The Fiscal Roots of Inflation

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Abstract

This paper studies the fiscal roots of inflation, how innovations to inflation correspond to innovations to the valuation equation for government debt. I pay particular attention to discount rates, how changes in expected bond returns raise and lower the value of government debt, and thereby account for inflation, without changes in expected primary surpluses. Though these calculations are particularly salient in the context of a fiscal theory of monetary policy, they can also fill out the nature of “passive” fiscal policy that must accompany an “active” monetary regime.

I derive a linearized identity, that unexpected inflation less the unexpected nominal return on government bonds, must equal the innovation in the sum of future surplus to GDP ratios less the sum of future real bond returns. I estimate the terms of the identity with a simple VAR. A 1% inflation shock corresponds to an 0.68% decline in future surpluses, a 0.68% rise in future discount rates, and a 0.36% decrease in the value of government bonds, which reflects future inflation. Discount rates are important in the latter two roles, and government bonds help. I examine responses to a recessionary deflation shock that moves both consumption and inflation, to a monetary policy shock that moves interest rates but not the present value of surpluses, and a fiscal shock that moves expected surpluses but not the interest rate. The disinflation of a recession shock corresponds entirely to a decline in discount rates, which raises the value of government debt. The monetary policy shock is super-Fisherian, raising inflation immediately. That inflation also corresponds entirely to a discount rate change. The fiscal shock sets off a protracted inflation, which lowers bond prices as well.

I also decompose the value of debt. 70% of variation in the value of debt corresponds to forecasts of future primary surpluses, with only 30% from variation in discount rates, and none from growth. The historical accounting of the WWII debt focusing on growth does not extend to the full sample.

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1. Introduction

This paper measures the fiscal foundations of inflation, and of the value of US federal debt.

The analysis is based on the government debt valuation equation, which states that the value of government debt equals the present value of primary surpluses. Discounting future surpluses by the return on the portfolio of government bonds, the valuation equation is an identity.

I linearize the valuation equation and take innovations. Then, unexpected inflation minus the unexpected real return of government bonds equals the unexpected decline in future primary surpluses plus the unexpected rise in future expected real government bond returns. I use a vector autoregression (VAR) to measure each component. I study ratios of debt and surplus to GDP, so returns also adjust for GDP growth.

Unexpectedly higher inflation devalues outstanding nominal government debt, and thus must correspond either to larger future deficits or to a higher real discount rate for government debt. Which is it? I find about half of each. Discount rates matter for government-debt present-value relations.

A decline in the present value of surpluses, coming from either surpluses or their discount rate, can also correspond to lower long-term bond prices, and thus an unexpected negative return on long term bonds. To what extent do shocks to the present value of surpluses correspond to declines in the nominal value of debt, rather than inflation? I find the answer is about a third. Long term debt acts as an important absorber for fiscal shocks.

A decline in bond prices reflects either future inflation or higher future real interest rates. I verify that bond price declines here are largely driven by future inflation. Thus, the decline in long-term bond prices, when the present value of debt declines, spreads over time inflation that would happen instantly otherwise.

Studying the present value relation underlying inflation leads to a natural puzzle – why do we see less inflation during recessions, with larger deficits, and why do we see more inflation during booms, with smaller deficits? Well, larger deficits may come with expectations of larger subsequent surpluses, to pay off the accumulated debts, and perhaps more, so the present value of all future surpluses does not necessarily decline when deficits rise in a recession. The larger current deficits in recessions also come with lower real interest rates, a lower discount rate for government debt. Lower discount rates make the stream of future surpluses more valuable, even
if current deficits are larger, and can account for the decline in inflation during high-deficit re-
cessions. I estimate that recession-related deficits are just about exactly paid off by subsequent
higher surpluses, and that the lower discount rate in recessions entirely accounts for the decline
of inflation in recessions, and contrariwise in booms. Discount rate variation produces this sort
of fiscal Phillips curve.

These questions are natural in the context of fiscal theory of the price level, in which case changes
in the present value of surpluses are thought to cause changes in inflation. However, the calcu-
lations make no assumption about active vs. passive policy configurations or the direction
of causality. If one assumes passive fiscal policy, and one assumes that monetary policy de-
termines the price level, those “passive” fiscal policy adjustments must take place. Whether a
monetary-induced unexpected inflation results in lower surpluses or higher real returns, and
when and how those adjustments happen is an important question for a well-specified mone-
tary regime. A theory paper may be happy with a little footnote that ex-post lump-sum taxes will
adjust to pay off any inflation-induced revaluations of government debt, but our economy does
not have infinite lump-sum taxes. A well-specified monetary regime must also have solid fiscal
foundations. So the only real question for fiscal vs. monetary theory of the price level is whether
I write that unexpected inflation “corresponds to” or “is caused by” shocks to the present value
of surpluses.

The valuation equation also allows us to ask how high debt to GDP ratios have resolved his-
torically – by high surpluses, by low real returns, expected or unexpected, or by large growth?
I confirm that growth largely paid off the WWII debts, but after about 1975 we entered a new
regime in which debts correspond to surpluses.

I then examine the forward-looking sources of variation in the value of government debt – on
average, do high values come from larger future surpluses, or lower future growth-adjusted dis-
count rates? I find that about 70% of the value of debt, and 50% of the value of innovations to
the debt correspond to variation in expected surpluses. Despite the WWII data point, the value
of debt is, on average, maintained by expectations of subsequent surpluses.

I do not test the valuation equation. This equation holds, in equilibrium, in part of every well-
specified macroeconomic model. (The equation requires that the present value of surpluses is
finite, loosely that \( r > g \), and so does not hold in dynamically inefficient models. I presuppose
the opposite case without further comment.) “Active” vs. “passive” specifications differ on how
the equilibrium forms, but not on the equations that hold in equilibrium. Testing an equation
that holds in every model does nothing to distinguish models and so is not interesting. Moreover, I discount surpluses at the ex-post return on the government debt portfolio, in which case the valuation equation is an identity, which has no testable content at all. But which element in an identity moves is an interesting calculation. The identity tells us that inflation must correspond to surpluses or discount rates. The identity does not tell us which one it is, or how quickly the surpluses and discount rates appear.

2. Literature

Much of the technique in this paper is imported from asset pricing. The general approach to linearizing the valuation identity owes much to the linearized present value identity for dividend yields in Campbell and Shiller (1988). I likewise linearize a return formula and iterate forward. I linearize in the level rather than the log of the surplus to GDP ratio, as that ratio is often negative. My accounting of the variance of unexpected inflation in a VAR context is similar to the return variance decomposition in Campbell and Ammer (1993). However, to avoid covariance terms, I focus on an extension of the decomposition of variance in Cochrane (1992) to a multivariate context. (With $x = y + z$, I explore $var(x) = cov(x, y) + cov(x, z)$ rather than $var(x) = var(y) + var(z) + cov(y, z)$.) The summary in Cochrane (2011b) and treatment of identities in Cochrane (2007) are obvious precursors. The uniting theme in the former is that discount rates matter. This paper is in many ways an arbitrage of that insight to government debt questions, where discount rate variation is largely ignored.

The analysis of government finances, how debt is paid off, grown out of, or inflated away, is a huge literature. Hall and Sargent (1997), Hall and Sargent (2011) is the most important precursor. Hall and Sargent emphasize the importance of the market value of debt, not the face value reported by Treasury, and consequent proper accounting for interest costs. Perhaps most importantly, I use data provided by Hall, Payne, and Sargent (2018). Bohn (2008) examines a long history of US debt, notes its value is stationary arguing that present values are finite, and shows that primary surpluses are higher following large debts, though growth also brings down debt to GDP ratios. This paper adds the asset pricing variance decompositions to this sort of exercise. Its primary contribution of studying inflation by looking at the innovation in the present value formula is more novel.

While the calculations do not require an active (Leeper (1991)) fiscal policy assumption, and
can be read as measuring the nature of passive fiscal response to an active monetary policy specification, they are most naturally read as an empirical investigation of the fiscal theory of monetary policy. By this term, I mean a model that adopts most of the standard new-Keynesian ingredients – intertemporal optimization, market clearing, and pricing frictions – but substitutes an active fiscal policy for active monetary policy to select equilibria.

As the simplest example, consider a frictionless model composed of only the Fisher equation

\[ i_t = r + E_t \pi_{t+1} \]

and the government debt valuation equation,

\[ \frac{B_{t-1}}{P_t} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} \]  

(1)

where \( i_t \) is the nominal interest rate, \( r \) is a constant real rate, \( \pi_t \) is inflation, \( B_{t-1} \) is one-period nominal government debt, \( P_t \) is the price level, \( \beta = 1/(1 + r) \), and \( s_t \) are real primary surpluses.

If the central bank sets an interest rate target, and with a passive fiscal policy in which \( s_t \) reacts ex-post to any price level, this model determines expected inflation but not unexpected inflation. There are multiple equilibria corresponding to any value of \( \pi_{t+1} - E_t \pi_{t+1} \).

The standard new-Keynesian approach solves this multiplicity with an active monetary policy

\[ i_t = r + \phi \pi_t, \phi > 1, \]

retaining passive fiscal policy, and a rule against nominal explosions (Woodford (2003), Cochrane (2011a)). Now only one value of \( \pi_{t+1} - E_t \pi_{t+1} \) remains.

A fiscal theory of monetary policy either specifies a “passive” \( \phi < 1 \) monetary policy or allows nominal explosions, but turns off the passive fiscal policy assumption. Splitting (1), one period forward, into expected and unexpected components,

\[ \frac{B_t}{P_t} (E_{t+1} - E_t) \left( \frac{P_t}{P_{t+1}} \right) = (E_{t+1} - E_t) \sum_{j=0}^{\infty} \beta^j s_{t+1+j} \]  

(2)

Now, the revision in the discounted value of future surpluses picks unexpected inflation. Expected inflation is still determined by the central bank’s interest rate target, which can be achieved
by adjusting nominal debt $B_t$ with no change in surpluses. Thus, monetary policy – changing quantities of government debt with no change in fiscal policy – sets interest rates and expected inflation, while fiscal policy chooses unexpected inflation.

In this framework, empirically tying unexpected inflation to revisions in the present value of primary surpluses obviously takes on a great significance. This fiscal theory of monetary policy also motivates my calculation of the responses to interest rate changes that hold future surpluses constant – monetary policy – and to surplus shocks that hold interest rates constant.

Naturally, we want to allow both theoretically and empirically for a more realistic model, including long-term debt in place of $B_{t-1}$, time-varying discount rates in place of $\beta^j$, and sticky prices. Examples of such a theory include Sims (2011) Cochrane (2017) and a detailed treatment in Cochrane (2019). In particular, the latter shows how long-term debt and sticky prices can generate a protracted inflation response to a fiscal shock, where (2) suggests that a fiscal shock can only create an instant price level jump. I look for and find such a protracted response below.

3. Linearized identities

The analysis is based on linearized flow and present value identities for government debt. Following the flow of money in a period, we have the linearized flow identity

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = s_{t+1} + v_{t+1}. \quad (3)$$

The debt to GDP ratio at the end of period $t+1$, $v_{t+1}$, is its value at the beginning of the period, $v_t$, increased by the real return on debt $r_{t+1}^n - \pi_{t+1}$ less GDP growth $g_{t+1}$, and decreased by a positive real primary surplus.

The symbol $s_t$ can stand for the ratio of the real primary surplus $sp_t$, divided by GDP $Y_t$ and adjusted for the steady state value of the debt to GDP ratio $V/Y$,

$$s_t = \frac{sp_t}{Y_t} \frac{1}{V/Y},$$
or it can stand for the ratio of real primary surplus to the previous period's debt,

$$s_t = sv_t = \frac{sp_t}{V_{t-1}}.$$  

Both definitions lead to the same identity (3). I will refer to $s_{t+1}$ calculated from (3) as just the “surplus.” Its units are surplus divided by market value of debt, or surplus to GDP ratio divided by debt to GDP ratio.

Iterating forward, we have the linearized present value identity,

$$v_t = \sum_{j=1}^{T} s_{t+j} - \sum_{j=1}^{T} (r^n_{t+j} - \pi_{t+j} - g_{t+j}) + v_{t+T}. \tag{4}$$

The value of government debt, divided by GDP, is the present value of future surplus to GDP ratios, discounted at the ex-post real return adjusted by GDP growth. In general, I’ll take the $T \rightarrow \infty$ limit and show that the last term vanishes.

Seigniorage and liquidity premiums for government debt are included in the possibility that $R^n$ is lower than corresponding real interest rates available to private investors. Relationship (4) holds ex-post, and therefore also ex-ante, with $E_t$ in front of the sum and any information set that includes the current real value of the debt.

### 3.1. Derivation

The symbols are as follows:

$$V_t = M_t + \sum_{j=0}^{\infty} Q_t^{(t+1+j)} B_t^{(t+1+j)}$$

is the nominal end-of-period market value of debt, with $M_t$ money, $B_t^{(t+j)}$ zero-coupon nominal debt outstanding at the end of period $t$ and due at the beginning of period $t + j$, $Q_t^{(t+j)}$ is the time $t$ price of that bond, with $Q_t^{(i)} = 1$. $Y_t$ is real GDP or another stationarity-inducing divisor (consumption, potential GDP, population, etc.) and $P_t$ is the price level, so

$$v_t \equiv \log \left( \frac{V_t}{Y_t P_t} \right)$$
is log market value of the debt divided by GDP.

\[ R^n_{t+1} = \frac{M_t + \sum_{j=1}^{\infty} Q^{(t+1+j)}_t B^{(t+1+j)}_t}{M_t + \sum_{j=0}^{\infty} Q^{(t+1+j)}_t B^{(t+1+j)}_t} \]

is the nominal return on the portfolio of government debt, i.e. overnight from the end of \( t-1 \) to the beginning of \( t \), and

\[ r^n_{t+1} \equiv \log(R^n_{t+1}) \]

is the log nominal return on that portfolio.

\[ \pi_t \equiv \log \left( \frac{P_t}{P_{t-1}} \right), \; g_t \equiv \log \left( \frac{Y_t}{Y_{t-1}} \right) \]

are log inflation and GDP growth rate. Let \( sp_t \) denote the real primary (not including interest payments) surplus or deficit.

Now, I establish the nonlinear flow and present value identities. In period \( t \), we have

\[ \sum_{j=0}^{\infty} Q^{(t+j)}_t B^{(t+j)}_{t-1} + M_{t-1} = P_t sp_t + \sum_{j=0}^{\infty} Q^{(t+j)}_t B^{(t+j)}_t + M_t. \]  

Money at the end of the period \( M_{t+1} \) is money brought in from the previous period \( M_{t-1} \) plus the effects of bond sales or purchases at price \( Q^{(t+j)}_t \), less money soaked up by primary surpluses. The left hand side is the beginning of period market value of debt, i.e. before debt sales or repurchases \( B^{(t+j)}_t - B^{(t+j)}_{t-1} \) have taken place. It turns out to be more convenient here to express equations in terms of the end-of-period market value of debt. To that end, multiply and divide, and shift the time index forward one period, to write

\[ \frac{M_t + \sum_{j=1}^{\infty} Q^{(t+j)}_t B^{(t+j)}_t}{P_t Y_t} \frac{P^{n}_{t+1}}{G_{t+1}} \frac{P_t}{P_{t+1}} = \frac{sp_{t+1} + \sum_{j=1}^{\infty} Q^{(t+1+j)}_{t+1} B^{(t+1+j)}_{t+1}}{P^{n+1}_{t+1} + \sum_{j=1}^{\infty} Q^{(t+1+j)}_{t+1} B^{(t+1+j)}_{t+1}}. \]  

We can iterate the flow identity (6) forward to express the nonlinear government debt valuation identity as

\[ \frac{M_t + \sum_{j=1}^{\infty} Q^{(t+j)}_t B^{(t+j)}_t}{P_t Y_t} \frac{s_{t+1}}{Y_{t+1}} = \sum_{j=1}^{\infty} \left( \prod_{k=1}^{j} \frac{1}{R_{t+k}/G_{t+k}} \right) \frac{s_{t+1}}{Y_{t+1}}. \]

The market value of government debt, as a fraction of GDP, equals the present value of primary surplus to GDP ratios, discounted at the government debt rate of return less the GDP growth rate. (I write the equation under the assumption that the right hand side converges.)
By using ex-post returns to discount, the flow and present value identities hold ex-post. One can add expectations to give ex-ante present value relations. When using other discount factors, only the ex-ante versions may be valid.

I linearize the flow equation (6) and then iterate forward. Write (6) as

\[ \frac{V_t}{P_t} R_{t+1} P_{t+1} Y_{t+1} = sp_{t+1} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}}. \]

Taking logs,

\[ v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log \left( \frac{sp_{t+1}}{Y_{t+1}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}} \right) \]

(8)

At this point I pursue two linearizations: I linearize in terms of the surplus to GDP ratio, and then in terms of the surplus to value ratio,

\[ sv_t \equiv \frac{sp_t}{V_{t-1}/P_{t-1}}. \]

(9)

The surplus to GDP ratio has some conceptual advantages, but the surplus to value ratio produces a more accurate approximation. Linearizing around \( r - g = 0 \), both approaches produce the same linearized answer, (3) and (4).

To approximate in terms of the surplus to value ratio, write (8) as

\[ v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log \left( \frac{V_t}{P_t} \frac{sp_{t+1}}{Y_{t+1}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}} \right) \]

\[ r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log \left( sv_{t+1} + e^{v_{t+1} - v_t} \right). \]

Denote steady state values without subscripts, \( r = r^n - \pi, g, sv \) and \( v \). At a steady state

\[ r - g = \log (1 + sv). \]

(10)

Taylor expanding around a steady state,

\[ r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log (1 + sv) + \frac{1}{(1 + sv)} (sv_{t+1} - sv + v_{t+1} - v_t) \]

\[ v_t + (1 + sv) \left[ r^n_{t+1} - \pi_{t+1} - g_{t+1} \right] = [(1 + sv) \log (1 + sv) - sv] + sv_{t+1} + v_{t+1} \]

(11)

Equation (3) follows, with the symbol \( st \) representing the surplus to value ratio \( st = sv_t \), if we suppress the constant, using deviations from means in the analysis, or if we use \( r = g \) or \( sv = 0 \).
as a point of expansion. The point of expansion need not be the sample mean.

To linearize in terms of the surplus/GDP ratio, start again from (8). Taylor expanding the last term directly,

\[ v_t + r_{t+1}^n - \pi_{t+1} + g_{t+1} = \log(e^v + sy) + \frac{e^v}{e^v + sy} (v_{t+1} - v) + \frac{1}{e^v + sy} (sy_{t+1} - sy) \]

where

\[ sy_{t+1} = \frac{sp_{t+1}}{Y_{t+1}}, \quad \text{(12)} \]

so

\[ v_t + r_{t+1}^n - \pi_{t+1} + g_{t+1} = \log(e^v + e^v sy) + \frac{e^v}{e^v + e^v sy} (v_{t+1} - v) + \frac{e^v}{e^v + e^v sy} \left( \frac{sy_{t+1}}{e^v} - sv \right) \]

\[ v_t + r_{t+1}^n - \pi_{t+1} + g_{t+1} = v + \log(1 + sv) + \frac{1}{1 + sv} (v_{t+1} - v) + \frac{1}{1 + sv} \left( \frac{sy_{t+1}}{e^v} - sv \right) \]

\[ v_t + r_{t+1}^n - \pi_{t+1} + g_{t+1} = \left[ \frac{sv}{1 + sv} (v - 1) + \log(1 + sv) \right] + \frac{1}{1 + sv} v_{t+1} + \frac{1}{1 + sv} \frac{sy_{t+1}}{e^v} \]

Suppressing the small constant, and thus interpreting variables as deviations from means, the linearized flow identity is

\[ v_t + r_{t+1}^n - \pi_{t+1} + g_{t+1} = \beta sy_{t+1} + \beta v_{t+1} \quad \text{(13)} \]

where

\[ \beta = e^{-(r-g)} = \frac{1}{1 + sv}, \quad \text{(14)} \]

Iterating forward, the present value identity is

\[ v_t = \sum_{j=1}^{T} \beta^{j-1} \left[ \frac{sy_{t+j}}{e^v} - \left( \frac{r_{t+j}^n - \pi_{t+j} + g_{t+j}}{e^v} \right) \right] + \beta^T v_T. \quad \text{(15)} \]

If we linearize around \( r - g = 0 \), then (13) reduces to (3) exactly, with the symbol \( s_t \) representing \( sy_t/e^v \).

The linearizations in terms of \( sv_t \) are more accurate. The units of the flow identities (3) (13) are rates of return. Dividing the surplus by the previous period’s value gives a better approximation to the change in value, when the value of debt is far from the steady state. Suppose debt to GDP is 100%, the real rate of return is 0, the surplus is -20% of GDP, and 20% of debt, so debt to GDP rises to 120%. If the steady state debt to GDP ratio is 50%, we count that surplus instead as a
40% change in the surplus to GDP ratio, resulting in a 20% error. Outside of WWII, most values are not this extreme, but the principle still holds.

However, the term $v_{t+T}$ does not vanish in (4), where the term $\beta^T v_{t+T}$ can vanish in (15). All discounting the former comes through the explicit return terms $r_{t+j}$, for any value of $r - g$. In this paper, the presence of the $v_{t+T}$ term is not a difficulty. I mostly study expectations $E_t v_{t+T}$ and $v_t$ is stationary, so $E_t v_{t+T} \rightarrow 0$.

More deeply, a constant ratio of surplus to market value of debt leads to a passive fiscal policy. A deflation raises the real value of debt, and if surpluses rise in response, they validate the lower price level. Thus, although on the equilibrium path one can describe dynamics via either linearization, if one wants to think about how fiscal-theory equilibria are formed, it is better to describe a surplus that does not react to price level changes, so only one value $v_t$ emerges, as is the case in (15). The analysis of this paper is about what happens in equilibrium, and does not require an active-fiscal assumption, so the difference is irrelevant here.

More deeply, I infer the surplus from the linearized flow identity (3) so which concept the surplus corresponds to makes exactly no difference whatsoever to the analysis. The difference is only the accuracy of approximation, how close the surplus recovered from the linearized flow identity corresponds to a surplus recovered from the nonlinear exact identity (8).

4. Inflation

To focus on the fiscal roots of inflation, and not other sources of variation in the value of debt $v_t$, I take innovations $\Delta E_{t+1} = E_{t+1} - E_t$ of (4), and rearrange, leaving

$$\Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} \left( r^n_{t+1} - g_{t+1} \right)$$

$$= - \sum_{j=1}^{\infty} \Delta E_{t+1}s_{t+j} + \sum_{j=1}^{\infty} \Delta E_{t+1} \left( r^n_{t+1+j} - \pi_{t+1+j} - g_{t+1+j} \right)$$

I forecast variables using a VAR. Since we start with an identity, the result holds for any information set. I do not thereby assume that agents observe only the variables in the VAR. However, the result is in terms of the VAR information set, so “innovation” refers to that set of variables only, and not the larger set that agents observe.
I put unexpected inflation $\Delta E_{t+1} \pi_{t+1}$ first in (16), as it is the variable I’m most interested in.

The second term $\Delta E_{t+1} (r^n_{t+1})$ is the unexpected return on the portfolio of government bonds. With one-period debt, this term is zero, as $r^n_t = i_{t-1}$ is known ahead of time. With long-term debt, this term measures how much news about the present value of surplus shows up in bond prices rather than current inflation. Higher future inflation leads to lower bond prices, so this term can measure how much a fiscal shock is spread out via long-term bonds to future inflation rather than immediate inflation.

The first term on the right hand side is the revision in expected future surpluses. Inflation or devaluation of government bonds comes from all future surpluses, not just current surpluses. If big deficits today are followed by large surpluses to pay off the debt, there need be no inflation today. The identity (16) lets us see if it happens.

The second term on the right hand side captures the discount rate effect: A rise in the expected return of government bonds lowers the value of future surpluses. This is an important and underappreciated effect. Most discussion of the valuation equation uses a constant discount rate for simplicity, and therefore ties variation in the value of government debt entirely to variation in prospective surpluses. Since the VAR is stationary, a potential last term $\lim_{T \to \infty} \Delta E_{t+1} v_{t+T}$ is zero.

The different treatment of the time-$t$ term involving inflation, return, and growth, and the future terms looks strained at first glance. One is tempted to treat all the $\pi_t$, $r^n_t$ and $g_t$ terms together, and write the model as a model of long-term average of inflation $\sum_{j=0}^{\infty} \pi_{t+j}$ instead. But separating out the first and subsequent terms makes sense. In a frictionless, constant real rate model, expected inflation and expected nominal bond returns move one-for-one. $E_{t+1} \left( r^n_{t+j} - \pi_{t+j} \right) = 0$ for $j > 1$, and $(E_{t+1} - E_t) \left( r^n_{t+j} - \pi_{t+j} \right) = 0$ always for $j > 1$. That is not true of the time $t+1$ term $(E_{t+1} - E_t) \left( r^n_{t+1} - \pi_{t+1} \right)$ on the left hand side. Even with constant real rates, ex-post returns can be large, as ex-post nominal returns and inflation can both move unexpectedly and in different directions. With varying real rates the future terms can still move, but much less and by a different economic mechanism than the current $t+1$ term: The revision in expected future inflation and returns revise the discount rate by which we discount future surpluses. The revision in current inflation and ex-post returns are a capital gain or loss that responds ex-post to that change in present value.
5. Estimates

5.1. Data

I use data on the market value of government debt held by the public, and the nominal rate of return of the government debt portfolio from Hall, Payne, and Sargent (2018). I do not attempt to subtract Fed holdings of treasuries, and add the monetary base to the definition of government debt, though one should ideally do that.

I infer the primary surplus from the flow identities. This calculation measures how much money the government actually borrows. NIPA surplus data does not obey the identity. I infer the surplus from the linearized identity (3), to produce a series that exactly matches that identity. Given that the linear approximation is good, it is clearer to bundle the approximation error in the surplus than to produce numbers that should add up but do not due to approximation error.

To measure the accuracy of this approximation, I also infer the real primary surplus from the exact nonlinear identity (5). I calculate the surplus in each month from the nonlinear flow identity (5). I then carry the surplus to the end of the year using the government bond return. This procedure produces an annual series for which the identity (5) continues to hold in annual data. I use standard BEA data for GDP and consumption, and the GDP deflator as my measure of inflation. I use CRSP data for the three-month Treasury rate.

I measure the debt to GDP and surplus to GDP ratios by the ratios of debt and surplus to personal consumption expenditures, times the average GDP to consumption ratio. Debt to GDP ratios are common measures across countries, but in our time-series application they introduce cyclical variation in GDP. We want only a detrending divisor, and an indicator of the economy’s long-run level of tax revenue and spending. Potential GDP would be better, but has a severe look forward bias as it is constructed ex post and looks just like a two-sided moving average. Consumption is a decent stochastic trend for GDP.

I use a data sample 1947-2014. The immense deficits of WWII would otherwise dominate the analysis, and one may well suspect that financing that war and resolving its debts follows a different pattern than the subsequent decades of largely cyclical deficits and inflation.

Figure 1 presents the surplus and compares three measures. The “Linear, \( s_t \)” line imputes the surplus from the linearized flow identity (3) directly at the one-year horizon, which is the mea-
Figure 1: Surplus. “Linear” is inferred from the linearized flow identity. “$sv$” is the ratio of the primary surplus to the previous year’s market value of the debt. “$sy$” is the ratio of surplus to GDP scaled by the average value of debt.

The first piece of news is that there are primary surpluses. One’s impression of endless deficits comes from the full deficit including interest payments on the debt. Even NIPA measures show regular positive primary surpluses. Steady primary surpluses from 1947 to 1975 helped to pay off WWII debt. 1975 to the early 1990s inaugurated a new era of primary deficits, interrupted by the strong surpluses of the late 1990s. On top of these trends, and outside of WWII, primary surpluses have a clear cyclical pattern. The primary surplus correlates very well with the unemployment rate (not shown).

The three measures in Figure 1 are quite close, and that is the point – the graph is a measure
Table 1: VAR estimate. One (two) stars means the estimate is one (two) standard errors above zero of the accuracy of the linearized identity (3), after aggregation to annual data. The linearized identity is a closer approximation to the surplus to value ratio $sv$. Those two lines are nearly indistinguishable in the post WWII data. It is a somewhat less good approximation to the surplus to GDP ratio $sy$. The difference is largest when the value of debt is far from its mean, both in WWII and in the 1970s.

5.2. Impulse-response for inflation shocks

Table 1 presents OLS estimates of the VAR coefficients. Each column is a separate regression. The VAR includes the central variables for the inflation identity – nominal return on the government bond portfolio $r^n$, consumption growth rate $g$, inflation $\pi$, surplus $s$ and value $v$. I include the three month interest rate $y$ as it is an important forecasting variable.

It is important to include the value of debt $vt$ in the VAR, even if we are calculating terms of the innovation identity (16) that does not reference that value. Related, it is tempting to exclude $vt$ from the VAR, compute the right hand side of (4), and test whether the value of the debt comes out equal to this present value of discounted expected surpluses. The fact that the identity is an identity, which holds in all models, ought to warn us against trying to test it. Identities hold. They are useful for measurement – which terms of the identity vary more or less? But anything that purports to test them must be a mistake. When we deduce from (4) expressions $vt = E_t(...)$, 

\[ \begin{array}{cccccc}
 r_{t+1}^n & gt+1 & \pi_{t+1} & st+1 & vt+1 & yt+1 \\
 r^n_t & -0.03 & 0.02 & -0.11^{**} & -0.31^* & 0.37^* & -0.07^* \\
 g_t & -0.28^* & 0.19^* & 0.16^* & 1.34^{**} & -1.98^{**} & 0.28^{**} \\
 \pi_t & -0.17 & -0.15^* & 0.54^{**} & -0.22 & -0.34 & 0.09 \\
 st & -0.04 & -0.01 & -0.02 & 0.40^{**} & -0.41^{**} & -0.05^* \\
 vt & 0.01 & -0.00 & -0.02^{**} & 0.05^* & 0.98^{**} & -0.01 \\
 yt & 1.23^{**} & 0.02 & 0.16^* & 0.44^* & 0.60^* & 0.82^{**} \\
100 \times std(\epsilon_{t+1}) & 2.76 & 1.64 & 1.16 & 4.49 & 6.69 & 1.31 \\
 R^2 & 0.54^* & 0.09^* & 0.72^* & 0.55^* & 0.97^* & 0.80^* \\
100 \times std(x) & 4.07 & 1.72 & 2.18 & 6.72 & 36.04 & 2.95 
\end{array} \]
we must include \( v_t \) in the information set that takes the expectation.

I compute standard errors from a Monte Carlo. The stars represent one or two standard errors above zero. Since we aren’t testing anything, this is just a visual way to show standard errors without another table.

Unsurprisingly, the three month treasury bill yield \( y_t \) helps a lot to predict the ex-post bond return \( r_{t+1}^n \), which is one reason I include the yield in the VAR. Inflation has a substantial autocorrelation (0.54) as we might expect, and the three month yield also helps to predict inflation. The surplus, value, and three month yield are each strongly autocorrelated. A larger surplus \( s_t \) results in less market value of debt, \( v_{t+1} \), (-0.41), as one would expect. The surplus responds to the value of the debt. 0.05 is economically important, as any passive value means that surpluses rise to pay off debt. This coefficient can be misinterpreted to measure a passive-fiscal regime. The active vs. passive fiscal question is how surpluses would respond to multiple-equilibrium variation in the value of debt, induced by surprise inflation. We do not measure off-equilibrium responses from data drawn from equilibrium.

To calculate the response to an inflation shock, I orthogonalize the inflation shock last – all other variables respond contemporaneously to whatever the shock is that moves inflation. This choice also means that any other orthogonalized shock will not impact unexpected inflation. To compute the inflation shock, I specify \( \varepsilon_\pi^t = 1 \). Then I fill in shocks to the other variables by running regressions of their shocks on the inflation shock. I run

\[
\varepsilon_{t+1}^x = b_{x,\pi} \varepsilon_{t+1}^\pi + \delta_{t+1}.
\]

Then I start the VAR at

\[
\varepsilon_1 = - \begin{bmatrix} b_{r,n,\pi} & b_{g,\pi} & \varepsilon_{1}^\pi = 1 & b_{s,\pi} & \ldots \end{bmatrix}.
\]

This procedure is equivalent to the usual orthogonalization of the shock covariance matrix, but it is more transparent and it generalizes more easily later.

The top panel of Figure 2 plots the responses of the main variables in the identity (16). Inflation \( \pi \) dies out slowly. A positive inflation shock coincides with a negative shock to the surplus \( s \), which is also persistent. However, surpluses eventually do rise – deficits now mean surpluses later to pay back the debt, at least partially. Mechanically, this rise comes from the surplus response to the rising debt, though again we don’t measure if this is a causal response or simply how debt evolves given its shock. The sum of all surplus responses is -0.68, so although the majority of the
Figure 2: Response to a 1% inflation shock.
5 years of negative surpluses is offset by subsequent positive values, not all of it is.

The short line marked \( r^n - g \) plots \( \Delta E_1 (r^n_1 - g_1) \), driven by the bond portfolio return \( r^n_1 \). The unexpected inflation, which devalues government debt directly, is accompanied by a decline in the nominal market value of government debt, which further devalues long-term debt. Both mechanisms soak up fiscal shocks. This line stops because it is uninteresting past the first observation.

The longer line marked \( r - g \) plots \( \Delta E_1 \left( r^n_{1+j} - \pi_{1+j} - g_{1+j} \right) \), the revision in the real discount rate. These are plotted at the time of the return, \( 1 + j \), so they are the expected return one period earlier at time \( j \). After one period, the discount rate rises. Thus, just as the surpluses out to year 7 are making government debt less valuable, the discount rates are doing the same thing. This line starts one period after the shock, as its impact response is not interesting.

The market value of the debt \( v_t \) initially declines, as there is a negative return shock. The negative surpluses (deficits) then soon pile up the debt, until surpluses reverse and the debt starts to decline. The rise in expected return adds a bit to the rise in market value of debt.

The lower panel of Figure 2 plots the response of rates of return in more detail, to give some intuition for the discount rate behavior of the previous figure.

The return \( r^n_t \) takes a large one-period fall, but then rises. This is the picture of an unexpected rise in bond yields, i.e. decline in bond prices. The three month bill yield rises throughout, and moves one period ahead of the return after the first period. The expectations hypothesis \( y_t = E_t r^n_{t+1} \) holds quite well, and certainly within a standard error, in this response.

The discount rate \( r^n_t - \pi_t - g_t \) has three components. The inflation shock \( \pi_t \) comes with a decline in the economy – the inflation shock is stagflationary. The slight rise in expected real return \( r - g \) in the previous figure then comes from a slightly higher nominal rate, much higher inflation, but somewhat lower growth rate, and the latter two effects offset.
5.3. A variance decomposition

My central calculation is a decomposition of the variance of unexpected inflation based on the present value identity (16), which I repeat for convenience:

\[
\Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} \left(r^n_{t+1} - g_{t+1}\right) \\
= - \sum_{j=1}^{\infty} \Delta E_{t+1} s_{t+j} + \sum_{j=1}^{\infty} \Delta E_{t+1} \left(r^n_{t+1+j} - \pi_{t+1+j} - g_{t+1+j}\right).
\]

We can think of the variance decomposition in two ways. First, just sum up the terms of the impulse response function as plotted in Figure 2. When there is a unit shock to inflation \(\Delta E_1 \pi_1\), how large are the other terms? Second, multiply both sides of (16) by \(\Delta E_{t+1} \pi_{t+1}\) and take expectations, giving

\[
\text{var}(\Delta E_{t+1} \pi_{t+1}) - \text{cov}\left[\Delta E_{t+1} \pi_{t+1}, \Delta E_{t+1} \left(r^n_{t+1} - g_{t+1}\right)\right] \\
= - \sum_{j=1}^{\infty} \text{cov}\left[\Delta E_{t+1} \pi_{t+1}, \Delta E_{t+1} s_{t+j}\right] + \sum_{j=1}^{\infty} \text{cov}\left[\Delta E_{t+1} \pi_{t+1}, \Delta E_{t+1} \left(r^n_{t+1+j} - \pi_{t+1+j} - g_{t+1+j}\right)\right].
\]

Unexpected inflation may only vary to the extent that it covaries with current bond returns, or if it forecasts surpluses or real discount rate. We should see that in the data, and we can measure which one it is.

Dividing by \(\text{var}(\Delta E_{t+1} \pi_{t+1})\), we can express this variance decomposition as a relation between regression coefficients, which give the fractions of variance accounted for by each term directly. This decomposition adds up to 100%, within the accuracy of approximation, but it is not an orthogonal decomposition, nor are all the elements necessarily positive.

The two approaches give exactly the same result – the terms of (17) are exactly the terms of the impulse-response function, to a shock that moves all variables including \(\Delta E_t \pi_{t+1}\) when all the other orthogonalized shocks leave \(\Delta E_t \pi_{t+1} = 0\). We can compute the variance decomposition either way.

We could just simulate the impulse response function out a long way and add up its terms. More elegantly, write the VAR

\[
x_{t+1} = Ax_t + \varepsilon_{t+1}
\]
so
\[ \Delta E_{t+1} \sum_{j=1}^{\infty} x_{t+j} = (I - A)^{-1} \epsilon_{t+1}. \]

Let \( a \) denote vectors which pull out each variable, i.e.
\[ \pi_t = a'_\pi x_t, \quad s_t = a'_s x_t, \]

etc. Then the present value identity (16) reads
\[ a'_\pi \epsilon_{t+1} - (a_r - a_g)' \epsilon_{t+1} = -a'_s (I - A)^{-1} \epsilon_{t+1} + (a_r - a_\pi - a_g)' (I - A)^{-1} A \epsilon_{t+1}. \]

We can calculate the variance decomposition (17) by
\[ a'_\pi \Sigma a_\pi - (a_r - a_g)' \Sigma a_\pi = -a'_s (I - A)^{-1} \Sigma a_\pi + (a_r - a_\pi - a_g)' (I - A)^{-1} A \Sigma a_\pi \]
where \( \Sigma = \text{cov}(\epsilon_{t+1}, \epsilon'_{t+1}) \), and then divide by \( a'_\pi \Sigma a_\pi \) to express the result as a fraction,
\[ 1 - (a_r - a_g)' \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi} = -a'_s (I - A)^{-1} \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi} + (a_r - a_\pi - a_g)' (I - A)^{-1} A \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi}. \]

To show that this formula is the same as the elements and sum of elements of the impulse-response function, note that the regression coefficient of other shocks on the inflation shock is
\[ b_{\epsilon_\pi, \epsilon_\pi} = \frac{\text{cov}(\epsilon^x_{t+1}, \epsilon^\pi_{t+1})}{\text{var}(\epsilon^\pi_{t+1})} = \frac{a'_\pi \Sigma a_\pi}{a'_\pi \Sigma a_\pi}, \]
so the VAR shock, consisting of a unit movement in inflation \( \epsilon^\pi_1 = 1 \) and movements \( \epsilon^x_1 = b_{\epsilon_\pi, \epsilon_\pi} \) in each of the other variables is given by
\[ \epsilon_1 = \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi}. \]

We recognize in (20) the responses and sums of responses to this shock.

Table 2 presents the terms of the variance decomposition. Figure 2 includes the sums of the future surplus and adjusted discount rate \( r - g \) responses.

In Table 2, inflation, \( \Delta E_1 \pi_1 \), is 1.0 by construction. The impulse response is based on a unit (one percent) shock, or we express each covariance term as a fraction of the inflation innovation variance.
### Table 2: Decomposition of unexpected inflation variance

<table>
<thead>
<tr>
<th>Component</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation $\pi_1$</td>
<td>1.00</td>
</tr>
<tr>
<td>Bond return $-(r^n_1 - g_1)$</td>
<td>0.36</td>
</tr>
<tr>
<td>of which $-r^n_1$</td>
<td>0.72</td>
</tr>
<tr>
<td>of which $g_1$</td>
<td>-0.36</td>
</tr>
<tr>
<td>Total current $\pi_1 - (r^n_1 - g_1)$</td>
<td>1.36</td>
</tr>
<tr>
<td>-Future $\Sigma s$</td>
<td>0.68</td>
</tr>
<tr>
<td>Future $\Sigma r - g$</td>
<td>0.68</td>
</tr>
<tr>
<td>Total future</td>
<td>1.36</td>
</tr>
<tr>
<td>Std dev $\pi \times 100$</td>
<td>1.16</td>
</tr>
</tbody>
</table>

Surplus responses $-\sum_{j=1}^{\infty} \Delta E_1 s_j$, contribute 0.68 to the variance decomposition. In Figure 2, we see negative near-term surpluses, but positive long-term surpluses. It was possible the latter equate to or outweigh the former, so that debt is paid off. They don’t.

- **Half of the decline in present value of surplus corresponding to an inflation shock comes from deficits that are not expected to be paid off, and so devalue outstanding bonds.**

The discount rate term, $\sum_{j=1}^{\infty} \Delta E_{t+1} \left( r^n_{t+1+j} - \pi_{t+1+j} - g_{t+1+j} \right)$ is also 0.68.

- **Half of the decline in the present value of surplus corresponding to an inflation shock comes from higher discount rates.**

The response $\Delta E_1 (r^n_1 - g_1)$ to the inflation shock is -0.36, so it contributes 0.36 to the variance decomposition. When the present value of surpluses declines 1.36%, inflation rises 1% and growth-adjusted return of nominal bonds declines 0.36%. Long-term debt soaks up (or corresponds to, in a passive fiscal reading) about a quarter of the fiscal shock.

- **A fourth of the decline in present value of surpluses corresponding to an inflation shock is soaked up by a decline in the adjusted value of long term bonds.**

Breaking out this return to its components, bonds $r^n_{t+1}$ fall by 0.73%. However, there is a fairly strong 0.36% decline in consumption as well, so the value of debt to consumption only falls by
0.36%.

To what extent does the decline in long term bond prices \( r_{t+1}^N \) represent future inflation? With short-term debt, only current inflation can devalue debt so a fiscal shock can only produce a one-period innovation in inflation \( \Delta E_1 \pi_1 \). The subsequent persistent inflation cannot unexpectedly devalue bonds, so it must come from another source such as monetary policy. With long-term debt, it is possible that a smooth inflation devalues debt, by devaluing nominal bonds. But nominal bond prices also fall if the expected real return rises. Which effect is at work? The drawn out inflation response in Figure 2 suggests that a good part of the nominal bond price fall is a response to expected future inflation.

To make a rough calculation, note the log price of an \( N \) year discount bond is

\[
P_t^{(N)} = -\sum_{j=1}^{N} r_{t+j}^{n(N-j+1)}
\]

where \( r_{t+1}^{n(N)} \) denotes the nominal log return at time \( t + 1 \) of a bond which has maturity \( N \) at time \( t \). Moving the time index forward, adding and subtracting inflation, and taking innovations, we can relate the unexpected return \( \Delta E_1 r_1^{n(N)} = \Delta E_1 p_1^{(N)} \) to innovations to expected returns and to expected inflation.

\[
\Delta E_1 \left( r_1^{n(N)} \right) = \Delta E_1 \sum_{j=1}^{N} \left[ -\left( r_{1+j}^{n(N-j+1)} - \pi_{1+j} \right) - \pi_{1+j} \right]
\]

This is the decomposition we're after: To what extent does the negative return coincident with inflation reflect future inflation, and to what extent does it reflect a rise in expected real returns, with a less easily interpretable source, such as the interaction of inflation with sticky prices?

Imposing the expectations hypothesis, we can use bonds of any maturity on the right hand side, and thus the government bond portfolio return,

\[
\Delta E_1 \left( r_1^n \right) = \Delta E_1 \sum_{j=1}^{N} \left[ -\left( r_{1+j}^n - \pi_{1+j} \right) - \pi_{1+j} \right]
\]
or we can use the short-run real interest rate included in the VAR,

$$\Delta E_1 (r^n_1) = \Delta E_1 \sum_{j=1}^{N} \left[ -(y_j - \pi_{1+j}) - \pi_{1+j} \right].$$  \hspace{1cm} (23)

To make a more accurate calculation, we would need to model the maturity structure of the debt, and to include long-term bond yields in the VAR. Without them, the underlying identity (22) need not hold in the impulse-responses. The negative autocorrelation of bond returns is more likely to be seen with their crucial state variable, yields, included. But all this takes us away from the task of interpreting the simple VAR.

For now, use (23) as inspiration, and let us simply examine the terms on the right hand side, as if US government debt were a single zero coupon bond, and as if the response functions without long-term yields as a state variable captured expectations fully, i.e. as if the expectations hypothesis held.

Table 3: Decomposition of the bond return shock to future inflation and future real returns

<table>
<thead>
<tr>
<th>N</th>
<th>Real, Real, Inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r^n - \pi$</td>
</tr>
<tr>
<td>2</td>
<td>0.39</td>
</tr>
<tr>
<td>3</td>
<td>0.45</td>
</tr>
<tr>
<td>4</td>
<td>0.36</td>
</tr>
<tr>
<td>$r^n_1$</td>
<td>-0.72</td>
</tr>
</tbody>
</table>

Table 3 presents the decomposition for a variety of $N$. The US maintains a relatively short maturity structure overall, so $N=2-3$ is a decent approximation. The bottom entry is $\Delta E_1 (r^n_1) = -0.72$, the response of unexpected bond returns to the inflation shock. The upper entries give, for each value of $N$, the amount of that shock attributed to innovations to real returns, measured as $\Delta E_1 \sum_{j=1}^{N} \left( r^n_{1+j} - \pi_{1+j} \right)$, and then measured as $\Delta E_1 \sum_{j=1}^{N} \left[ -(y_j - \pi_{1+j}) \right]$, and the amount attributed to innovations to future inflation, $\Delta E_1 \sum_{j=1}^{N} \left[ -\pi_{1+j} \right]$.

Real returns contribute in the wrong direction. The first few responses have a rise in nominal returns less than the rise in inflation, so the rise in real returns is negative, and accounts for a negative amount of the (negative) change in bond prices.

The rise in inflation correspondingly explains more than all of the decline in bond prices, with
a value of -0.61 to -1.10 to account for a bond return of -0.72. The magnitude is also about
double to triple that of real returns. On this back of the envelope basis, then, we can conclude
that higher future inflation is the major driver of the lower bond return \( r_{n+1} \), and thus,

- **By maintaining a maturity structure with about three years duration, and allowing long-
term bond prices to decline when there are shocks to the present value of surpluses, the US spreads the inflationary impact of changes in the present value of surpluses forward for about three years.**

### 5.4. Recession shocks

We can use the same procedure to understand the fiscal underpinnings responses to other shocks.
For any \( \varepsilon_1 \), we can compute impulse responses \( A^j \varepsilon_1 \), and thereby the terms of the decomposition (16). If shock \( \varepsilon_1 \) happens, how do its effects distribute across the terms of the identity (16)? We can consider the calculation as a decomposition of the covariance of inflation with the shock \( \varepsilon_1 \).

I start with a recession shock. The response to an inflation shock, in Figure 2, is stagflation-
ary, in that consumption \((g)\) falls when inflation rises. Unexpected inflation is, in this sample,
negatively correlated with unexpected consumption (and also GDP) growth. The stagflationary
episodes outweigh the simple Phillips curve episodes. However, I started with a puzzle about
events in which inflation falls during a recession such as 2008, though there are contemporane-
ous deficits. What are the fiscal underpinnings of such episodes?

To answer that question, we want to study a shock in which inflation and GDP go in the same
direction. We can simply create such shock – we can specify \( \varepsilon_1^\pi = -1, \varepsilon_1^g = -1 \). (The model is
linear, so the sign doesn’t matter, but the story is clearer for a recession.) Again, we want shocks
to other variables to have whatever value they have, on average, conditional on the inflation and
output shock. To fill out the other shocks, then, I run a multiple regression

\[
\varepsilon_{t+1}^x = b_{x,\pi} \varepsilon_{t+1}^\pi + b_{x,g} \varepsilon_{t+1}^g + \delta_{t+1}
\]

and then I fill in the other shocks at time 1 from their predicted variables given \( \varepsilon_1^\pi = -1 \) and
### Table 4: Inflation identity terms for monetary, fiscal and recession shocks

<table>
<thead>
<tr>
<th>Component</th>
<th>Recession</th>
<th>Monetary</th>
<th>Fiscal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation $\pi_1$</td>
<td>-1.00</td>
<td>0.71</td>
<td>0.25</td>
</tr>
<tr>
<td>Bond return $(r^n_1 - g_1)$</td>
<td>1.93</td>
<td>-3.27</td>
<td>-0.75</td>
</tr>
<tr>
<td>of which $r^n_1$</td>
<td>0.93</td>
<td>-2.72</td>
<td>-0.74</td>
</tr>
<tr>
<td>of which $g_1$</td>
<td>-1.00</td>
<td>0.55</td>
<td>0.01</td>
</tr>
<tr>
<td>Total current $\pi_1 - (r^n_1 - g_1)$</td>
<td>-2.93</td>
<td>3.98</td>
<td>1.01</td>
</tr>
<tr>
<td>Future $\sum s$</td>
<td>-0.15</td>
<td>0.00</td>
<td>-1.00</td>
</tr>
<tr>
<td>Future $\sum (r - g)$</td>
<td>-3.07</td>
<td>3.99</td>
<td>0.01</td>
</tr>
<tr>
<td>Total future $- \sum s + \sum (r - g)$</td>
<td>-2.93</td>
<td>3.98</td>
<td>1.01</td>
</tr>
</tbody>
</table>

Figure 3 presents responses to this recession shock, and Table 4 collects the inflation decomposition elements.

Both inflation $\pi$ and growth $g$ responses start at -1, by construction. Consumption growth returns rapidly, though the level of consumption does not recover much at all. Consumption is roughly a random walk. The nominal interest rate $y$ falls in the recession, and recovers slowly. The persistent fall in interest rate corresponds to a large positive ex-post bond return $r^n$ on impact. Again subsequent expected returns follow the short rate, almost exactly in an expectations-hypothesis pattern. In short, we see very sensible and standard picture of a recession.

On the fiscal side, the recession includes a deficit $s$, which continues for three years. The deficits imply a large rise in the value of debt, $v$. Surpluses subsequently turn positive paying down much of the debt.

Discount rates are still an essential part of the story however. The value of debt rises by even more than the deficits, because of the large ex-post return $r^n_1 - g_1$.

To quantify the various components, turn to the accounting of inflation $\pi_1$ as summarized in Table 4. Though there is a large deficit, the following surpluses pay it off almost entirely – of

$$\varepsilon^g_1 = -1. I start the VAR at$$

$$\varepsilon_1 = - \left[ b_{r^n,\pi} + b_{r^n,g} \varepsilon^{g}_1 = 1 \varepsilon^{\pi}_1 = 1 b_{s,\pi} + b_{s,g} \ldots \right]^\prime.$$
Figure 3: Response to a recession shock, $\varepsilon_t^r = \varepsilon_t^g = -1$. 
the -1.00 percent inflation shock, future surpluses account for only -0.15. The total discount rate term is -3.07. The lower interest rates discount the essentially unchanged surpluses by a lot. This increase in the present value of the debt is met 1/3 by the current disinflation, and 2/3 (1.93) by the higher value of long term debt.

In sum,

- **Disinflation shocks coincident with a recession are driven entirely by a higher discount rate.** For each one percent disinflation shock, the expected return on bonds falls so much that the present value of debt rises by three percent. Two of those percentage points are soaked up in the rise in bond values, which reflect the persistence of the disinflation.

The opposite conclusions hold of inflationary shocks in a boom.

### 5.5. Monetary and fiscal shocks

Monetary policy moves interest rates, but cannot tax or spend – monetary policy cannot on its own move the primary surplus. I’d like to define a monetary policy shock as one that does not move surplus at any horizon, but the VAR is not complex enough for that. So, define a monetary policy shock as one that moves interest rates \( y_t \), but does not affect the sum of current and future surpluses, \( \Delta E_1 \sum_{j=0}^{\infty} s_{1+j} = 0 \).

This is not an “exogenous” monetary policy shock in the traditional sense, and may well happen in response to other variables. Here, I ask a different question: if monetary policy does change, whether endogenously or exogenously, and if it is not accompanied by a contemporaneous change in future fiscal surpluses, what are the fiscal underpinnings of any resultant inflation? The usual identification tries to find shocks that are exogenous, to measure the causal effect of a monetary policy change, but allow a coincident shocks to surpluses.

Conversely, I define a fiscal shock as a movement in current and expected primary surpluses that comes with no movement in the short-run interest rate. Yes, the central bank will typically move the short run rate in response to fiscal shocks but we want to know, what happens to inflation if there is a pure fiscal shock that the central bank does not respond to?
In response to a shock \( \varepsilon_1 \), the response of the sum of future surpluses is

\[
\Delta E_1 \sum_{j=0}^{\infty} s_{1+j} = a_s'(I - A)^{-1} \varepsilon_1.
\]

Thus, to calculate monetary and fiscal shocks, I run multiple regressions

\[
\varepsilon^x_{t+1} = b_{x,y} \varepsilon^y_{t+1} + b_{x,pv} a_s'(I - A)^{-1} \varepsilon_{t+1} + \delta_{t+1}.
\] (24)

The monetary policy shock uses \( \varepsilon^y_1 = 1, a_s'(I - A)^{-1} \varepsilon_1 = 0 \), while the fiscal policy shock makes the opposite assumption, \( \varepsilon^y_1 = 1, a_s'(I - A)^{-1} \varepsilon_1 = 0 \). Thus, we start the monetary policy impulse-response function with

\[
\varepsilon_1 = \begin{bmatrix} b_{r^n,y} & b_{g,y} & b_{\pi,y} & \ldots \end{bmatrix}',
\]

while we start the fiscal shock impulse-response function with

\[
\varepsilon_1 = \begin{bmatrix} b_{r^n,pv} & b_{g,pv} & b_{\pi,pv} & \ldots \end{bmatrix}'.
\]

These coefficients are by construction 1 and 0 in the appropriate entries.

Figure 4 presents the responses to the monetary policy shock. Table 4 collects relevant magnitudes of the responses.

Figure 4 shows that

- The response of inflation to a monetary policy shock, with no change in fiscal policy is super-Fisherian.

A “Fisherian” response has come to mean that if the central bank raises the interest rate \( y_t \), then inflation follows with a one period lag, fulfilling the simple Fisher relation \( y_t = r + E_t \pi_{t+1} \). A “super-Fisherian” response is one in which raising the interest rate \( y_t \) raises inflation \( \pi_t \) contemporaneously. That is the pattern shown in Figure 4. Standard empirical results to the contrary do not specify fiscal policy, and thus they allow an implicit contemporaneous fiscal policy shock.

The monetary policy shock also coincides with a one-period rise in consumption \( g \). Again this “shock” makes no attempt to be exogenous, orthogonalized to other shocks, or to net out correlation induced by a Taylor rule, so this correlation should not be read causally.
Figure 4: Response to a monetary policy shock – a movement in the interest rate $y_1$ with no movement in the sum of future surpluses $\sum_{1}^{\infty} s_{1+j}$
Figure 4 shows that, although the sum $\sum_{j=1}^{\infty} \Delta E_1 s_j$ does not change by construction, near-term surpluses increase, long-term surpluses decrease. Coincident with the rise in interest rates, which comes with a rise in growth, surpluses increase. Higher surpluses lower the value of the debt. But then the government dips into this savings with a long string of later deficits, bringing the value of the debt back again.

With no change to future surpluses $\Delta E_1 \sum_{j=1}^{\infty} s_{1+j}$, where does the fiscal backing for the super-Fisherian inflation innovation $\Delta E_1 \pi_1$ come from? Discount rates. Figure 4 shows that future $r - g$ rises uniformly and persistently, lowering the present value of the debt. The sum of the future $r - g$ terms, 3.99, is even greater than the inflation shock, $E_1 \pi_1 = 0.71$. This large discount rate effect shows up in a large current loss, $\Delta E_1 (r_1^n - g_1) = -3.27$. Of this $\Delta E_1 g_1 = 0.55$, so most of the action comes from a rise in nominal rates.

An interest rate shock is persistent. Prices are sticky, so the real rate rises when the nominal rate rises. The higher real rate discounts surpluses more heavily. Some of the higher future nominal rates results in a negative bond return. A smaller part results in current inflation. We should not be surprised by discount rate effects that explain more than all of the inflation shock. When interest rates rise, bond prices go down.

The “Monetary” column of Table 4 collects the terms of the innovation present value decomposition. The monetary policy shock raises inflation by 0.71. Surpluses, by construction, contribute nothing. The discount rate effect on present values contributes 3.98, of which 3.27 is soaked up by the decline in bond prices.

- A monetary shock, defined as an interest rate rise with no effect on surpluses, gives rise to an immediate and proportionate increase in inflation. This inflation all comes from a rise in discount rates. For every 4 percentage points of decline in present value of surpluses, 0.71 percentage points results in immediate inflation, and 3.27 percentage points in lower bond prices, which signal a drawn out inflation response.

Figure 5 presents responses to fiscal policy shocks, a negative unit shock to future surpluses $\Delta E_1 \sum_{j=1}^{\infty} s_{j+1}$ while holding the current interest rate $\Delta E_1 y_1 = e_1^y = 0$. In the top panel, though the sum of all $s$ terms is -1.00 by construction, again near-term surpluses rise, and the long term surpluses fall even more.

In the bottom panel, the contemporaneous interest rate response to the fiscal shock is $\Delta E_1 y_1 = 0$, by construction. There is essentially no discount rate effect at all. The 1.0 future surplus
Figure 5: Fiscal policy response. Response to a shock to expected surpluses $\Delta E_{t+1} \sum_{j=1}^{\infty} sv_{t+j} = 1$, with no interest rate shock $v_{t+1}^y = 0$. 

$s, \Sigma = -1.00$ 

$r^n - g, \Sigma = 1.24$
A fiscal shock sets off a protracted inflation. Three quarters of the fiscal shock is transmitted to future inflation via a decline in long-term bond prices.

In modeling this sort of event one might include a Taylor rule for monetary policy. Such a rule will drag out the inflation response to a fiscal shock even more.

6. Debt

Using the value identity (4), which I repeat for convenience,

\[ v_t = \sum_{j=1}^{T} s_{t+j} - \sum_{j=1}^{T} \left( r_{t+j}^n - \pi_{t+j} - g_{t+j} \right) + v_{t+T}, \]

we can characterize the forces behind the value of debt.

6.1. The history of debt

Figure 6 presents a backwards value decomposition. At each date it plots the terms of

\[ v_t = v_0 - \sum_{j=1}^{t} s_j + \sum_{j=1}^{t} \left( r_{t-j}^n - \pi_t - g_t \right) \]

It answers, “where did each date’s market value of debt come from?” The latter terms add \( v_0 \), so each line presents what debt \( v_t \) would be at date \( t \), starting from \( v_0 \), if that were the only term.

In \( v_t \) we see the evolution of the market value of debt to GDP ratio since 1930. There are three distinct periods, with different behavior: the Great Depression and WWII, WWII to 1980, and the post 1980 period.

Debt rises in the great depression and in WWII, to a debt-to-GDP ratio greater than one i.e. \( 0 = \log(1) \). The \( s \) line here is the negative of surplus, i.e. how surplus contributes to debt. The rise in debt largely parallels the surplus line, i.e. the rise in market value of debt is driven by the deficits.
of the 1930s and WWII. That is not entirely the case however: About half of the rise in market value of debt to GDP ratio in the great depression comes from the $r - g$ term, and that almost all from the fall in GDP and deflation of the early 1930s. The rise in market value of debt to GDP ratio in WWII was a good deal lower than the cumulative deficits. Here the rise in GDP helped a lot.

From the end of WWII to about 1975, the value of debt fell steadily. About a third of this fall comes from primary surpluses, as shown by the fall in the $-s$ line. The rest came from the sharp fall in $r - g$. Breaking the latter out to nominal returns, inflation, and growth, we see that nominal returns and inflation largely cancel, leading to the standard conclusion, that about 2/3 of the fall in debt to GDP ratio came from growth, rising GDP. In fact, ex-post, $g > r$ in this period.

The inflation of the 1970s is visible, in a speeding up of the $\pi$ line. But ex post returns rise as well, the $r^n$ line rises, so the net effect $r = r^n - \pi$ is not as large as one would suppose. To devalue outstanding debt, inflation must either come swiftly, devaluing short term debt, or it must produce a negative return on long-term debt. Expected inflation cannot devalue short term debt. The short maturity structure of debt in the 1970s means that current inflation did not really do that much to lower the value of debt.
That even the inflation of the 1970s is not a large phenomenon here helps to explain why I focus the previous section, and this paper, on the inflation innovation accounting, rather than value of the debt accounting. Variation in the debt comes largely from surpluses, growth, and real returns. Inflation, in the US, in this sample, is a smaller effect. By taking innovations above, I focus on inflation and eliminate the value of the debt from the identity.

A sharp break occurs in 1980. First, there are two waves of deficits, with an interlude of surpluses in the 1990s. The rise and fall of the value of debt largely reflects these surpluses and deficits. The cumulative effect of \( r - g \) is very small. However, that cumulative effect includes a change in behavior. The nominal \( r^n \) and real \( r^n - \pi \) returns rise – bonds did well in the 1980s and 1990s. This rise in real return means that continuing growth does not bring down the debt to GDP ratio as it once did.

There is some feeling that the US has entered a period with \( r < g \) so the debt to GDP ratio can withstand primary deficits. The graph does not yet show such behavior.

Figure 7 presents a forward-looking value decomposition,

\[
v_t = \sum_{j=1}^{T} s_{t+j} - \sum_{j=1}^{T} \left( r^n_{t+j} - \pi_{t+j} - g_{t+j} \right) + v_{t+T}.
\]

Figure 7: Forward decomposition of the market value of debt to GDP ratio.
Regarding the value of the debt as the present value of future surpluses, this decomposition asks, “suppose people knew the future, and discounted the actual ex-post surpluses using actual ex-post real returns. How would those ex-post expectations account for the value of the debt at each date?”

Mirroring the break seen in the previous graph, we see that after 1980, the value of the debt largely follows the value of future surpluses, with little contribution from \( r - g \). Why, for example, is the value of debt low in 1980? A big part of that is foreknowledge that the value of the debt will be high at the end of the sample – there will not be a default, an inflation, a growth disaster, or another big hit to \( r - g \). Beyond that, the value of debt is low because on net, there will be a string of primary deficits in the 1980s and after 2018, so the \( s \) line is negative throughout. Time series variation in the value of deficits to come drives time series variation in the value of debt.

From WWII to 1975, we see a different picture. Now, assuming again perfect foresight, we tell the same story backwards. Why was the value of debt so high at the end of WWII? About 2/3 of it was low growth-adjusted discount rates, with about 1/3 knowledge of primary surpluses to come.

Of course, it is quite possible that this data point contains events people did not expect. One may speculate that people expected lower growth, positive real interest rates, and therefore a long period of primary surpluses. Ex-post rational calculations are what they are, a bound on rational expectations. That low returns, some inflation, and high growth brought down the debt to GDP ratio once in the past does not imply that people will hold large debts expecting the same in the future.

### 6.2. A variance decomposition for the value of the debt

We can construct a variance decomposition for the value of debt, to see how real values of the debt are resolved on average, not just with the ex-post luck of one data point, and under rational expectations we can interpret that calculation to tell us why real values are what they are. (This calculation adapts the analysis of the volatility of price/dividend ratios reviewed in the literature section. It also extends the covariance based approach – if \( x = y + z \), then \( \text{var}(x) = \text{cov}(x, y) + \text{cov}(x, z) \) – to multiple regression forecasts, VARs, filtered data and innovations of \( x \) in a way that has not been pursued yet in the asset pricing literature.)
Multiplying (4) by \( v_t - E(v_t) \) and taking expectations, we obtain a variance decomposition

\[
\text{var}(v_t) = \sum_{j=1}^{T} \text{cov}(v_t, s_{t+j}) - \sum_{j=1}^{T} \text{cov}[v_t, (r^n_{t+j} - \pi_{t+j} - g_{t+j})] + \text{cov}(v_t, v_{t+T}).
\]

Dividing by \( \text{var}(v_t) \), we can express the result as fractions of variance due to each component. The components are also the coefficients of single regressions by which \( v_t \) forecasts the other terms:

\[
1 = \sum_{j=1}^{T} b_{s_{t+j}, v_t} - \sum_{j=1}^{T} (b_{r_{t+j}, v_t} - b_{s_{t+j}, v_t} - b_{g_{t+j}, v_t}) + b_{v_{t+j}, v_t},
\]

(25)

where \( b \) denote regression coefficients, e.g.

\[
s_{t+j} = a + b_{s_{t+j}, v_t} v_t + \varepsilon_{t+j}.
\]

Debt may only vary to the extent that it forecasts surpluses, discount rates, or itself. Or, multiplying both sides by \( v_t \),

\[
v_t = \sum_{j=1}^{T} E(s_{t+j}|v_t) - \sum_{j=1}^{T} E(r_{t+j} - \pi_{t+j} - g_{t+j}|v_t) + R(v_{t+j}|v_t)
\]

Thus, regression coefficients answer directly, “What fraction of the variation in the value of debt comes from forecasts of future surpluses, vs. future discount rates, or ever-rising debts?”

To include additional information, we can forecast using additional variables \( x_t \), e.g.

\[
s_{t+j} = a + b_{s_{t+j}, v_t} v_t + b_{s_{t+j}, x_t} + \varepsilon_{t+j}.
\]

Now, taking expectations of (4), we obtain

\[
v_t = \sum_{j=1}^{T} E(s_{t+j}|v_t, x_t) - \sum_{j=1}^{T} E(r_{t+j} - \pi_{t+j} - g_{t+j}|v_t, x_t) + E(v_{t+j}|v_t, x_t).
\]

We can plot these components as well, and we can create a variance decomposition with

\[
\text{var}(v_t) = \text{cov} \left[ v_t, \sum_{j=1}^{T} E(s_{t+j}|v_t, x_t) \right] - \text{cov} \left[ v_t \sum_{j=1}^{T} E(r_{t+j} - \pi_{t+j} - g_{t+j}|v_t, x_t) \right] + \text{cov} \left[ E(v_{t+j}|v_t, x_t) \right].
\]
Equation (25) still holds for multiple regression coefficients $b_{-,vt}$, while

$$0 = \sum_{j=1}^{T} b_{s_{t+j},x_t} - \sum_{j=1}^{T} (b_{r_{t+j},x_t} - b_{\pi_{t+j},x_t} - b_{g_{t+j},x_t}) + b_{v_{t+j},x_t}.$$ 

If a variable $x_t$ helps to predict surpluses, given the value of debt $v_t$, then that variable must also help to predict discount rates or long-term value. Since the value of debt $v_t$ is what it is, any marginal forecast power in additional variables must offset.

We can make estimates of these decompositions directly, running long horizon regressions, e.g.

$$\sum_{j=1}^{T} s_{t+j} = a + b^l_{s,v} v_t + b^l_{s,x} x_t + \varepsilon_{t+T}.$$ 

We can also infer the long-horizon coefficients from a VAR. The VAR makes it easier to run the sums out to infinity rather than terminate.

We can also filter. The quick look at the data above reveals an obvious trouble: US Debt is dominated by the runup in WWII, its steady decline, and it so-far-unresolved second runup. Two to three data points are not much on which to take averages. It is also economically interesting to separate debt variation into components. Perhaps war debts are paid off one way, but business cycle debts are paid off another way. For example, perhaps business cycle variation in the value of debt corresponds to interest rate variation, but secular debts are paid by surpluses or growth.

Since we start with an ex-post identity, we can first filter and then take expectations. We can meaningfully write

$$\theta(L)v_t = E_t \sum_{j=1}^{T} \theta(L)s_{t+j} - E_t \sum_{j=1}^{T} \theta(L) (r^n_{t+j} - \pi_{t+j} - g_{t+j}) + E_t \theta(L)v_{t+T},$$

and analyze, say, business cycle components of debt in relation to business cycle components of surpluses and returns.

Filtering valuation equations is delicate. If we start with

$$p_t = E_t (m_{t+1} x_{t+1}),$$
we can write
\[ \theta(L)p_t = \theta(L)E_t \left( m_{t+1}x_{t+1} \right), \]
but we cannot write
\[ \theta(L)p_t = E_t \left\{ \theta(L) \left[ m_{t+1}x_{t+1} \right] \right\}. \]

The lag operator must apply to the expectations on the outside. For example, if \( x_t \) is i.i.d.,
\[ 0 = E_t(x_{t+1}) \]
then \( E_t(x_{t+1} - x_t) = -x_t \) but \( E_t(x_{t+1} - E_{t-1}x_t) = 0 \). For this reason, we cannot say that a business cycle component of price corresponds to a business cycle component of discounted dividends. But since we start with an ex-post identity, we can first filter and then take expectations, so we do not make this mistake. We could also apply the innovation identity \( \Delta E_{t+1}(\cdot) \) as in (16) to filtered data, but I find it less interesting to do so. (The filtered data approach is less useful for asset pricing applications, as the hypothesis that filtered returns are unpredictable is not particularly interesting. The corresponding hypothesis is not relevant here.)

It is easiest to simply rerun the VAR with filtered data,
\[ \theta(L)x_{t+1} = A\theta(L)x_t + \eta_{t+1} \]
with
\[ E(\eta_{t+1}\eta'_{t+1}) = \Omega. \]

Define
\[ W = \text{cov}(\theta(L)x_t, \theta(L)x'_{t}) = (I - A)^{-1}\Omega(I - A)^{-1}. \]

Then the filtered present value identity (26) leads to
\[ a'_v\theta(L)x_t = a'_s(I - A)^{-1}A\theta(L)x_t - (a_{r^u} - a_{\pi} - a_g)'(I - A)^{-1}A\theta(L)x_t. \]

We can plot both sides. Taking covariance of both sides with \( \theta(L)v_t = a'_v\theta(L)x_t \), the terms of the variance decomposition are
\[ \text{var} [\theta(L)v_t] = a'_vWa_v = a'_s(I - A)^{-1}AWa_v - (a_{r^u} - a_{\pi} - a_g)'(I - A)^{-1}AWa_v. \]

Last, we can apply the variance decomposition to innovations in the value of debt. As we took time-t + 1 innovations of (4) to derive the inflation innovation identity (16), now take time t
innovations of the present value identity (4), so the $v_t$ term does not cancel:

$$(E_t - E_{t-1})v_t = (E_t - E_{t-1}) \sum_{j=1}^{\infty} s_{t+j} - (E_t - E_{t-1}) \sum_{j=1}^{\infty} (r_{t+j}^n - \pi_{t+j} - g_{t+j})$$

$$\text{var} \ (\Delta E_t v_t) = \text{cov} \left( \Delta E_t v_t, \Delta E_t \sum_{j=1}^{\infty} s_{t+j} \right) - \text{cov} \left( \Delta E_t v_t, \Delta E_t \sum_{j=1}^{\infty} (r_{t+j}^n - \pi_{t+j} - g_{t+j}) \right)$$

$$a_v' \Sigma a_v = a_v' (I - A)^{-1} A \Sigma a_v - (a_{rn} - a_{\pi} - a_g)' (I - A)^{-1} A \Sigma a_v$$

The variance of the unexpected market value of debt to GDP ratio equals the covariance, or regression coefficient, of surplus and discount rate innovations with the unexpected value.

6.3. Variance of value estimates

In addition to analyzing the log value of debt to GDP, I filter by using the difference between the log value of debt and three lags, $v^f_t = v_t - \frac{1}{3} (v_{t-1} + v_{t-2} + v_{t-3})$, and I use the VAR innovation in the value of debt, $\Delta E_t (v_t) = v_t - E_{t-1} (v_t)$.

Figure 8: Filtered value of debt $v^f_t$ (symbols) and the VAR innovation in the value of debt, $v_t - E_{t-1} (v_t)$. The filter is $v_t - \frac{1}{3} (v_{t-1} + v_{t-2} + v_{t-3})$. The vertical dashed line denotes the beginning of the sample used for VAR and statistical analysis.
Figure 8 presents the filtered and VAR innovation in the value of debt. Both measures pick up familiar cyclical movements, and deemphasize the long-term trend. For example, you can see the big increase in debt following the 2008 financial crisis, the decrease of the 1990s, the buildup in the early 1980s, and variation in debt through the recessions of the 1970s.

<table>
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<th>Component</th>
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<tr>
<td>$\text{var}(v)$</td>
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<tr>
<td>$\text{cov}(v, \Sigma s)$</td>
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</tr>
<tr>
<td>$-\text{cov}(v, \Sigma (r - g))$</td>
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</tr>
<tr>
<td>$\text{cov}(v, \Sigma r^n)$</td>
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</tr>
<tr>
<td>$\text{cov}(v, \Sigma \pi)$</td>
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</tr>
<tr>
<td>$\text{cov}(v, \Sigma (r^n - \pi))$</td>
<td>-0.28</td>
</tr>
<tr>
<td>$\text{cov}(v, \Sigma g)$</td>
<td>0.02</td>
</tr>
<tr>
<td>$\sigma(v)$</td>
<td>0.36</td>
</tr>
</tbody>
</table>

Table 5: Decomposition of the variance of the value of debt.

Table 5 presents the decompositions of the variance of each of these measures of the value of debt $v_t$. The table entries are fractions of the variance of each measure of debt.

The last row gives the standard deviation of each measure of the value of debt. “Plain,” the standard deviation of the log market value of debt to GDP $v_t$ is 0.36 – the debt to GDP ratio varies by 36 percentage points, a lot. The innovation and filtered variances are lower, as they eliminate the large low-frequency variation, leaving 7 (innovation) and 15 (filtered) percentage points respectively.

The second and third rows give the variance decompositions. Looking down the first column,

- **71% of the variance of debt $\text{var}(v_t)$ does in fact correspond to forecasts of future surpluses.**
  
  *A substantial component, of that variance 29%, comes from (forecasted) discount rate variation. The breakdown shows that practically all of this discount rate variation, 28%, comes from real returns, and only 1% from growth.*

Some of the return effect is mechanical, a reflection of the fact that I use the market value of debt. It is nonetheless an important mechanism. If the real yield of government bonds declines, their value rises, and the market value of debt rises, followed by lower real returns, on average.
Some of the return effect, however, reflects greater issuance of debt during times of low interest rates.

The point of taking the average, of a variance decomposition, is to avoid ex-post luck. One particular debt episode may be resolved by poor ex-post returns (i.e. default or devaluation via inflation), but investors should not buy debt ex-ante anticipating such an event. So, if the sample is long enough, we obtain estimates of rationally expected discount rate variation rather than ex-post luck. Of course this sample is small, and standard errors are large, which motivates my look at the filtered and innovation measures.

That growth $g$ is not important, on average, illustrates the important difference between this variance accounting and the previous ex-post history. As above, the WWII debt was repaid, ex-post, by a lot of growth. However, the subsequent debt episodes did not resolve by a lot of ex-post growth, so on average expected growth is not an important driver of the value of debt from an ex-ante point of view. The postwar growth was unexpected, which is reasonable given writings at the time warning of secular stagnation.

The result that forecasted variation in primary surpluses contributes the bulk of the variation of the value of debt stands in contrast to the usual result in asset pricing that discount rates account for all variation in valuation ratios. One big difference is important however: Here I account for variation in the total market value of debt, including (and dominated by) issuances and redemptions. The standard asset pricing calculation accounts for the variation of the value of a particular security, how the value of one government bond varies over time, or, here, perhaps, how the value of one dollar invested in the government bond portfolio varies over time. The variation in such individual bond values is, since the US has not defaulted in this sample, 100% due to variation in nominal discount rates and none to variation in nominal cash flows. The variation of the real value of individual bonds is, I suppose, dominated by real interest rate variation and less by unexpected real values of cash flows, eroded by inflation. The main point: variation in total value is a different object than variation in the value of one security, so there is no necessary puzzle.

Turning to the “Innovation” and “Filtered” columns, we see overall not much difference.

- The filtered value of debt corresponds to even more, 78%, future surpluses, and 22% discount rates. Debt innovations are slightly less, 51%, due to future surpluses and somewhat more, 48%, due to discount rates.
The latter comparison makes sense, as revaluations due to changes in bond prices should make a larger difference at higher frequencies. The former tells us that recession-related, and somewhat longer (Vietnam, 1980s deficits, 1990s surpluses) variation in debt is not fundamentally different from the really long-term variation, and indeed the value of debt varies because people do see surpluses ahead.

Turning to the breakdowns, however, we see a major difference between the filtered, innovation and the overall variance decomposition. For innovations, the 45% of variance that comes from real returns \( r_n - \pi \), comes from 124% \( r_n \) less 79% \( \pi \), a large movement in nominal rates offset by a large movement in inflation. For the filtered returns, nominal returns only move 12%, reinforced by a 7.5% rise in inflation.

7. **Standard errors**

I have delayed a discussion of standard errors because there is nothing important to test. Identities are identities, if \( x = y + z \) and \( x \) moves, \( y \) or \( z \) must move, and all we can do is to measure which one. In addition, no important economic hypothesis rests on whether it is \( y \) or \( z \) that moves. (In asset pricing, whether discount rates or cashflows account for variation in values is a much more meaningful hypothesis.) Standard errors only give us a sense of how accurate the measurement is.

To evaluate sampling distributions I run a Monte Carlo. I simulate the errors of the VAR, and repeat the analysis over and over. Most of the interesting statistics – variance decompositions, impulse response functions, \((I - A)^{-1}\), etc. – are nonlinear functions of the underlying data, and the near-unit root in the evolution of value also induces non-normal distributions. For these reasons, I largely characterize the sampling distribution by the interquartile range – the 25% and 75% points of the sampling distribution. This roughly corresponds to one-standard error bands.

Table 6 collects the sampling quantiles for the variance decomposition of Table 2. Figure 9 presents the main components of the impulse-response function relevant to the inflation variance decomposition. The bands are 25% and 75% points of the sampling distribution, the dashed line is the median, and the solid line is the estimate.

As shown in Table 6, the future surplus contribution to the inflation decomposition of 0.68 has
quartiles of 0.32 and 1.18, and the future return contribution of 0.68 has even larger quartiles of 0.10 and 1.10. Even the instantaneous return contribution of 0.36 has quartiles of 0.11 to 0.60.

There are several sources of this rather large sampling variation. First, the shocks are large. A shown in Table 1, the surplus innovation $\varepsilon_s$ has a 4.49 percentage point standard deviation, and value 6.69 percentage point, compared to 1.16 for inflation and 1-2 for the other variables. Our friend $\sigma/\sqrt{T}$ starts off badly.

Second, the shocks are poorly correlated. This matters, because in each case I find movements in other variables contemporaneous with the shock of interest by running a regression of the other shocks on the shock of interest. Table 7 includes the regression of other shocks on inflation shock that starts off the main inflation decomposition and the response function of Figures 2 and 9, and the correlation matrix of the shocks. The instantaneous response of the surplus to an inflation shock is clearly important. Yet its value, -0.42, is measured with a 0.5 standard error, and the corresponding correlation is only -0.11. We see a correspondingly wide band around the initial surplus response in Figure 9.

Figure 9: Distribution of the impulse response function, to an inflation shock. The bands are 25% and 75% points of the sampling distribution, the dashed line is the median, and the solid line is the estimate.
Table 6: Decomposition of unexpected inflation variance – distribution quantiles. No b2 holds the initial response constant across trials. No A holds the VAR regression coefficients constant across trials.

<table>
<thead>
<tr>
<th>Component</th>
<th>Fraction</th>
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<th>75%</th>
<th>25%</th>
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</thead>
<tbody>
<tr>
<td>Inflation $\pi_1$</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bond return $-(r_1^n - g_1)$</td>
<td>0.36</td>
<td>0.11</td>
<td>0.60</td>
<td>0.36</td>
<td>0.36</td>
<td>0.11</td>
<td>0.60</td>
</tr>
<tr>
<td>- Future $\Sigma s$</td>
<td>0.68</td>
<td>0.35</td>
<td>1.18</td>
<td>0.44</td>
<td>1.09</td>
<td>0.41</td>
<td>0.97</td>
</tr>
<tr>
<td>Future $\Sigma r - g$</td>
<td>0.68</td>
<td>0.10</td>
<td>1.08</td>
<td>0.28</td>
<td>0.93</td>
<td>0.27</td>
<td>1.05</td>
</tr>
</tbody>
</table>

Table 7: Regression of other shocks on inflation shock, and correlation matrix of VAR shocks.

<table>
<thead>
<tr>
<th></th>
<th>$r^n$</th>
<th>$g$</th>
<th>$\pi$</th>
<th>$s$</th>
<th>$v$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression of other shocks on inflation shock</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coefficient</td>
<td>-0.72</td>
<td>-0.36</td>
<td>1.00</td>
<td>-0.42</td>
<td>-0.94</td>
<td>0.22</td>
</tr>
<tr>
<td>Std. err.</td>
<td>(0.30)</td>
<td>(0.18)</td>
<td>(0.00)</td>
<td>(0.51)</td>
<td>(0.75)</td>
<td>(0.15)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$r^n$</th>
<th>$g$</th>
<th>$\pi$</th>
<th>$s$</th>
<th>$v$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correlation matrix of VAR shocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^n$</td>
<td>1.00</td>
<td>-0.01</td>
<td>-0.30</td>
<td>-0.27</td>
<td>0.65</td>
<td>-0.54</td>
</tr>
<tr>
<td>$g$</td>
<td>-0.01</td>
<td>1.00</td>
<td>-0.26</td>
<td>0.42</td>
<td>-0.49</td>
<td>0.42</td>
</tr>
<tr>
<td>$\pi$</td>
<td>-0.30</td>
<td>-0.26</td>
<td>1.00</td>
<td>-0.11</td>
<td>-0.16</td>
<td>0.19</td>
</tr>
<tr>
<td>$s$</td>
<td>-0.27</td>
<td>0.42</td>
<td>-0.11</td>
<td>1.00</td>
<td>-0.87</td>
<td>0.44</td>
</tr>
<tr>
<td>$v$</td>
<td>0.65</td>
<td>-0.49</td>
<td>-0.16</td>
<td>-0.87</td>
<td>1.00</td>
<td>-0.65</td>
</tr>
<tr>
<td>$y$</td>
<td>-0.54</td>
<td>0.42</td>
<td>0.19</td>
<td>0.44</td>
<td>-0.65</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 7: Regression of other shocks on inflation shock, and correlation matrix of VAR shocks.
There is hope in this observation, however. Higher frequency data can better identify shock correlations, at the cost that one must model the strong seasonal in Federal borrowing. Moreover, other shock identifications may have better measured correlations. The correlation of more inflation with lower surpluses adds recessionary shocks that go the other way with stagflationary shocks that go in this direction. Separating the two may lead to better measured, and more interesting, analysis.

Third, where the surplus starts doesn’t really matter, but the sum of future surpluses matters. The value of the debt is the main state variable, and uncertainty about its evolution adds to the uncertainty about the sum of surpluses. Its coefficient on its own lag is 0.98 in Table 1, so small variations in that value lead to large variation in $(I - A)^{-1}$ sums.

To measure the contribution of the latter two sources of variation, Table 6 includes two other sampling calculations. The “no b2” columns resample data using the original regression of shocks $\varepsilon_{t+1}$ on inflation shocks $\varepsilon_{\pi,t+1}$, the top row of Table 7, in each sample. The VAR coefficients still vary across samples, but the identification of the inflation shock does not. The “no A” columns likewise keep constant the VAR regression coefficients, but reestimate the shock regression in each sample. Turning off either source of sampling variation reduces that variation, but not as much as you might think. Sampling variation is still large in either case.

Table 8 presents quantiles of the inflation decomposition for recession, monetary, and fiscal shocks of Table 4. The conclusion that recessionary inflation shocks have no surplus implications is subject to a good deal of uncertainty, from -0.58 to 0.996 (and an illustration of asymmetry of sampling distributions). The corresponding large discount rate effect, -3.07 is at least within a quartile range -3.52 to -1.86 away from zero, as is the large 1.93 contemporaneous return. That recessionary inflation shocks come from large discount rate effects is on firmer
ground. Similarly, the positive inflation effect and strong discount rate effects of monetary policy shocks are well away from zero, as is the negative inflation effect and contemporaneous bond return effect, reflecting future inflation, of the fiscal shock. As suggested above, by looking at shock identifications that are more constrained, the recession, monetary and fiscal shocks are a bit better measured.

Overall, the inflation decomposition is not well measured. However, remember that the terms are related by an identity that sums to one. Though we often cannot reject zero for each item individually, the hypothesis that all of the elements are zero is nonsense, and easily rejected. The cross-correlation of the individual estimates would quickly reveal that fact if one were to compute such a test, as the covariance matrices are always singular. The uncertainty is over which component accounts for inflation, not whether the three components together do so.

8. Concluding comments

This analysis evidently just scratches the surface. Different definitions of inflation, and a parallel investigation of exchange rates beckons, following Jiang (2019a), Jiang (2019b). Quarterly data, which offer better measurement of correlations but require modeling seasonality, are attractive. Debt data go back centuries, allowing and requiring us to think what is the same and different across different periods of history. Inflation through wars and under the gold standard may well have different fiscal foundations than in the postwar environment. A narrative counterpart, especially for big episodes such as the 1970s and 1980s awaits. Different countries under different monetary and exchange rate regimes and different fiscal constraints will behave differently. One could define shocks in many additional interesting ways. Perhaps most of all, linking these theory-free characterizations to explicit fiscal theory of monetary policy models, such as the impulse-response functions in Cochrane (2019), or at least explicit models of discount rates and long-term debt management policies, is an obviously important step.
References


