t, even when the shocks θ_t and χ_t are functions of s_t only. The firm *chooses* the more complex distribution of the actual technology shock. Also, keep in mind that capital is determined one period ahead of time, $k_t = k_t(s_{t-1})$.

---

[8] To motivate such solutions, consider a finite-period version of the model. Since output can be transformed across dates and states, $k_{T+1} = 0$. Then, the shock constraint (44) implies

$$\sum_{s_T} \Pr(s_T \mid s_{T-1})\left(\frac{-k_T(1-\delta) + \gamma\chi(s_T, s_{T-1})}{\theta(s_T, s_{T-1})^{1-\eta}k_T^{\eta}}\right)^{\alpha} = 1$$

The left hand side is a monotonically declining function of k_T which starts at $+\infty$ and ends at 0 when $y_T = 0$. It implies a unique value of k_T for each s_{T-1}. Thus, k_T is a function of $s_{T-1}, k_T(s_{T-1})$.

Continuing, the shock constraint (44) implies

$$\sum_{s_T} \Pr(s_{T-1} \mid s_{T-2})\left(\frac{Gk_T(s_{T-1}) - k_{T-1}(1-\delta) + \gamma\chi(s_{T-1}, s_{T-2})}{\theta(s_{T-1}, s_{T-2})^{1-\eta}k_{T-1}^{\eta}}\right)^{\alpha} \leq 1$$

Again, this equation can be solved for k_{T-1}, and k_{T-1} is a function of s_{T-2}. If γ is not too high (if the solution is feasible), this process converges to a stationary solution $k(s)$.

6.2.2 Find Asset Prices

The producer's first order conditions (9) imply that the marginal rate of transformation is

$$m(s,s') = \frac{\epsilon(s,s')^{(\alpha-1)}/\theta(s,s')^{\alpha(1-\eta)}}{\eta k(s)^{\eta-1} + (1-\delta)E(\epsilon(s,s')^{(\alpha-1)}/\theta(s,s')^{\alpha(1-\eta)} \mid s)}$$

Other asset prices and returns follow from the discount factor m. The slope of the conditional mean-standard deviation frontier is $\sigma_t(m)/E_t(m)$. The slope of the unconditional mean-standard deviation frontier is $\sigma(m)/E(m)$. The real risk-free rate is $R_t^f = 1/E_t(m_{t+1})$.

A claim to the detrended consumption stream[9] has price

$$p_t^{cd} = E_t \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} m_{t+k} \right) c_{t+j}$$

Hence,

$$p^{cd}(s) = \sum_{s'} Pr(s' \mid s) m(s,s')(p^{cd}(s') + c(s,s'))$$

Letting $\mathbf{p}^{cd}$ denote the vector of $p^{cd}(s)$ over states s, $\mathbf{p}^{cd}$ obeys

$$\mathbf{p}^{cd} = A\mathbf{p}^{cd} + (A. * C)\mathbf{1}$$

where

$$A_{s,s'} \equiv \Pr(s' \mid s) m(s',s),\ C_{s,s'} \equiv c(s,s'),$$

and $.*$ denotes element by element multiplication. (If $c(s,s')$ is only a function of s', then $\mathbf{p}^{cd} = A\mathbf{p}^{cd} + A\mathbf{c}$.) Hence, the price can be found from

$$\mathbf{p}^{cd} = (I-A)^{-1}(A. * C)\mathbf{1}$$

[9]The price of the actual consumption stream is infinite in the nonstochastic version of the model, since the plan – maximize consumption – yields an interest rate equal to the growth rate. In stochastic versions of the model, the price is no longer infinite, but still uncomfortably large, on the order of 10^9. The returns of the claim to detrended consumption are quite similar, but the prices are more reasonable, on the order of the inverse growth rate, or 50. For this reason, I value the detrended consumption stream rather than the actual consumption stream.

and the return follows.

A perpetuity can stand for long-term bonds. Letting $\mathbf{p}^b$ denote a vector of the perpetuity price in each state $p^b(s)$, the perpetuity price obeys

$$\mathbf{p}^b = A(\mathbf{p}^b + 1)$$

so

$$\mathbf{p}^b = (I - A)^{-1} A$$

and the return follows.

In addition, we can track the investment return, without explicitly finding the price of the capital stock, from

$$R^I_{t+1} = \epsilon_{t+1} \eta k^{\eta-1}_{t+1} + (1 - \delta).$$

6.2.3 Numerical procedure.

I solve the model by numerically searching for a solution to the first order conditions of the planning problem (45). The choice objects are the capital stock $k(s)$ in each state, and the consumption multiplier γ. Starting at the analytical solution to the first-order conditions in the nonstochastic case, GAUSS's equation solver NLSYS is able to find solutions to the first order conditions in a few iterations.

6.3 simulation results

As explained above, our job is to find parameterizations of the model that 1) generate an slope of the mean-standard deviation frontier or "risk-premium" of roughly 0.4 as in the data, 2) generate expected returns on the investment and consumption-claim return that are greater than the risk-free rate, mimicking stock returns, and 3) generate expected returns on the perpetuity that are roughly the same as the risk-free rate. In addition, the model should 4) replicate the volatility and autocorrelation of consumption, output, investment, and capital.

Two-state iid model with no underlying productivity shock

I start with a very simple model. It does not replicate any of the above desiderata, but its failings help to motivate the features of models that do. There are no underlying technology shocks, so $\theta(s, s') = 1.0$. The preference shock is only a function of the current state, $\chi(s) = 1.02$ or 0.98. Probabilities $Pr(s' \mid s)$ depend only on the final state, not on the initial state. Table 7 presents the resulting equilibrium quantities, asset prices and returns.

Consumption in each state is simply proportional to the consumption shock in that state. We will need to add autocorrelation to the consumption shocks to get autocorrelated consumption.

In this model, the future looks the same no matter what the current state. Therefore, capital is the same for all states. The shock is chosen so that output is higher in states with a higher consumption shock and lower in states with a lower consumption shock. Investment is the same in all states, so that capital stays the same in all states.

Since output, the chosen shock ϵ and the productivity shock θ all depend only on the final state, so does the marginal rate of substitution. The slope of the mean-standard deviation frontier $\sigma(m)/E(m)$ is 0.014, much less than in the data. This can be cured by a higher α, and also will rise when other features, such as persistence in shocks and variation in the underlying technology shocks, are added to the model. The risk-free rate is a constant, equal to the growth rate of the economy.

The investment return R^I and return on the claim to detrended consumption do vary. However, their standard deviations (less than a percent) are trivial compared to the standard deviation of stock returns (roughly 20% on an annual basis.)

The low standard deviation of investment returns is expected, and is not likely to be solved by plausible parameterizations of this model. The investment return is $\epsilon f'(k) + (1 - \delta)$. Variation in the marginal product of capital or technology shock cannot plausibly account for more than a percent or two variation in investment return. As another way to see the problem, consumption can be freely traded for capital in a model with no adjustment costs, so the price of installed capital is always *one*. To generate realistic standard deviations of the investment return, one needs to add adjustment costs or some other wedge between installed and uninstalled capital.

The expected values of the investment return and return to a claim on detrended consumption are both *lower* than the risk-free rate. The ratios $\frac{E_t(R^e)/\sigma_t(R^e)}{\sigma(m)/E(m)}$ tell us where the return R^e is relative to the minimum variance frontier. Since the ratios are -1.00, the returns are on the lower portion of the frontier. To explain this fact, note that both returns are positively correlated with m as in Figure 8.

The claim to the perpetuity has constant price, and hence pays the same return as the risk-free rate.

Adding uncorrelated technology shocks

Table 8 presents the results from a model with four states. I add a technology shock of +/- 3%. I start with a technology shock that is uncorrelated with the preference shock.

Since the future still looks the same from each state, capital and investment are again constant. Surprisingly, output still varies *only* in response to the preference shock χ. The firm does not take advantage of the fact that it is easier to produce in some states than in others. There is no point to doing so, since the objective is to maximize the minimum consumption across states. If the firm planned to produce more in high θ states and less in low θ states, a succession of low θ states would force it to lower consumption. The Lagrange multiplier on the technology constraint λ does vary as the productivity shock θ varies.

The addition of the technology shock raises the slope of the mean-standard deviation frontier $\sigma(m)/E(m)$ to 0.04, but this is still too low. Again, a

Table 7: Simulation of Mehra-Prescott style model. No θ shock, probabilities do not depend on initial state. $\alpha = 2$, $\eta = 0.3$.

	From state	To state s'=hi	To state s'=lo
Assumptions			
ξ		1.02	0.98
θ		1.00	1.00
$Pr(s' = \mid s)$	all	0.4	0.6
Equilibrium quantities			
Capital k		3.70	3.70
Output y	all	1.51	1.46
Investment i	all	0.44	0.44
mrt	all	0.997	0.969
$\sigma_t(m)/E_t(m)$		0.014	0.014
Asset prices and returns			
Risk-free (%)		2.00	2.00
Investment R^I (%)	all	2.20	1.863
$E_t(R^I)$		1.998	1.998
$E_t(R^{Ie})$		-0.0023	-0.0023
$(E_t/\sigma_t(R^{Ie}))$ / $(\sigma_t/E_t(m))$		-1.00	-1.00
Detr. cons. p^{cd}		51.8	51.8
Return (%)	all	2.05	1.97
$E_t(R^{cd})$		1.9995	1.9995
$E_t(R^{cde})$		-0.0005	-0.0005
$(E_t/\sigma_t(R^{cde}))$ / $(\sigma_t/E_t(m))$		-1.00	-1.00
Perpetuity price p^b		50.0	50.0
Return R^b (%)	all	2.00	2.00

higher transformation coefficient α and other modifications can easily raise this number.

The risk-free rate and perpetuity rate are still 2%, equal to the growth rate, and prices of all assets are still constant (since the future still looks the same from every date). The investment return and perpetuity return are now no longer on the bottom half of the minimum variance frontier, but still have mean returns less than the risk free rate. Their Sharpe ratio $E(R^e)/\sigma(R^e)$ is -32% of the slope of the mean-standard deviation frontier.

Correlated preference, technology shocks

As suggested above, productivity shocks that are positively correlated with the "preference shocks" are a device that can lead to expected returns greater than the risk free rate. Table 9 presents a model just like that of table 7, except that a +/- 3% technology shock has been added to the model. Since there are only two states, the production and technology shocks are perfectly correlated. The table only shows results that are substantially different from those in table 7.

The slope of the mean-standard deviation frontier is raised to about 0.027. Most importantly, the investment and detrended consumption claim returns now are negatively correlated with the marginal rate of transformation, and lie on the upper half of the minimum variance frontier. Thus, a positively correlated technology and preference shock have just the effect suggested by figure 7

Adding autocorrelated shocks

The previous models show no autocorrelation in any of the macroeconomic variables. Next, I modify the model by selecting the transition matrix to match the autocorrelation of detrended consumption observed in the data (0.92). Tables 10 and 11 present the results.

Now the future does not look the same from both states: since the persistence is positive, it is much more likely that the "hi" state will follow the "hi" state. Thus, capital, output and investment all vary across states.

Table 8: Simulation of Mehra-Prescott style model. Four state model with uncorrelated preference, technology shocks; no serial correlation. $\alpha = 2$, $\eta = 0.3$.

	From state	To state s'=h,h	s'=h,l	s'=l,h	s'=l,l
Assumptions					
ξ		1.02	1.02	0.98	0.98
θ		1.03	0.97	1.03	0.97
$Pr(s' =\mid s)$	all	0.25	0.25	0.25	0.25
Equilibrium quantities					
Capital k		3.70	3.70	3.70	3.70
$\lambda(s)$		0.180	0.195	0.175	0.190
Output y	all	1.50	1.50	1.46	1.46
Investment i	all	0.44	0.44	0.44	0.44
m.r.t m	all	0.95	1.04	0.93	1.01
$\sigma_t(m)/E_t(m)$		0.044	0.044	0.044	0.044
Asset prices and returns					
Risk-free (%)		2.00	2.00	2.00	2.00
Investment R^I	all	2.17	2.17	1.83	1.83
$E_t(R^I)$ (%)		1.998	1.998	1.998	1.998
$(E_t/\sigma_t(R^{Ie}))$ / $(\sigma_t/E_t(m))$		-0.32	-0.32	-0.32	-0.32
Detr. cons. $p^c d$		51.8	51.8	51.8	51.8
Return R^{cd} (%)	all	2.04	2.04	1.96	1.96
$E_t(R^{cd}$ (%)		1.999	1.999	1.999	1.999
$(E_t/\sigma_t(R^{cde}))$ / $(\sigma_t/E_t(m))$		-0.32	-0.32	-0.32	-0.32
Perpetuity price p^b		50	50	50	50
Return R^b (%)	all	2.00	2.00	2.00	2.00

Table 9: Simulation of Mehra-Prescott style model. Productivity and preference shocks are perfectly correlated. Probabilities do not depend on initial state.

	From state	To state s'=hi	s'=lo
Assumptions			
ξ		1.02	0.98
θ		1.03	0.97
Equilibrium quantities			
$\sigma(m)/E(m)$		0.027	0.027
Asset prices and returns			
Investment R^I (%)			
$(E_t/\sigma_t(R^{Ie}))$ / $(\sigma_t/E_t(m))$		+1.00	+1.00
Detr. cons. p^{cd}			
$(E_t/\sigma_t(R^{cde}))$ / $(\sigma_t/E_t(m))$		+1.00	+1.00

Investment is much more volatile (10.4%) than output (2.72%)

The marginal rate of substitution is still about 1/10 as volatile as it should be: $\sigma(m)/E(m) = 0.042$ rather than 0.20. A value of α near 5 can raise $\sigma(m)/E(m)$ to 0.4 without much effect on the other results. Though the actual values of m vary by much more than 0.04, the probability of changing states is so low, that $\sigma(m)$ is low.

The risk-free rate now varies as well with a 0.5 % standard deviation.

All the asset returns now vary as a function of initial as well as final states. However, all three assets have conditional means lower than the risk-free rate and are again on the lower half of the minimum variance frontier.

Thus, one can introduce autocorrelation through autocorrelation of the shocks. It improves the fit of the quantity dynamics, but reintroduces expected returns less than the risk-free rate, even with perfectly correlated shocks.

Table 10: Simulation of Mehra-Prescott style model. Productivity and preference shocks are perfectly correlated. Probability matrix replicates persistence of consumption.

	From state	To state s'=hi	To state s'=lo	uncond. moments
Assumptions				
ξ		1.02	0.98	
θ		1.03	0.97	
$Pr(s' =\| s)$	hi	0.96	0.04	
	lo	0.04	0.96	
Equilibrium quantities				
Capital k		3.54	3.76	
Output y	hi	1.48	1.67	
	lo	1.28	1.47	
$\sigma(y)/E(y)$ (%)				2.72
Investment i	hi	0.42	0.66	
	lo	0.22	0.45	
$\sigma(i)/E(i)$ (%)				10.4
m.r.t. m	hi	0.97	1.19	
	lo	0.79	0.99	
$\sigma_t(m)/E_t(m)$		0.044	0.040	0.042

Table 11: Continuation of last table.

	From state	To state s'=hi	To state s'=lo	uncond. moments
Asset prices and returns				
Risk-free $R_t^f = 1/E_t(m)$ (%)		2.63	1.63	2.13
$\sigma(R_t^f)$ (%)				0.50
Investment R^I (%)	hi	2.54	4.16	
	lo	0.16	1.68	
$E_t(R^I)$ (%)		2.61	1.62	2.11
$E_t(R^{Ie})$		-0.014	-0.012	-0.13
$(E_t/\sigma_t(R^{Ie}))$ / $(\sigma_t/E_t(m))$		-1.00	-1.00	-0.99
Detr. cons. p^{cd}		48.6	53.2	
Return (%)	hi	2.18	11.53	
	lo	-6.64	1.91	
$E_t(R^{cd})$		2.55	1.56	2.06
$E_t(R^{cde})$		-0.08	-0.06	-0.07
$(E_t/\sigma_t(R^{cde}))$ / $(\sigma_t/E_t(m))$		-1.00	-1.00	-0.99
Perpetuity price p^b		47.0	51.8	
Return R^b (%)	hi	2.13	12.4	
	lo	-7.37	1.93	
$E_t(R^b)$		2.54	1.56	2.05
$E_t(R^{be})$		-0.089	-0.071	-0.080
$(E_t/\sigma_t(R^{be}))$ / $(\sigma_t/E_t(m))$		-1.00	-1.00	-0.99

7 A production based asset pricing model

The empirical exercise that motivates this paper is the construction of a production-based asset pricing model, i.e. specification and testing of a model for the marginal rate of transformation to be used in $1 = E(mR)$ for a vector of asset returns R.

7.1 Specification

I use the technology with adjustment costs specified in section 3. In the general equilibrium models studied in the last two sections, we could specify a process for the shock θ, and judge success by how well the model replicated features of the observable data. That approach is not feasible here. With an arbitrary shock θ, we can pick its value at each data point to make the model fit exactly. Unless we severely restrict other parts of the model, for example by making all variables functions of a low-dimensional Markov state vector there is no way to identify θ. One must make *some* distributional or other assumptions on θ for the model to have any content.

In this section, I let θ_{t+1} be a random variable known at time t. This choice specifies that there is *no* underlying technology shock, i.e., that it is not fundamentally easier to produce in one state rather than another. As a result, the model must include labor supply, and rely on procyclical labor to generate positive expected excess returns and a positive correlation between returns and macroeconomic variables, as discussed in section 4.6 above.

I would like to accommodate growth, via the usual device of a technology shock that follows a random walk with drift. Thus, $\ln(\epsilon_{t+1}) = \ln(\mu) + \ln(\epsilon_t) +$ iid shock should be a possible choice for the firm. Equivalently, $\epsilon_{t+1}/\epsilon_t =$ i.i.d. shock should be feasible. In addition, θ_{t+1} is known at time t. The natural way to accomplish both objectives is to let $\theta_{t+1} = \mu\epsilon_t$.

Substituting in (15), we have

$$m* = \frac{\epsilon_{t+1}^{\alpha}}{y_{t+1}\epsilon_t^{\alpha}}.$$

Finally, it is convenient for $m*$ to be a stationary variable. Exploiting the fact that we can multiply $m*$ by any variable known at time t without changing the pricing implications, we have

$$m* = \frac{y_t}{y_{t+1}}\left(\frac{\epsilon_{t+1}}{\epsilon_t}\right)^{\alpha}. \tag{46}$$

Together with the moment conditions

$$0 = E_t(m_{t+1}R_{t+1}^{e})$$

for a vector of asset and investment excess returns R^e, (46), (17), (18) define the testable equations for the model[10]. This is a standard GMM problem. The parameters to be estimated are $\delta, \beta, \eta,$and θ.

7.2 Results

(To be completed.)

[10]Since the production function is linearly homogeneous (average q equals marginal q) this model also predicts that the investment return is equal to the market return on a claim to the firm's capital stock, ex-post as well as ex-ante. Cochrane (1991) exploits this prediction.

Appendix.

A Production sets.

A.1 Axioms for production sets.

In a two-date, finite-state economy, the commodity space is $\Re^{S+1}$ – consumption today, and consumption tomorrow in each state. Thus, production possibility sets are composed of elements

$$\{-k,\, y(s)\} = \{-k,\, y(1),\, y(2), \ldots,\, y(S)\}$$

In a continuous-state economy, y is a continuously-valued random variable, so elements of the production possibility sets can be written

$$\{-k,\, y(\omega)\}$$

where $y(\omega)$ denotes a random variable.

Following the standard textbook treatment (for example, Varian (198x)) we define the production possibility set Y as all the elements $\{-k,\, y\}$ that the firm can achieve. I will make the following assumptions about production possibility sets.

A1. Free disposal.

$$\text{If } \{-k,\, y\} \in Y, \text{ then } y' \leq y \text{ and } k' \geq k \text{ implies } \{-k',\, y'\} \in Y.$$

This assumption just states that the firm can throw away inputs or outputs.

A2. Convexity.

$$\begin{aligned}&\text{If } \{-k_1,\, y_1\} \in Y, \{-k_2,\, y_2\} \in Y \text{ and } 0 \leq \alpha \leq 1,\\ &\text{then } \{-(\alpha k_1 + (1-\alpha)k_2),\, \alpha y_1 + (1-\alpha)y_2\} \in Y.\end{aligned}$$

This assumption is less obvious. For example, it says that if the firm can achieve two distributions of technology shocks, then it can achieve any technology shock that is a linear combination of the first two.

For motivation, note that the production sets derived by aggregating fixed-coefficient production functions as in the last section are convex:

Lemma: Free disposal (A1) and convex $f(k)$ imply convex Y in the above aggregation model.

Proof: Recall that total output is given by

$$y(\omega) = \int dz\ \lambda(\omega, z) f(k(z)).$$

Constructing the right hand side for a linear combination of inputs k_1 and k_2, the firm can produce an amount

$$y'(\omega) = \int dz\ \lambda(\omega, z) f(\alpha k_1(z) + (1-\alpha) k_2(z)).$$

Since f is assumed concave,

$$f(\alpha k_1(z) + (1-\alpha) k_2(z)) \geq \alpha f(k_1(z)) + (1-\alpha) f(k_2(z))$$

Hence,

$$y'(\omega) \geq \int dz\ \lambda(\omega, z)[\alpha f(k_1(z) + (1-\alpha) f(k_2(z))] = \alpha y_1(\omega) + (1-\alpha) y_2(\omega).$$

By free disposal, the amount $\alpha y_1(\omega) + (1-\alpha) y_2(\omega)$ must therefore be feasible.

□

So far, production sets with kinks such as in figure 1 are still accommodated. The next assumption establishes the existence of at least some smoothness across states.

The firm can replicate assets by making small marginal investments. The production set is smooth when the cost to the firm of replicating a small long position is the same as its gain from replicating a small short position. To make this notion precise, let p_l(l for long) denote the least cost to the firm to replicate a small fraction of a payoff $x(\omega)$:

$$p_l = \inf_{k', y'(\omega), \delta} \left(\frac{k' - k}{\delta}\right) \text{ s.t.}$$

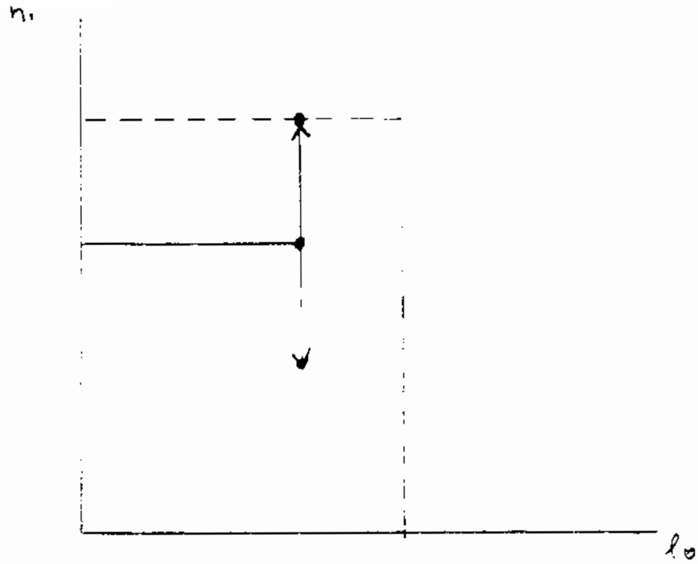


Figure 11: Synthesizing a contingent claim with kinked technology

$$y'(\omega) - y(\omega) \geq \delta x(\omega),\ \{-k,\ y(\omega)\} \in Y,\ \{-k',\ y'(\omega)\} \in Y, \delta > 0.$$

Similarly, let p_s(s for "short") denote the greatest savings for the firm to replicate a small fraction of the negative of $x(\omega)$,

$$p_s = \inf_{k', y'(\omega), \delta} (\frac{k - k''}{\delta})$$

s.t.

$$y(\omega) - y''(\omega) \geq \delta x(w),\ \{-k,\ y(\omega)\} \in Y,\ \{-k'',\ y''(\omega)\} \in Y, \delta > 0.$$

Note that p_l and p_s may depend on where the firm is currently producing.

In general, p_l and p_s are not equal. For example, consider a firm with a kinked production technology as in figure 11. Here, the firm is trying to synthesize contingent claims to the h state. To sell such a claim, is must produce more, and then discard any l state output. To buy such a claim, it only needs to throw away some l state output, with no change in inputs. Thus, p_l is positive and p_s is zero. With a smooth production set, as in figure 12, p_l and p_s are the same.

Define X to be the set of payoffs the firm can price uniquely,

$$X \equiv \{x(\omega) \text{ s.t. } p_l = p_s\}.$$

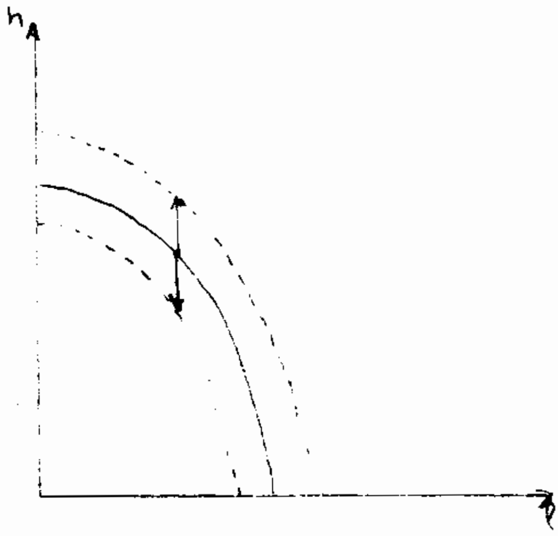


Figure 12: Synthesizing a contingent claim with smooth technology

With a kinked production set, only claims whose payoffs are proportional to the production shock have the same buy and sell prices p_l and p_s.

$$\text{A3a: } X = \{x(\omega) : \ x(\omega) = \alpha\epsilon(\omega)\}$$

The opposite extreme is that the firm can modify its shock to uniquely price any possible asset:

$$\text{A3b: } X = L^2$$

where L^2 represents the space of all (finite second moment) random variables[11].

Assumption A3b is too much to reasonably hope for. Thinking in terms of the aggregation model of section 2, if a random variable is independent of the production shocks of all the underlying technologies (this week's lotto drawing, phase of the moon) then we cannot hope that the firm can rearrange its production to make a marginal change in total output whose distribution matches that random variable.

However, we are often interested in pricing much smaller sets of state-contingent securities. For example, for pricing NYSE stocks, we only need to assume

$$\text{A3c: } X = \{c \cdot R, \ R = \text{ vector of N NYSE asset returns }\}$$

[11]One often limits the commodity space to finite-second moment random variables in asset pricing applications to ensure that prices, represented by second moments, exist.

Even this assumption may be more stringent than needed for many applications. A large body of research in finance examines "arbitrage pricing" models, in which the pricing of a large number of assets can be reduced to pricing a few systematic macroeconomic "factors". Asset returns are expressed as a linear combination of the factors $c \cdot f$ plus zero-price residuals. In this case, we could uniquely price assets assuming only

$$\text{A3d: } X = \{c \cdot f,\ f = k\text{-dimensional vector of factors}\}$$

A.2 First order conditions for optimization

Given a production set that satisfies axioms $A1 - A3$, we can derive relations between production variables and asset payoffs from firms' first order conditions. The simplest statement given the current set-up is, *the firm removes any arbitrage opportunities between physical investment and asset payoffs.*

More precisely, suppose the firm can trade an asset with payoff x. Recalling the definition of p_l and p_s as the smallest price at which the firm can synthesize a marginal long position and largest price at which it can synthesize a marginal short position with payoff x, we must have that

$$p_s \leq \text{ price of } x \leq p_l.$$

If not, the firm would synthesize such marginal assets and sell them, making arbitrage profits. Obviously, when $p_l = p_s$, we can assign a unique price to the payoff x from knowledge of where the firm is operating in its production set.

A.3 Existence and Differentiability of production functions.

Assumptions A1 and A2 on production sets imply the existence of a function

$$g(y(s),\ -k) : \Re^{S+1} \rightarrow \Re$$

that is monotone and convex, and $g(\cdot) \geq 0$ for $\{y, -k\} \in Y, g(\cdot) < 0$ for $\{y, -k\}$ not in Y.

Assumptions A3 translate into the *differentiability* of the production functions. If all derivatives of g exist, then we can price any payoff as in A3b. Totally differentiating the production function

$$\sum_s \partial g/\partial y(s) dy(s) = \partial g/\partial k dk.$$

To price a payoff $x(s)$, consider changing output y in the direction x, so $dy(s) = \delta\, x(s)$. Then the price of x must be dk/δ or

$$\sum_s \partial g/\partial y(s) x(s)/(\partial g/\partial k).$$

Production functions that price only payoffs in $X \subset \Re^S$ have only directional derivatives in X.

B Estimating production functions

In estimating standard production functions $y = \epsilon f(k,l)$, one often runs a regression such as $\ln y_t = \eta \ln k_t + (1-\eta) \ln \epsilon_t$, or similar nonparametric procedures. It would seem that production function estimation is relatively straightforward. This approach does not easily extend to estimation of the shock choice set, unless we observe the entire information set that the firm observes.

Suppose we have a long dataset of inputs and outputs. Consider a few special cases:

Case 1: Contingent claims prices (equivalently, the conditional distribution of all asset returns) are constant over time.

Since prices are constant, inputs k and the output random variable $y(\omega)$ will be constant over time. Thus, in this case, the data simply fill out *one* point in the production possibility set: *one* input k, and *one random variable* for output y. Without *some* variation in prices to induce some variation in the firm's choice of input and output, we won't map out the set of random variables for output.

Case 2: The conditional distribution of asset returns varies with a vector of instruments z, which we and the firm observe.

As z varies over time, the chosen input k, $k \mid z$, and random variable for output $y(\omega) \mid z$ will change too. Now, we can see some changes in the random output chosen by firms, so we can price some asset payoffs. However, there is no reason to expect that the firm will choose *every* point in its production set for some value of z. Hence, we may see only a subset of the asset pricing implications.

Case 3: The conditional distribution of asset returns varies with an instrument z, which the firm observes, and which we do not.

Since we do not observe z, this situation looks to us like case 1, no change in the conditional distribution of asset returns. But now the random variable $y*$ we observe is a mixture of the random variables actually chosen by the firm.

$$y*(\omega) = \int dPr(z)\ (y(\omega) \mid z).$$

The assumption of convexity (A2) implies that $\{k,\ y*(\omega)\}$ so defined is also in the production set. Thus, though the firm will have the ability to vary its output random variable more than we think, the production set we can see $\{k,\ y*(\omega)\}$ may be a stricct subset of the true production set. The elements $\{k,\ y*(\omega)\}$ we observe will typically not be on the boundary of the production set.