Decomposing the Yield Curve

John H. Cochrane and Monika Piazzesi

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Objective and motivation

- Yield curve: expected interest rates or risk premiums?

  **Yield:** \[ y_t^{(n)} = \frac{1}{n} E_t \left( y_t^{(1)} + y_{t+1}^{(1)} + \ldots + y_{t+n-1}^{(1)} \right) + rpy_t^{(n)} \]

  **Forward:** \[ f_t^{(n)} = E_t \left( y_t^{(1)} + \ldots + y_t^{(1)} + rpf_t^{(n)} \right) \]

  **Returns:** \[ E_t \left( r_t^{(n)} \right) = y_t^{(1)} + rpr_t^{(n)} \]

- Current risk premium or expected future premium? Term structure of risk premiums?

  \[ rpy_t^{(n)} = \frac{1}{n} \left[ E_t \left( r_{t+1}^{(n)} \right) + E_t \left( r_{t+2}^{(n-1)} \right) + \ldots + E_t \left( r_{t+n-1}^{(2)} \right) \right] \]

  → Affine model with a lot of attention to risk premiums
Affine model structure

Factors, e.g.: \( X_t = [x_t \ level_t \ slope_t \ curve_t]' \)

Real factor dynamics: \( X_{t+1} = \mu + \phi X_t + \nu_{t+1}; \ E(\nu_{t+1}\nu'_{t+1}) = V \)

Affine model: \( f^{(n)}_t = E^*_t(y^{(1)}_{t+n-1}) = (\cdot) + \delta_1 \phi^{n-1} X_t \)

- \( \phi^* \) is easy to fit, pure cross section. Need \( \phi \) for forecasts, premiums

Market prices \( \lambda \): \( \phi^* \equiv \phi - V\lambda_1 \)

Market prices \( \lambda \): \( E_t(\cdot + cov(\cdot, \cdot)) (\lambda_0 + \lambda_1 X_t) \)

\( \lambda_t = (\lambda_0 + \lambda_1 X_t) = \begin{bmatrix} \lambda_0^{(x)} \\
\lambda_0^{(\text{level})} \\
\lambda_0^{(\text{slope})} \\
\lambda_0^{(\text{curve})} \end{bmatrix} + \begin{bmatrix} \lambda_1^{(x,x)} & \lambda_1^{(x,l)} & \cdot & \cdot \\
\lambda_1^{(l,x)} & \lambda_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_t \\
level_t \\
slope_t \\
curve_t \end{bmatrix} \)

- Two issues: 1) How does \( \lambda_t \) vary over time (columns)? 2) Covariance with which shocks generates a premium (rows)?
- Can we simplify estimation of 20 unknown parameters, please?
Single-factor model for expected excess returns

\[ r_{x_t}^{(n)} = a_n + b_1 y_{t}^{(1)} + b_2 f_t^{(2)} + \ldots + b_5 f_t^{(5)} + \epsilon_{t+1}^{(n)} \]

- Single-factor model for expected excess returns

\[ E_t \left( r_{x_t}^{(n)} \right) = b_n \left( \gamma' f_t \right) = b_n x_t \]

- Paper: Eigenvalue decompose covariance matrix of expected returns; 
  \( x_t = \gamma' f_t \) is the dominant (>99%) eigenvector.
CP to affine model

Data: $E_t \left( r_{x_t+1}^{(n)} \right) = b_n (\gamma' f_t) = b_n x_t$

Model: $E_t (r_{x_t+1}) = (\cdot) + \text{cov}(r_{x_t+1}, v'_{t+1}) (\lambda_0 + \lambda_1 X_t)$

- All variation through time in market prices of risk is carried by $x_t$

\[
\lambda_t = (\lambda_0 + \lambda_1 X_t) = \begin{bmatrix}
\lambda^{(x)}_0 \\
\lambda^{(\text{level})}_0 \\
\lambda^{(\text{slope})}_0 \\
\lambda^{(\text{curve})}_0 \\
\lambda^{(x)}_1 \\
\lambda^{(l,x)}_1 \\
\lambda^{(s,x)}_1 \\
\lambda^{(c,x)}_1 \\
\end{bmatrix} + \begin{bmatrix}
\lambda^{(x,x)}_0 & 0 & 0 & 0 \\
\lambda^{(l,x)}_1 & 0 & 0 & 0 \\
\lambda^{(s,x)}_1 & 0 & 0 & 0 \\
\lambda^{(c,x)}_1 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
x_t \\
\text{level}_t \\
\text{slope}_t \\
\text{curve}_t \\
\end{bmatrix}
\]
\[ E_t \left( r_{x_t+1}^{(n)} \right) = (\cdot) + \text{cov}(r_{x_t+1}^{(n)}, v_{t+1}') (\lambda_0 + \lambda_1 X_t) \]

\[ b_n x_t = \text{cov}(r_{x_t+1}^{(n)}, v_x^x) \lambda_1^{(x,x)} x_t + \text{cov}(r_{x_t+1}^{(n)}, v_{t+1}') \lambda_1^{(l,x)} x_t + \ldots \]

- Market prices of risk correspond entirely to covariance with the level shock.

You can estimate \( \lambda_1 \) with a “cross-sectional regression”
Market price of risk summary

- Market price of risk only *varies over time* in response to one state variable, $x_t$, and *not* to level, slope and curvature.
- Risk premium is only earned in return for exposure to term-structure *level* shocks $v_{t+1}^l$. The premium for $x$, slope, curvature risk is zero.
- Dramatic simplification. *Two* parameters to estimate, and cross-sectional regression method to do so!

\[
\phi = \phi^* + V\lambda_1 \\
E_t(rx_{t+1}) = (\cdot) + \text{cov}(rx_{t+1}, v_{t+1}')\lambda_t \\
\lambda_t = \lambda_0 + \lambda_1 X_t
\]

\[
\lambda_t = \begin{bmatrix} 0 \\ \lambda_{0l} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_{1l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \text{level}_t \\ \text{slope}_t \\ \text{curve}_t \end{bmatrix}
\]
Find risk-neutral $\phi^*$ to fit cross section

$$f_t^{(n)} = (\cdot) + \delta_1' \phi^{n-1} X_t (+\epsilon_t)$$

$$\min_{\{\phi^*\}} \sum_{n=1}^{N} \sum_{t=1}^{T} \left( (\cdot) + \delta_1' \phi^{n-1} X_t - f_t^{(n)} \right)^2$$

- No forecasting information in risk-neutral transition matrix $\phi^*$.
- As usual, very close fit.

Use cross-sectional regression estimate $\lambda$ to find real $\phi$

$$\phi = \phi^* + V \lambda_1$$

$$\phi = \phi^* + V \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Zero restrictions mean that all but one column of $\phi$ is estimated from the cross-section alone!
The risk-neutral $\phi^*$ from the cross-section = a lot of information about the true $\phi$!

0.98 does not change. Near unit-root estimation problems are solved. The root is identified from the cross section.
- True dynamics $\phi$. $x$ is not an AR(1). Slope, curve $\rightarrow x$. Can expect future risk premium without current; term-structure of risk premiums.
Term structure of risk premiums

Resp. to x shock – Return-forecast

Resp. to level shock

Resp. to slope shock

Resp. to curve shock
5 year forward decomposition, Return-forecast model

- 1-Yr Yield
- 5-Yr fwd
- Expected Y (1)

Cochrane/Piazzesi ()

Yield Curve