NBER WORKING PAPER SERIES

PRODUCTION BASED ASSET PRICING

John H. Cochrane

Working Paper No. 2776

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
November 1988

This research is part of NBER's research program in Economic Fluctuations. Any opinions expressed are those of the author not those of the National Bureau of Economic Research.

NBER Working Paper #2776
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ABSTRACT

This paper exploits producer's first order conditions to link asset prices to data on investment, output, etc. through marginal rates of transformation, just as consumer's first order conditions are commonly used to link asset prices to consumption data or proxies through marginal rates of substitution. It presents simulation economies analogous to the consumption based models of Mehra and Prescott (1985) and Backus, Gregory and Zin (1986) that capture the size of the equity premium and the size and cyclical timing of the forward rate term premium.

John H. Cochrane
University of Chicago
Department of Economics
1126 E. 59th Street
Chicago, IL 60637

Production Based Asset Pricing

1. Introduction

Much of the current theory of asset pricing is based on the consumer's first order conditions,

$$1 = \rho\, E_t \left[R_{t+1} \frac{u'(c_{t+1})}{u'(c_t)} \right] \tag{1}$$

where R_{t+1} is the return on an asset and ρ is the consumer's discount factor. This equation can be derived from the statement that the consumer's marginal rate of substitution between date and state contingent claims should be equal to the price ratio of such claims. This paper explores the equivalent equation for producers, which can be derived from the condition that the marginal rate of *transformation* between date and state contingent claims should also equal their price ratio. Its purpose is to provide an empirically tractable framework for linking asset returns to macroeconomic fluctuations.

There is a great deal of evidence that such links exist: term premia in the bond market, futures premia in the foreign exchange market, and risk premia in the stock market vary through time and are correlated with macroeconomic variables. (See, among others, Hansen and Hodrick (1983), Fama and Bliss (1987), Chen, Roll and Ross (1986), and Ferson (1986)). The most common approach to explaining these results uses (1) to link asset price phenomena to consumption data or state variables presumed to determine consumption, but this approach has not been particularly successful to date. As a result, a great deal of work on the specification of the consumption based model is in progress, including durability, continuous time, a variety of consumption goods, new forms for the utility function (state nonseparability, habit persistence), lack of perfect consumption insurance, heterogeneous consumers, borrowing constraints, money, etc.

One difficulty in empirically implementing consumption based models is that they require information on how the *conditional distribution* of consumption varies over time and in response to the events of the business cycle. But business cycles are a prominent phenomenon of output, durables purchases, investment, inventories, employment etc., not of nondurable or services consumption. To illustrate the relative smoothness of consumption, Fig. 1 presents the log values of consumption of nondurables and services together with gross fixed private investment and purchases of consumer durables. Also, as Wilcox (1988) has recently emphasized, the concept of consumption in the theoretical models corresponds weakly to the quantities used in the income and product accounts, especially when they are seasonally adjusted. Furthermore, deviations from optimal decision rules that are large enough to destroy empirically useful predictions about the cyclical relation between consumption and asset returns can imply minute utility costs. Cochrane (1988) calculates this utility loss for a variety of models, and finds that it is typically on the order of 1¢-10¢ per quarter, which is interpreted to say that the predictions of the theory are sensitive to the modelling of small (1¢-10¢/quarter) costs of information acquisition or processing, transactions, etc..

This paper doesn't attempt to solve these or other problems with consumption based models. Instead it ties the important cyclical variables such as output, investment, etc. directly to asset prices via the producer's first order conditions, completely ignoring their tie to (or through) nondurable consumption using the consumer's first order conditions.

Of course, one hopes to eventually produce an empirically tractable general equilibrium model of asset prices and economic fluctuations, one that includes consumer and producer first order conditions and market equilibrium, because partial equilibrium models cannot explore the effects of fundamental sources of uncertainty, such as changes in technology or government policy. (Brock (1982) is an example of a prototype general equilibrium model with production, and the model in this paper is closely related to the production side of Brock's model.)

However, the behavior of asset prices in general equilibrium depends on a mixture of preference and technological parameters, so it is hard to understand the mapping between the structure of a general equilibrium model and the qualitative features of the asset prices that result. Hence, as a great many insights into the behavior of a complete general equilibrium model appear from the consumer's first order conditions alone, ignoring producers, so many insights may appear from the corresponding producer's first order conditions, ignoring consumers. By studying the empirical implications of each set of first order conditions separately before uniting them, we can learn about preferences and technology in isolation, and this should simplify the task of producing empirically useful general equilibrium models.

Either consumer or producer first order conditions describe restrictions between the stochastic processes followed by real variables and asset prices or returns. Just as (1) describes a relation that must hold between asset returns and consumption no matter what the production technology, so producer first order conditions [(10) below] describe a relation that must hold between asset returns and production variables no matter what the preferences.

These restrictions can be exploited empirically in three ways. First, we can model the stochastic process for quantities, derive the process for prices, and compare those prices or the corresponding returns to data. This is Mehra and Prescott's (1985) approach, or the approach of any empirical implementation of Lucas' (1978) asset pricing model to data generated by a production economy. (Models that are based on (1) are often called "general equilibrium" following Lucas. However, the stochastic process for consumption in an economy with storage or production is not exogenous as in Lucas' model, so when these models are applied to real data, they in fact only exploit partial equilibrium relationships.) Second, we can model the stochastic process for prices, derive what the quantity process should be, and compare those quantities to data. This is the approach of permanent income theory and the Q-theory of investment. Third, we can model the joint stochastic process for prices and quantities and test whether the restrictions implied by the first order condition hold. This is Hansen and Singleton's (1983) approach.

The simulations of sections 3 and 4 use the first approach to see if the size of the equity premium and the size and cyclical timing of the forward rate term premium are consistent with data on investment and consumer durables purchases, using a simple specification of technology. The purpose of this paper, like that of Mehra and Prescott's paper, is to see whether there *exist* simple and approximate models with "reasonable" parameters and functional forms that are capable of *explaining* a few well-documented phenomena, before proceeding to the construction of detailed (and hence, unavoidably, complex) models that can be formally tested. This paper finds that there *are* such production-based models, as Mehra and Prescott argued that there *are no* such consumption based models.

A *test* assesses whether a model, including its auxiliary specification and statistical assumptions, is capable of explaining *all* phenomena for which one can derive predictions. In particular, the model presented in this paper and Mehra and Prescott's model have implications for asset prices beyond the first two moments, and those implications are ignored. In both cases, the models are so simple that one can derive the prices for contingent claims, and thus one can deduce all moments of asset returns and the prices of all derivative securities. A literally-minded formal test of either model would investigate whether all these predictions are satisfied, and such a test would certainly reject the models in sec. 3 and 4 of this paper as well as Mehra and Prescott's model. The exercise of both papers is a prelude to, and not a substitute for, formal testing and the search for specification and statistical assumptions that can hope to pass such formal testing. Such a test is under construction, in the form of a test whether the physical rates of return on a few technologies can act as factors for stock returns.

2. Asset Prices and Producer's First Order Conditions

Producers' first order conditions state that the marginal rate of *transformation* between state and date contingent claims achievable by varying investments in a variety of technology must be equal to the price ratios of such claims implicit in asset prices and payoffs. Alternately, they state that firms should adjust investment, production, etc. until they can no longer short a portfolio of assets that mimics the pattern of returns across states of nature provided by a marginal unit of investment in their technology, invest the proceeds in their technology, and make a sure (marginal) profit. Thus, they state that physical returns must lie in the space of asset returns, and so act as factors for asset returns. This section reviews these statements of producer's first order conditions in a simple environment with discrete time, a finite number of states, and complete markets. None of these elements are essential, but they simplify the mathematics and they establish the formulas used in the simulation economies that follow.

Uncertainty comes from a state variable s_t. s_t can take one of S values, $\{\lambda_1, \lambda_2, \ldots, \lambda_S\}$. The cumulative history of shocks at time t is denoted $s^t = \{s_0, s_1, s_2, \ldots, s_t\}$. $s^t s_{t+1}$ denotes the states s^{t+1} which follow a given state s^t. $P(s^t)$ is the time 0 price to a claim to a unit of a single consumption good $c(s^t)$ delivered at time t in state s^t. An asset is a claim to a contingent stream of payoffs $\{d(s^1), d(s^2) \ldots\}$, where the list extends over all dates and states. The asset's price at time t in state s^t (i.e. with $c(s^t)$ as numeraire) is

$$P^A(s^t) = \sum_{\tau>t} \sum_{s^\tau = s^t s_{t+1} \ldots s_\tau} \frac{P(s^\tau)}{P(s_t)} d(s^\tau) \tag{2}$$

The notation under the second sum indicates that it is taken over all states s^τ which follow s^t. Let

$$p(s^t, s_{t+1}) = \frac{P(s^t s_{t+1})}{P(s^t)}$$

denote the one period ahead contingent claims price, and let

$$R^A(s^t, s_{t+1}) = \frac{P^A(s^t s_{t+1}) + d(s^t s_{t+1})}{P^A(s^t)}$$

denote a one period asset return. Define the vector $\underline{p}(s^t) = [p(s^t, \lambda_1)\ p(s^t, \lambda_2)\ \ldots\ p(s^t, \lambda_S)]'$ and $\underline{R}^A(s^t) = [R^A(s^t, \lambda_1)\ R^A(s^t, \lambda_2)\ \ldots\ R^A(s^t, \lambda_S)]'$. Then, (2) implies that returns lie in a conditional linear space,

$$1 = \sum_{s_{t+1}} p(s^t, s_{t+1})\ R^A(s^t, s_{t+1}) = \underline{p}(s^t) \cdot \underline{R}^A(s^t). \tag{3}$$

Fig. 2 illustrates (3) for the case S=3. The axes are returns or payoffs in each state at t+1, so contingent claim prices and asset returns are points in $\mathbb{R}^3$. (3) implies that all returns lie on a plane, characterized by its orthogonality to the vector of contingent claims prices. Hansen and Richard (1987) derive this representation in a more general setting. Here it is just an accounting relation that must hold between asset prices and contingent claim prices. The calculations that follow use either consumer or producer first order conditions to identify the contingent claim prices, and then characterize asset returns from the contingent claim prices via (3).[1]

A firm has access to N technologies i= 1,2,...N with which it can transfer some of the consumption good forward through time. The firm chooses a production plan { $c(s^t)$ $I_i(s^t)$ $k_i(s^t)$ } (the list extends across all dates, states and technologies) to maximize its contingent claim value

$$\max \sum_{\tau} \sum_{s^{\tau}} P(s^{\tau})\, c(s^{\tau})$$

subject to the constraints

$$y_t = \sum_{j=1}^{N} f_j(k_{jt}, s_t) \tag{4}$$

$$y_t = c_t + \sum_{j=1}^{N} I_{jt} \tag{5}$$

$$k_{jt+1} = g_j(k_{jt}, I_{jt}) \qquad j = 1 , \ldots , N \tag{6}$$

k_0 given, and k_t, $c_t \geq 0$ for all t. Here and below, I omit the dependence on state where it's not necessary, to keep the notation simpler. k_{jt} is really $k_j(s^t)$, etc.. k_{jt} denotes the jth capital stock, so (4) describes the production function. I_{jt} denotes investment in the jth technology, to (5) is a resource constraint. (6) is the capital accumulation rule. The function g_j allows for adjustment costs in investment. This could be achieved equivalently with investment in the production function, but the above form turns out to be more convenient.

The first order conditions to this maximization are

$$1 = \sum_{s_{t+1}} p(s^t, s_{t+1}) \left[\frac{\partial f_i(t+1)}{\partial k_{t+1}} + \frac{\partial g_i(t+1) \;/\; \partial k_i}{\partial g_i(t+1) \;/\; dI_i} \right] \frac{\partial g_i(t)}{\partial I_{it}} \qquad i=1, \ldots N \tag{7}$$

where the notation (t+1) means "evaluated with respect to the appropriate arguments at time t+1 in state s^{t+1}". (Throughout, I assume that the inequality constraints $k_t>0$ $I_t>0$ and $c_t>0$ are not binding.)

Let $R_i(s^t, s_{t+1})$ denote the physical rate of return from state s^t to state $s^t s_{t+1}$ available from investment in the ith technology, or

$$R_i(s^t, s_{t+1}) = \left[\frac{\partial f_i(t+1)}{\partial k_{t+1}} + \frac{\partial g_i(t+1) / \partial k_i}{\partial g_i(t+1) / dI_i} \right] \frac{\partial g_i(k_{it}, I_{it})}{\partial I_{it}} . \tag{9}$$

If the producer invests one extra unit in technology i at time t, he can lower investment in that technology by $R_i(s^t, s_{t+1})$ at date t+1 in state $s^t s_{t+1}$, and leave his future production plan unchanged. Let $\underline{R}_i(s^t)$ denote a vector of $R_i(s^t, s_{t+1})$ over the S states s_{t+1}, as $\underline{R}^A(s^t)$ is defined above (3). With these definitions, the first order conditions (7) become

$$1 = \sum_{s_{t+1}} p(s^t, s_{t+1})\, R_i(s^t, s_{t+1}) = \underline{p}(s^t) \cdot \underline{R}_i(s^t) \qquad i = 1 , \ldots , N \tag{10}$$

The cost of investing one unit of consumption in the ith technology is 1, using $c(s^t)$ as numeraire. The benefits (evaluated at time t) are $\Sigma_{s_{t+1}} p(s^t, s_{t+1}) R_i(s^t, s_{t+1})$. Hence, the first order conditions just say to operate each technology up to the point where the marginal cost equals the marginal benefits, and direct the firm to adjust investment so that the physical returns $\underline{R}_i(s^t)$ lie in the space of asset returns $\underline{R}^A(s^t)$ defined by (3). This is illustrated in Fig. 2.

Consumer first order conditions state that

$$p(s^t, s_{t+1}) = \rho\, \pi(s_{t+1} | s^t) \frac{u'(c(s^t s_{t+1})}{u'(s^t)} \tag{11}$$

with obvious modifications as the utility function is varied. Using (11) we can identify the contingent claims prices from the consumption process, and then characterize asset returns using (3). This is how (1) is derived.

When there are N technologies, the producer first order conditions (10) allow us to identify an N-1 dimensional subspace of the S-1 dimensional space of asset returns at each date, so long as the physical returns $\underline{R}_i(s^t)$ are linearly independent. The simulation economies that follow use a number of technologies N equal to the multiplicity of states S, in which case we can recover contingent claims prices from the returns on technologies, and construct asset returns from those contingent claim prices. Since by (3) asset returns lie in a plane orthogonal to the vector of contingent claims prices (see Fig. 2), one finds contingent claims prices analytically by finding the vector orthogonal to the given plane. Let $\bar{R}(s^t)$ be an NxS matrix, $\bar{R}_{ij}(s^t) = R_i(s^t, s^t\lambda_j)$, and let $\bar{1}$ be an Sx1 matrix of 1's. With this notation, (10) can be rewritten

$$\bar{1} = \bar{R}(s^t)\ \underline{p}(s^t)$$

Then, we can solve (10) for the contingent claims prices,

$$\underline{p}(s^t) = \bar{R}(s^t)^{-1}\ \bar{1} \tag{12}$$

We can use (12) to produce an equation that looks like (1), to illustrate the equivalence of consumer and producer based models in this case,

$$1 = E(m_{t+1}R_{t+1}|s^t)$$

where

$$m_{t+1} = m(s^t, s_{t+1}) = p(s^t, s_{t+1})/\pi(s^t, s_{t+1}).$$

and $p(s^t, s_{t+1})$ is derived from production data by (12).

The simulation economies that follow adopt a parametric form of the above model, with two technologies for transferring consumption across dates, fixed investment and purchases of consumer durables, and two states λ_1 and λ_2 at each date, high and low investment growth. Each technology has constant returns to scale in production, but an adjustment cost to investment. The technologies are:

$$\begin{aligned}
y_t &= c_t + I_{kt} + I_{dt} \qquad\qquad (13)\\
y_t &= \text{mpk}\ k_{kt} + \text{mpd}\ k_{dt}\\
k_{kt+1} &= (1-\delta_k)\ [\ k_{kt} + (1 - .5\alpha_k(I_{kt}/k_{kt})^2\ )I_{kt}\ ]\\
k_{dt+1} &= (1-\delta_d)\ [\ k_{dt} + (1 - .5\alpha_d(I_{dt}/k_{dt})^2\ )I_{dt}\ ]
\end{aligned}$$

k_k and k_d are the stocks of physical capital and durable goods, and mpk and mpd their marginal products. As one can include preference shocks in consumption based models, the marginal products could be made stochastic by making them depend on the state.

The physical rates of return available through technology are, from (9),

$$R_k(s^t, s_{t+1}) = (1-\delta_k)\left(\ \text{mpk} + \frac{1 + \alpha_k\left(\frac{I_{kt+1}}{k_{kt+1}}\right)^3}{1 - \frac{3\alpha_k}{2}\left(\frac{I_{kt+1}}{k_{kt+1}}\right)^2}\right)\left(1 - \frac{3\alpha_k}{2}\left(\frac{I_{kt}}{k_{kt}}\right)^2\ \right) \qquad (14)$$

Here I have again dropped the dependence on state for simplicity--I_{t+1} is $I(s^t s_{t+1})$, etc. $\underline{R}_d$ looks exactly the same, with d's in the place of k's.

Given $\underline{R}_k$, $\underline{R}_d$ we can find contingent claims prices using (12),

$$\begin{bmatrix} p(s^t,\lambda_1) & p(s^t,\lambda_2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} R_k(s^t,\lambda_1) & R_d(s^t,\lambda_1) \\ R_k(s^t,\lambda_2) & R_d(s^t,\lambda_2) \end{bmatrix}^{-1} \tag{15}$$

Unlike the Mehra-Prescott model, it is not sufficient to characterize the current state history s^t by just the current draw of the investment growth state s_t, because the capital stock at any date is a function of a long past moving average of investment. $\{s_t, I_{kt}, K_{kt}, I_{dt}, K_{dt}\}$ are sufficient state variables for s^t. Hence, over many dates, many values of each physical return $\underline{R}_k$, $\underline{R}_d$ and the resulting asset returns will be observed.

The simulations proceed as follows. 1) Model the stochastic process of investment growth as a two state Markov process; 2) use (14) to find the one or many period ahead physical returns at each date; 3) use (15) to derive one or several period ahead contingent claim prices; 4) use the contingent claim prices to construct the asset returns of interest.

3. Equity Premium

A well documented puzzle of the consumption based asset pricing model is that the difference in mean returns between stocks and relatively risk free bonds is higher than predicted, without resorting to implausibly high risk aversion and discount factors greater than 1. Shiller (1982), Mehra and Prescott (1985) and Hansen and Jagannathan (1988) contain successively sharper statements of this puzzle.

The consumer's first order conditions (1) imply that higher variance of marginal rates of substitution $(u'(c_{t+1})/u'(c_t))$ generate a more steeply sloped mean-standard deviation frontier of asset returns. Observations on the returns of a stock portfolio and a risk-free rate imply a lower bound on the slope of the mean-standard deviation frontier--lower because the stock portfolio may not be efficient, or perfectly correlated with consumption growth. Hence one can deduce a lower bound on the volatility of marginal rates of substitution and (along with data on consumption growth) the coefficient of risk aversion from the slope of the mean-standard deviation frontier. This is an attractive statement of the Mehra-Prescott puzzle, because it does not require us to identify stocks with an asset that pays a dividend equal to aggregate consumption, as Mehra and Prescott assumed.

In this section, I'll contrast an approach to this puzzle using consumer vs. producer first order conditions. It will be simpler and will highlight the symmetry of the two approaches to first derive the slope of the mean standard deviation frontier given contingent claims prices, and then use consumer's or producer's first order conditions to identify those contingent claims prices from consumption or production data.

Since markets are complete, there is a risk free rate,

$$R^f(s^t) = \left[\sum_{s_{t+1}} p(s^t, s_{t+1}) \right]^{-1}$$

Define the excess return on an asset as its return minus the risk free rate.

$$R^e(s^t, s_{t+1}) = R^A(s^t, s_{t+1}) - R^f(s^t)$$

Plugging into (3) we find that the excess returns must be orthogonal to the contingent claims prices:

$$0 = \sum_{s_{t+1}} p(s^t, s_{t+1}) R^e(s^t, s_{t+1}) = \underline{p}(s^t)\cdot\underline{R}^e(s^t) \tag{16}$$

The unconditional mean-standard deviation frontier is found by the minimum variance excess return, among all excess returns with a given mean and that satisfy (16):

$$\min \sum_{s^t s_{t+1}} \pi(s^t s_{t+1}) \left[R^e(s^t, s_{t+1}) - E(R^e | s^t) \right]^2$$

subject to :

$$\mu = \sum_{s^t s_{t+1}} \pi(s^t s_{t+1}) R^e(s^t, s_{t+1})$$

and (16) for each s^t. This is a straightforward Lagrangian maximization. The resulting slope is:

$$\frac{\mu^2}{\sigma^2} = \left[\sum_{s^t} \frac{\left[\Sigma_{s_{t+1}} p(s^t, s_{t+1}) \right]^2}{\Sigma_{s_{t+1}} p(s^t, s_{t+1})^2 \,/\, \pi(s^t s_{t+1})} \right]^{-1} - 1 . \tag{17}$$

The corresponding formula for the conditional mean-standard deviation frontier can be found by using probabilities conditional on state s^t. The result is:

$$\frac{\mu^2(s^t)}{\sigma^2(s^t)} = \frac{\Sigma_{s_{t+1}} p(s^t, s_{t+1})^2 \,/\, \pi(s_{t+1} | s^t)}{\left[\Sigma_{s_{t+1}} p(s^t, s_{t+1}) \right]^2} - 1 . \tag{18}$$

The conditional and unconditional mean and variance are related by:

$$\frac{\sigma^2}{\mu^2 + \sigma^2} = \sum_{s^t} \pi(s^t) \frac{\sigma^2(s^t)}{\mu^2(s^t) + \sigma^2(s^t)} . \tag{19}$$

Note that the frontiers are linear, as we expect given the existence of a risk free rate. Also, if consumers are risk neutral, so that state prices are proportional to probabilities--the price and probability vectors are colinear--the slope of the mean-standard deviation frontier reduces to 0, as it should. (To see this, substitute $p(s^t, s_{t+1}) = \gamma\ \pi(s_{t+1}|s^t)$, where γ is an arbitrary constant, into (17) or (18). Hansen and Richard (1987) give an similar characterization of the conditional and unconditional mean-standard deviation frontier.)

Using Consumption to Identify the Frontier

Substituting the consumer's first order conditions (11) in for the contingent claims prices in (18), we relate the slope of the conditional mean-standard deviation frontier to the variance of marginal rates of substitution:

$$\frac{\mu(s^t)}{\sigma(s^t)} = \frac{\operatorname{var}(\ u'(c_{t+1})/u'(c_t)\ |s^t\ )^{1/2}}{E(\ u'(c_{t+1})/u'(c_t)\ |s^t\ )} \tag{20}$$

We can use (20) to calculate bounds on the coefficient of risk aversion that ignore conditioning information.[2] With CRRA utility, $u'(c) = c^{-\alpha}$, we can approximate marginal utility growth as $(c_{t+1}/c_t)^{-\alpha} \cong (1-\alpha\Delta c/c)$, so (20) becomes

$$\frac{\mu}{\sigma} \cong \frac{\alpha\ sd(\Delta c/c)}{1 - \alpha\ E(\Delta c/c)} \quad \text{or} \quad \alpha \cong \frac{1}{sd(\Delta c/c) \left(\dfrac{\sigma}{\mu} + \dfrac{E(\Delta c/c)}{sd(\Delta c/c)}\right)} \tag{21}$$

Table 1 presents summary statistics on CRSP stock portfolio returns and consumption growth in postwar quarterly data. Each stock return in Table 1 yields a ratio of mean excess return to its standard deviation μ/σ of about .2. Shiller (1982) reports $\mu/\sigma = .279$ for the S&P500. Using $\mu/\sigma = .2$ and the consumption growth statistics from table 1, (21) implies

$$\alpha \cong \frac{1}{.00854\ (5 + .712)} = 20.6.$$

If the risk aversion coefficient really is 20.6, then we need to reconcile

$$\rho R^f = 1/E\left[(c_{t+1}/c_t)^{-\alpha}\right] \cong 1/(1-\alpha E(\Delta c/c)) = 1/(1-(20.6)(.00854)) = 1.21.$$

With $R^f = 1.0019$ (the T-bill rate - CPI in table 1), this requires that the discount factor $\rho \cong 1.21$. or a *neagtive* discount rate of about -21% per quarter. A more conventional value for ρ, near but below 1, requires a mean real interest rate near 21% per quarter.

To perform a similar calculation that recognizes the difference between conditional and unconditional probabilities, I fit a two state Markov process to consumption growth rates, following Mehra and Prescott. The two states are growth above average and growth below average. I picked the value of consumption growth in each state to be one standard deviation above or below the mean. Since the unconditional frequencies of each state are .50 in the data, this choice of growth rates maintains the unconditional mean and variance of growth rates in the data.

With the probability structure and growth rates of consumption in each state specified, we know $c(s^t s_{t+1})$ for each state $s^t s_{t+1}$ that follows s^t.

Then, we can identify contingent claims prices from consumption using (11), plug (11) into to (18) to get the (conditional) slope of the mean-standard deviation frontier at each date, and then use either (17) or (19) to get the unconditional slope. Table 2 summarizes the results of this simulation for various values of the risk aversion parameter, from 1 to 45. Values of the unconditional slope (μ/σ) around .2 as in table 1 correspond to α more than 20. Table 2 also presents the calculation of ρ times the conditional risk free rate from the consumption model. Again, values of α that generate the observed slope of the mean standard deviation frontier also generate a "too large" ρR_f. Furthermore, values of the α that reconcile the slope of the mean-standard deviation frontier generate real interest rates that vary over time or across states far more than is observed.

Using The Firm's First Order Conditions to Identify the Frontier

The technologies are specified in (13). Table 3 presents the Markov matrix for investment growth used in the simulations. I chose the values of the growth rates in each state to match the unconditional mean and variance of investment growth in the postwar period. This choice requires

$$\pi_1 g_1 + \pi_2 g_2 = \dot{g} \tag{22}$$

and

$$\pi_1 (g_1 - \dot{g})^2 + \pi_2 (g_2 - \dot{g})^2 = \text{var } g \tag{23}$$

where π_1 and π_2 are the unconditional probabilities of each state, g_1 and g_2 are investment growth rates in each state, $\dot{g}$ is the unconditional mean, and var g is the unconditional variance of growth rates. (22) and (23) imply that the growth rates in the two states are

$$g_1 = (\bar{g} + \left[\frac{\pi_2}{\pi_1} \text{var } g\right]^{1/2}) \; ; \; g_2 = (\bar{g} - \left[\frac{\pi_1}{\pi_2} \text{var } g\right]^{1/2})$$

Before proceeding with simulations, it's worthwhile to see how qualitative features of this technology and probability specification map into slopes of the conditional and unconditional mean-standard deviation frontiers. The first task is to find the qualitative behavior of the physical returns $\underline{R}_k$ and $\underline{R}_d$ defined in (14). When adjustment costs $\alpha = 0$ or the investment to capital ratio $(I_t/k_t) = 0$, the physical returns collapse to $(1+mp)(1-\delta)$ in both states (I eliminate the k or d subscript when referring to either k or d simultaneously). As the investment to capital ratio I_t/k_t rises, the last term in (14) lowers the physical return $\underline{R}$ into both states, because it raises current period adjustment costs. As tomorrow's investment to capital ratio I_{t+1}/k_{t+1} rises, the first term in (14) *raises* the physical return $\underline{R}$, because the firm gets the benefit of higher adjustment costs tomorrow when disinvesting. Hence, R(high growth) > R(low growth). Fig. 3 illustrates $R(s^t, s_{t+1})$. $\underline{R}$ is an element of $\mathbb{R}^2$, corresponding to the two possible states at t+1. R(high growth) > R(low growth) implies that $\underline{R}$ will lie below the 45^0 line, as in Fig. 3.

We want to derive the behavior of physical returns $\underline{R}$ as a function of state variables at time t. A higher investment to capital ratio today I_t/k_t implies a higher investment to capital ratio tomorrow I_{t+1}/k_{t+1} as well because investment *growth* is the state variable. The partial effects of raising both investment to capital ratios, I_t/k_t and I_{t+1}/k_{t+1}, cancel to first order, as explained in the last paragraph. The major effect of raising the investment to capital ratio today I_t/k_t is thus to increase the disparity between the returns in the two states tomorrow, $R(s^t,\lambda_1)$ and $R(s^t,\lambda_2)$, rather than affecting the level of both. Hence, $\underline{R}(I/k)$ behaves as follows (refer to Fig. 3). When $I/k = 0$, $\underline{R}$ starts at $R^0_k = (1+mp)(1-\delta)$ in both states, or a point on the 45^0 line in Fig. 3. As the I_t/k_t ratio is increased, $\underline{R}$ moves sideways, increasing the disparity between $\underline{R}$ in the two states, rather than toward or away from the origin, which would reflect an overall decrease or increase in return.

The adjustment cost parameter α governs the sensitivity of the physical return $\underline{R}$ to the investment to capital ratio I/k. Depreciation δ and marginal product mpk or mpd only enter together in the determination of the no-adjustment-cost returns $R^0_k = (1+mpk)(1-\delta_k)$ and $R^0_d = (1+mpd)(1-\delta_d)$, shown as the intersection with the 45^o line in Fig. 3. They have a smaller secondary effect, because the determine how capital is accumulated from investment.

Now we are in a position to graphically evaluate how big a slope of the mean-standard deviation frontier will be generated by given parameters: we know how production function parameters map into the position of $\underline{R}_k$ and $\underline{R}_d$ on a graph like Fig. 3; given $\underline{R}_k$ and $\underline{R}_d$, all assets lie on the line connecting them; the contingent claims prices are orthogonal to that line; and the slope of the mean-standard deviation frontier is proportional to the deviation of the contingent claims price vector from the probability vector.

We're looking for evidence that the price and probability vectors diverge enough to generate the observed slope of the mean-standard deviation frontier, so it is useful to start with parameterizations that ensure no premium, and characterize more realistic parameterizations by their departure from a no-premium configuration. Since the probabilities of various states do not directly enter the technology as they do into expected utility, there is no way as simple or as general as setting the coefficient of risk aversion to 0 to guarantee that price and probability are colinear.

The following choice of parameters guarantees that contingent claims prices will be colinear with probabilities and hence that there will be no equity premium. First, assume that there are no adjustment costs to investment in consumer durables, giving investment in consumer durables a risk free return, like storage. The physical return for consumer durable simplifies to:

$$R_d(s^t, s_{t+1}) = (1 + mpd)(1 - \delta_d) = R_d^0$$

Second, linearize the return of capital about the mean growth rate of investment. Fig. 3 suggests that this is not a bad approximation for small α_k and (I_k/k_k). Let $\bar{g}$ = the mean growth rate and $g(\lambda_1) = \bar{g} + dg_1$, $g(\lambda_2) = \bar{g} + dg_2$. The form of this linearization is then

$$\left[R_k(s^t,\lambda_1) \; R_k(s^t,\lambda_2) \right] \cong \underline{R}_k(s^t,\bar{g}) + \underline{R}_k'(s^t,\bar{g}) \cdot \left[dg_1 \; dg_2 \right] \tag{24}$$

$\underline{R}_k(s^t,\bar{g})$ is $\underline{R}_k$ (14) and $\underline{R}_k'(s^t,\bar{g})$ is the derivative of $\underline{R}_k$ with respect to g, both evaluated at $I_{kt+1} = \bar{g}\, I_{kt}$. This is just an analytical statement of replacing the slightly curved $\underline{R}(I/k)$ path in Fig. 3 by a straight line. Third, choose mpk so that $R_k(s^t,\bar{g}) = R_d^0 = (1+mpd)(1-\delta_d)$. This assumption says that if the uncertainty is turned off in a specific way ($dg = 0$), then the rates of return of the two technologies are the same. Thus, $\underline{R}_k$ and $\underline{R}_d$ start on the same point on the 45^0 line in Fig. 3, and $\underline{R}_k$ moves linearly to the southeast as I_k/k_k is increased.

The contingent claims prices are orthogonal to the line connecting $\underline{R}_k$ and $\underline{R}_d$, or

$$\left[p(s^t,\lambda_1)\ p(s^t,\lambda_2)\right] \propto \left[R_d^0 - R_k(s^t,\lambda_2) \qquad R_d^0 - R_k(s^t,\lambda_1)\right]$$

($\propto$ stands for "is proportional to"). Using (24),

$$\left[p(s^t,\lambda_1)\ p(s^t,\lambda_2)\right] \propto \left[\ -dg_2\ dg_1\right]$$

Now, what choices of $[dg_1\ \ dg_2]$ give rise to prices colinear with probabilities? We need

$$\left[\ -dg_2\ dg_1\ \right] \propto \left[\ \pi(\lambda_1|s^t) \quad \pi(\lambda_2|s^t)\ \right]$$

or

$$\pi(\lambda_1|s^t)\ dg_1 + \pi(\lambda_2|s^t)\ dg_2 = 0$$

Comparing to (22), *choices of dg_1 and dg_2 that maintain the mean growth rate at $\bar{g}$ also ensure that prices and probabilities are collinear.*

Now we can examine what features of a more realistic technology account for deviation of the price and probability vectors. First, the linearization in (24) may fail. This has a very small effect for a wide range of reasonable parameters, as reflected in the small curvature of the $\underline{R}(I/k)$ line in Fig. 3. Second, the "persistence effect": as the Markov matrix governing investment growth displays more persistence[3], the conditional probabilities diverge from the unconditional probabilities and the contingent claim prices, raising the equity premium. Third, the "shift effect": the constant term $\underline{R}_k(s^t,\bar{g})$ in (24) will vary over time as capital is accumulated, even if it is identical to $R_k^0 = (1 - \delta_d)(1 + mpd)$ at one date. Changes in this term shift the point about which the deviations $[dg_1\ dg_2]$ are taken, so that even if investment growth is independent over time, contingent claim prices and probabilities will diverge.

With this in mind, we can turn to some simulations. I examined the values $\alpha_k = 1$, $\alpha_k = 10$ and $\alpha_d = 0$. The Q-theory literature supports much higher values for α_k, but the high values common in that literature are regarded as a puzzle. I picked a value $\delta_d = .08$ for the depreciation rate of durables, and mpd = .10 for the marginal product of consumer durables. These always enter together; they imply that R_d^0 and the risk free rate are $(1+mpd)(1-\delta_d) = 1.012$ (4% per year), and have no other effects. I chose the depreciation rate of capital to be $\delta_k = .025$ (10% per year). I chose the marginal product of capital so that if capital is at its steady state value $k_k = (1-\delta_k)/\delta_k\ I_k$, then the constant term $R_k(s^t,\bar{g})$ would equal R_d^0. Thus in the steady state, there is no "shift effect". With the other parameter values, it implies mpk = .038144. I found very little difference in experimenting with a wide range of mpk and δ_k so long as their product $(1+mpk)(1-\delta_k)$ is the same.

I calculated the conditional and unconditional slopes of the mean standard deviation frontier in the following way: I simulated investment growth from a random number generator; then at each date, I accumulated capital, calculated $\underline{R}_k$ and $\underline{R}_d$ using (14), calculated the contingent claims prices each date using (15), and calculated the slope of the conditional mean-standard deviation frontier using (18). Then I calculated the unconditional slope using (19), averaging the conditional moments over time. This procedure yields small variations in the unconditional slope each time due to sampling variation, so I report the average of ten such simulations for each parameter choice. Table 3 presents a flowchart of this simulation.

Table 3 also presents the results of the simulations. The column marked "force" indicates whether the certainty return on investment $\underline{R}_k(s^t,\bar{g})$ in (24), is forced to be $R_d^0 = (1+\text{mpd})(1-\delta_d)$ at each date to eliminate the "shift effect". This is achieved by multiplying $\underline{R}_k$ by a suitable constant at each date. The column marked "persistence" indicates whether the assumed Markov matrix is the actual one ("yes") or a matrix with no persistence, formed by the unconditional probabilities of each state ("no"). When "force" is "yes" and "persistence" is "no", the only thing that makes price and probability vectors diverge is nonlinearities in $\underline{R}_k$. As the table shows, these have very small effects. With "force" = "no" and "persistence" = "yes", we see that persistence in investment growth *alone* raises the unconditional slope to the range of its observed values. "Force" by itself has a smaller effect. Finally, with "force" = "yes" and "persistence" = "yes", we again get results similar to those estimated from the data. This slope is slightly *less* than the value for "force" = "no" and "persistence" = "yes", which initially seems puzzling. There are states where the conditional slope is very low--prices and conditional probabilities are essentially colinear. Adopting "force" = "yes" to minimize the risk premium based on *unconditional* probabilities raises the *conditional* risk premium in these states.

These simulations are analogous to Mehra and Prescott's. Yet the observed value of the equity premium is not at all hard to reconcile with production data through this adjustment cost technology. Note that $\alpha_k = 10$ instead of $\alpha_k = 1$ has very little effect on the results. Increasing α_k has the effect of moving $\underline{R}_k$ away from R_d^0 while maintaining constant the slope of the line connecting $\underline{R}_k$ and $\underline{R}_d$, and hence the contingent claims prices and the equity premium. Thus, the crucial observation is the *persistence* of investment growth, as widely varying (and low) values of the *curvature* parameter α_k result in about the same equity premium. Note also that the conditional and unconditional risk free rates are constant at an arbitrary value of 4%, so the puzzle that values of the parameters that explain the equity premium predict strange risk free rates does not appear in this model.

4. Forward Rate Term Premium

The equity premium simulation investigates the ability of a production-based model to capture the unconditional level of a risk premium. This second set of simulations is designed to capture the cyclical behavior of a risk premium as well as its unconditional value.

Fama and Bliss (1987) regressed current year term premia (the ex-post return from holding an X year bond for one year minus the return from holding a one year bond) on forward rate term premia (forward rate - spot rate). They found coefficients near 1.0 for long maturities. They concluded that long (X) maturity forward rate term premia move one for one with one year expected term premia and have little forecast power for one year changes in long (X-1) year rates. Fama and Bliss also pointed out that forward rate term premia also display an enticing cyclical pattern. Fig. 4 presents the 5 year forward rate term premium and gross fixed investment. In the 70's the forward rate moved slightly before business cycles in investment; while in the 60's and in 1979, it moved contemporaneously. Table 5 part 1 presents some regressions that quantify the cyclical correlations between the term premium, investment, and durable goods purchases. While the 5 year forward premium is negatively correlated with both the investment/output ratio and the durable/output ratio taken alone, in a multiple regression the durable/output ratio is positively correlated with the 5 year forward premium, and these multiple correlations are more stable through the sample than the single correlations. I take the correlations documented by these regressions as the stylized facts to be explained.

Since the model as developed so far is entirely real, Fama and Bliss' evidence that variation in the forward rate term premium is almost entirely due to variation in a (real) risk premium and that the risk premium has an enticing cyclical correlation with production variables make it an attractive quantity for a simulation exercise with this model. In this section, I'll present simulations designed to replicate this behavior of the forward rate term premium, in a model similar to the one presented for the equity premium. Backus Gregory and Zin (1986) present an analogous Mehra-Prescott style model of the term structure using consumer's first order conditions. Stambaugh (1987) ties the forward rate term premium to conditional moments of consumption.

Given contingent claims prices, we can calculate multiperiod bond prices as follows. The price of a one period bond is

$$P^{(1)}(s^t) = \sum_{s_{t+1}} p(s^t, s_{t+1})$$

Then, the price of a two period bond is

$$P^{(2)}(s^t) = \sum_{s_{t+1}} \sum_{s_{t+2}} \frac{P(s^t s_{t+1} s_{t+2})}{P(s^t)} = \sum_{s_{t+1}} p(s^t, s_{t+1})\, P^{(1)}(s^t s_{t+1}).$$

We can continue this process, leading to

$$P^{(k)}(s^t) = \sum_{s_{t+1}} p(s^t, s_{t+1})\, P^{(k-1)}(s^t s_{t+1}) \quad . \tag{25}$$

The forward rate term premium--the excess of the forward rate from t+x-1 to t+x over the spot rate is then:

$$f^{(x)}(s^t) = \frac{P^{(x)}(s^t)}{P^{(x-1)}(s^t)} - \frac{1}{P^{(1)}(s^t)} \tag{26}$$

I use the same technology as the last section, and the same two state Markov model for investment growth. In this model, investment and consumer durable purchases are always in the same growth state. This is a poor approximation to quarterly data, in which investment and consumer durables grow above or below their mean growth rate contemporaneously only 78% of the time (47,1-86,4). However, in annual data there are only two years in the last forty in which consumer durable growth was above its mean with investment growth below its mean or vice versa. For this reason (as well as the computational difficulty of examining all 2^{20} states 5 years ahead in quarterly data rather than the 2^5 states 5 years ahead in annual data), I will compare the model to actual data at annual frequencies. Alternately, one could add more states, such as λ_3: durables grow and investment declines.

The physical returns are given by (14) as before. $\{I_{kt}, k_{kt}, I_{dt}, k_{dt}, g_t\}$ are state variables for s^t, so $\underline{R}_k$ and $\underline{R}_d$, contingent claims prices, interest rates, etc., will be functions of these state variables. With physical returns (14), we can calculate contingent claims prices using (15), and then multi-period bond prices and term premia from (25) - (26). As before, I start with a description of the parameter choices that yield desirable qualitative behavior.

Recall that the one period interest rate is constructed as in Fig. 5, by the intersection of a line connecting $\underline{R}_k$ and $\underline{R}_d$ with the 45^o line. This construction mimics the creation of a risk free asset (one that pays off equally in either state) from the two risky technologies. Obviously, we cannot pick $\alpha_d = 0$ as before, or there will be no variation in interest rates. To match the stylized fact (see table 5 part 1) that interest rates rise and forward premia decline (forecasting long horizon declines in the interest rate) when investment rises but vice versa for consumer durables, we must pick an arrangement like Fig. 5. With the $\underline{R}_k$ line to the right of the $\underline{R}_d$ line, increases in investment raise the one year rate, while increases in consumer durables purchases lower it. I assure this behavior by picking mpk, mpd, δ_k and δ_d so that the return R_k^0 corresponding to $I_k/k_k = 0$ is further out from the origin than the corresponding return for $\underline{R}_d$, as in Fig. 5.

I searched for parameters that produced a high correlation between actual and simulated forward premia, within the space of "sensible" parameters: δ_k and $\delta_d < .2$; all returns positive, etc. and while maintaining the geometry of Fig. 5. I calculated the correlation without removing a mean as

$$\sum_t (\text{real}_t \text{sim}_t) / (\sum \text{real}_t^2 \sum \text{sim}_t^2)^{.5}$$

This objective prizes a match with the level of the actual term premium as well as matching the cyclical fluctuations.

A short manual search produced the following parameters, which I use in the simulations below. (A subsequent automated search produced only slightly different parameters and a slight improvement in correlation, from .48 to .499.)

$$\begin{array}{ll} \alpha_k = 5 & \alpha_d = 2 \\ \delta_k = .1 & \delta_d = .1 \\ (1+mpk) = 1.05 * (1-\delta_k)^{-1} & (1+mpd) = 1.02 * (1-\delta_d)^{-1} \end{array}$$

To simulate the model, I followed the following procedure (Table 4 presents a flowchart). 1) I produced a capital stock series at each date by accumulating capital according to the technology (13), starting with "long run values" $k = (1-\delta)/\delta \ I$; 2) I constructed the physical returns $\underline{R}_k$ and $\underline{R}_d$ at each date using the observed values for investment and durable goods, and the accumulated capital stocks; 3) I constructed the implied yield curve and forward premia five years forward at each date from (25)-(26). Note that this procedure is a little different from a pure simulation of the model, because at each date I use the actual investment and durables numbers rather than a the value given by a simulation of the Markov model for investment growth. However, at each date potential *future* values of investment and durables purchases needed to calculate the future physical returns are given by the Markov model for investment growth.

There are several ways to evaluate the accuracy of the simulations. First, Fig. 6 presents the simulated and actual 5 year forward rate term premium. (Keep in mind that the simulated premium is only a function of investment data, no asset information goes into its construction.) The simulation at least picks up the level and the cyclical timing of the actual premium. Second, since the model predicts that all rates are functions of $\{I_{kt}, k_{kt}, I_{dt}, k_{dt}\}$, we can evaluate how well the simulated forward premium matches the actual forward premium as functions of these state variables. Table 5 parts 2, 3, 4 below give regressions of the actual 5 year forward premium on the investment and capital stock variables in a variety of specifications, and the corresponding regressions using the simulated premium. In both Fig. 6 and in table 5 (especially part 4) it's clear that the simulations *over*state the cyclical sensitivity of forward rate term premia. On the other hand, the mean value of the simulated term premium is slightly less than the actual. An objective of R^2 in deviations from the mean produces coefficients that match the cyclical pattern better, at the expense of matching the level.

The model predicts an *exact* relation between asset prices (the 5 year forward rate term premium, for example) and the state variables $\{I_{kt}, k_{kt}, I_{dt}, k_{dt}, g(s_t)\}$, so *any* deviation of the actual and simulated data is a formal statistical rejection of the model. Though the model is formally rejected, it replicates certain interesting stylized facts of the data. In particular, it gives an account of the puzzling *negative* partial correlation between investment and the forward rate term premium, implying a *positive* partial correlation with real interest rates; it gives an account of the *different sign* of the partial correlations of forward rate term premia with durables purchases and fixed investment; and it gives a quantitative account of the cyclical movement in the forward rate term premium. Also, there are (in retrospect) potential patterns in the data that this model could not have replicated. For example, it's clear from Fig. 5 that no arrangement would deliver positive partial correlations of *both* forms of investment with forward rate term premia.

This example also contains some lessons for the theory of investment, if we regard asset returns as given and ask what are the firm's investment patterns. For example, the partial correlation of investment in physical capital and the risk free rate of interest is *positive* in this model, and both the mean rates of return and the rates of return on each technology in each state are much higher than the risk free rate (see Fig. 5). Thus this model has the potential to explain some puzzling bad fits of the Q theory of investment, since it considers the risk considerations in forming a "portfolio" of investments rather than just equations like $1+r = E(f'(k))$.

The biggest weakness of the model so far is that I required different parameters to match the equity premium and the forward rates.[4] The essential problem was matching the *level* (not the cyclical pattern) of forward rates. Matching the level of forward rates required the large difference (1.05 vs. 1.02) between R_k^0 and R_d^0. In turn, this implies that the line (Fig. 5) connecting $\underline{R}_k$ and $\underline{R}_d$ is steeply sloped compared to the probability vectors. Hence, the parameters that work for the term premium yield enormous equity premia. If we move R_k^0 closer to R_d^0, the equity premium declines towards the values of Table 3 and the cyclical pattern of the forward rate term premium is preserved, but the *level* of the forward rates declines. These observations provide some of the important lessons we would hope to learn from formal testing--the dimensions along which the model succeeds and fails, and an understanding of the improvements we must seek.

Table 1

Summary statistics of quarterly returns and consumption growth

	Cons.	VWR	VW	EWR	EW	CPI	T-Bill
mean	0.608	1.781	2.788	2.240	3.248	1.024	1.215
standard deviation	0.854	7.839	7.583	9.891	9.706	0.949	0.821
mean/std. dev.	0.712	0.227	0.368	0.226	0.335	1.079	1.480
mean excess return		1.590	1.574	2.049	2.033		
standard deviation		7.777	7.737	9.835	9.808		
mean/std. dev.		0.204	0.203	0.208	0.207		

Note: Cons. and CPI are the first difference of log quarterly real nondurable consumption and CPI respectively. VWR, VW, EWR, EW are the (log) quarterly return on the CRSP value and equally weighted real and nominal portfolios. T-Bill is the quarterly average T-Bill rate divided by 4. Data sources: Citibase and CRSP. Excess returns for nominal returns are (return-TBill), for real returns they are (Return-Tbill+CPI).

Table 2

Slope of mean standard deviation frontier with two state markov consumption

A. Nondurable Consumption Process

		growth state	
		high	low
Unconditional probabilities of each state		0.5	0.5
Consumption growth in each state		1.5%	-.2%
Markov matrix	high	0.565	0.435
	low	0.435	0.565

B. Effects of Risk Aversion on Slope of Mean-Standard Deviation Frontier

Coefficient of risk aversion α		1	2	5	10	15	20
(Risk free rate + discount rate) $R_f\rho$	low	1.007	1.015	1.037	1.078	1.121	1.168
	high	1.005	1.010	1.025	1.052	1.081	1.110
Conditional slopes	low	0.009	0.017	0.044	0.091	0.142	0.198
	high	0.009	0.017	0.043	0.089	0.137	0.188
Unconditional slope		0.009	0.017	0.044	0.090	0.140	0.193

α		25	30	35	40	45
(Risk free rate + discount rate) $R_f\rho$	low	1.219	1.275	1.336	1.404	1.479
	high	1.142	1.175	1.211	1.248	1.288
Conditional slopes	low	0.258	0.324	0.396	0.476	0.563
	high	0.242	0.299	0.359	0.423	0.491
Unconditional slope		0.250	0.311	0.378	0.449	0.527

Note: The conditional slope is calculated from (11) and (18); then (19) is used to calculate the conditional slope.

Table 3

Slope of Mean Variance Frontier with two state Markov investment

A. Flowchart of simulations

Specify parameters α_k, α_d, mpk, mpd, δ_k, δ_d
↓
Specify Markov process for investment.
↓
Do for trial = 1 to 10
↓
Do for t= 1947,1 to 1986,4
↓
Simulate markov process, find state g(t),
↓
Find investment from $I_t = g(t)\, I_{t-1}$,
↓
Accumulate capital stock $k_{t+1} = (1-\delta_k)(k_t + (1- .5\, \alpha_k (I_t/k_t)^2) I_t)$
↓
Generate $\underline{R}$ at each date from I_t, k_t equation (14)
↓
Find contingent claims prices at t, $P = [1\ 1]R^{-1}$, equation (15)
↓
Find slope of μ-σ frontier at each date, using conditional π (18)
↓
End date do, average conditional slopes over all dates (19)
↓
End trials do, average unconditional slopes over all trials

B. Markov Process

		growth state high	growth state low
Unconditional probabilities of each state		.48	.53
Markov matrix	high	.61	.39
	low	.35	.66
"No persistence" markov matrix	high	.48	.53
	low	.48	.53

B. Simulated slopes

α_k	Force $\underline{R}_k(\bar{g})=R^0_d$	persistent I growth	slope of frontier	α_k	Force $\underline{R}_k(\bar{g})=R^0_d$	persistent I growth	slope of frontier
1	yes	no	.015	10	yes	no	.016
1	yes	yes	.273	10	yes	yes	.272
1	no	no	.124	10	no	no	.116
1	no	yes	.250	10	no	yes	.250

Slope in postwar data ≅ .2

Table 4
Flowchart for term premium simulations

Specify markov process for investment.

↓

Specify parameters α_k, α_d, mpk, mpd, δ_k, δ_d

↓

→ Do for t= 1947,1 to 1986,4

↓

Use actual investment at t;
Accumulate capital stock $k_{t+1} = (1-\delta_k)(k_t + (1- .5\ \alpha_k^2\ (I_t/k_t))I_t)$

↓

Simulate markov process 5 steps ahead, find I, k for each date-state

↓

Generate $\underline{R}$ at each date 5 steps ahead from I_t, k_t equation (14)

↓

Find contingent claims 5 steps ahead $P = [1\ 1]R^{-1}$, equation (15)

↓

find 1 - 5 period bond prices, forward rates (25)-(26)

↓

← End date do

↓

(Optional: calculate R^2 of actual and simulated premia, pick new parameters)

Table 5. Term Premium Regressions

1. Forward rate term premium on investment/output ratio and consumption/output ratio. (quarterly data, 53:2 - 86:4)

$$\underset{(2.3)}{f5 = 4.9} - \underset{(17.8)}{57.8}\,(I_k/GNP) + \underset{(13.5)}{70.3}\,(I_d/GNP)$$

2. Forward rate term premium on investment, capital, durables purchases, and durable stock. (Annual data 1953 - 1985)

$$\text{real } f5 = \underset{(2.3)}{6.5} - \underset{(.006)}{.041}\, I_k - \underset{(.0029)}{.0017}\, k_k + \underset{(.014)}{.086}\, I_d - \underset{(.0048)}{.0007}\, k_d$$

$$\text{sim. } f5 = \quad - .064\, I_k - .0108\, k_k + .099\, I_d - .0176\, k_d$$

$$\text{sim. } f5 = -1.7 - .064\, I_k - .0130\, k_k + .098\, I_d - .0208\, k_d$$

3. Forward rate term premium on logs of investment, capital, durables purchases and durables stock. (Annual data 1953 - 1985.)

$$\text{real } f5 = \underset{(61.9)}{91.1} - \underset{(3.7)}{17.4}\log(I_k) - \underset{(19.1)}{15.4}\log(k_k) + \underset{(3.76)}{14.5}\log(I_d) + \underset{(12.88)}{8.4}\log(k_d)$$

$$\text{sim. } f5 = -24.71 - 29.3\log(I_k) + 32.2\log(k_k) + 21.9\log(I_d) - 23.7\log(k_d)$$

$$\text{sim. } f5 = \quad - 29.9\log(I_k) + 24.7\log(k_k) + 22.6\log(I_d) - 18.8\log(k_d)$$

4. Forward rate term premium on investment/capital ratio and durables purchases/durables stock ratio. (Annual data 1953 - 1985)

$$\text{real } f5 = \underset{(1.6)}{3.2} - \underset{(22.7)}{100.4}\,(I_k/k_k) + \underset{(21.0)}{77.3}\,(I_d/k_d)$$

$$\text{sim. } f5 = 8.0 - 198.2\,(I_k/k_k) + 142.2\,(I_d/k_d)$$

$$\text{sim. } f5 = 3.2 - 167.0\,(I_k/k_k) + 162.5\,(I_d/k_d)$$

(OLS standard errors in parentheses.)

"real F5 " = actual data.

"sim. F5 " = artificial data, simulated from I, k using model.

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Footnotes

[*] I thank Gene Fama, Lars Hansen, Ed Prescott and an anonymous referee for valuable comments on an earlier draft of this paper, and Gene Fama for his generous permission to use the term structure data. This research was partially supported by a grant from the National Science Foundation.

[1] These formulas are often written in terms of normalized prices $q(s^t) = P(s^t)/\rho^t \pi(s^t)$,

$$1 = \sum_{s_{t+1}} \frac{\pi(s^t s_{t+1})\rho^{t+1} q(s^t s_{t+1})}{\pi(s^t)\rho^t q(s^t)} R^A(s^t, s_{t+1}) = \rho\, E\left[\frac{q_{t+1}}{q_t} R^A_{t+1} \mid s^t \right]$$

and

$$P^A(s^t) = E\left[\sum_{\tau > t} \rho^{\tau - t} \frac{q_\tau}{q_t} d_\tau \mid s^t \right]$$

Using this notation, the firm's problem could be written as an expected present value with no change of content. The equivalent form given in the text turns out to be notationally simpler to use for producers, since probabilities do not enter production functions as they do expected utility functions.

[2] More precisely, the unconditional mean-standard deviation frontier is found by substituting (3.13) in (3.10),

$$\mu^2/\sigma^2 = \left\{ E\left[\frac{[E(m|s^t)]^2}{E(m^2|s^t)} \right] \right\}^{-1} - 1$$

where $m = u'(c_{t+1})/u'(c_t)$. If the conditional moments are constant across s^t, this reduces to an unconditional version of (3.14) as used in the following paragraphs.

[3] One way to quantify the persistence of a Markov process is with Markov matrices of the form

$$\begin{bmatrix} (1-\theta)\pi_1 + \theta & (1-\theta)\pi_1 \\ (1-\theta)\pi_2 & (1-\theta)\pi_2 + \theta \end{bmatrix}$$

where π_1 and π_2 are unconditional probabilities of each state. Persistence parameters $\theta = .13$ and $\theta = .264$ for consumption and investment respectively produce transition matrices very close to those estimated from postwar data. In experiments with several Markov matrices of this form, the slope of the mean-standard deviation frontier scaled with the persistence parameter θ.

[4] A second weakness is that the high forward rate term premia are associated with a falling real term structure and vice versa. Subsequent evidence by Fama (1988) suggests the opposite: high forward rate term premia correspond to a *rising* real term structure, correlated with even more strongly declining inflation forecasts.

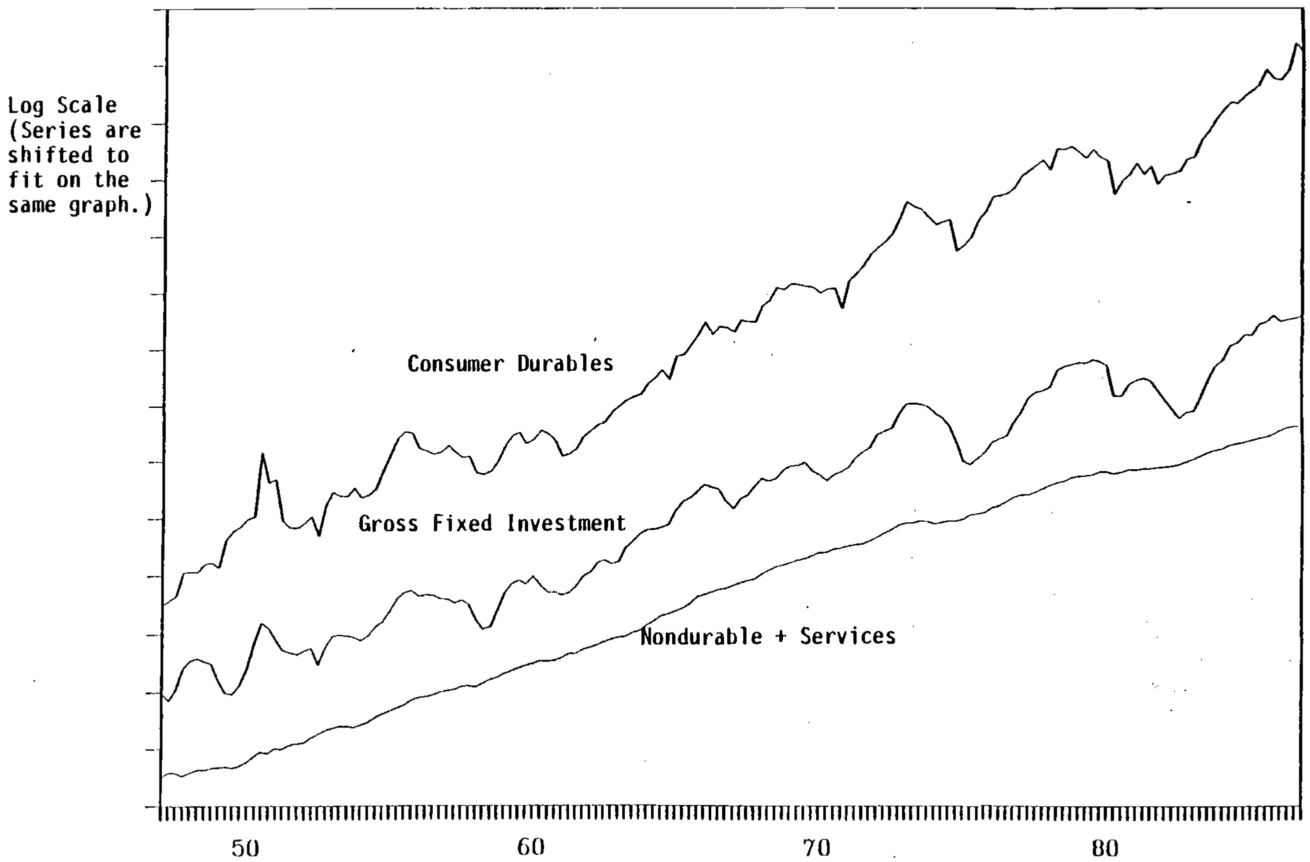
Log Scale
(Series are
shifted to
fit on the
same graph.)
Consumer Durables
Gross Fixed Investment
Nondurable + Services
50
60
70
80

Fig. 1

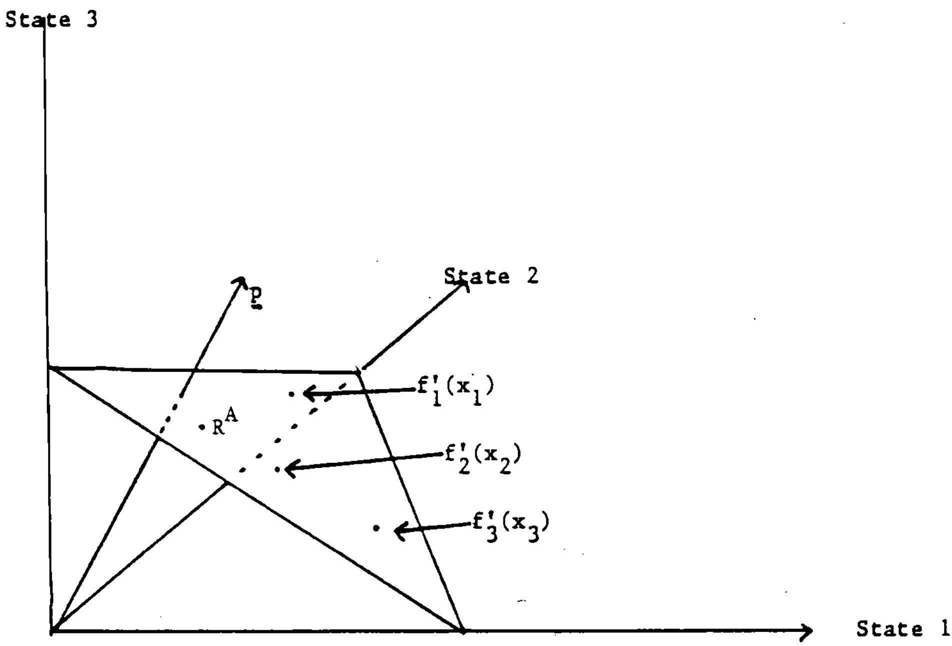

Fig. 2

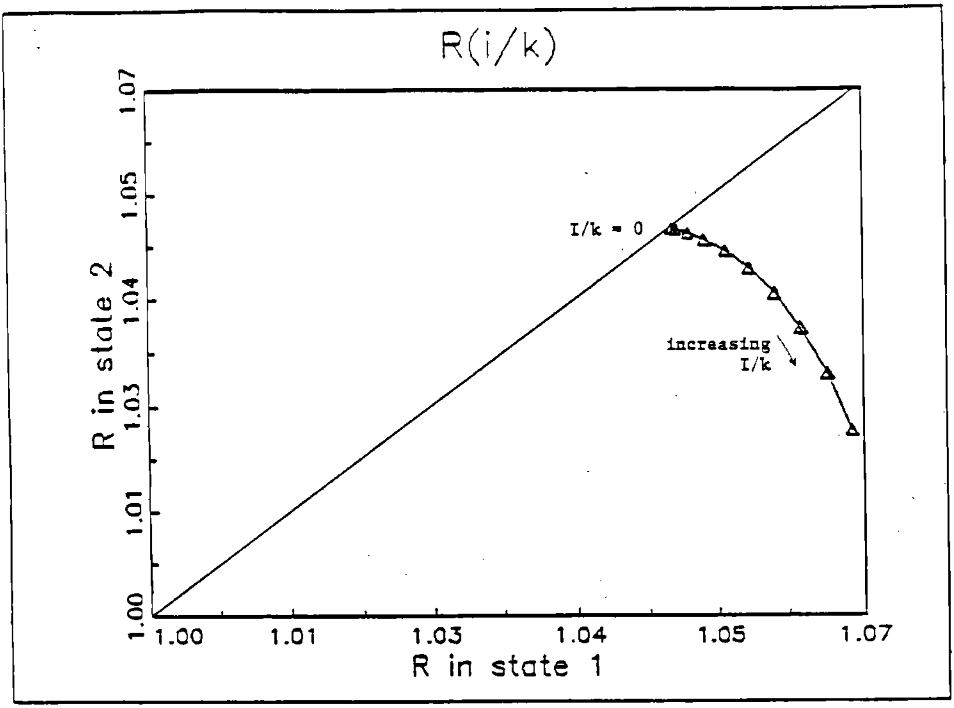

Fig. 3

Geometry of R as a function of I_t/k_t.

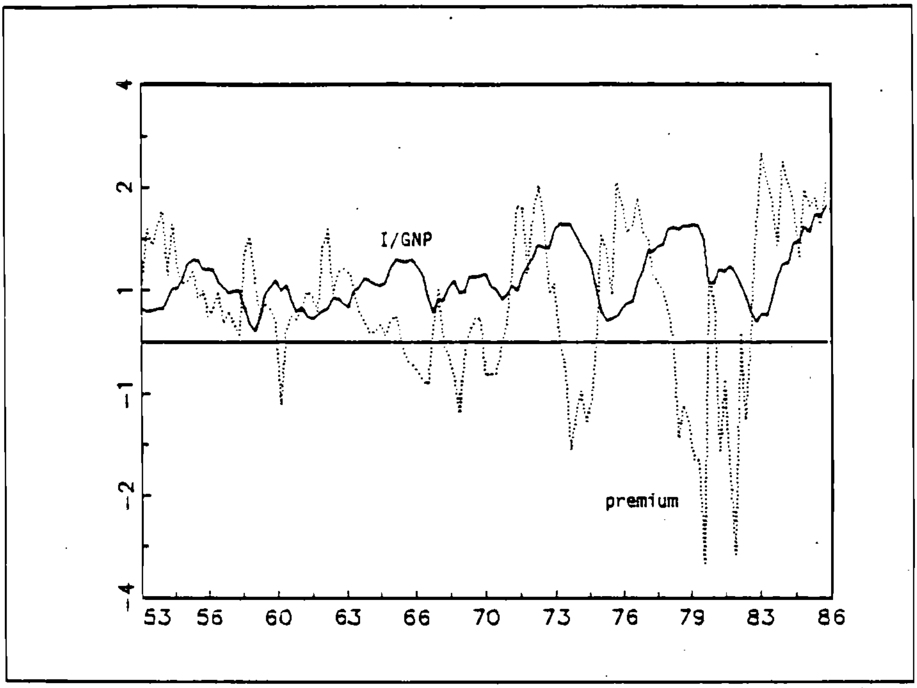

Fig. 4

5 year forward rate term premium and Investment/GNP ratio

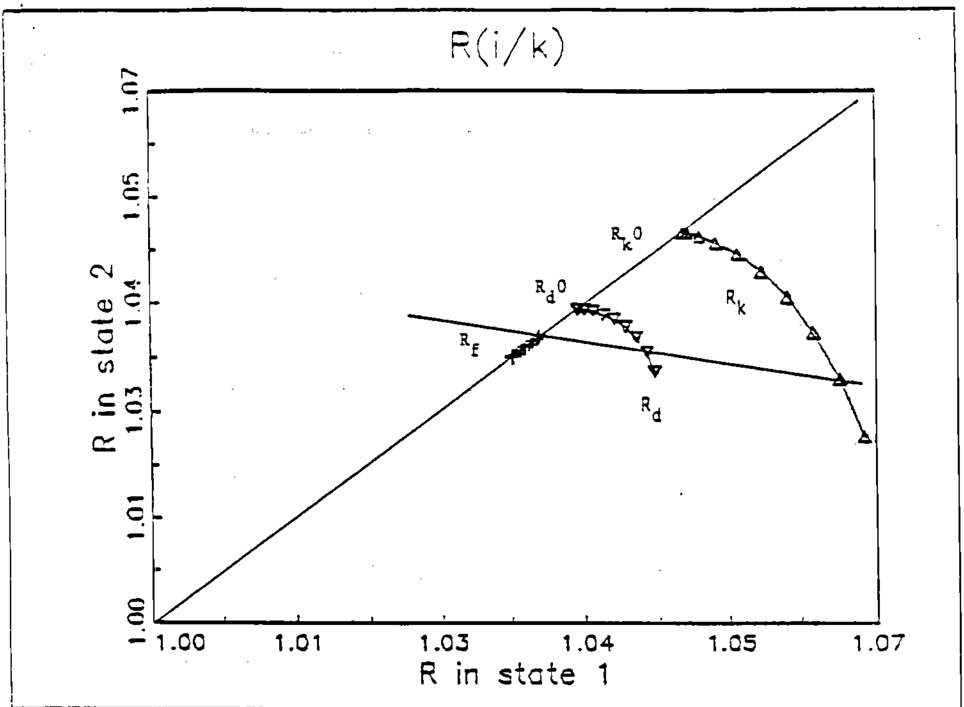

Fig. 5

Geometry of the parameterization for term premium simulations

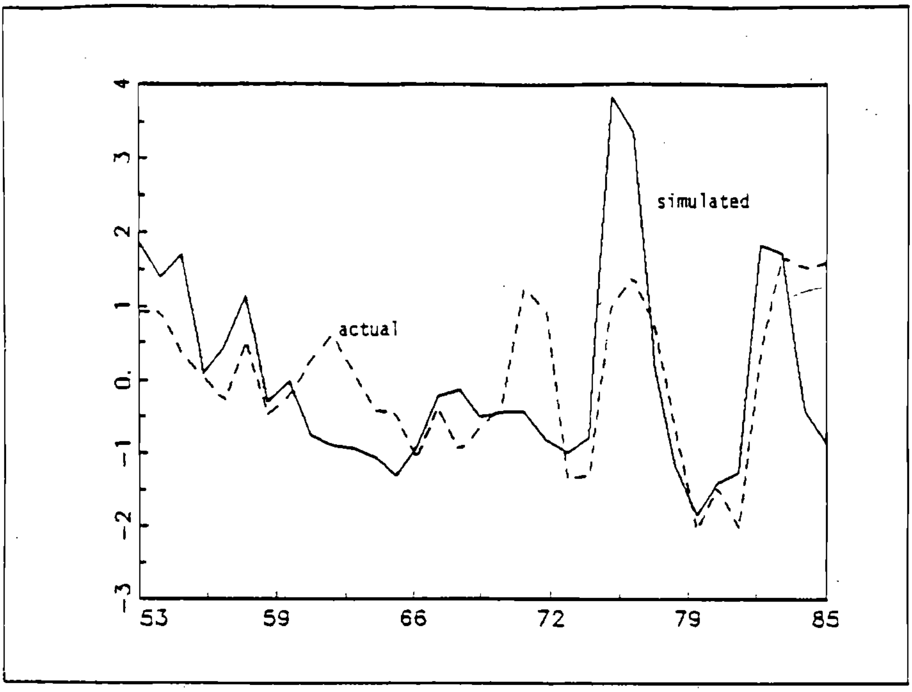

Fig. 6

Actual and simulated 5 year forward rate term premium