Do Higher Interest Rates Raise or Lower Inflation?

John H. Cochrane*

February 10, 2016

Abstract

The standard "new-Keynesian" model accounts well for the fact that inflation has been stable at a zero interest rate peg. However, If the Fed raises nominal interest rates, the same model model predicts that inflation will smoothly rise, both in the short run and long run. This paper presents a series of failed attempts to escape this prediction. Sticky prices, money, backward-looking Phillips curves, alternative equilibrium selection rules, and active Taylor rules do not convincingly overturn the result. The evidence for lower inflation is weak. Perhaps both theory and data are trying to tell us that, when conditions including adequate fiscal-monetary coordination operate, pegs can be stable and inflation responds positively to nominal interest rate increases.

*Hoover Institution and NBER. The current version of this paper is posted at http://faculty.chicagobooth.edu/john.cochrane/research/papers/fisher.pdf Please do not post copies. Post a link instead so that people find the most recent version. I thank Edward Nelson for helpful comments.

Contents

1. Introduction

What happens to inflation if the Federal Reserve raises interest rates? Does it matter that the Fed will do so by raising interest on abundant excess reserves, rather than rationing non-interest-bearing reserves as in the past?

The recent history of zero interest rates with low and stable inflation in the US (Figure 20) and Europe, and longer experience in Japan (Figure 21), has important implications for these questions. The simplest interpretation of these episodes is that inflation can be stable under an interest rate peg. That interpretation contradicts long-standing contrary doctrine, from Milton Friedman's (Friedman (1968)) warning that a peg leads to ever-increasing inflation, to widespread warnings of a deflationary spiral when Japan and later the US and Europe hit the zero bound.

"Can" is an important qualifier. The fact that our recent pegs appear to be stable does not mean pegs are always and everywhere stable. Other pegs have fallen apart. So we learn that pegs can be stable if other preconditions are met. The theory outlined below stresses the solvency of fiscal policy as one of those preconditions. Also, our policy is a one-sided peg – interest rates cannot go significantly below zero. They can and likely would rise quickly if inflation rose. Yet the theory is linear, so if a zero bound does not produce a deflation spiral, it follows that a full peg would not produce an upward spiral.

Most theories contain the Fisher relation that the nominal interest rate equals the real rate plus expected inflation, $i_t = r_t + E_t\pi_{t+1}$, so they contain a steady state in which higher interest rates correspond to higher inflation. Traditional theories predict that it is an unstable steady state however, so the Fed must continually adjust interest rates to produce stable inflation. The recent stability of inflation at the zero bound suggests that the Fisher relation can be instead describe a stable steady state.

But if inflation is stable under an interest rate peg, then were the Fed to raise interest rates, sooner or later inflation must rise. This prediction has been dubbed the "neo-Fisherian" hypothesis. One may distinguish a "long run" hypothesis, that a rise in an interest rate peg will eventually raise inflation, allowing a short-run movement in the other direction, from a "short-run" version in which inflation rises immediately.

A frictionless model incorporating a Fisher relation $i_t = r_t + E_t\pi_{t+1}$ (and solvent fiscal policy) predicts that an interest rate peg is stable. In a frictionless model, the real interest rate r_t is unrelated to monetary policy and inflation. So, if the Fed sets the nominal rate i_t, expected inflation must follow. The impulse-response function to a step-function rise in interest rates generates a path $\{E_t\pi_{t+j}\}$ that matches the step in interest rates.

This paper is devoted to trying to escape this prediction; to establish the conventional belief that a rise in nominal interest rates lowers inflation, at least temporarily, in a simple modern economic model of monetary policy following an interest rate target that is consistent with the experience of stable inflation at the zero bound.

This quest has a larger methodological purpose. One might think the answer to the latter quest is obvious: the standard three-equation model consisting of an intertemporal substitution equation, a forward looking Phillips curve, and an active Taylor rule, as set forth, say, in Woodford (2003). But if one accepts the critiques in Cochrane (2011) and Cochrane (2014c), that model relies on the Fed to induce instability in an otherwise stable economy, and to explode the economy for any but its desired equilibrium. Such a threat is not visible in any data drawn from an equilibrium, nor believable. It violates the qualifier "simple," in that its monetary policy is really equilibrium-selection policy. An interest rate rise does not directly lower inflation via supply and demand. Instead it induces the economy to jump from one to another of multiple equilibria.

If one accepts that critique, what should rise in its place? This paper offers a constructive answer. The answer is mostly that one can use the same economic model, but accept a passive Taylor rule or time-varying and state-varying pegs, and find other ways to deal with the multiple equilibrium problem. I offer several suggestions, including limiting the fiscal implications of expectational shocks and the backwards-stability criterion. They all point approximately to the same answer.

That answer, however, leads to different conclusions about the effect of policy. The conventional prediction that a rise in interest rates temporarily lowers inflation in forward-looking models comes deeply from the Fed inducing the economy to jump to a different equilibrium. Removing such equilibrium-selection policy, we are left with solutions in which inflation rises when monetary policy raises interest rates.

This paper's quest is only to map out logical possibilities, not to argue for a result. Perhaps the answer *is* to abandon one of the qualifiers simple, modern, or economic in the baseline model of monetary policy. Perhaps the conventional negative sign is true, and perhaps it does come from Fed-induced jumps between multiple equilibria, extensive frictions or non-economic behavior, irrationalities, and so forth, and cannot be captured in simple economics. But if so that knowledge is surely important – and quite a challenge for any honest public discussion of policy.

1.1. Outline

Since the results are unusual, I use simple and standard off-the-shelf ingredients.

I start with the conventional sticky-price model consisting of an intertemporal substitution (IS) first order condition and a forward looking Phillips curve. Under an interest rate peg, this model produces stable inflation. Therefore, it is consistent with recent experience, in a way that traditional unstable models are not. That is a big point in its favor.

However, in response to an increase in interest rates, this model produces a steady rise in inflation and a transitory output decline (Figure 1), in both the short and long run.

The first objection to the Fisherian frictionless result is to suggest that price stickiness will give at least a short-run conventional prediction, that higher interest rates lower inflation. Price stickiness, of this standard form, does not deliver even a short-run negative sign. It just slows down the frictionless model's dynamics, turning a step function rate rise into an S-shaped inflation rise.

The model's prediction of a small output decline is, however, consistent with standard intuition, VAR evidence, and anecdotal experience of recent interest rate rises.

The bulk of the paper is devoted to efforts to reverse the Fisherian inflation prediction.

I introduce a monetary distortion, via nonseparable money in the utility function. The model with money also addresses an important policy question. If the Fed raises interest rates by paying higher interest on abundant reserves, the mechanism will be fundamentally different than the conventional story told of the past, that the Fed raised interest rates by rationing a small quantity of non-interest-bearing reserves. Does experience from the past regime bear on a prediction for the future regime? If the model with money can deliver a temporary inflation decline, that finding could help to reconcile the model with evidence from the past. But it would warn us that future monetary policy via interest on reserves may no longer have the traditional effect.

The level of output, loosely "demand," this class of models comes from intertemporal substitution: higher real interest rates induce consumers to shift consumption from the present to the future. The model with money, complementary to consumption, generates an extra term in the IS equation, whereby expected *changes* in interest rates affect intertemporal substitution. A time of higher interest rates is, other things equal, a time with less money, or higher interest costs of holding money; it is a worse time to consume. An expected rise in nominal interest rates thus induces consumers to consume more now, and less in the future, than they would otherwise do. Money in the utility function has no effect on the form of the Phillips curve. But through the Phillips curve, the distortion of the intertemporal allocation of consumption can lower inflation around the time of the rate rise, even producing a temporary decline in inflation. (Figure 4.)

Alas, this modification does not bear out quantitatively. The intertemporal substitution effects of the level of interest rates are too large relative to the monetary distortion implied by expected changes in interest rates. I am only able to produce a temporary decline in inflation by assuming counterfactually large money holdings or intertemporal substitution elasticities, $M/PY = 2$ with $\sigma = 1$ (Figure 4) or $M/PY = 1$ with $\sigma = 3$ (Figure 5).

In this model, unexpected changes in money or interest rates have no effects, since they do not distort the forward looking intertemporal substitution mechanism. Thus, this model has precisely the opposite of Lucas (1972) style models: only expected monetary policy matters.

Therefore, adding money does little to reconcile standard VAR estimates, which identify unexpected changes in interest rates. And the model with money makes matters worse in response to a temporary change in interest rates, as also found by most VAR analysis (Figure 6). A temporary unexpected rise in rates sets off a chain of expected declines in interest rates. These expected declines depress output further, emphasizing the model's ability to produce lower output in response to rate changes, but they raise inflation even more.

In a small but important detour from the Fisherian quest, the model with money allows me to analyze policy in which the Fed changes the interest on reserves while leaving the level of interest rates alone. This sort of policy isolates the monetary distortion, while leaving unchanged the direct intertemporal substitution mechanism. The Fed is considering such policy, stated as I have, or stated as balance-sheet policy in which the Fed will target the size of reserves as well as market interest rates, and thus work down a demand curve which will alter interest on reserves relative to market interest rates.

An expected step function rise in the interest paid on reserves, holding the level of interest rates constant, induces a pure spurt of inflation. It induces consumers to shift from before the rate change to afterwards. Lower output before the rate change means rising inflation with a forward looking Phillips curve, and vice versa after the rate change. An unexpected rate change again has no effect. (Figure 8.)

The purely forward looking Phillips curve is a weak ingredient of the model. Empirical Phillips curves find effects of past inflation. Perhaps adding reactions to past inflation can restore the traditional signs?

To address this question, I include backward looking terms in the Phillips curve. Alas, even when the Phillips curve is entirely backward looking, the model still delivers both short-run and long-run rises in inflation when the Fed raises interest rates (Figure 12). Even mixing ingredients, with money and a backward looking Phillips curve, does not restore the desired signs (Figure 13).

I examine multiple equilibria of the conventional model, and I conclude that this avenue is not promising. (Figure 14 and Figure 15.) Expressing monetary policy as an active Taylor rule makes no difference at all to the analysis, as one can construct an active Taylor rule to justify any equilibrium choice.

What does it take? A simple old-Keynesian model with adaptive expectations and static demand produces all the standard signs: A rise in interest rates lowers inflation and output, immediately and in the long run. This model is also unstable: If the nominal interest rate is constant, any small deviation from a steady state leads to spiraling inflation and deflation. (Figure 18.)

However, its instability means that this model cannot account for the stability of inflation at the zero bound. And, it does not at this stage deserve to be called an economic model. It uses adaptive expectations in the relationship between interest rates and inflation, $i_t = r_t + \pi_t$, not $i_t = r_t + E_t\pi_{t+1}$. Its IS curve is static, not intertemporal. In both ways it is not derived from optimizing behavior. And its Phillips curve is purely backward looking, relating output mechanically to changes in inflation. So it fails my quest for a simple economic model that delivers the standard signs.

The natural next steps in this quest might be to add frictions so that the forward-looking model looks more old-Keynesian. One might add informational, market, and other frictions; hand-to-mouth consumption, borrowing constraints, non-rational expectations, alternative equilibrium concepts, and so forth. But that course admits that there is no simple benchmark model that produces the desired outcomes. That course admits that the proposition that higher interest rates lower inflation rests squarely on a more complex set of frictions, beyond price stickiness and a demand for money. So the quest of this paper is over.

Perhaps, instead, we should take the neo-Fisherian predictions seriously. What is the strong evidence that raising interest rates lowers inflation, or at least did so when monetary frictions mattered? I review VAR literature, finding that the evidence is weak. VAR estimates have long featured a “price puzzle,” that raises in interest rates lead to increased inflation. Efforts to modify the specification of VARs to deliver the desired result have not, in recent reviews, produced strong evidence that interest rate rises produce lower inflation. The data are much clearer that interest rate increases produce lower output, but even the simple model of Figure 1 confirms that view.

Even without a price puzzle, the VAR literature may be interpreted in ways consistent with a Fisherian response. The simple model produces the standard sign – temporarily lower inflation – in response to a joint monetary-fiscal contraction. VARs typically do not orthogonalize monetary and fiscal policy shocks – which would be very hard to do. So if VARs are picking up joint fiscal and monetary policy shocks – really coordinated responses to other variables not seen by the econometrician, such as a political decision to really fight inflation – then those are the wrong shocks with which to evaluate the question at hand, a pure monetary policy increase.

Similarly, historical episodes, such as the successful US disinflation of the early 1980s, or unsuccessful disinflations and pegs, represent joint fiscal and monetary policy. So, more deeply and constructively, this line of thought emphasizes that a policy which does affect inflation must combine fiscal and monetary policy, and that the attractive features of the current zero interest rate peg – we get to live the Friedman (1969) optimum quantity of money – depend on fiscal policy as well as monetary policy.

1.2. Literature

The observation that interest rate pegs are stable in forward looking new Keynesian models goes back a long way.

Woodford (1995) discusses the issue. Woodford (2001) is a clear statement, analyzing interest rate pegs such as the WWII US price support regime, showing they are stable so long as fiscal policy cooperates.

Benhabib, Schmitt-Grohé, and Uribe (2002) is a classic treatment of the zero-rate liquidity trap. They note that the zero bound means there must be an equilibrium with a locally passive $\phi_\pi < 1$ Taylor rule, with multiple stable equilibria. However, they view this state as a pathology, not a realization of the optimal quantity of money, and devote the main point of the paper to escaping the trap via fiscal policy.

The realization that stability implies that the Fed could raise the peg and therefore raise inflation came later. Schmitt-Grohé and Uribe (2014) express this as another possibility for escaping a liquidity trap. They write

> The paper... shows that raising the nominal interest rate to its intended target for an extended period of time, rather than exacerbating the recession as conventional wisdom would have it, can boost inflationary expectations and thereby foster employment.

The simple model here disagrees that raising inflation raises unemployment.

Belaygorod and Dueker (2009) and Castelnuovo and Surico (2010) estimate new-Keynesian / DSGE models allowing for switches between determinacy and indeterminacy. They find that the model displays the price puzzle – interest rate shocks lead to rising inflation, starting immediately – in the indeterminacy region $\phi_\pi < 1$, as I do.

These papers also view the Fisher proposition as only holding when Taylor rules are passive, as in a liquidity trap. I show here that the argument holds also for active Taylor rules.

The possibility that a peg at zero causes deflation, so raising interest rates raises inflation, has had a larger airing in speeches and the blogosphere. Minneapolis Federal Reserve Chairman Narayana Kocherlakota suggested it in a famous speech, Kocherlakota (2010), including

> Long-run monetary neutrality is an uncontroversial, simple, but nonetheless profound proposition. In particular, it implies that if the FOMC maintains the fed funds rate at its current level of 0-25 basis points for too long, both anticipated and actual inflation have to become negative. Why? It's simple arithmetic. Let's say that the real rate of return on safe investments is 1 percent and we need to add an amount of anticipated inflation that will result in a fed funds rate of 0.25 percent. The only way to get that is to add a negative number – in this case, 0.75 percent.
>
> To sum up, over the long run, a low fed funds rate must lead to consistent–but low–levels of deflation.

To be clear, Friedman (1968) disagrees. Friedman views the Fisher equation as an unstable steady state, and Kocherlakota, seeing recent experience, views it as a stable one.

In the blogosphere, Williamson (2013) and a series including Cochrane (2013, 2014b), raise the possibility that pegs are stable, and emphasize the implication that raising rates will raise inflation. Smith (2014) provides a good overview.

Cochrane (2014a) works out a model with fiscal price determination, an interest rate target, and simple k-period price stickiness. Higher interest rates raise inflation in the short and long run, just as in this paper, but the k-period stickiness leads to unrealistic dynamics.

Following Woodford (2003), many authors have also tried putting money back into sticky-price models. **?** and **?** study a CES money in the utility function specification as here, in a detailed model applied to the Eurozone. They find that adding money makes small but important differences to the estimated model dynamics.

Ireland (2004) also adds money in the utility function. In his model, money also spills over into the Phillips curve. He writes, (p. 974) "... optimizing firms set prices on the basis of marginal costs; hence, the measure of real economic activity that belongs in a forward-looking Phillips curve ... is a measure of real marginal costs, rather than a measure of detrended output ... in this model, real marginal costs depend on real wages, which are in turn linked to the optimizing household's marginal rate of substitution between consumption and leisure. Once again, when utility is nonseparable, real balances affect this marginal rate of substitution; hence, in this case, they also appear in the Phillips curve." However, he finds that maximum likelihood estimates lead to very small influences of money, a very small if not zero cross partial derivative u_{cm}.

Where Ireland's Phillips curve comes from quadratic adjustment costs, Andrés, López-Salido, and Vallés (2006) find a similar result from a Calvo-style pricing model. Their estimate also finds no effects of money on model dynamics.

2. The simple model

I start with a simple standard optimizing sticky-price model,

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) \tag{1}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \tag{2}$$

where x_t denotes the output gap, i_t is the nominal interest rate, and π_t is inflation.

The solution for a given interest rate path is derived in the appendix. Inflation and output are two-sided geometrically-weighted distributed lags of the interest rate path,

$$\pi_{t+1} = \frac{\kappa\sigma}{\lambda_1 - \lambda_2}\left[i_t + \sum_{j=1}^{\infty}\lambda_1^{-j} i_{t-j} + \sum_{j=1}^{\infty}\lambda_2^{j} E_{t+1} i_{t+j}\right] + \sum_{j=0}^{\infty}\lambda_1^{-j}\delta_{t+1-j} \tag{3}$$

$$\kappa x_{t+1} = \frac{\kappa\sigma}{\lambda_1 - \lambda_2}\left[\left(1-\beta\lambda_1^{-1}\right)\sum_{j=0}^{\infty}\lambda_1^{-j} i_{t-j} + \left(1-\beta\lambda_2^{-1}\right)\sum_{j=1}^{\infty}\lambda_2^{j} E_{t+1} i_{t+j}\right]$$
$$+ (1-\beta\lambda_1^{-1})\sum_{j=0}^{\infty}\lambda_1^{-j}\delta_{t+1-j}, \tag{4}$$

where

$$\lambda_1 = \frac{(1+\beta+\kappa\sigma) + \sqrt{(1+\beta+\kappa\sigma)^2 - 4\beta}}{2} > 1 \tag{5}$$
$$\lambda_2 = \frac{(1+\beta+\kappa\sigma) - \sqrt{(1+\beta+\kappa\sigma)^2 - 4\beta}}{2} < 1 \tag{6}$$

Here, δ_{t+1} is an expectational shock indexing multiple equilibria. From (2), the model only determines $E_t\pi_{t+1}$. Hence, actual inflation is

$$\pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{k}{\beta}x_t + \delta_{t+1}.$$

Without further specification, δ_{t+1} is arbitrary so long as it is unforecastable. I discuss equilibrium selection below.

2.1. Basic impulse-response function

We can calculate the impulse-response function easily by putting the path $\{i_t\}$ after the announced policy change on the right hand side of (3), with $i_t = 0$ before the announced policy change. The date of announcement need not coincide with the date of first raising interest rates.

Figure 1 presents the response of inflation and the output gap to a step function rise in the interest rate, using (3)-(4), and choosing the basic solution $\delta_0 = 0$. I discuss alternative δ_0 choices below, concluding that $\delta_0 = 0$ or nearby choices are the most reasonable. Throughout, unless otherwise noted, I use parameters

$$\beta = 0.97, \kappa = 0.2, \sigma = 1. \tag{7}$$

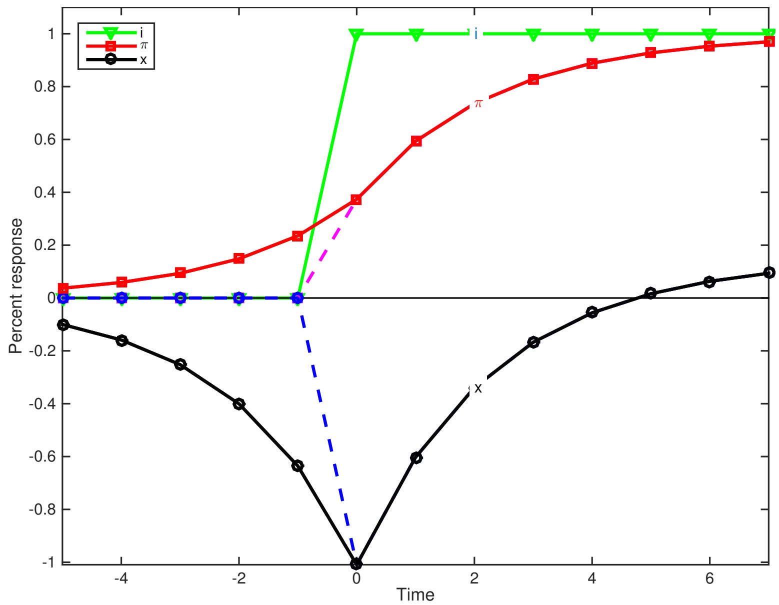


Figure 1: Response of inflation and output to a step function interest rate change. The solid lines show the response to an expected change. The dashed lines show the response to an unexpected change.

The solid lines and dashed lines of Figure 1 plot the responses to a pre-announced and to an unexpected interest rate rise respectively. The responses are the same after the announcement day, so you can no longer distinguish dashed and solid lines for $t > 0$. More generally, the response to this policy announced at any time before zero jumps up to match the anticipated-policy reaction on the day of announcement.

Inflation rises throughout the episode. Mathematically, that is a result of a two-sided moving average with positive weights in (3).

Output declines around the interest rate rise. When the nominal interest rate is higher than the inflation rate, the real rate is high. Output is low when current and future real interest rates are high via intertemporal substitution. Equivalently, the forward looking Phillips curve (2) says that output is low when inflation is low relative to future inflation, i.e. when inflation is increasing.

Output eventually rises slightly, as the steady state of the Phillips curve (2) gives a slight increase in the level of output when inflation increases permanently.

In sum, this simple standard model gives a smoothed Fisherian inflation response to interest rate changes. One might have hoped that price stickiness would deliver the traditional view of a temporary decline in inflation. It does not.

The model does, however, generate the output decline that conventional intuition and most empirical work associates with monetary policy tightening. It therefore suggests a novel picture of monetary policy. Raising interest rates to cool off a booming economy, and lowering interest rates to stimulate a slow economy may make sense. Doing so just has a different effect on inflation than we might have thought. In fact, lowering rates actually contributes to the decline in inflation that we see in recessions. And raising rates may create the very inflation that the Fed will then pat itself on the back for foreseeing and offsetting. It paints a picture, not unlike recent experience, in which monetary policy is primarily about manipulating *output*, not inflation.

These results are not much affected by changes in the parameters. There isn't much you can do to an S shape. The parameters κ and σ enter together in the inflation response. Larger values speed up the dynamics, approaching the step function of a frictionless model as their product rises. Larger values of the parameter β slightly slow down the dynamics. Larger σ gives larger output effects with the same pattern.

Expected and unexpected policy have similar responses because the interest rate shock $i_t - E_{t-1}i_t$ does not appear as a separate right hand variable in the model's solutions (3)-(4), as it does in information-based Phillips curves such as Lucas (1972). As a result, in this class of models, expected monetary policy matters.

Also, though VARs often focus on the responses to unexpected policies, our Fed telegraphs its intentions, often far in advance. So the expected policy change case is important to study.

Output and inflation move ahead of the expected policy change. This fact reminds us that "forward guidance" matters, and that outcomes are affected by expectations, even when those expectations do not bear out.

2.2. Mean-reverting and stairstep rates

Empirical impulse-response functions usually find that the response of interest rates to an interest rate shock is mean-reverting, not a pure random walk as is the conceptual experiment of Figure 1. To think about that case, Figure 2 plots responses to an AR(1) interest rate shock.

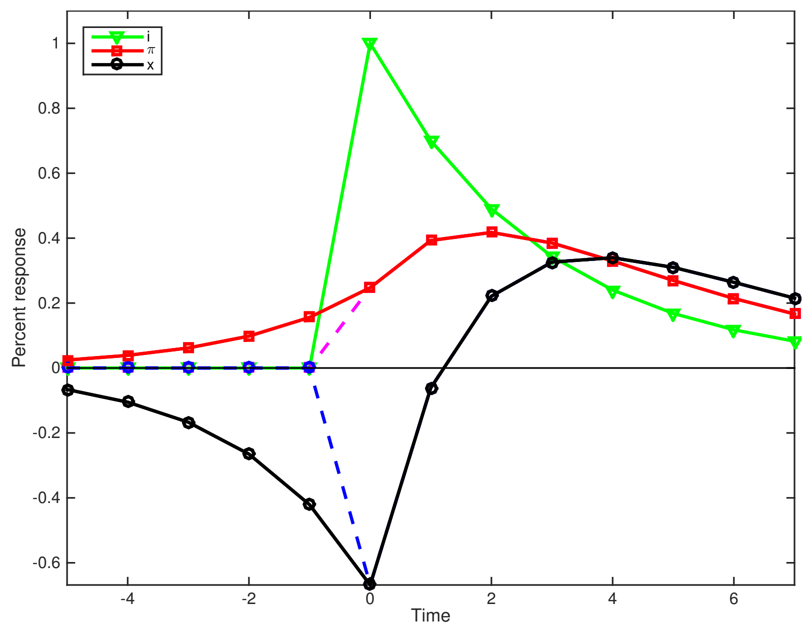


Figure 2: Response of inflation and output to a mean-reverting interest-rate path. Dashed lines are the response to an unexpected change. Solid lines are the response to an expected change.

One might have hoped that, since an expected rise in interest rates raises inflation, the expected declines in interest rates set off by the initial shock might have a contrary effect, depressing inflation or maybe even giving rise to a negative movement. Alas, that hope does not bear out. The responses in Figure 2 are similar to those of Figure 1 in the short run, with a long-run return to zero.

Figure 2 serves as an important reminder though: VARs that estimate transitory responses of interest rates to interest rates do not give us evidence on the long-run Fisher hypothesis. If the Fed does not raise interest long enough, we get no evidence on the eventual response of inflation to a change in interest rates. The zero bound experience tells us something that we could not observe in the transitory interest rate changes typical of the previous era.

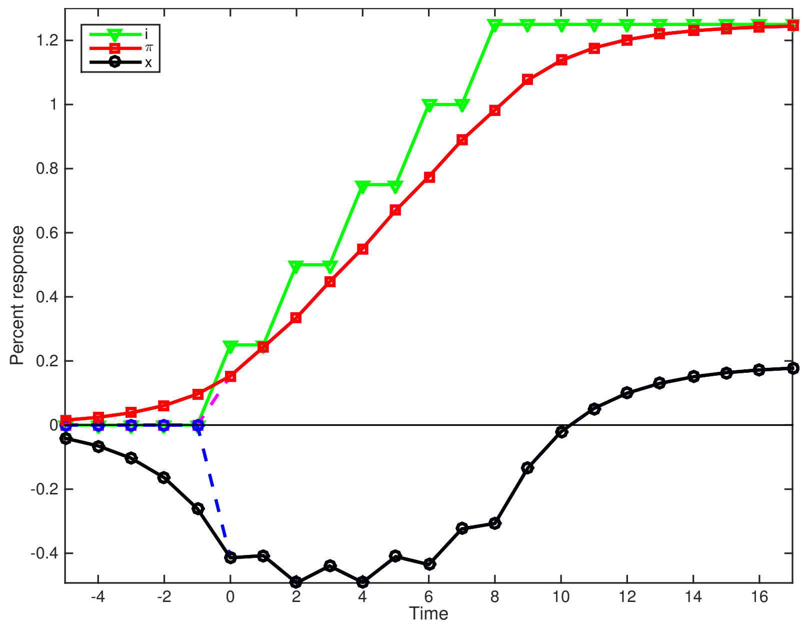


Figure 3: Response of inflation and output to a stairstep interest-rate path

Federal Reserve tightening typically takes the form of a well-anticipated steady set of stair step interest rate rises. Figure 3 presents the effects of such a policy. The result is qualitatively predictable from the other figures, though the smoothness of the inflation and output effects is noteworthy.

2.3. Multiple equilibria and Taylor rules preview

By solving for inflation and output given the equilibrium interest rate sequence $\{i_t\}$ I appear to assume that the Fed follows a time-varying peg. This is not the case.

The series $\{i_t\}$ represents a conjectured path for the equilibrium interest rate. Equations (3)-(4) tell us that *if* an equilibrium has an interest rate sequence $\{i_t\}$, then its inflation and output paths $\{\pi_t,\ x_t\}$ must follow (3)-(4). The "impulse response functions" are really the response of equilibrium inflation and equilibrium output to a policy change that also produces a step function rise in equilibrium interest rates.

To match empirical VARs, and to understand how inflation and output would respond if the Fed engineers a step function path of the nominal interest rate, this is exactly the question one wants to answer. Just how the Fed engineers the equilibrium interest rate path is not important, so long as it can do so.

In most models of Fed policy, it can. The Fed can follow a time-varying or state-varying peg, and simply set the nominal interest rate path. The Fed can also follow a Taylor rule with an active inflation response. I show below how to construct such a Taylor rule. Werning (2012) innovated this idea of first finding equilibrium inflation and output given equilibrium interest rate paths, and then constructing the underlying Taylor rule.

There are multiple equilibrium responses of $\{E_t\pi_{t+j}\}$, $\{E_t x_{t+j}\}$ corresponding to each $\{E_t i_{t+j}\}$, indexed by δ_t, as shown in (3)-(4), where t is the date of the shock and j describes the path of expectations. The previous figures only study the case $\delta_t = 0$.

One may hope to focus on a different equilibrium via the assumption of a Taylor rule, or that a different equilibrium choice δ_t would produce the desired responses. I investigate both issues below, and conclude that neither hope works out, and that equilibria near $\delta_t = 0$ are the most reasonable choices. Since that analysis is negative, long, and less novel than the next few sections, I focus first on the $\delta_t = 0$ solutions with other modifications.

3. Money

Perhaps monetary distortions, in addition to pricing distortions, will give us the traditional result. Perhaps when interest rate increases were accomplished by reducing the supply of non-interest-bearing reserves, that reduction in money produced a temporary decline in inflation that simply raising the interest rate on excess reserves will not produce.

Such a finding would help to give an interpretation to decades of formal and informal empirical work and widespread intuition. It would, however, suggest that raising interest rates by simply raising the rate paid on abundant excess reserves would not have the same temporary disinflationary effect as past history suggests.

I introduce money in the utility function, nonseparable from consumption, so that changes in money, induced by interest rate changes, affect the marginal utility of consumption, and thus the intertemporal-substitution equation.

Woodford (2003) (p. 111) begins an analysis of this specification. But Woodford quickly abandons money to produce a theory that is independent of monetary frictions, and does not work out the effects of monetary policy with money. If theory following that choice now does not produce the desired outcome, perhaps we should revisit the decision to drop money from the analysis.

The detailed presentation is in the Appendix. The bottom line is a generalization of the intertemporal-substitution condition (1):

$$x_t = E_t x_{t+1} + (\sigma - \xi)\left(\frac{m}{c}\right) E_t\left[\left(i_{t+1} - i_{t+1}^m\right) - \left(i_t - i_t^m\right)\right] - \sigma\left(i_t - E_t \pi_{t+1}\right). \quad (8)$$

The presence of money in the utility function has no effect on firm pricing decisions and hence on the Phillips curve (2). Here, ξ is the interest-elasticity of money demand. Evidence and literature surveyed in the Appendix suggests $d\log(m)/d\log(i) = \xi \approx 0.1$. The value m/c is the steady state ratio of real money holdings to consumption. The larger this value, the more important monetary distortions. The quantity i_t^m is the interest rate paid on money.

Equation (8) differs from its standard counterpart (1) by the middle, change in interest rate term. (The term is the time-derivative of the interest rate in the continuous-time expression, equation (54) in the Appendix.) Equation (8) reverts to (1) if utility is separable $(\sigma - \xi) = 0$, if m/c goes to zero, or if money pays the same interest rate as other assets.

The expression $m/c\ (i_t - i_t^m)$ represents the proportional interest costs of holding money. The middle term following $(\sigma - \xi)$ represents the expected change in those proportional interest costs. An expected increase in interest costs of holding money, a complement to consumption, induces the consumer to shift consumption from the future towards the present, just like a lower real interest rate.

The presence of expected changes in interest rates brings to the model a mechanism that one can detect in verbal commentary: the sense that changes in interest rates affect the economy as well as the level of interest rates.

However, monetary distortions only matter in this model if there is an *expected change in future* interest rate differentials. Expected, change, and future are all crucial modifiers. A higher or lower steady state level of the interest spread does not raise or depress today's consumption relative to future consumption. An unexpected change in interest costs has no monetary effect at all, since $E_t\left(i_{t+1} - i_t\right) = 0$ throughout. Equation (8) is an intertemporal substitution relationship; in this model "demand" changes come entirely from changes in the intertemporal allocation of consumption.

If one has intuition to the contrary, that unexpected changes matter, or that steady but higher interest differentials matter, that intuition must correspond to a fundamentally different model. One needs some other fundamental source of "demand" than intertemporal substitution, or one needs monetary distortions to affect price-setting behavior.

The model solution is essentially unchanged. The extra term in the intertemporal substitution equation (8) amounts to a slightly more complex forcing process involving expected changes in interest rates as well as the level of interest rates. One simply replaces i_t in (3)-(4) with z_t defined by

$$z_t \equiv i_t - \left(\frac{\sigma - \xi}{\sigma}\right)\left(\frac{m}{c}\right) E_t\left[\left(i_{t+1} - i^m_{t+1}\right) - \left(i_t - i^m_t\right)\right].$$

The slight subtlety is that this forcing process is the change in *expected* interest differentials. The lag operators must apply to the E_t as well as what's inside. Inflation depends on past expectations of interest rate changes, not to past interest rate changes themselves.

3.1. Impulse-response functions

I start with the traditional specification that the interest on money $i_t^m = 0$, so that increases in the nominal interest rate are synonymous with monetary distortions. Figure 4 plots the response function to our expected and unexpected interest rate step with money distortions $m/c = 0,\ 2,\ 4$.

For the unexpected interest rate rise, shown in dashed lines, the presence of money makes no difference at all. The dashed lines are the same for all values of m/c, and all the same as previously, and the model remains stubbornly Fisherian. This is an important negative result. Money can only affect the response to *expected* interest rate changes.

The response to an expected interest rate rise, shown in solid lines, is affected by the monetary distortion. As we increase the size of the monetary distortion m/c, inflation is lower in the short run. For $m/c = 4$, we get the "right" impulse response function. The announced interest rate rise produces a temporary decline in inflation, and then eventually the Fisher effect takes over and inflation increases.

The only time-difference in interest costs comes at time 0. Larger and larger m/c induces the consumer to shift consumption forwards in time relative to time 0. Output is high when inflation is decreasing, and vice versa, so this pattern of output corresponds to lower inflation before time 0 and higher inflation afterward.

The $m/c = 4$ curve seems like a great success, until one ponders the size of the monetary distortion – four years of output. This model is not carefully calibrated, but $m/c = 4$ is still a lot.

Equation (8) suggests that raising σ, which multiplies m/c, may substitute for a large m/c, by magnifying the effect on consumption of a given monetary distortion. Now, higher σ also magnifies the last term, which induces Fisherian dynamics. But in our response functions, the middle term multiplies a one-time shock, where the last term multiplies the entire higher step. Thus, raising σ can raise the relative importance of the one-time shock in the dynamics of inflation.

Figure 5 investigates the effect of changing the intertemporal substitution elasticity σ. Since an unexpected interest rate rise again has no monetary effect, I present only

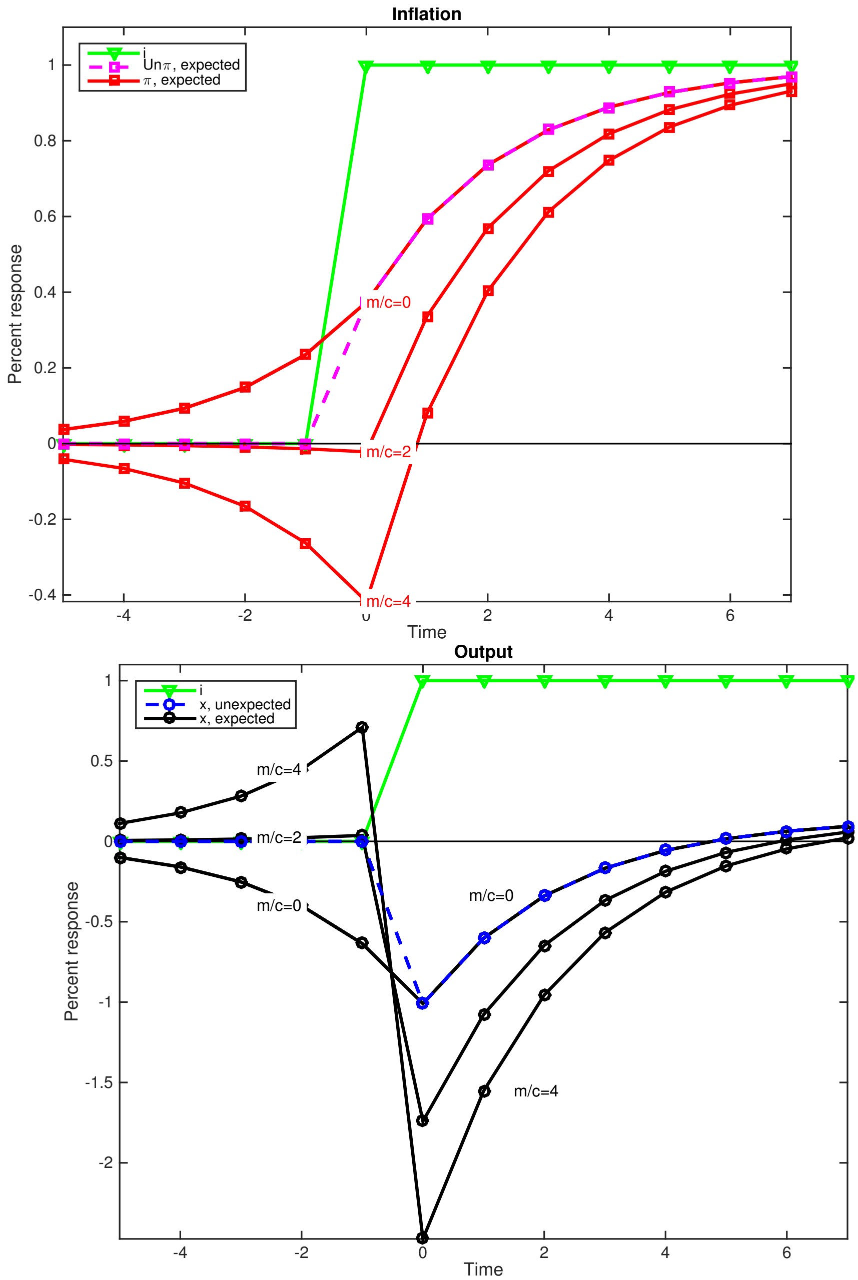


Figure 4: Response of inflation and output to an interest rate rise; model with money. The three cases are $m/c = 0,\ 2,\ 4$. Solid lines are an expected interest rate rise, dashed lines are an unexpected rise.

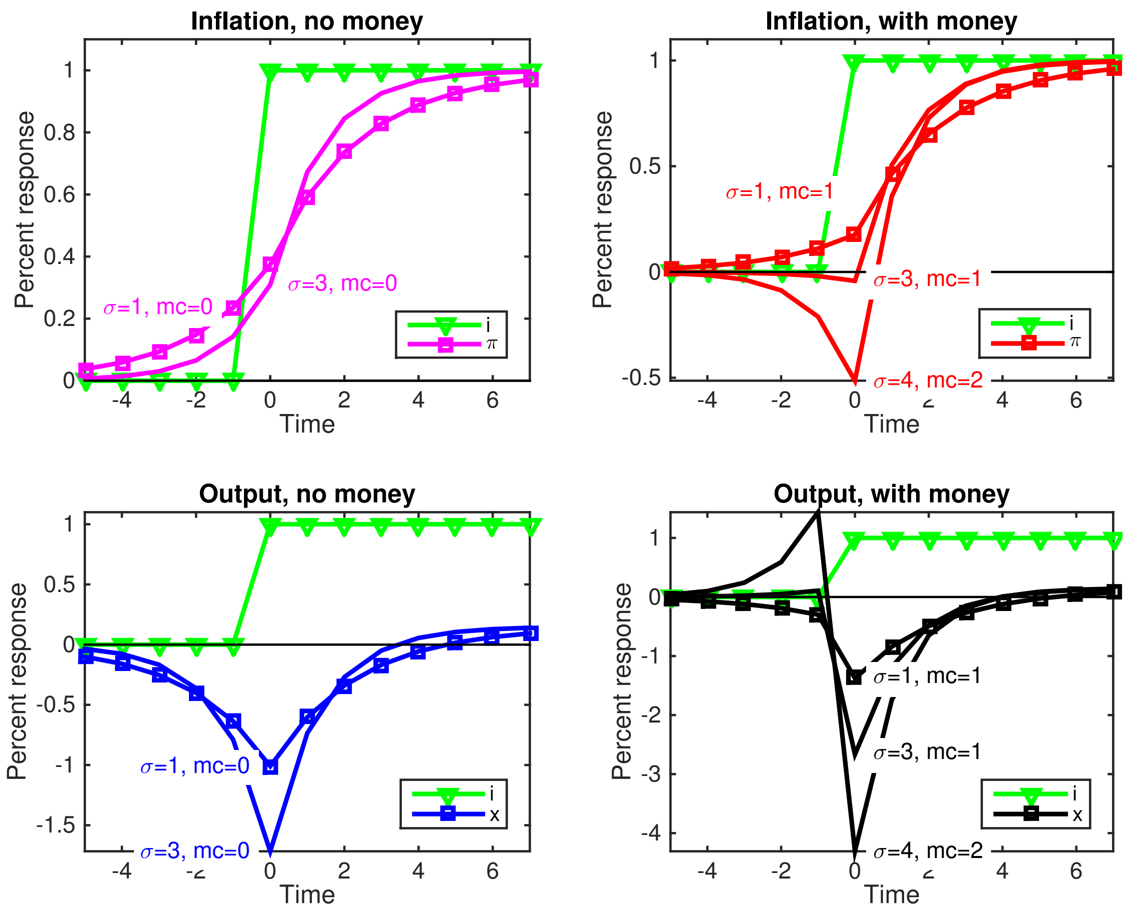


Figure 5: Response of output and inflation to an expected interest rate step; model with money and varying intertemporal substitution elasticity σ.

the case of an expected interest rate rise.

The left two panels, labeled $mc = 0$ in Figure 5, show the effect of varying σ in the model without money. In this model, σ and κ enter symmetrically in the determination of inflation. Raising σ increases the speed of the dynamics, pulling the S shaped response closer to the step that will hold in a frictionless model. Raising the speed of the dynamics has the effect of lowering inflation in the early period, a step in the direction of the conventional belief. But raising σ without money can never produce a negative effect on inflation.

The right two panels of Figure 5 with $m/c > 0$ show how increasing σ can work together with a monetary friction. At $m/c = 1$, increasing σ from $\sigma = 1$ to $\sigma = 3$ produces a slight decline in inflation before the inevitable rise. The subsequent rise is quicker; the main effect here has been to borrow inflation from the future. To get a substantial negative effect, one must increase either σ or m/c even more. The line $\sigma = 4,\ m/c = 2$ produces about the same inflation decline as $\sigma = 1, m/c = 4$ produced in Figure 4.

So, higher σ can help to produce a temporary dip in inflation, largely by speeding up dynamics. Alas, $\sigma = 1$ was already above most estimates and calibrations. A coefficient $\sigma = 3$ implies that a one percentage point increase in the real interest rate induces a three percentage point increase in consumption growth, which is well beyond most estimates. And $m/c = 1$ is already at least twice as big as one can reasonably defend.

In sum, these calculations show what it takes to produce the standard view: For an anticipated interest rate rise, money in the model can induce lower inflation than a frictionless model produces. If we either have very large money holdings subject to the distortion, or a very large intertemporal substitution elasticity, the effect can be large enough to produce a short-run decline in inflation.

Adding money to the model in this way has absolutely no effect on responses to an unexpected permanent interest rate rise.

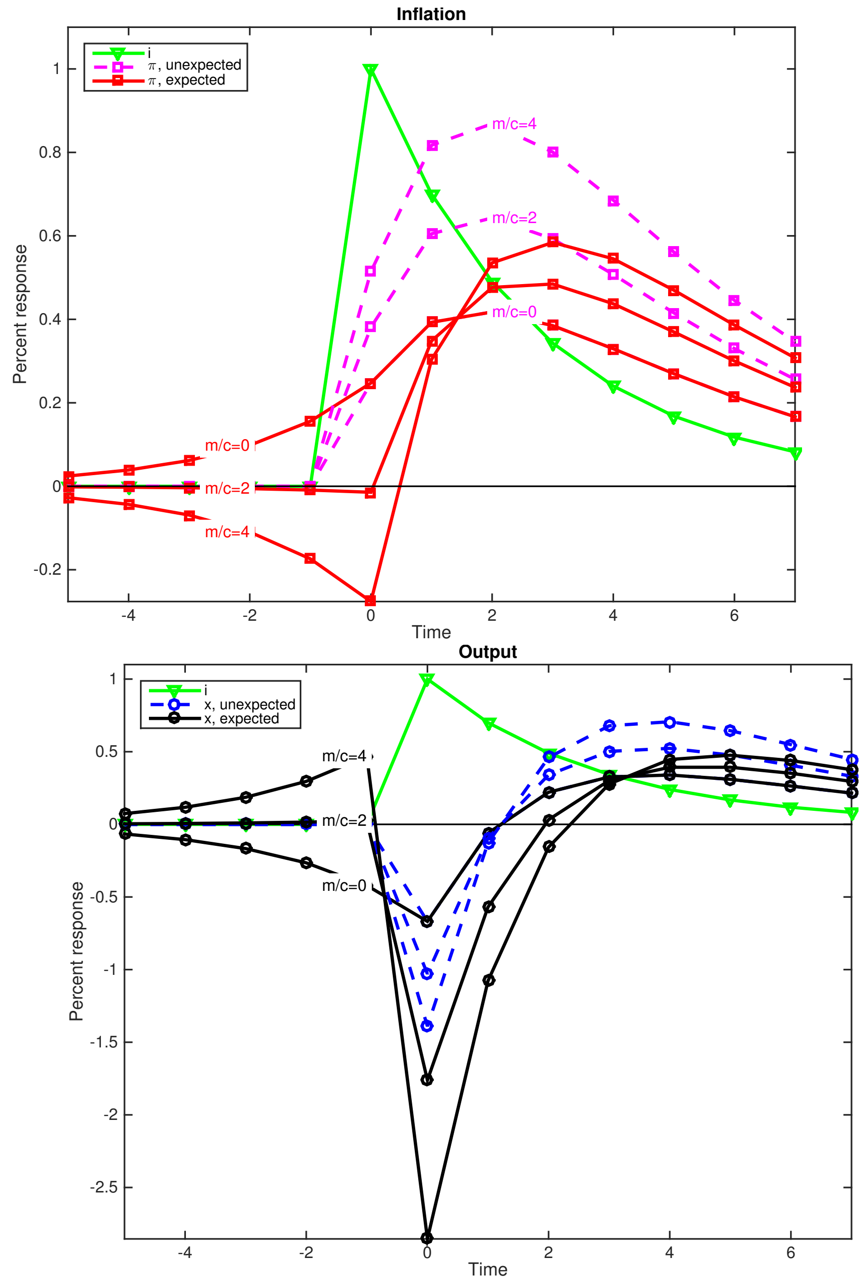


Figure 6: Response of inflation and output to a temporary rate rise, model with money. Dashed lines are the response to an unexpected rise, solid lines are the response to an expected rise.

3.2. Money and transitory rate shocks

The step function interest rate rise is an extreme example. In most VARs, and in most policy interventions, interest rate shocks do not last forever. After the initial rise, interest rates are expected to decline back to where they started. The expected decline in rates following an unexpected shock may affect inflation and output where the permanent shock had no such effect.

Figure 6 shows the result of such an event, in which interest rates decay back to zero with an AR(1) pattern after the initial rise. In this case, money does affect the response functions. And, that effect is uniformly to *raise* inflation. The expected decline in interest costs posed by the AR(1) reversion after the shock shifts consumption from the present to the future, and inflation rises when output is low.

VAR impulse-response functions are often hump-shaped: An initial rate rise leads to more rises. Perhaps for the unexpected shock, the fact that interest rates are expected to rise further before declining will produce some lower inflation.

To explore this idea, Figure 7 presents the response to a hump-shaped interest rate path, typical of many VARs. The path is $i_t = (0.7^{(t-1)} - 0.67^{(t-1)}/(0.7 - 0.67)$. The results are similar to the AR(1) interest rate path, but with smoother dynamics. The "best" case is the anticipated effect with a large m/c. The drawn out period of expected interest rate rise produces a bit less inflation initially. But this effect is too small to make much difference. Overall, the unexpected interest rate path still gives a Fisherian response

3.3. Interest spread policy

The Federal Reserve is contemplating varying the interest it pays on reserves as separate policy tool. By changing the interest on reserves, the Fed can affect money demand without changing the nominal rate. Thus, it can focus on the monetary effects on demand without the direct intertemporal substitution effects. This framework is well suited to analyze that sort of policy.

Figure 8 presents a calculation. Here, the Fed raises the interest on reserves i^m by one percentage point, with no change in the nominal interest rate i. Thus, only the

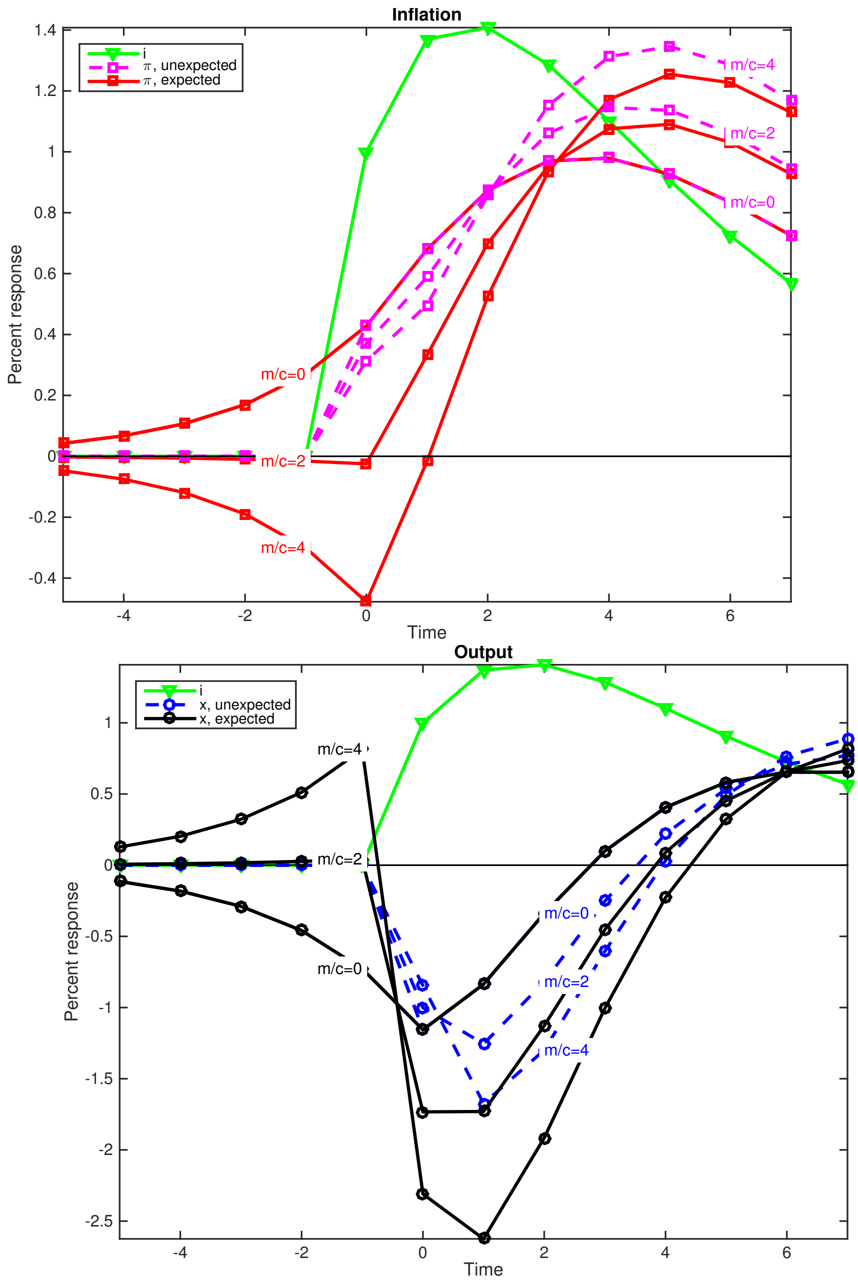


Figure 7: Response to a hump-shaped interest rate path.

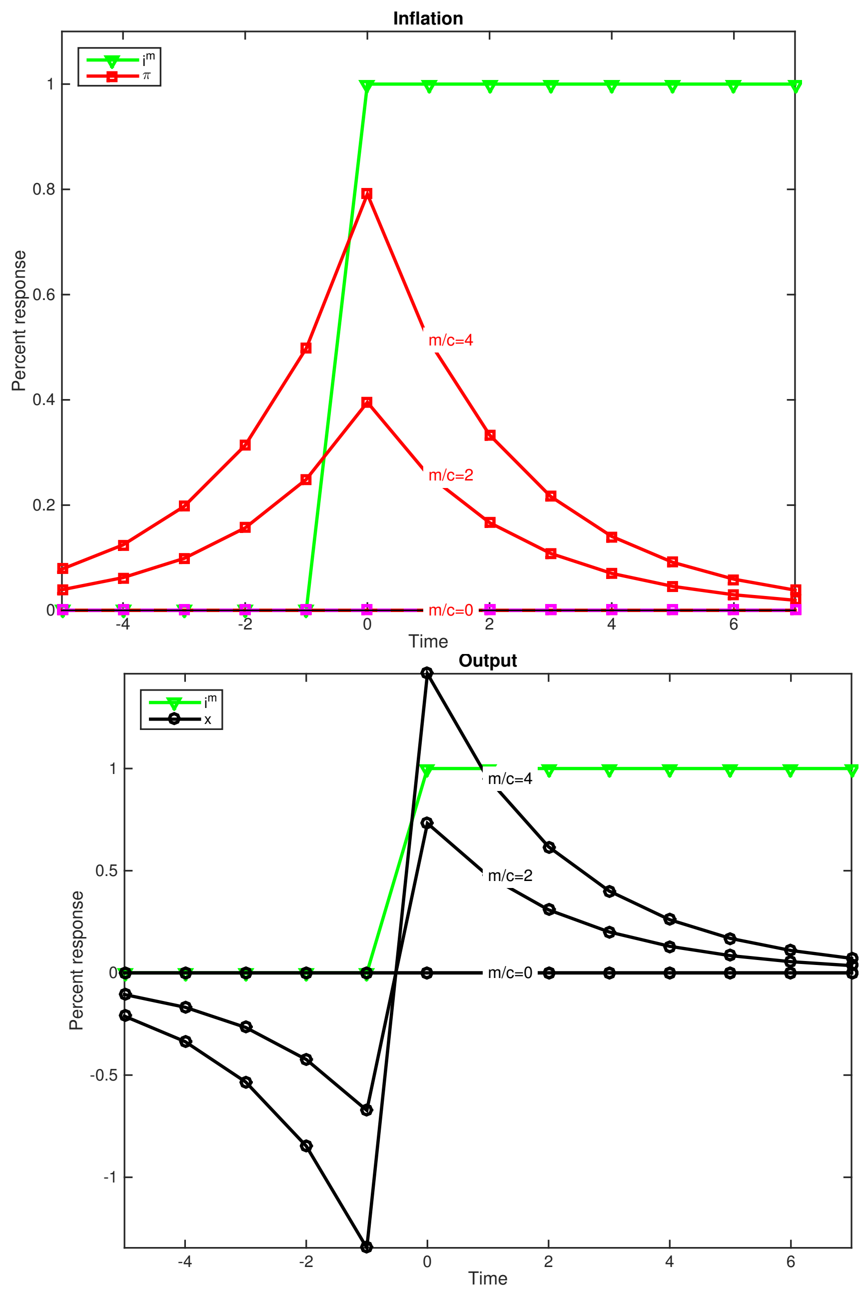


Figure 8: Response to a permanent rise in the rate paid on reserves, holding the nominal interest rate constant.

monetary $i - i^m$ term of (8) has any effect on demand. The last pure intertemporal-substitution term is absent.

One can debate whether we should call a rise in interest on reserves "expansionary" or "contractionary" policy a priori. Raising interest on reserves is often considered contractionary, as it encourages banks to sit on reserves rather than aggressively to pursue lending. On the other hand, raising interest on reserves lowers the spread between reserves and other instruments, and so encourages the accumulation of money, which one might consider to be expansionary.

Again, the response to an unexpected rise in the interest on reserves is exactly zero. The intertemporal substitution mechanism only operates when the expected future is different from the present.

Figure 8 shows that an expected rise in the interest on reserves raises inflation throughout. Output declines ahead of the change, and rises after the change. Money is cheaper to hold after the rise, encouraging consumers to postpone consumption. With a forward looking Phillips curve, lower output corresponds to rising inflation, and vice versa.

To match the sort of policy that the Fed is more likely to pursue, Figure 9 graphs the effects of a temporary increase in the interest on reserves, which is expected to die out with an AR(1) pattern.

In this case, even the unexpected change (solid lines) affects inflation and output, because the reversion is expected. Here we get a simple decline in inflation. The unexpected change induces, first, a rise in inflation due to the expected rise in interest on reserves, and then a decline in inflation due to the expected declines in interest on reserves. The output paths similarly reflect the now clear intertemporal substitution motive. In all cases a rise in interest on reserves coincides with expansion. That expansion is drawn from either future or past periods.

The bottom line of these exercises reinforces the basic message of this model of money. The basic mechanism for "demand" in this, as in all new-Keynesian models, is intertemporal substitution, changes in the margin of current versus future consumption. So "demand" can only be affected by a *change* in expected future monetary distortions. Intuition that the level of monetary distortions matters for "demand" must come

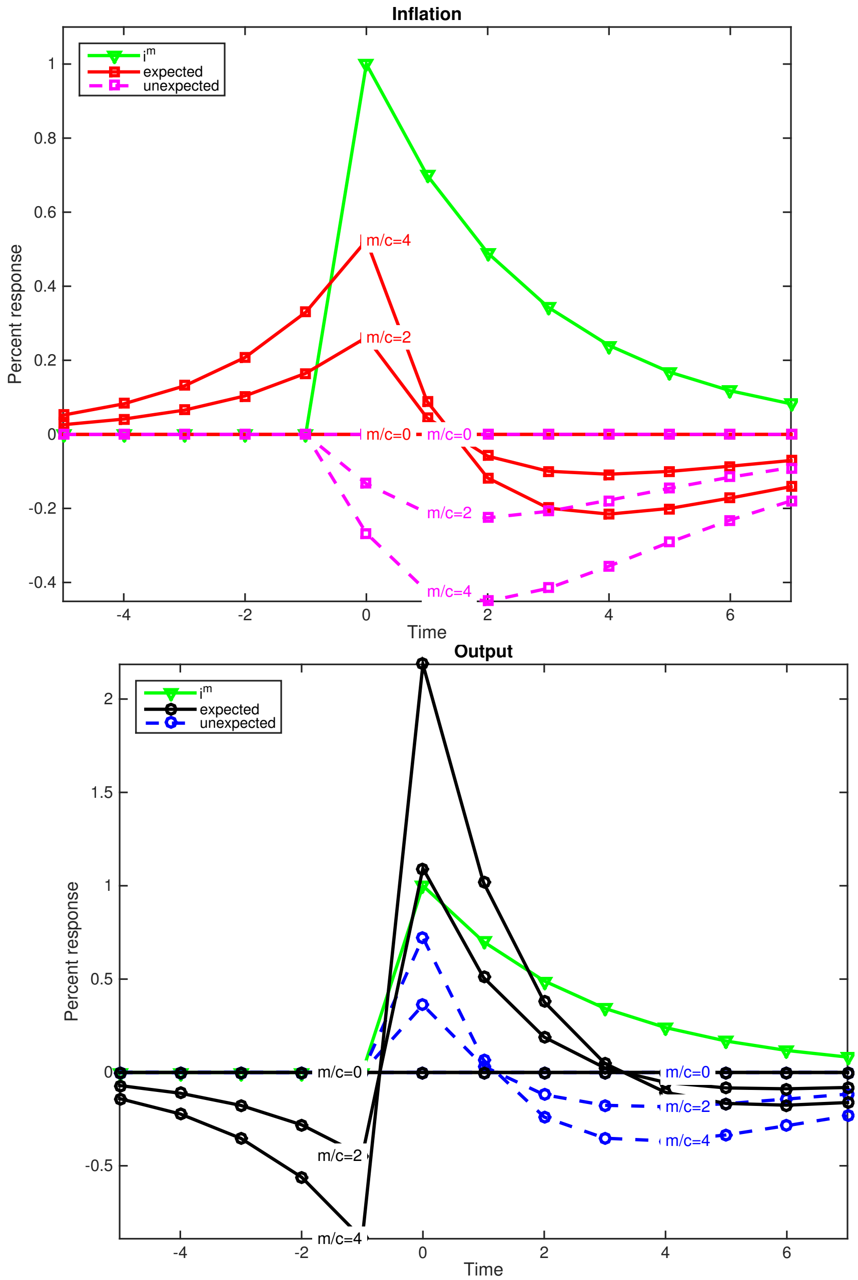


Figure 9: Response to a transitory rise in the rate paid on reserves, holding the nominal interest rate constant. Solid lines are the response to an expected change; dashed lines are the response to an unexpected change.

from some other source of demand, that allows some static effects.

4. Lagged inflation in the Phillips curve

Empirically, lags seem important in Phillips curves. The forward looking Phillips curve (2) specifies that output is higher when inflation is high relative to future inflation, i.e. when inflation is declining. Though Phillips curves fit the data poorly, especially recently, output is better related to high inflation relative to past inflation, i.e. when inflation is rising (Mankiw and Reis (2002)).

Theoretically, the pure forward looking Phillips curve is not central. We should check if the short or long-run neo-Fisherian conclusions can be escaped by adding past inflation to the Phillips curve. The forward looking IS curve is, by contrast, a much more robust part of an optimizing economic model.

4.1. A Phillips curve with lags

The usual Phillips curve (2) is forward looking:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \tag{9}$$

or equivalently

$$\pi_t = \kappa \sum_{j=0}^{\infty} \beta^j E_t x_{t+j}. \tag{10}$$

To incorporate lagged inflation, I write instead

$$\pi_t = \kappa \left(x_t + \sum_{j=1}^{\infty} \phi^j E_t x_{t+j} + \sum_{j=1}^{\infty} \rho^j x_{t-j} \right) \tag{11}$$

or, in autoregressive form,

$$\pi_t = \frac{\phi}{1+\rho\phi} E_t \pi_{t+1} + \frac{\rho}{1+\rho\phi} \pi_{t-1} + \frac{(1-\phi\rho)}{(1+\phi\rho)} \kappa x_t.$$

So that the sum of coefficients on the right hand side of (11) is the same as it is in (10), and hence so that the steady state relationship between output and inflation remains unchanged, I constrain the backward and forward looking coefficients ρ and ϕ to satisfy

$$\frac{(1-\phi)(1-\rho)}{(1-\phi\rho)} = (1-\beta)\,. \tag{12}$$

Repeating the model solution, inflation is again a two-sided moving average of interest rates, and in the presence of money of expected changes in interest rates,

$$\pi_{t+1} = \kappa\sigma\frac{(1-\phi\rho)}{\phi}\frac{\lambda_3}{\left(1-\lambda_3\lambda_1^{-1}\right)\left(1-\lambda_3\lambda_2^{-1}\right)} \times \left(E_t z_t + \sum_{j=1}^{\infty}\lambda_3^j E_t z_{t+j} + \right.$$

$$\left. + \frac{\lambda_1^{-1}\left(1-\lambda_2^{-1}\lambda_3\right)}{\left(\lambda_1^{-1}-\lambda_2^{-1}\right)}\sum_{j=1}^{\infty}\lambda_1^{-j}E_{t-j}z_{t-j} - \frac{\lambda_2^{-1}\left(1-\lambda_1^{-1}\lambda_3\right)}{\left(\lambda_1^{-1}-\lambda_2^{-1}\right)}\sum_{j=0}^{\infty}\lambda_2^{-j}E_{t-j}z_{t-j}\right) \tag{13}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the polynomial in

$$-\frac{\rho}{\phi} + \frac{(1+\rho(1+\phi))}{\phi}L^{-1} - \frac{(1+\phi+\kappa\sigma+\phi\rho(1-\kappa\sigma))}{\phi}L^{-2} + L^{-3}$$

$$= \left(L^{-1}-\lambda_1^{-1}\right)\left(L^{-1}-\lambda_2^{-1}\right)\left(L^{-1}-\lambda_3^{-1}\right)$$

The expression (13) now has two backward looking moving averages as well as one forward looking term. The long-run response of inflation to interest rates remains one.

4.2. Roots, moving average, and response

Figure 10 displays for each value of the backward looking weight ρ in the Phillips curve with lagged inflation (11), the forward looking weight ϕ given by restriction (12), and the three roots $\lambda_1^{-1}, \lambda_2^{-1}$ and λ_3 that govern the moving averages (13).

The left hand column of Figure 11 presents the moving average representation of the Phillips curve (11), and the right hand column presents the moving average coefficients of the solution (13), for specific choices of ρ and the consequent ϕ. These specific values of $\rho,\ \phi$ are represented as dark circles in Figure 10.

Starting at the left of Figure 10, and the top of Figure 11, we have the previous case

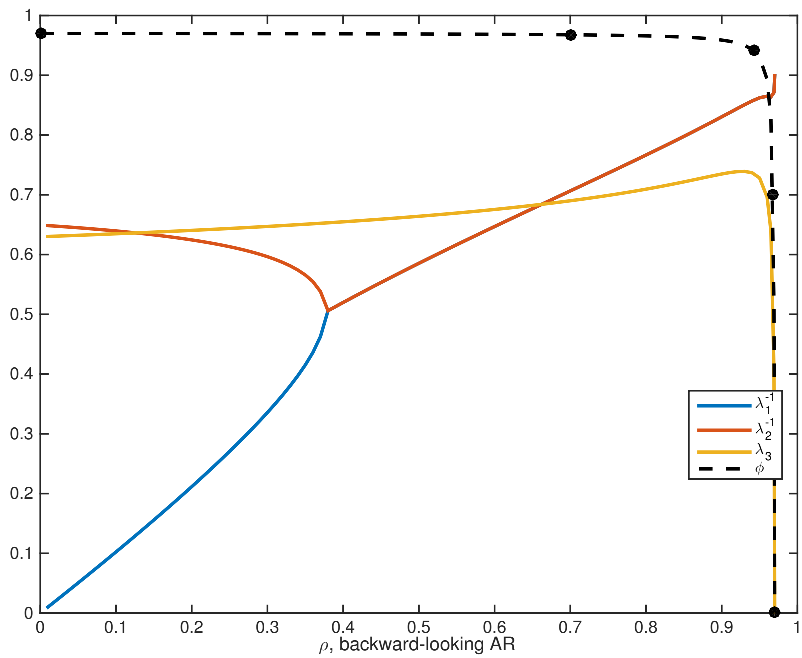


Figure 10: Roots λ of the impulse response function, and forward looking Phillips curve parameter ϕ, for each choice of the backward looking Phillips curve parameter ρ.

$\rho = 0$, $\phi = \beta$ of a purely forward looking Phillips curve. Figure 10 shows the nearly equal forward and backward looking roots λ_2^{-1} and λ_3, and $\phi = \beta$. The top left element of Figure 11 shows the purely forward looking weights of the Phillips curve, while the top right element shows the nearly equal forward and backward moving average weights of the model's solution for inflation as a function of interest rates.

As we raise the backward looking coefficient ρ, Figure 10 shows that the forward looking coefficient ϕ, the forward looking root in the solution λ_3 and the original backward looking root λ_2^{-1} change little. We bring in a second backward looking root, roughly equal to ρ itself. Around $\rho = 0.55$, the two backward looking roots become complex. Their magnitude is still less than one, but the complex nature will generate a damped sinusoidal response function.

The second row of Figure 11 shows the Phillips weights and response function for a backward Phillips curve coefficient $\rho = 0.7$. The forward looking coefficient $\phi \approx \beta$ is still large, so this case captures a small amount of backward looking behavior, and helps

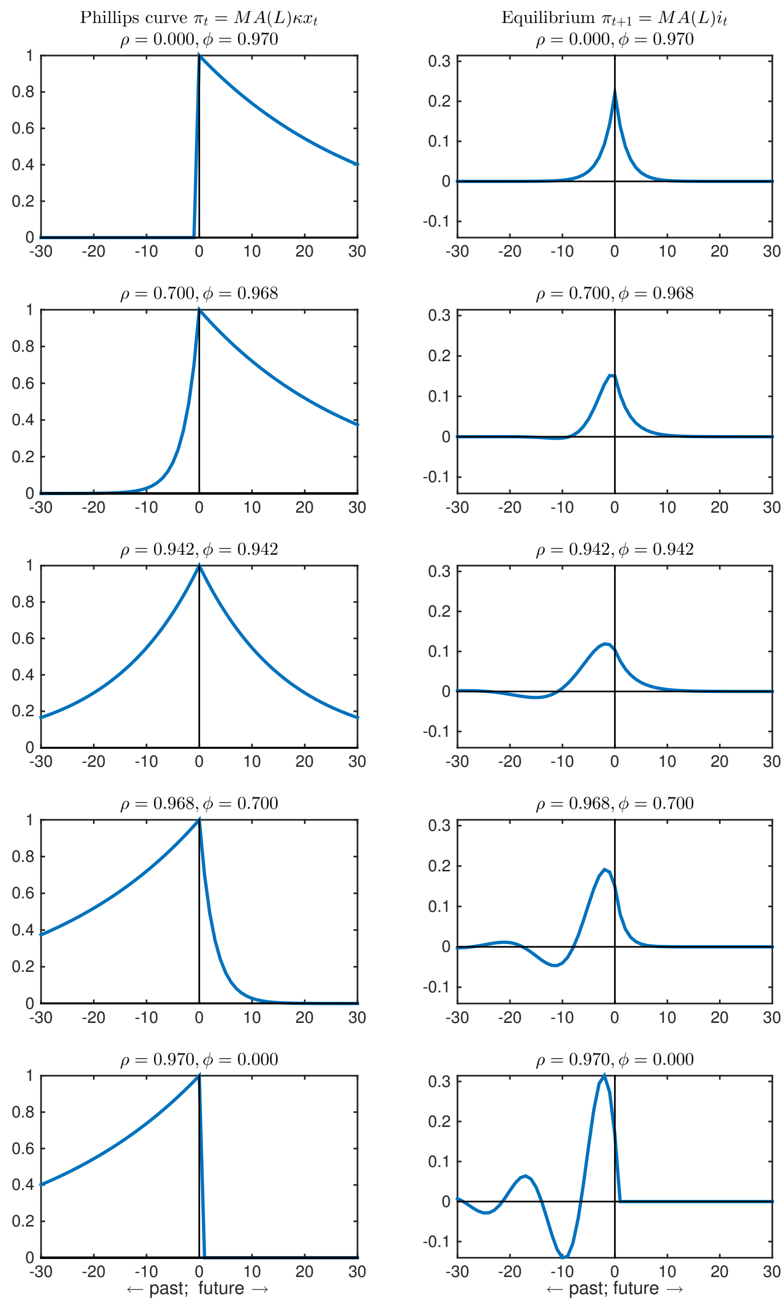


Figure 11: Moving average representation of the two-sided Phillips curve, and corresponding moving-average response of inflation to interest rates.

us to assess if a small amount of such behavior can substantially change results. The moving average solution in the right column is still basically two-sided and positive. It begins to weight the past more than the future. The small change in this moving average previews the result below that this small amount of backward looking behavior in the Phillips curve will not materially affect the neo-Fisherian response to an interest rate rise.

Continuing to the right in Figure 10, the backward looking roots continue to grow nearly linearly with the backward looking Phillips parameter ρ. The forward looking coefficient ϕ and the forward looking root λ_3 remain nearly unaffected however, for even very large values of ρ.

The third row of Figure 11 shows the case $\rho = \phi$ that the Phillips curve is equally backward- and forward looking. The right column shows that the response function is now weighted more to the past than the future. In addition, negative coefficients are starting to show up, giving us some hope that higher interest rates can result in lower inflation at some point along the dynamic path.

The fourth and fifth rows of Figure 11 show cases in which the Phillips curve becomes more and more backward looking. The fourth row shows a forward weight reduced to $\phi = 0.7$, and the fifth row shows the purely backward looking case $\rho = 0$, $\phi = \beta$. Figure 10 shows that only for very large values of the backward looking coefficient ρ near $\rho = \beta$ do ϕ and the forward looking root λ_3 substantially decline. At that limit, both forward looking terms disappear, and the two complex backward looking roots remain. Figure 11 shows that the moving average solution becomes more and more weighted to past values, with larger sinusoidal movements.

Figure 12 presents the response of inflation, on the left, and output, on the right, to the standard step function interest rate path, for the same choices of forward and backward looking Phillips curve parameters as in the last two figures. The dashed line in each case is the unexpected case, verifying that once again expected and unexpected paths are the same for dates after the announcement.

Starting at the top of Figure 12, we have the purely forward looking case $\rho = 0,\ \phi = \beta$, and the same result as before. Inflation rises smoothly to meet the higher interest rate, and the Phillips curve produces a small output reduction on the whole path.

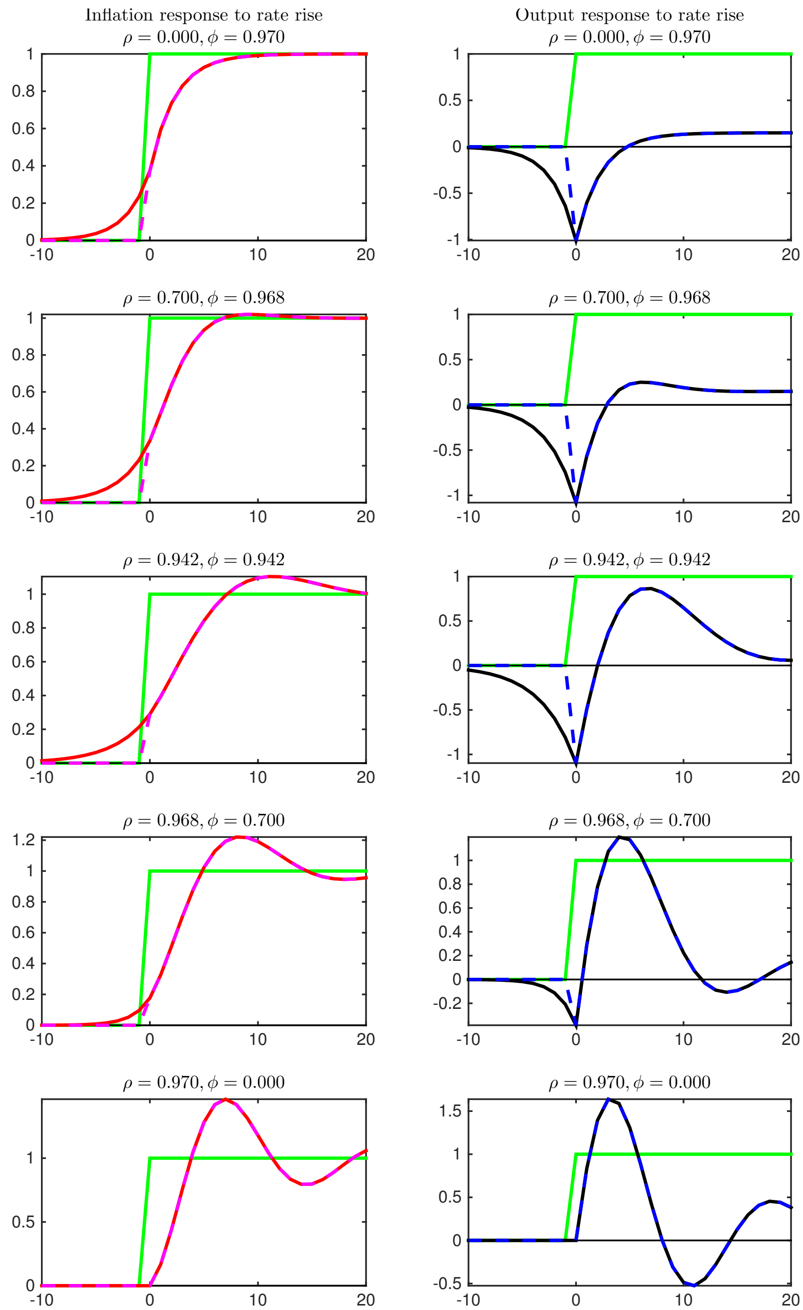


Figure 12: Response of inflation and output to a step-function rise in interest rates, with lagged inflation in the Phillips curve. Solid lines are the response to an expected change, dashed lines are the response to an unexpected change. The backward-looking Phillips parameter is ρ, and ϕ is the forward-looking parameter.

In the second row, a little bit of backward looking behavior $\rho = 0.7$ produces a plot that is almost visually indistinguishable. Inflation rises throughout, and output is still depressed until the long-run inflation-output tradeoff of this model takes hold. So, the basic result is robust to adding backward looking behavior.

As we go to more and more backward looking behavior in the remaining rows of Figure 12, inflation and output cease to respond ahead of the funds rate rise. Backward looking Phillips curves mean that forward guidance has less and less effect.

However, in none of the cases does a rise in interest rates provoke a decline in inflation. We can see the reason in the moving average coefficients of Figure 11. Though that figure does have some negative coefficients, in which past interest rates lower current inflation, the coefficients at low lags are always positive, and outweigh the negative coefficients further in the past. Integrating, we see the overshooting behavior in the bottom of Figure 12.

As the Phillips curve becomes more backward looking, the output decline with an interest rate rise weakens, and eventually becomes an output rise. While the forward looking Phillips curve gives higher output when inflation is declining, the backward looking Phillips curve gives higher output when inflation is rising. In this experiment inflation does rise, so output rises as well. Moving to a backward looking Phillips curve, we did turn around a sign: the "right" decline in output turned into a "wrong" rise in output, leaving the "wrong" rise in inflation alone.

Together, then, the backward looking Phillips curve and the neo-Fisherian behavior of inflation mean that in interest rate rise looks much like what is conventionally expected of a monetary *expansion*, not a contraction, plus some interesting slow sinusoidal dynamics.

Figure 13 adds both money and a purely backward looking Phillips curve. Compare this result to Figure 4 for money with a forward looking Phillips curve, and to the bottom row of Figure 12 for a backward looking Phillips curve without money.

Figure 13 produces something like the standard intuition, for inflation, at last. The unanticipated rate rise still does not interact with money at all, so it produces the same response for all values of money m/c. But the anticipated rate rise now benefits from

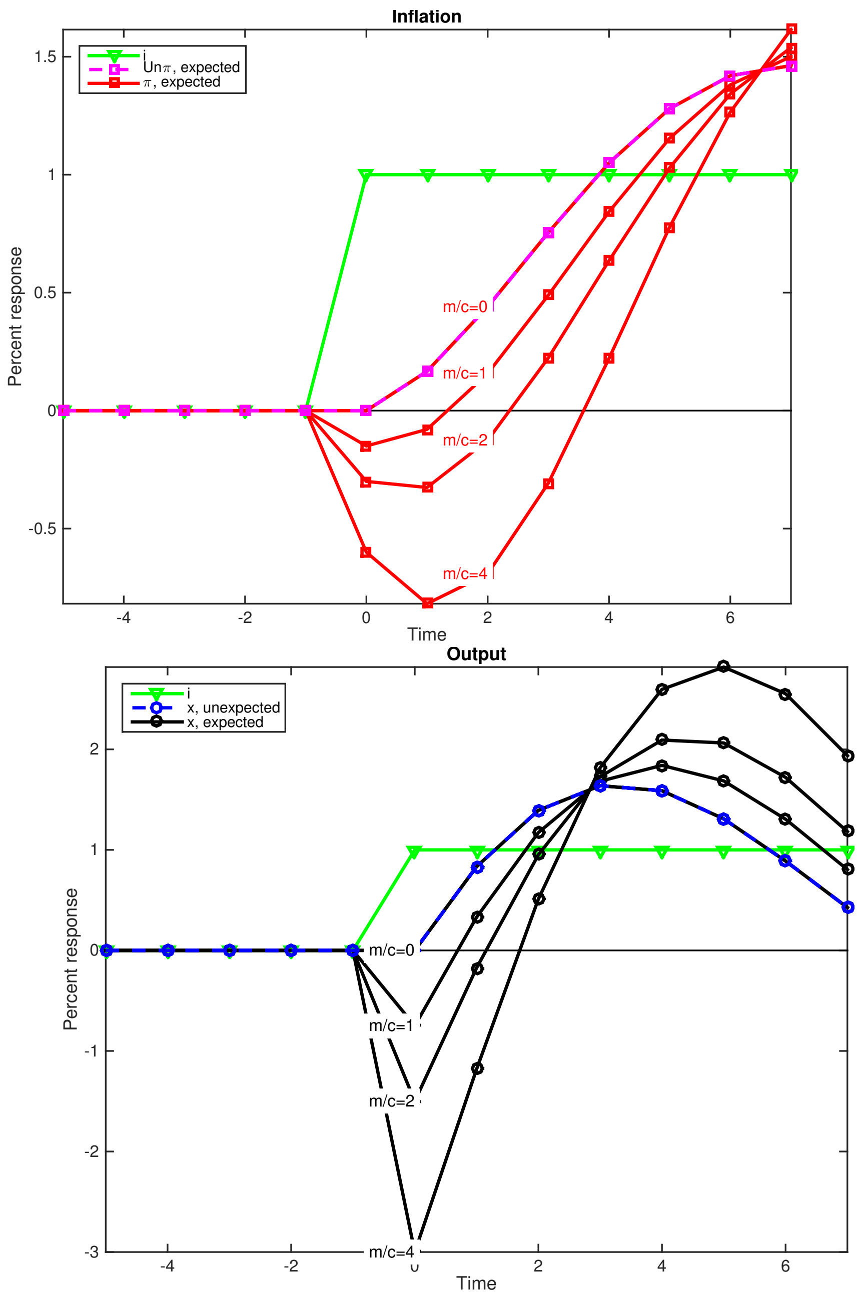


Figure 13: Response to expected and unexpected interest rate rise, with money and a purely backward looking Phillips curve $\rho = \beta$, $\phi = 0$.

the postponement of its response, from the backward looking Phillips curve, and also the reductions in near-term inflation from adding money. Now there is a temporary reduction in inflation before inflation rises to join the interest rate, for any positive value of money m/c.

Output now also falls, before rising with inflation. The backward looking Phillips curve generates low output when inflation is decreasing, and high output when inflation is increasing.

However, comparing the results to Figure 4 with a forward looking Phillips curve, the benefit is small. That figure already showed a temporary inflation decline, and no oscillating dynamics. The inflation decline is larger now, and smoother, but not dramatically different. We still need substantial monetary distortions $m/c > 1$ to obtain a quantitatively interesting response, or large $\sigma > 1$ (not shown).

The conventional sign of the short-run output response along with lower short-run inflation is perhaps the greatest benefit.

But the cost is throwing out all of the forward looking optimizing microfoundations of the forward looking Phillips curve. Anything much less that purely backward looking behavior (not shown) does not produce significant improvements.

Also, since unanticipated interest rate rises have no interaction with money, this modification does not help to match VAR evidence and intuition that focuses on unanticipated changes in interest rates.

Finally, the long-run responses still defy conventional intuition, losing the smooth decline present in the simple model of Figure 1. The disinflation and output cooling are borrowed from future inflation and an output boom.

5. Choosing equilibria

As usual in this class of models, there are multiple equilibria, indexed by the expectational shock $\{\delta_t\}$. Can one recover a short-run negative inflation response by other equilibrium choices? Yes. Are those choices sensible? I will argue, no.

A wide variety of equilibrium selection schemes are advocated for this class of models. To evaluate policy or compare models, rather than to formally test a specific model, given the state of the art, I think it is better to exhibit the range of possible equilibria and consider their plausibility in a variety of ways rather than to rigidly impose one or the other selection criterion.

5.1. Multiple equilibria

An impulse-response function studies the path of $E_t \pi_{t+j},\ j = 0, 1, 2, 3...$ to an announcement made at time t of a new interest rate path. Therefore, values of $\delta_{t-j},\ j > 0$ in the solutions (3)-(4) do not matter. They are the same pre- and post-announcement. Values of $\delta_{t+j},\ j > 0$ for the purposes of an impulse-response function are zero after the announcement, $E_t \delta_{t+j} = 0,\ j > 0$. Thus, the indeterminacy of equilibria comes down to the possibility of a single shock δ_t on the date t of the announcement.

Returning to the simple model of equations (1)-(2) and response function displayed for $\delta_0 = 0$ in Figure 1, Figure 14 plots a range of such multiple equilibrium responses to the unanticipated step function in interest rates. Each equilibrium is generated by a different choice of the expectational shock δ_0 that coincides with the monetary policy shock at date zero.

Equilibrium A has $\delta_0 = 1\%$. Equilibrium B chooses δ_0 to produce 1% inflation at time 0, $\pi_0 = 1\%$. Equilibrium C chooses δ_0 to have no fiscal consequences, as explained below. Equilibrium D chooses δ_0 to produce no inflation at time 0, $\pi_0 = 0$. Equilibrium E chooses $\delta_0 = -1\%$.

The figure shows graphically that the model may have too many equilibria, but all of them are stable, and all of them are Fisherian in the long run, with inflation converging to the higher nominal interest rate.

Equilibrium E verifies that the model can produce a temporary decline in inflation in response to the interest rate rise. Equilibrium E achieves that result by pairing a negative expectational or ex-post inflation shock with the positive interest rate or expected inflation shock.

The other possibilities are informative as well. In equilibrium B inflation jumps instantly to the full increase in nominal interest rates, and stays there throughout. Output

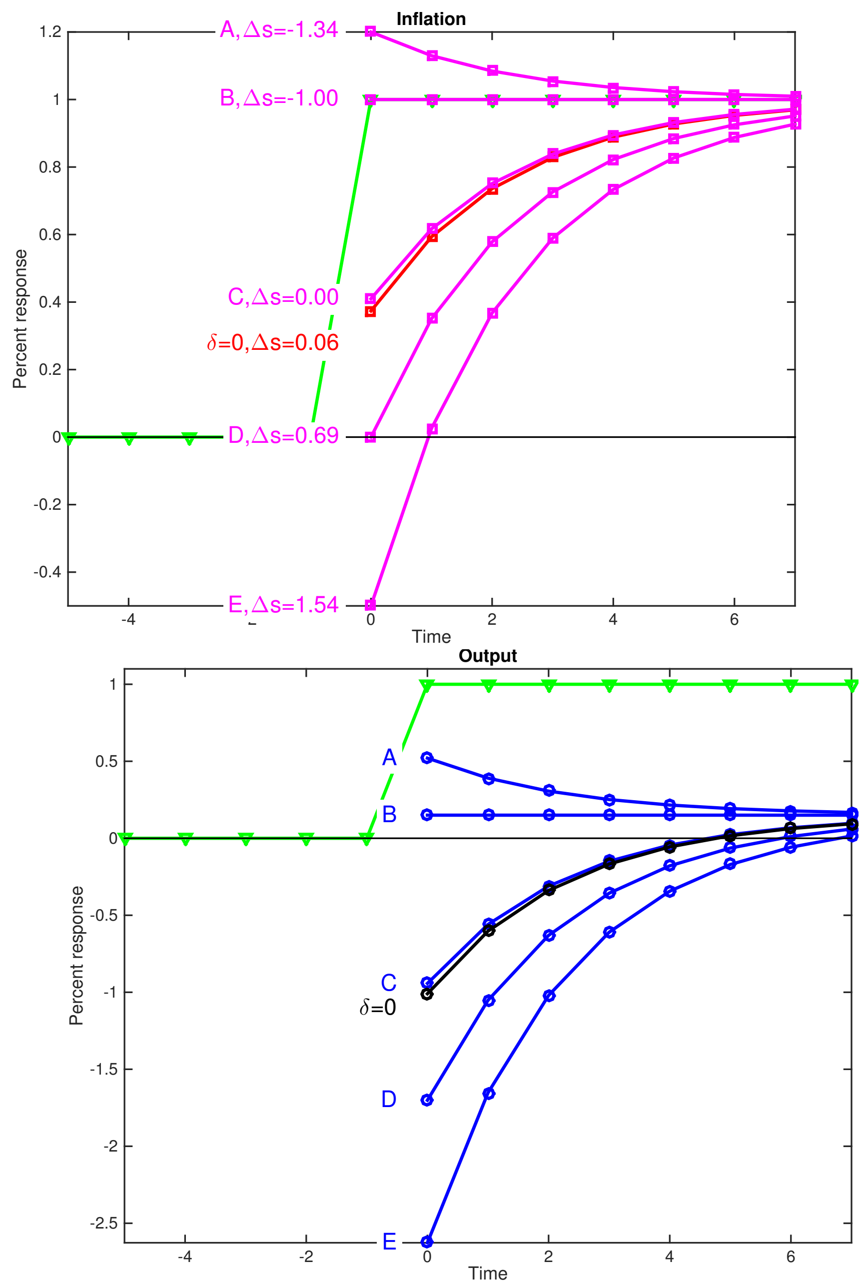


Figure 14: Multiple equilibrium responses to an unexpected interest rate rise. The solid green line gives the interest rate path. Letters identify different equilibria for discussion. The original case is $\delta_0 = 0$.

also jumps immediately to the steady-state value. Thus, despite price stickiness, the model can produce a super-neutral or super-Fisherian response, in which an interest rate rise instantly implies inflation with no output change!

Equilibrium A shows that even more inflation is possible. With a sufficiently large expectational shock, inflation can actually increase by more than the interest rate change, and then settle down, and output can increase as well.

Equilibrium D adds a small negative expectational shock δ_0, so that the initial inflation response is precisely zero. One may be troubled by inflation jumps, since inflation seems to have inertia in the data. It can be inertial in the model as well.

5.2. Choosing equilibria

The central question of this section is to ask whether there is a convincing argument to prefer equilibrium E, and to view this result as an embodiment of the conventional belief that raising interest rates temporarily lowers inflation.

The issue is not what shock δ_t we will see on a particular date. The question is what shock δ_t we will expect to see *on average* in response to announcements at date t of an interest rate rise–what shock to unexpected inflation should regularly and systematically accompany the announcement of an interest rate rise and consequent positive shock to expected inflation. It would be most convincing if there were a reason that raising interest rates at time t could be thought to *induce* a given shock δ_t.

So far, there isn't a strong argument for equilibrium E. So far, all the equilibria are possible, and there is no mechanism linking shocks to inflation to shocks to expected inflation. One could make an empirical argument that this is what we seem to see in the data. But the point of this paper is to find economics for an inflation decline, not to fit the most central prediction of monetary economics through a free parameter, the correlation of expected and unexpected inflation shocks.

So, let us add additional considerations to choose among the equilibria.

5.3. Anticipated movements and backwards stability

Most monetary policy tightenings are expected, and as above, this kind of model is well suited to studying expected monetary policy changes.

To display multiple equilibrium responses to an anticipated monetary policy change, Figure 15 pushes the announcement of the monetary policy change back to $t = -3$, three periods before the actual interest rate change at time $t = 0$. All responses except equilibrium C are the same as in Figure 14 for $t \geq 0$. Equilibrium C is recalculated to give zero fiscal effect at time $t = -3$ rather than $t = 0$.

Figure 15 cautions us on the apparent success of equilibrium E. That success relies crucially on matching the announcement of the interest rate change with the actual change. Figure 15 tells us that if a tightening is expected, as tightenings usually are, then inflation and output should both drop the most on the *announcement* of a tightening, not later when rates actually rise. The classic intuition might allow small announcement effects, but the bulk of inflation and output reactions should at least coincide with if not follow actual interest rate rises. Equilibrium D makes this behavior more apparent: Inflation jumps down on the announcement but (by construction) is zero on the day of the actual rate rise and positive thereafter.

This is an instance of a more general behavior. All the alternative equilibria explode backwards; they imply bigger inflation shocks on the day of the announcement than at $t = 0$. Equivalently, they have the property that news about events further in the future has larger effects today. The original $\delta_t = 0$ equilibrium choice has the "backwards-stable" property emphasized in Cochrane (2014c), that it does not explode backwards, and that news about events further in the future has less and less effect today.

5.4. Fiscal index

Each equilibrium choice has a fiscal policy consequence. Unexpected inflation devalues outstanding nominal debt, and thus lowers the long-run financing costs of the debt. Higher real interest rates raise financing costs. For each equilibrium, then, I calculate the percentage amount by which long-run real primary surpluses must rise or fall in that equilibrium. That number is presented alongside the initial inflation value of each

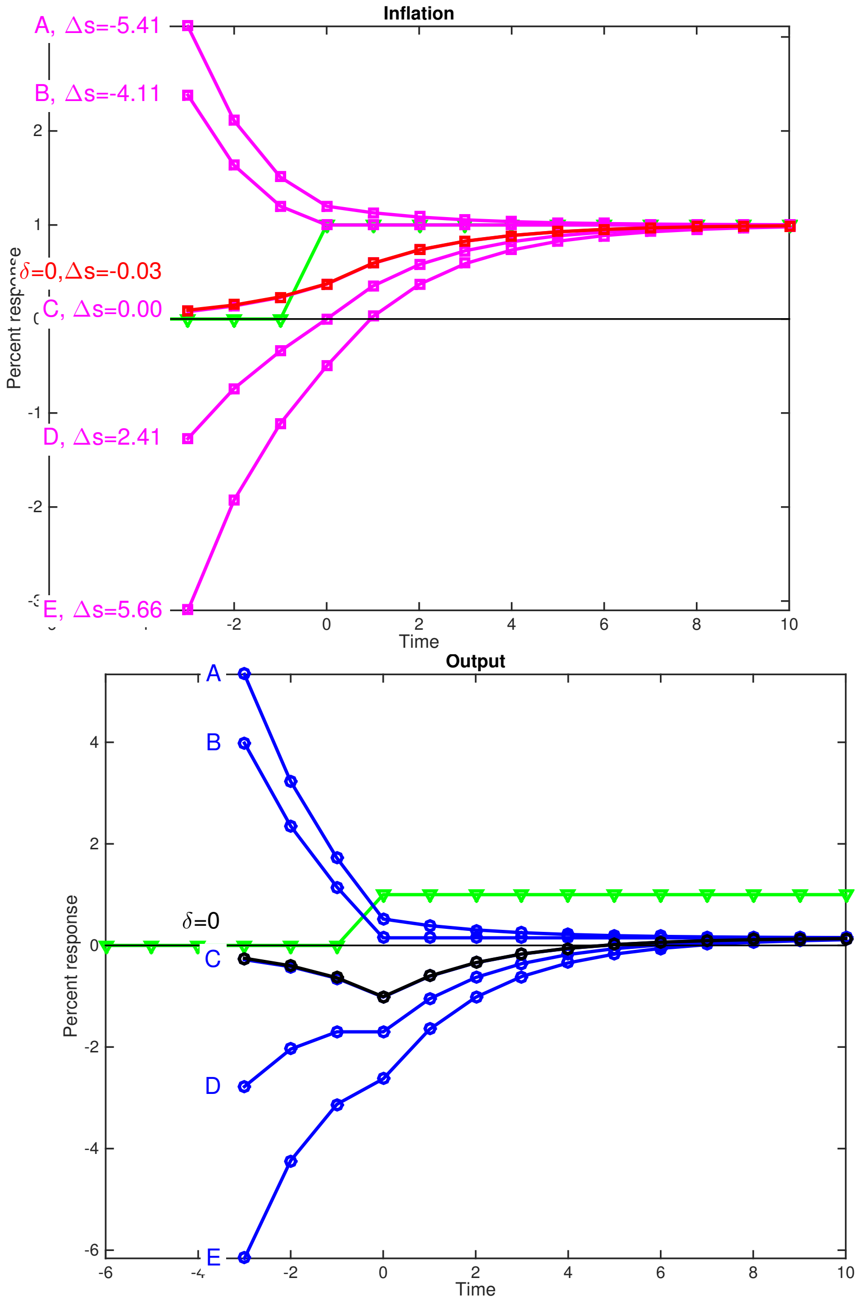


Figure 15: Multiple equilibrium responses to an anticipated interest rate change. The numbers $\Delta s =$ give the percent change in steady state surpluses required to achieve each equilibrium.

equilibrium in Figure 14 and Figure 15.

To make this calculation, I start with the valuation equation for government debt, which states that the real value of nominal debt must correspond to the present value of primary surpluses,

$$\frac{B_{t-1}}{P_t} = E_t\left[\sum_{j=0}^{\infty}\beta^j \frac{u'(C_{t+j})}{u'(C_t)} S_{t+j}\right], \tag{14}$$

where B_{t-1} denotes the face value of debt outstanding at the end of period $t-1$ and beginning of period t, P_t is the price level and S_t is the real net primary surplus. To keep the analysis simple, I specify one-period nominal debt.

Starting from a steady state with constant surplus S, I calculate the fractional permanent change in surplus Δs, i. e. $S_t = S^{\Delta s}$, that is required of the right hand side of expression (14) for each response function. Linearizing, I obtain in the appendix

$$\Delta s \approx -\Delta E_t\left(\pi_t\right) + \frac{1-\beta}{\sigma}\sum_{j=0}^{\infty}\beta^j \Delta E_t\left(x_{t+j} - x_t\right) \tag{15}$$

where $\Delta E_t \equiv E_t - E_{t-1}$ and t is the date of the announcement of a new policy.

The first term of (15) captures the fact that unexpected inflation devalues outstanding government debt. In the second term, $(x_{t+j} - x_t)/\sigma$ is the real interest rate between time t and time $t+j$. So this term captures the fact that if real rates rise, the government must pay more interest on the debt.

This calculation is simplified in many ways. First, the U.S. has considerable long-term debt. Long-term debt means that the US does not gain as much as one might think by a jump in the price level, and the Treasury is insured a bit against interest rate increases. Second, output changes directly affect primary surpluses, as taxes rise more than spending in booms and fall more than spending in recessions. Third, inflation also raises revenue due to a poorly indexed tax code. But some of these effects may represent a change in timing of surpluses – borrowing during recessions that is repaid later during booms – rather than permanent changes that affect the real value of government debt. A serious calculation of the fiscal impacts of monetary policy requires considerable detail on these lines. As with the rest of the model, the point here is not quantitative realism, but to capture some of the important effects and to show how one can use fiscal considerations to evaluate different equilibrium possibilities.

The super-neutral equilibrium B in which inflation rises instantly by 1%, also marked "$\Delta s = -1.00$" in Figure 14, implies a 1% decline in long-run surpluses. The 1% jump in inflation devalues outstanding nominal debt by 1%, and since output is constant after the shock there is no real interest rate change. Equilibrium A, with a larger inflation shock, has even larger fiscal implications, corresponding to a larger than 1% decline in long-run surpluses.

Further down in Figure 14, equilibrium D has no change in inflation at time 0, and so there is no devaluation of outstanding nominal debt. However, the rise in real interest rates means that the government incurs greater financing costs. These costs require a small permanent rise in surpluses.

In between, at equilibrium C, I find the shock δ_0 that requires no change in fiscal policy at all, so $\Delta s = 0$ by construction. Here, the devaluation effect of an inflation shock just matches the higher financing costs imposed by higher real interest rates. Cochrane (2014a) ignores real interest rate effects, and identifies an equilibrium that requires no change in fiscal policy $\Delta s = 0$ as the equilibrium with no inflation shock. Here, by allowing real interest rate effects, the equilibrium with no inflation shock D is not quite fiscally neutral. However, at least in this back of the envelope calibration, the difference is not large.

The original equilibrium with no expectational shock, $\delta_0 = 0$, implies a small but nonzero change in surpluses.

Equilibrium E, in which inflation temporarily declines half a percentage point after the interest rate shock, requires an 1.54% rise in permanent fiscal net-of-interest surpluses. Disinflation *raises* the value of nominal debt, which must be paid. And the rise in real interest rates also means fiscal surpluses must rise.

Turning to the anticipated shocks of Figure 15, the larger inflation shocks at time $t = -3$, and the longer periods of high real interest rates, mean that the fiscal changes required to support most of the equilibria increase as we move the announcement back in time. For example, the originally super-neutral equilibrium which required a 1% decline in surpluses in Figure 14 now requires a 4.11% decline, because of the larger inflation shock. And equilibrium E, selected to generate a 1% decline in inflation when interest rates rise 1%, now requires a 5.6% permanent rise in fiscal surpluses rather than 1.54%. The exceptions to this rule are the original equilibrium choice $\delta = 0$, and the equilibrium choice C or $\Delta s = 0$ with no fiscal impact.

5.5. Using fiscal policy to choose equilibria

To produce the standard view that raising interest rates lowers inflation, even temporarily, as in equilibrium E, we must accompany the rate rise with a fiscal tightening. Disinflation implies an unexpected present to holders of nominal government debt. Higher real interest rates also imply higher debt service payments. Fiscal authorities must be expected to raise taxes or to cut spending to make those payments.

An event such as equilibrium E is therefore a joint fiscal-monetary tightening. It provides useful guidance about what would happen if fiscal and monetary policy act together. Historic successful disinflation programs have typically combined monetary tightening with fiscal reform, which produces a rise in future (though often not current) surpluses. And unexpected monetary policy changes are often responses to unexpected economic or political conditions (getting sick of inflation), which trigger fiscal policy reactions as well.

So equilibrium E of Figure 14 – or its counterpart in the announced tightening of Figure 15, where inflation drops on the mere announcement of the policy change – might be useful when one wishes to match data that potentially combine a monetary and a fiscal policy shock.

But our question is to evaluate the hypothetical effects of monetary policy alone. For that question, and given the possibility of any of these equilibria, it is not compelling that we should pair the monetary policy shock (rise in interest rates) with a substantial fiscal policy shock.

Monetary policy has fiscal implications, and can thereby affect inflation. The $\Delta s = 0$ equilibrium C captures that fact: monetary policy raises real interest rates which impacts the budget. This equilibrium supposes just enough unexpected inflation to devalue outstanding debt and thereby offset the adverse effect of higher interest rates on the budget. That effect produces a slight reduction in inflation – the C line is below the $\Delta s = 0$ line in Figure 14. But that effect is not anywhere near large enough to produce a decline in inflation.

The fiscal calculation can serve as a formal equilibrium-selection mechanism. One may choose to pair the monetary policy shock with a particular fiscal shock, and then impose the fiscal theory of the price level to select one equilibrium as the globally unique equilibrium. The fiscal theory of the price level equilibrium-selection mechanism, and the definition that a "monetary policy shock" means one with no contemporaneous fiscal policy response $\Delta s = 0$, selects equilibrium C. It is very close to the original and backward-stable equilibrium $\delta_0 = 0$.

However, one can also keep an entirely passive-fiscal view. In that case, the fiscal index merely reveals how much fiscal policy must "passively" adjust to whichever equilibrium is selected by some other means. One can then decide if the required fiscal adjustment is reasonable or not. If criterion x selects a path that requires the "passive" fiscal authority to raise 200% of GDP in taxes, it's not going to happen. Fiscal policy *indexes* and *characterizes* equilibria even if it does not *select* equilibria.

In this vein, one might argue for equilibria that have limited or small fiscal requirements, rather than equilibria which require large changes in surpluses to be generated by the "passive" fiscal authorities, or insist on equilibria with exactly zero fiscal implications. That argument, which we might call fiscal theory lite, puts us in a range around the $\Delta s = 0$ equilibrium, and still limits our ability to produce disinflation.

Pushing the announcement date back as in Figure 15 enlarges these fiscal considerations. The equilibria that are not backward-stable all have larger and larger fiscal policy consequences as the announcement is pushed back. Conversely, the backward-stable $\delta = 0$ and no fiscal impact $\Delta s = 0$ equilibria converge as the announcement is pushed back. By an announcement $t = -3$ shown in Figure 15, the $\delta = 0$ equilibrium and the $\Delta s = 0$ equilibrium are already visually indistinguishable. The principle "pick the equilibrium with no fiscal impact" is a backward-stable equilibrium-selection procedure.

Equilibria with no jump in inflation are also attractive. Equilibrium D in Figure 14 has this property, and one can construct an equilibrium with no change in inflation upon announcement for the $t = -3$ shock of Figure 15. We do not see inflation jumps in the data, and new-Keynesian models are often specified so that inflation must be set one or more periods in advance.

The no inflation jump criterion results in a positive response of inflation once it moves, however, adding to that list.

In sum, the principles of small fiscal requirements, sensible behavior as announcements come earlier than actual rate changes, or limited jumps in inflation all push ones to the view that equilibria near the original $\delta = 0$ equilibrium are sensible, and the others less so.

The absence of the affirmative is more important here than the negative. We have not found a strong economic reason that we *should* pair large negative expectational, equilibrium-changing, or fiscal-policy-induced shocks δ_0 with announcements of interest rate rises, or that such shocks represent a natural interpretation of the economics by which interest rates affect the economy – rather than representing a hypothesized Fed equilibrium-selection policy or fiscal tightenings that historically coincide with monetary tightenings. This discussion finds no economic mechanism for producing a large unexpected inflation shock, except fiscal policy, which suggests no such shock. So, this discussion leads me to look away from paths such as equilibrium E as the device to generate a temporary decline in inflation when interest rates rise, and to look elsewhere.

6. Taylor rules

Taylor rules with active responses to inflation are usually invoked to prune equilibria, and to deliver a short-run negative inflation response. Can writing policy in terms of a Taylor rule help us to choose among the possible equilibria displayed in Figures 14 and 15? The conclusion of this section is, no.

6.1. Constructing Taylor rules

As mentioned above, the solution method using equations (3)-(4) does not assume a peg. We can *construct* a Taylor rule that supports any of the equilibria, as follows. Assume interest rate policy is

$$i_t = i_t^* + \phi_\pi(\pi_t - \pi_t^*) \tag{16}$$

where i_t^* is the step-function or other desired equilibrium interest rate path, π_t^* is the equilibrium path of inflation, i.e. one of the choices displayed in Figure 14 or Figure 15, and ϕ_π is arbitrary. If $\phi_\pi > 1$, then the desired path $\{i_t^*,\ \pi_t^*,\ x_t^*\}$ is the unique locally-bounded (nonexplosive) equilibrium.

Traditionally, one solves this kind of model by adding to (1)-(2) a monetary policy rule, say

$$i_t = \hat{i}_t + \phi_\pi \pi_t \tag{17}$$

and then solving for equilibrium $\{i_t,\ \pi_t,\ x_t\}$ given shocks including $\hat{i}_t$. To produce an impulse-response function, as I have, one must find a monetary policy disturbance sequence $\{\hat{i}_t\}$ that produces the desired response of equilibrium interest rates $\{i_t\}$. In general, the disturbance sequence $\{\hat{i}_t\}$ is different from the interest-rate response. For example, given the S-shaped pattern of $\{\pi_t\}$ in Figure 1, you can see quickly in (17) that a step-function $\{\hat{i}_t\}$ will not produce a step-function $\{i_t\}$.

Equations (16) and (17) are the same for

$$\hat{i}_t = i_t^* - \phi_\pi \pi^*. \tag{18}$$

In this context, then, my procedure – solving for output and inflation given the desired equilibrium interest rate path, and then constructing monetary policy that supports the desired equilibrium by (17), or by (18) – amounts simply to a way to avoid the unpleasant search for the monetary policy shock disturbance $\{\hat{i}_t\}$ that produces the desired equilibrium interest rate path. This clever approach and interpretation is due to Werning (2012).

Expressing the Taylor rule as in (16) emphasizes that the active Taylor rule includes two policy settings. The rule consists of an interest rate target, $\{i_t^*\}$, and an equilibrium-selection rule, the choice of ϕ_π and $\{\pi_t^*\}$ from the set of equilibrium $\{\pi_t\}$ consistent with the interest rate target. The interest rate target determines the path of equilibrium interest rates. The selection rule specifies a set of off-equilibrium threats or beliefs, that rules out all but the desired equilibrium path of inflation. (Many other equilibrium selection schemes achieve the same purpose, for example see **?** and the discussion in the online appendix to **?**.)

This construction (16) and its equivalence with (17) addresses the first question: Does the assumption of a Taylor rule solve the equilibrium selection problem? No. Via (16), all of the equilibria, any choice of δ_0 such as graphed in Figure 14 and Figure 15, are consistent with an active Taylor rule, and equation (16) shows how to construct the Taylor rule assumption that generates any desired equilibrium. The fact of adding a Taylor rule, by itself, doesn't help us at all to choose among equilibria.

6.2. Reasonable disturbances

Models based on disturbances to other equations sometimes use the restriction of no monetary policy disturbance at all, $i_t = \phi_\pi \pi_t$ to generate unique solutions. However, to generate responses to changes in monetary policy, we need some monetary policy disturbance.

Perhaps, however, translating from (16) $i_t = i_t^* + \phi_\pi(\pi_t - \pi_t^*)$ to (17) $i_t = \hat{i}_t + \phi_\pi \pi_t$ via (18) $\hat{i}_t = i_t^* - \phi_\pi \pi^*$ will indicate that one or another equilibrium results from a more sensible monetary policy disturbance.

Figure 16 gives the monetary policy disturbance $\hat{i}_t$ in a Taylor rule $i_t = \hat{i}_t + 1.5\pi_t$ needed to produce the step function rise in equilibrium interest rates and each of the possible inflation outcomes from Figures 14 (unexpected policy, solid) and 15 (expected policy, dashed).

The equilibria with large positive inflation shocks such as A result from negative monetary policy disturbances, and vice versa. To produce a larger change in inflation π_t with the same equilibrium interest rate i_t via $i_t = \hat{i}_t + \phi_\pi \pi_t$, you need a smaller and eventually negative $\hat{i}_t$, and vice versa.

All the disturbances end up at $\hat{i}_t = -0.5$, since they all end up with $i_t = 1$ and $\pi_t = 1$,

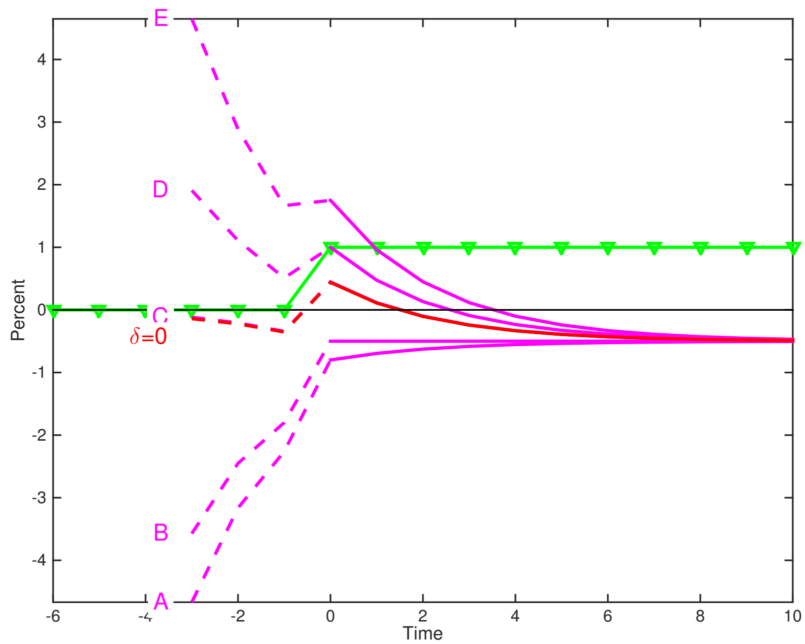


Figure 16: Monetary policy disturbance $\hat{i}_t = i_t^* - \phi_\pi \pi_t^*$ that produces each equilibrium with Taylor parameter $\phi_\pi = 1.5$. Dashed lines give values for the announcement at time $t = -3$. Solid lines give values for the announcement at the same time as the rate rise at time $t = 0$. Letters and $\delta = 0$ correspond to the equilibria shown in previous figures.

and $1.0 = -0.5 + 1.5 \times 1.0$.

To produce the baseline $\delta_0 = 0$ inflation pattern in the unanticipated (solid) case, the disturbance $\{\hat{i}_t\}$ follows a pattern with geometric decay. This pattern mirrors the geometric rise of inflation, relative to the step function rise in observed interest rates. This looks pretty reasonable, but it does not produce the desired decline in inflation. To produce an inflation decline, one needs the larger disturbance of equilibrium E. That disturbance also melts away in geometric pattern. Comparing the two $\{\hat{i}_t\}$ it's hard to say one is a lot more reasonable than the other. One may object that the $\delta = 0$ disturbance crosses the zero line, being first positive and then negative. But the E disturbance does this as well.

The dashed lines, showing the monetary policy disturbances necessary to produce the responses to an *anticipated* rise in equilibrium interest rates are wilder. Viewed through the lens of a Taylor rule, the Fed does not simply announce that rates will rise in the future – the Fed must take strong action. The Fed announces a monetary policy shock $\{\hat{i}_t\}$. Inflation moves so much, however, that the systematic component of monetary policy $\phi_\pi \pi_t$ exactly offsets the monetary policy shock $\{\hat{i}_t\}$, producing a change in inflation with no change at all in the actual interest rate.

For the anticipated rate rise (dash), there is no equilibrium in which $\hat{i}_t$ does not move ahead of the actual interest rate rise, as there is no equilibrium in which inflation does not move ahead of an anticipated interest rate rise. But the baseline equilibrium $\delta = 0$ and the equilibrium C with no fiscal consequence $\Delta s = 0$ at least have disturbances $\hat{i}_t$ that are small and that decline as the policy announcement moves back in time. By contrast, the disturbance E grows as the announcement time moves back.

In sum, if one pursues "reasonable" specifications for the monetary policy disturbance, in the context of a Taylor rule of the form $i_t = \hat{i}_t + \phi_\pi \pi_t$, as an equilibrium selection device, that path does not strongly suggest equilibria such as D and E in which inflation declines temporarily. In fact, the view that the Fed makes big monetary policy shocks that induce big changes in inflation, which through the systematic component of policy $\phi_\pi \pi_t$ then just offset the monetary policy shock and produce no change in interest rate, may seem the more far-fetched assumption.

6.3. Open-mouth policy

Suppose this is the answer; suppose that the Fed follows an equilibrium-selection policy implemented by an active Taylor rule $\phi_\pi > 1$, and we observe a short-run negative response of inflation to interest rates because the Fed induces a large negative jump δ_0 along with the news about interest rates.

If this is the case, there is no reason for the Fed to bother with interest-rate policy. There is no reason to pair the equilibrium-selection policy that gives rise to a shift in unexpected inflation with a change in interest rates. If the Fed wants to induce temporarily lower inflation, all it need to is to announce a new inflation target.

To be specific, suppose the Fed follows a Taylor rule

$$i_t = i_t^* + \phi_\pi(\pi_t - \pi_t^*). \tag{19}$$

Suppose the Fed, starting at $i_t^* = 0,\ \pi_t^* = 0$ for $t < 0$, leaves i_t^* alone, but shocks monetary policy for $t \geq 0$ to

$$\pi_t^* = \delta_0 \lambda_1^{-t}. \tag{20}$$

Here, δ_0 is just a constant indexing how large the monetary policy shock will be. This is a pure, temporary, change in the Fed's inflation target.

Equivalently, suppose the Fed follows a Taylor rule

$$i_t = \hat{i}_t + \phi_\pi \pi_t. \tag{21}$$

Suppose that the Fed, starting at $\hat{i}_t = 0$ for $t < 0$, shocks monetary policy for $t \geq 0$ to

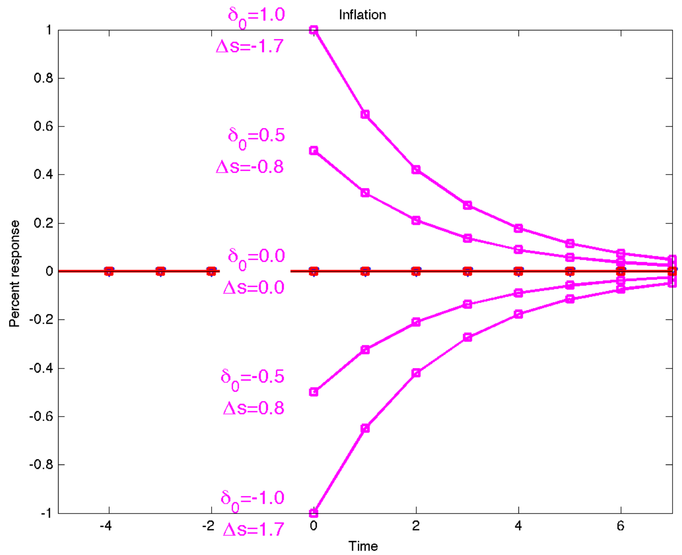
$$\hat{i}_t = -\delta_0 \phi_\pi \lambda_1^{-t}. \tag{22}$$

This is a pure, temporary, monetary policy disturbance.

Figure 17 plots the responses of inflation and output to these monetary policy disturbances. Inflation and output move, but interest rates are constant throughout the episode.

Intuitively, in response to a shock $\hat{i}_0$, and its expected subsequent values $\{\hat{i}_t\}$, inflation jumps down just enough so that the systematic component of policy in (21) exactly offsets the shock, and the actual interest rate i_t does not change at all. In response to the shock π_0^*, and to the expected subsequent values $\{\pi_t^*\}$, inflation jumps to $\pi_0 = \pi_0^*$. Via the Taylor rule (19), this change in inflation is is just enough so that actual interest rates do not change.

All the Fed has to do is to announce a new inflation target π_t^*, or announce the equivalent monetary policy disturbance $\hat{i}_t = -\phi_\pi \pi_t^*$, and the inflation arises with no change at all in interest rates. The monetary policy shock δ_0 induces an expectational shock δ_0. Monetary policy, divorced from money in this standard model, can also be divorced from interest rates! One might call this a pure equilibrium-selection policy shock.



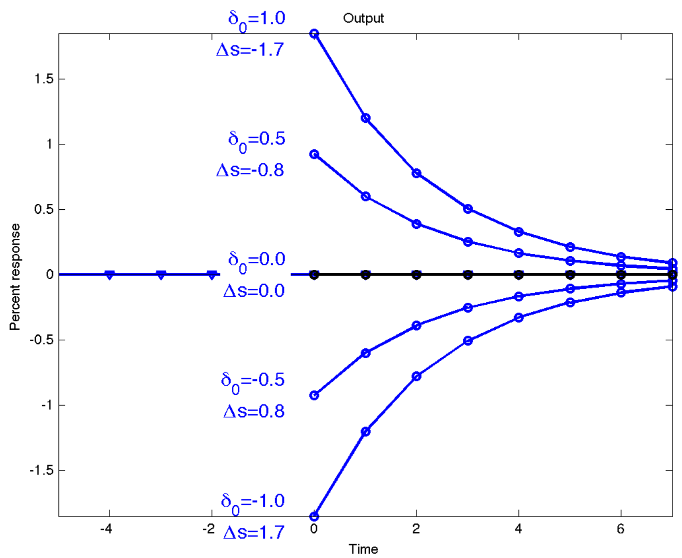


Figure 17: Response of inflation and output to a shift in inflation target with no shift in interest rate target.

The paths graphed in Figure 17 are given by

$$\pi_t = \delta_0 \lambda_1^{-t}, \tag{23}$$

$$\kappa x_t = \delta_0 (1 - \beta \lambda_1^{-1}) \lambda_1^{-t} \tag{24}$$

for $t \geq 0$. These are the last, expectational shock, terms of the solutions (3)-(4). Conjecture these solutions. Then, since $\pi_t = \pi_t^*$ in (19), interest rates do not move at all, confirming the conjecture.

One can restate and solve this example in standard form rather than use (3)-(4). Suppose then that we solve the model (1)-(2) plus Taylor rules (19) or (21) and monetary policy shock processes (20) or (22) for the unique locally bounded paths of inflation, output and interest rates, by any of the usual methods. Figure 17 is the result of that exercise. The open-mouth operation result that interest rates do not move at all is a special case that the persistence of the standard monetary policy disturbance is just equal to the system eigenvalue λ_1^{-1} defined in (5).

Deriving this standard solution takes a bit of algebra. But one can verify this claim easily. The inflation and output paths of Figure 17 are, for $t \geq 0$, given by (**??**)-(23). Plug these values, along with the interest rate rules (19) or (21), into the model (1)-(2) and you will verify the latter hold so long as λ_1 is given by (5).

Once we see how pure open-mouth / equilibrium-selection operations can induce the shocks δ_0, pairing shocks to inflation as in Figure 17 with shocks to interest rates is arbitrary. The Fed could just as easily pair the negative inflation shock with a change in flags flying on top of the Board of Governors building. One could imagine some sort of game-theoretic or signaling equilibrium in which the important action – a change in the inflation target – is tied to an otherwise unimportant action – announced changes in interest rates. But it's not clear why we or the Fed would pair the actions in exactly this way.

Indeed, pairing a negative $\delta_0 < 0$ shock with a rise in interest rates is counterproductive: the rise in interest rates on its own raises inflation, posing a needless headwind to the desired temporary inflation decline. When the Fed wants to reduce inflation, it usually wants to do so in the short and long run. Then, it should pair a $\delta < 0$ shock lowering short-run or ex-post inflation with an interest rate *decrease* to lower long-run or expected inflation.

Now, perhaps this is our world. Monetary policy at the zero bound has seemed to evolve into central banker statements accompanied by no actual changes in interest rates or asset purchases.[1] Perhaps inflation really has little to do with economics; supply and demand, intertemporal substitution, money, and so forth. Perhaps inflation really is predominantly a multiple-equilibrium question. Perhaps "monetary" policy affects inflation entirely by government officials making statements, with implicit unobserved off-equilibrium threats, that cause jumps from one equilibrium to another. Perhaps the analysis of monetary policy should go back where it left off in the 1950s and 1960s, in which inflation was largely thought to comprise "wage-price spirals," and inflation policy centered on talk not action.

If so, though, the quest of this paper – an economic model of the effect of interest rates on inflation – is moot. So, in pursuit of that quest, I must look for other ways to think about picking equilibria.

Figure 17 includes the change in long-run surpluses needed to validate each equilibrium. I include this number as a reminder that it is there. If one takes the fiscal-passive view, these are the resources that the "passive" Treasury will need to come up with to validate the Fed's "active" equilibrium-selection policy.

If one takes the fiscal theory of the price level view, these calculations have a much simpler interpretation. In this case, the indicated change in expected future surpluses results in one of these equilibria as the unique equilibrium. These are movements in inflation achievable by a pure change in *fiscal* policy, when monetary policy leaves nominal interest rates unchanged. In the fiscal theory of the price level, the Fed still sets *expected* inflation freely by setting nominal interest rates, while fiscal policy uniquely chooses *unexpected* inflation, and hence δ_0. (Cochrane (2014a) explains this division in detail.)

[1]The original "open-mouth" operation, as described by Reserve Bank of New Zealand Governor Donald Brash (Brash (2002)) is the observation that he seemed to be able to move interest rates by simply talking, without conducting open market operations. This open mouth operation is doubly removed from action, since the central bank can apparently move inflation without moving interest rates.

6.4. Avoiding explosion-inducing Taylor rules

In addition, Cochrane (2011) criticizes forward looking models with the Taylor principle $\phi_\pi > 1$. The main point: such models presume that the Fed induces instability in an otherwise stable economy, a non-credible off-equilibrium threat to hyperinflate the economy for all but one chosen equilibrium. The quest of this paper is also to respond constructively to that critique, again motivating a look past active Taylor rules for equilibrium selection.

One may object that regressions such as Clarida, Galí, and Gertler (2000) of interest rates on inflation find coefficients greater than one, at least in some subsamples. But such regressions are irrelevant to the point. We only see $\pi_t = \pi_t^*$ in equilibrium, so data from an equilibrium cannot tell us what happens in response to deviations of π_t from π_t^*. We cannot measure the response of interest rates to an off-equilibrium inflation from data that represents an equilibrium. Equivalently, we cannot tell whether monetary policy responds to inflation itself or whether it responds to the structural disturbances that produce inflation. If $\pi_t^* = \nu_t$, and $i_t^* = 1.5 \times \nu_t$, we will observe $i_t^* = 1.5 \times \pi_t^*$, for any value of ϕ_π in $i_t = i_t^* + \phi_\pi(\pi_t - \pi_t^*)$, as all samples have $i_t = i_t^*,\ \pi_t = \pi_t^*$.

Let us then think of monetary policy as a time-and state-varying peg. It responds to shocks in the economy, including the shocks that produce inflation. Woodford (2003) advocates such "Wicksellian" policy for the disturbance $\hat{i}_t$. We will just leave out the destabilizing response to off-equilibrium inflation; the idea that the Fed follows an "equilibrium-selection" policy. The argument can quickly be generalized to think of "passive" Taylor rules with $\phi_\pi < 1$ that do respond to inflation itself, but not enough to de-stabilize the economy.

Analyzing this regime is of practical importance as well. Even if one regards the Fed as having followed an active Taylor rule in the past, the Fed may well follow a passive policy in the future. The Fed followed a pure peg the 1940s and early 1950s, and the Fed is conventionally (Clarida, Galí, and Gertler (2000)) thought to have followed $\phi_\pi < 1$ in the late 1960s and 1970s as well. The ongoing zero bound represents at least a local peg, and policy may be inertial to the point of being indistinguishable from a peg in the future. We need to be able to say more than "indeterminate"about such episodes.

If we follow this course, all the equilibria, or values of δ_0 accompanying news of an interest rate change, remain viable. One can add backward-stability, fiscal theory of the price level, or no inflation jumps to select equilibria, as I have suggested, or other considerations. The long discussion of multiple equilibria here suggests that, at least for the narrow question of the response of inflation and output to interest rate changes, the exact equilibrium-selection rule is not that important. Just about any sensible criterion leads to small jumps δ_0 accompanying monetary policy changes, on average.

But if we take that course, we are still stuck with the model's prediction that rises in nominal interest rates raise inflation in both short run and long run.

7. Old-Keynesian Models

Old-Keynesian models give the standard intuition: A rise in interest rates lowers inflation at all horizons, and interest rate pegs are unstable. Old-Keynesian models are determinate – there is one equilibrium. But as a result of their inherent instability, the Fed must actively move interest rates to counteract shocks to inflation. The Fed's interest rates must be the jump variable to offset explosions, as consumption – no longer forward looking – or expectations of other state variables – no longer rational – can no longer serve that role. A Taylor rule serves to make the economy *stable*, where in new-Keynesian models the same Taylor rule serves to make the economy *unstable* and hence determinate.

A simple version of such a model, inspired by Taylor (1999), who wanted to display a model with these standard features, gives a clear example of a model in the traditional style that produces the standard results. In place of (1)-(2), write

$$x_t = -\sigma(i_t - \pi_t) + u_{xt} \tag{25}$$

$$\pi_t = \pi_{t-1} + \kappa x_t + u_{\pi t}. \tag{26}$$

(Taylor (1999) has x_{t-1} in the Phillips curve, which makes no difference for the points here, so I simplify by using x_t instead.)

The solution for inflation given the interest rate path is

$$\pi_t = \frac{1}{1-\sigma\kappa}\pi_{t-1} - \frac{\sigma\kappa}{1-\sigma\kappa}i_t + \frac{1}{1-\sigma\kappa}\left(\kappa u_t^x + u_t^\pi\right). \tag{27}$$

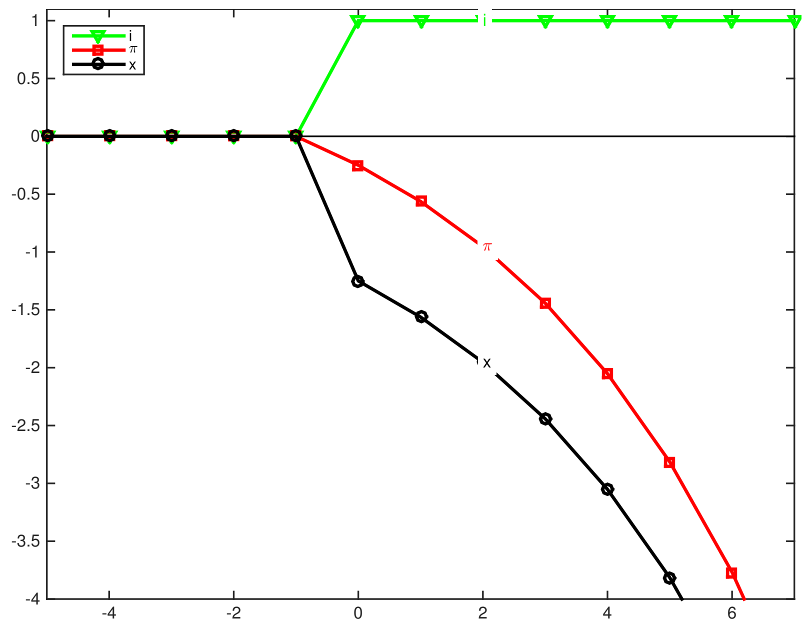


Figure 18: Response of inflation to an interest rate rise, old-Keynesian model.

Figure 18 presents the inflation response to the step-function interest rate rise, starting from a steady state, in this model. The response has the standard negative sign in both the short and the long run. The inflation response is

$$\pi_t = 1 - \frac{1}{(1-\sigma\kappa)^{t+1}}. \tag{28}$$

The steady state is occurs where $\pi = i - r$ (I have simplified to $r = 0$) as before. But now the steady state is unstable. If the Fed were to try a peg, after inflation or deflation spirals away, the Fed would have to move interest rates a great deal to stop the spiral. The model does not give a description of how a deflation spiral might stop at the zero bound. For this reason, this model is not consistent with the stability of inflation at the zero bound observed in the U.S., Europe, and Japan.

7.1. Taylor rules in old-Keynesian models

To produce stable inflation in this model, the Fed must actively move interest rates. One way to accomplish this is with a Taylor rule,

$$i_t = \phi_\pi \pi_t + u_t^i. \tag{29}$$

Adding that specification to (25)-(26), inflation follows

$$\pi_t = \frac{1}{1+\kappa\sigma(\phi_\pi - 1)}\pi_{t-1} + \frac{1}{1+\kappa\sigma(\phi_\pi - 1)}(\kappa u_t^x + u_t^\pi - \sigma\kappa u_t^i)$$

The Taylor rule $\phi_\pi > 1$ induces *stability* in an otherwise unstable but already determinate model. This is a central point of Taylor (1999). In the forward looking model (1)-(2), the Fed induces local *determinacy* by making an otherwise stable model unstable.

Adding such a Taylor rule does not help our quest, however. As with the new-Keynesian models, Figure 18 describes the dynamics of inflation given an equilibrium path, no matter what policy produces that interest rate path. In this case, there aren't even multiple equilibria to play with. To produce a step function interest rate path via a Taylor rule (29), the monetary policy disturbance must be explosive

$$u_t^i = 1 - \phi\left(1 - \frac{1}{(1-\sigma\kappa)^{t+1}}\right)$$

A tightening in this model cannot take the form of a higher and steady interest rate, because that path for rates must lead to unstable inflation. The Fed must raise rates, but then quickly lower then to head off inflation.

Figure 19 plots a more sensible exercise in this context – a step function rise in the monetary policy disturbance u_t^i, or equivalently a step function decline in the inflation target $\pi^T t$, if we write $i_t = \phi_\pi(\pi_t - \pi_t^T)$.

Here, the tightening sends inflation uniformly down. But after a quick rise, interest rates quickly fall to forestall the deflation which would otherwise spiral. So even this model has a long-run Fisherian implication.

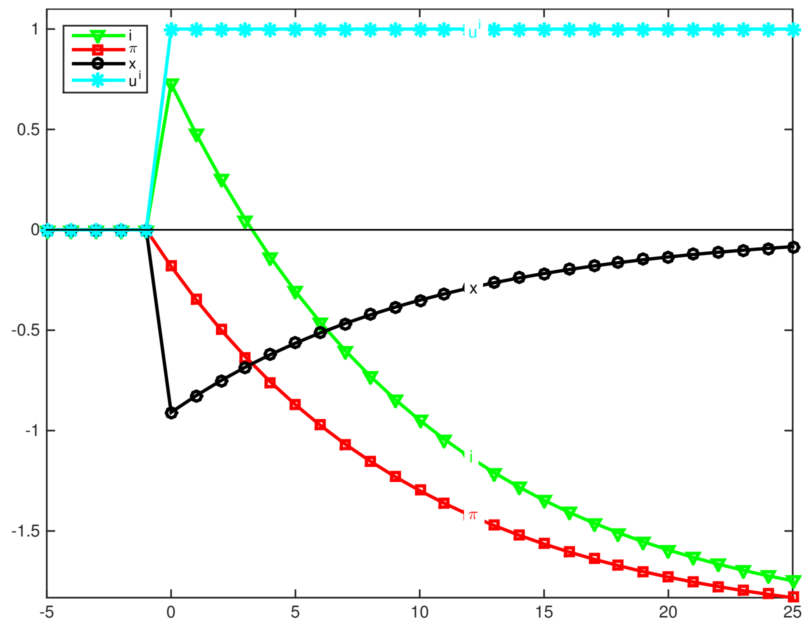


Figure 19: Response to a permanent tightening u_t^i in the Taylor rule $i_t = \phi_\pi \pi_t + u_t^i$, applied to the old-Keynesian model

But the model is simply inconsistent with an interest rate path following a shock that has only one sign. Even a transitory shock (not shown) must decline below zero after first rising, to keep inflation from spiraling away. That is again a reflection of the underlying instability, belied by our time at the zero bound.

7.2. Old-Keynesian models?

Three crucial differences make this model "old-Keynesian:" First, there is no $E_t x_{t+1}$ in the IS equation (25) as there is in (1), so this equation does not rely at all on intertemporal substitution to generate lower current output with a higher real interest rate. The permanent income hypothesis is absent here, and pure in (1). Second, current rather than expected future inflation appears in the IS equation (25) to translate nominal interest rates into real interest rates. One might think of this as adaptive expectations. Third, past inflation π_{t-1} appears in the Phillips curve rather than expected current inflation $E_{t-1}\pi_t$ in the rational-expectations tradition (Lucas (1972)) or expected future inflation $\beta E_t \pi_{t+1}$ in the new-Keynesian tradition as in (2).

One might also call the model "old-Monetarist," as Friedman (1968) predicts the same instability under an interest rate peg. He writes (p.5) that Monetary policy "cannot peg interest rates for more than very limited periods..." Friedman's prediction also comes from adaptive expectations: (p. 5-6): "Let the higher rate of monetary growth produce rising prices, and let the public come to expect that prices will continue to rise. Borrowers will then be willing to pay and lenders will then demand higher interest rates-as Irving Fisher pointed out decades ago. This price expectation effect is slow to develop and also slow to disappear."

Adaptive, backward-looking expectations make price dynamics unstable, like driving a car by looking in the rear-view mirror. Rational, forward-looking expectations make price dynamics stable as when drivers look forward and veer back on the road without outside help.

The absence of expectational terms in (25)-(26) is, deeply, the cause of its instability. The original (1)-(2) also have an unstable root, which we solved forward. Why not do the same for the difference equation (27), i.e. writing it as

$$\pi_{t-1} = (1-\sigma\kappa)\,\pi_t + i_t - (\kappa u_t^x + u_t^\pi) \tag{30}$$

and hence

$$\pi_{t-1} = \sum_{j=0}^{\infty} (1-\sigma\kappa)^j\, i_{t+j} - \left(\kappa u_{t+j}^x + u_{t+j}^\pi\right)? \tag{31}$$

By writing it out, the answer is clear. There are no expectations in (30) or (31), so ex-post values would have to go on the right hand side. Deeply, in a properly formed model such as (25), consumption jumps to adjust to the present value of future income, so wealth does not explode. With no expectations, there is no variable in (25)-(26) that can jump to offset explosions.

With all these departures, then, the model also fails the quest for a simple modern and *economic* model that generates the standard sign.

8. Other modeling directions

We have searched for a simple modern economic model, consistent with the observed stability of inflation at the zero bound, that restores the traditional view in which a rise in interest rates produces at least a temporary decline in inflation. The result is, so far, negative. Price stickiness, money, backward looking Phillips curves, alternative equilibrium choices and active Taylor rules do not provide a convincing basis to overturn the short-run Fisherian predictions of the frictionless model. They do not begin to overturn its long-run Fisherian prediction.

The next directions one might go to reestablish the conventional view all involve abandoning one of the qualifiers simple, modern, or economic.

In order to produce the standard signs, my brief excursion into ad-hoc old-Keynesian modeling suggests that one might add ingredients to micro-found that model's basic structure, to restore the "modern" and "economic" adjectives, while also somehow repairing the model's prediction that the zero bound is unstable. Since the basic problem is in the forward looking and intertemporal IS equation, one might add extensive borrowing or collateral constraints, hand-to-mouth consumers, irrational expectations or other irrational behavior (Gabaix (2015) is a concrete example; severely downweighting $E_t x_{t+1}$ in (1), one can produce traditional explosive dynamics), a lending channel, or other frictions, continuing the 60-year old quest to undermine the permanent income hypothesis. It is also possible that alternative models of pricing frictions will do the trick, though my excursion to a simple ad-hoc backward-looking Phillips curve failed.

Similarly, the models in this paper are quite simple by the standards of calibrated or estimated new-Keynesian models, such as Smets and Wouters (2003), Christiano, Eichenbaum, and Evans (2005) and their descendants. Perhaps adding the range of complexity in such models – habits, labor/leisure, production, capital, variable capital utilization, adjustment costs, or informational, market, payments, monetary, and other frictions – will perturb dynamics in the right way.

But following this path abandons the qualifier "simple." Doing so admits that the standard view rests squarely on the complexity of much more complex models. It admits that there is no economic model one can put on a blackboard, teach to undergraduates, summarize in a few paragraphs, or refer to in policy discussions to explain at least the signs and rough outlines of the standard view. It admits that nothing like, say, the stirring and simple description of monetary policy in Friedman (1968) can emerge.

Such an outcome would be unusual in macroeconomics. The standard new Keynesian approach views the complex models as refinements, building on (1)-(2) plus a Taylor rule, which help to match the details of model dynamics with those observed in the data, but views the simple model as capturing the basic message, signs, and intuition. The standard real business cycle approach views complex models as refinements, building on the stochastic growth model, but that simple model can still capture the basic story. The large multi-equation Keynesian models developed in the 1970s built on simple ISLM models to better match details of the data, but modelers felt that the simple ISLM model captured the basic signs and mechanisms.

This was all healthy. Economic models are quantitative parables, and one rightly distrusts predictions that crucially rely on the specific form of frictions, especially frictions that have little microeconomic validation.

So, if we go down the route of much greater complexity, and especially of adding many frictions unrelated to money, we have already admitted an important defeat.

García-Schmidt and Woodford (2015) address the Fisherian paradox differently, by fundamentally changing the nature of equilibrium. Here, I have stuck to the standard concepts of rational expectations or perfect foresight. Similarly, if one has to fundamentally change the nature of economic equilibrium, that means the search in this paper is over.

9. Evidence and VARs

If theory and experience point to a positive reaction of inflation to interest rates, perhaps we should revisit the empirical evidence behind the standard view to the contrary.

The main evidence we have for the effects of monetary policy comes from vector autoregressions (VARs). There are several problems with this evidence.

First, the VAR literature almost always pairs the announcement of a new policy with the change in the policy instrument, i.e. an unexpected shock to interest rates. That habit makes most sense in the context of models following Lucas (1972) in which only unanticipated monetary policy has real effects, and in the context of regressions of output on money in which VARs developed (Sims (1980)).

But in the model presented here, anticipated monetary policy has strong effects, and most monetary policy changes are widely anticipated. VARs may still want to find unexpected announcements, as part of an identification strategy to find changes in policy that are not driven by changes of the Fed's expectations of future output and inflation. But there is no reason, either theoretical, empirical, or for policy relevance, to focus so much on the few and small events in which the announcement surprise and the rate change happen simultaneously.

More deeply, the models with money presented here only had a chance of delivering the standard decline of inflation in response to raising interest rates if the interest rate rise was anticipated. An empirical technique that isolates unexpected interest rate rises cannot find or verify that theoretical prediction.

Second, the analysis of multiple equilibria in Figure 14 and Figure 15 found that inflation declines occur when an interest rate rise is paired with a fiscal policy tightening. It is plausible that inflation-fighting programs consist of such joint fiscal-monetary contractions. But the question is to find the response of a pure monetary policy change. VARs have to date made no attempt to orthogonalize monetary policy shocks with respect to fiscal policy, especially expected future fiscal policy which is what matters here. So one may well take the view that the VARs measure a joint fiscal-monetary policy shock that does not capture the effects of a pure monetary policy shock.

Third, VARs almost always find that the interest rate responses to an interest rate shock are transitory, as are those of Figure 2. As a result, they provide no evidence on the long-run response of inflation to permanent interest rate increases.

Most of all, the evidence for a negative sign is not strong, and one can read much of the evidence as supporting a positive sign. From the beginning, VARs have produced *increases* in inflation following increases in interest rates, a phenomenon dubbed the "price puzzle" by Eichenbaum (1992). A great deal of effort has been devoted to modifying the specification of VARs so that they can produce the desired result, that a rise in interest rates lowers inflation.

Sims (1992), studying VARs in 5 countries notes that "the responses of prices to interest rate shocks show some consistency - they are all initially positive." He also speculated that the central banks may have information about future inflation, so the response represents in fact reverse causality.

Christiano, Eichenbaum, and Evans (1999) took that suggestion. They put shocks in the order output Y, GDP deflator P, commodity prices PCOM, Federal Funds FF, Total Reserves TR, Nonborrowed reserves NBR, and M1 M (p. 83). Their idea is that commodity prices capture information about future inflation that the Fed may be reacting to, so including commodity prices first isolates policy shocks that are not reactions to inflation. Of course, this ordering also assumes that policy cannot affect output, inflation, or commodity prices for a quarter. With this specification (their Figure 2, top left), positive interest rate shocks reduce output. But even with the careful ordering, interest rate increases have no effect on inflation for a year and a half. The price level then gently declines, but remains within the confidence interval of zero throughout. Their Figure 5, p.100, shows nicely how sensitive even this much evidence is to the shock identification assumptions. If the monetary policy shock is ordered first, prices go up uniformly. (The inflation response in Christiano, Eichenbaum, and Evans (2005) also displays a short run price puzzle, and is never more than two standard errors from zero.)

Even this much success remains controversial. Hanson (2004) points out that commodity prices which solve the price puzzle don't forecast inflation and vice versa. He also finds that the ability of commodity prices to solve the price puzzle does not work after 1979. Sims (1992) was already troubled that commodities are usually globally traded, so while interest rate increases seem to lower commodity prices, it's hard to see how that could be the effect of monetary policy.

Ramey (2015) surveys and reproduces much of the exhaustive modern literature. She finds that "The pesky price puzzle keeps popping up." Of 9 different identification methods, only two present a statistically significant decline in inflation, and those only after four or more years of no effect have passed. Four methods have essentially no effect on inflation at all, and two show strong, statistically significant Fisher (positive) effects, which start without delay. Strong empirical evidence for a short-term (within 4 years) negative inflation effect is absent in her survey.

The Christiano, Eichenbaum, and Evans (1999) procedure may seem fishy already, in that so much of the identification choice was clearly made in order to produce the desired answer, that higher interest rates lead to lower inflation. Nobody spent the same amount of effort seeing if the output decline represented Fed reaction to news, because the output decline fit priors so well. As Uhlig (2006) points out, however, that procedure makes sense. *If* one has strong theoretical priors that positive interest rate shocks cause inflation to decline, then it makes sense to impose that view as part of shock identification, in order to better measure that and other responses. (Uhlig's eloquent introduction is worth reading and contains an extensive literature review that I will not repeat.)

But that only makes sense when one has that strong theoretical prior; when, as when these papers were written, existing theory uniformly specified a negative inflation response and nobody was even considering the opposite. In the context of this paper, when theory specifies a positive response, when I can't find a good theory with the opposite prediction, and we are looking for empirical evidence on the sign, following identification procedures that implicitly or explicitly throw out positive signs does not make sense. And even imposing the sign prior, Uhlig like many others finds that "The GDP price deflator falls only slowly following a contractionary monetary policy shock."

With less strong priors, positive signs are starting to show up. Belaygorod and Dueker (2009) and Castelnuovo and Surico (2010) find that VAR estimates produce positive inflation responses in the periods of estimated indeterminacy. Belaygorod and Dueker (2009) connect estimates to the robust facts one can see in simple plots: Through the 1970s and early 1980s, federal funds rates clearly lead inflation movements. (Dueker (2006) summarizes.)

10. Concluding comments

The observation that inflation has been stable or gently declining at the zero bound, suggests that an interest rate peg can lead to stable inflation. If that is true, then raising the interest rate peg should raise inflation.

Conventional "new-Keynesian" models predict that inflation is stable. It's there in the equations, though the literature using those models has preferred to try to escape the low inflation, low rate equilibrium.

Those models also predict that raising interest rates will raise inflation, both in the long and short run. My attempts to escape this prediction by adding money, backward looking Phillips curves, multiple equilibria or Taylor rules all fail.

This paper was also a search for a simple model that captures the effects of monetary policy, but overcomes the critiques of active and instability-inducing Taylor rules in forward-looking models. The models in this paper satisfy that criterion, but produce a higher inflation rate in response to a higher interest rate. To produce the negative sign, it seems one must rely centrally on more complex ingredients.

A review of the empirical evidence finds very weak support for the standard theoretical view that raising interest rates lowers inflation, and much of that evidence is colored by the imposition of strong priors of that sign.

I conclude that a positive reaction of inflation to interest rate changes is a possibility we, and central bankers, ought to begin to take seriously.

References

Andrés, Javier, J. David López-Salido, and Javier Vallés. 2006. "Money in an Estimated Business Cycle Model of the Euro Area." *The Economic Journal* 116 (511):457–477. URL http://dx.doi.org/10.1111/j.1468-0297.2006.01088.x.

Belaygorod, Anatoliy and Michael Dueker. 2009. "Indeterminacy, Change Points and the Price Puzzle in an Estimated DSGE Model." *Journal of Economic Dynamics and Control* 33 (3):624–648.

Benchimol, Jonathan and André Fourçansyz. 2012. "Money and risk in a DSGE framework: A Bayesian application to the Eurozone." *Journal of Macroeconomics* 34:95–111.

———. 2015. "Money and monetary policy in the Eurozone: an empirical analysis during crises." *Macroeconomic Dynamics* Forthcoming.

Benhabib, Jess, Stephanie Schmitt-Grohé, and Martín Uribe. 2002. "Avoiding Liquidity Traps." *Journal of Political Economy* 110:535–563.

Brash, Donald T. 2002. "Inflation Targeting 14 Years On." URL http://www.bis.org/review/r020121b.pdf. Speech given at 2002 American Economics Association Conference.

Castelnuovo, Efrem and Paolo Surico. 2010. "Monetary Policy, Inflation Expectations and The Price Puzzle." *The Economic Journal* 120:1262–1283. URL doi:10.1111/j.1468-0297.2010.02368.x.

Christiano, Lawrence, Martin Eichenbaum, and Charles Evans. 1999. "Monetary policy shocks: what have we learned and to what end?" In *Handbook of Macroeconomics*, edited by Michael Woodford and John B. Taylor. Amsterdam: North-Holland, 65–148.

———. 2005. "Nominal Rigidities and the Dynamic Effects of a Shock to Montary Policy." *Journal of Political Economy* 113, (1–45).

Clarida, Richard, Jordi Galí, and Mark Gertler. 2000. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory,." *Quarterly Journal of Economics* 115:147–180.

Cochrane, John H. 2011. "Determinacy and Identification with Taylor Rules." *Journal of Political Economy* 119 (565-615).

———. 2013. "What if we got the sign wrong on monetary policy?" URL http://johnhcochrane.blogspot.com/2013/12/what-if-we-got-sign-wrong-on-monetary.html. "Grumpy Economist" Blog Post.

———. 2014a. "Monetary Policy with Interest on Reserves." *Journal of Economic Dynamics and Control* 49:74–108.

———. 2014b. "The neo-Fisherian Question." URL http://johnhcochrane.blogspot.com/2014/11/the-neo-fisherian-question.html. "Grumpy Economist" Blog Post.

———. 2014c. "The New-Keynesian Liquidity Trap." URL http://faculty.chicagobooth.edu/john.cochrane/research/papers/zero_bound_2.pdf. Manuscript.

D'Amico, Stefania and Thomas B. King. 2013. "Flow and stock effects of large-scale treasury purchases: Evidence on the importance of local supply." *Journal of Financial Economics* 108 (2):425–448. URL doi:10.1016/j.jfineco.2012.11.007.

Dueker, Michael J. 2006. "The Price Puzzle: An Update and a Lesson." *Economic Synopses* 4:1. URL https://research.stlouisfed.org/publications/es/06/ES0624.pdf.

Eichenbaum, Martin. 1992. "Comment on Interpreting the macroeconomic time series facts: the effects of monetary policy." *European Economic Review,* 36 (5):1001–1011.

Friedman, Milton. 1968. "The Role of Monetary Policy." *The American Economic Review* 58:1–17.

———. 1969. "The Optimum Quantity of Money." In *The Optimum Quantity of Money and Other Essays*. Chicago, IL: Aldine, 1–50.

Gabaix, Xavier. 2015. "Behavioral Macroeconomics ViaSparse Dynamic Programming." Manuscript, NYU.

García-Schmidt, Mariana and Michael Woodford. 2015. "Are Low Interest Rates Deflationary? A Paradox of Perfect-Foresight Analysis." Manuscript.

Hanson, Michael. 2004. "The 'Price Puzzle' Reconsidered." *Journal of Monetary Economics* 51. URL http://www.sciencedirect.com/science/article/pii/S0304393204000959.

Ireland, Peter. 2004. "Money's Role in the Monetary Business Cycle." *Journal of Money, Credit, and Banking* 36 (6):969–983. URL http://irelandp.com/pubs/role.pdf.

Kocherlakota, Narayana. 2010. "Inside the FOMC." URL https://www.minneapolisfed.org/news-and-events/presidents-speeches/inside-the-fomc. Speech.

Krishnamurthy, Arvind and Annette Vissing-Jorgensen. 2011. "The Effects of Quantitative Easing on Interest Rates: Channels and Implications for Policy." *Brookings Papers on Economic Activity* :215–265URL http://www.brookings.edu/~/media/projects/bpea/fall-2011/2011b_bpea_krishnamurthy.pdf.

Lucas, Robert E. 1972. "Expectations and the Neutrality of Money." *Journal of Economic Theory* 4:103–124.

Lucas, Robert E. 1988. "Money Demand in the United States: A Quantitative Review." *Carnegie-Rochester Conference Series on Public Policy* 29:137–167.

Mankiw, N. Gregory and Ricardo Reis. 2002. "Sticky Information versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve." *Quarterly Journal of Economics* 117:1295–1328.

Ramey, Valerie. 2015. "Macroeconomic Shocks and Their Propagation." Manuscript.

Schmitt-Grohé, Stephanie and Martín Uribe. 2014. "Liquidity Traps: An Interest-Rate-Based Exit Strategy." *The Manchester School* 81 (S1):1–14. URL doi:10.1111/manc.12065.

Sims, Christopher. 1992. "Interpreting the macroeconomic time series facts: The effects of monetary policy." *European Economic Review* 36 (5):975–1000.

Sims, Christopher A. 1980. "Macroeconomics and reality." *Econometrica* :1–48.

Smets, Frank and Raf Wouters. 2003. "An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area,." *Journal of the European Economic Association* 1 (5):1123–1175.

Smith, Noah. 2014. "The Neo-Fisherite Rebellion." URL http://noahpinionblog.blogspot.com/2014/04/the-neo-fisherite-rebellion.html. Noahpinion Blog Post.

Taylor, John B. 1999. "The Robustness and Efficiency of Monetary Policy Rules as Guidelines for Interest Rate Setting by the European Central Bank." *Journal of Monetary Economics* 43:655–679.

Uhlig, Harald. 2006. "What are the effects of monetary policy? Results from an agnostic identification procedure." *Journal of Monetary Economics* 52 (2):381–419.

Werning, Iván. 2012. "Managing a Liquidity Trap: Monetary and Fiscal Policy." URL https://dl.dropboxusercontent.com/u/125966/zero_bound_2011.pdf. Manuscript, MIT.

Williamson, Steven. 2013. "Phillips Curves and Fisher Relations." URL http://newmonetarism.blogspot.com/2013/12/phillips-curves-and-fisher-relations.html. Blog Post.

Woodford, Michael. 1995. "Price-Level Determinacy Without Control of a Monetary Aggregate." *Carnegie-Rochester Conference Series on Public Policy* 43:1–46.

———. 2001. "Fiscal Requirements for Price Stability." *Journal of Money Credit and Banking* 33 (669-728).

———. 2003. *Interest and prices.* Princeton N. J.: Princeton University Press.

11. Appendix

11.1. Recent history

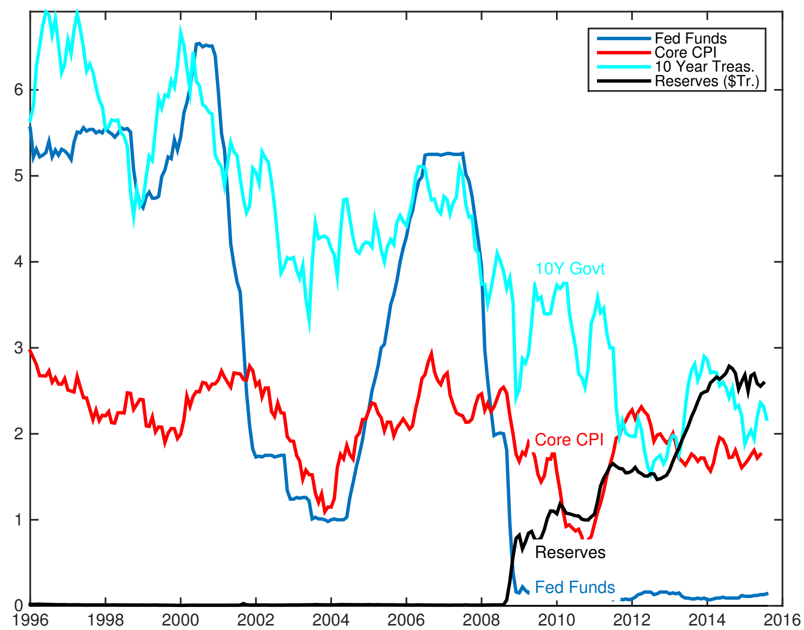


Figure 20: Federal Funds Rate, Core CPI, 10 Year government bond rate, and reserves

I referred in the text to the recent stability of inflation at the zero bound, and how that experience suggests the economy can be stable under an interest rate peg. That experience both voids the "deflationary spiral" warnings given as the zero bound approached, as well as the warning in Friedman (1968) that a peg must lead to an inflationary spiral. Finally, recent experience argues that people (banks) will hold any amount of reserves that pay market interest, i.e. money demand is satiated and open market purchases have no effect, and $MV = PY$ becomes $V = PY/M$. Here, I present some graphs to summarize that experience.

Figure 20 presents the last 20 years of data on the Federal Funds rate, the core CPI, the 10 year government bond rate, and reserves.

Inflation is dominated by a slow 20 year downward trend. On top of that trend there is the usual business cycle pattern, that inflation declines in a recession and then rises again when the recession is over.

The 10 year bond rate captures future expected inflation, expected real rates, and a troublesome risk premium. The 10 year bond rate is also on a steady downward trend throughout this period, plus regular small variations following the funds rate and recessions.

The interesting point of this graph is that inflation dynamics are so utterly unaffected by tumultuous events surrounding them.

The pattern following 2008 begins by mirroring that following 2002. But in 2008, interest rates hit zero. Conventional Keynesian models and their interpreters predicted deflation "spirals" or "vortices." Slight deflation would produce high real interest rates. High real rates would reduce aggregate demand, increase deflation, produce still higher real interest rates still, lower aggregate demand, and so forth, as in Figure 18. It never happened. Inflation recovered in 2011-2012 just as it did in 2004-2005.

In 2010-2012, following the normal timing of things and the normal Taylor rule, interest rates should have risen. They did not, leading many to fear inflation would break out. Friedman (1968) describes vividly how of inflation should break out under an interest rate peg. Again, nothing happened. Inflation recovered and then resumed its 20 year downward trend, just as it did in 2006.

The Fed then dramatically increased bank reserves, from about $50 billion to $2,000 billion, in three quantitative easing (QE) operations. Conventional monetarist models and their interpreters predicted inflation or even hyperinflation. Once again, nothing happened.

The three quantitative easing episodes are visible in the graph of reserves, which rises three times. QE2 is associated with a rise in inflation, but QE1 and QE2 are associated with a decline in inflation, and the QE2 rise looks just like the normal rise in 2004-2005 coming out of a recession.

QE2 and QE3 were supposed to lower long-term interest rates. Yet the 20 year downward trend in long term rates is essentially unaffected by QE. At best, one might try to ascribe the pause in 10 year rates between 2011 and 2013 to quantitative easing. But the timing is wrong. Interest rates were higher during each QE episode (rising reserves).

The 2012-2013 period of low 10 year rates coincides with a pause in QE (flat reserves) not with aggressive purchases (rising reserves).

Econometric evaluation of QE, such as Krishnamurthy and Vissing-Jorgensen (2011) and D'Amico and King (2013) centers around whether QE announcements lowered long-term bond yields a few tens of basis points. In general they do not evaluate how long such effects last, nor do they confront the puzzle that actual QE purchases coincide with interest rate increases. Krishnamurthy and Vissing-Jorgensen (2011) find about half of the effect of QE occurs by signaling a longer period of low interest rates, rather than any direct effect. D'Amico and King (2013) show that the biggest effects are in illiquid off-the-run issues, raising the question just how much effect purchases have on the market-clearing interest rates in the rest of the economy.

But the QE operations were enormous. Reserves increased from $50 billion to $3,000 billion. If MV=PY, we should be evaluating the size of the hyperinflation, not 10 to 30 basis point announcement effects in specific bond issues, invisible in the plot of 10 year rates of Figure 20 .

There is no sign of greater volatility of inflation at the zero bound either. Growth, while unusually slow, has been remarkably stable as well. If active interest rate movements are necessary to stabilize otherwise unstable inflation and growth, or even to offset shocks, we should see more volatile inflation and growth when their stabilizer is stuck at zero.

Theories fail publicly when they predict nothing and big things happen. This was the case in the 1970s, when unexpected stagflation broke out. Theories fail no less when they predict enormous movements, and nothing happens. That is the case now. It's just less public.

In sum, Figure 20 makes a suggestive case that inflation is stable with an interest rate peg, and that money and bonds are perfect substitutes when they pay the same interest rate and reserve requirements are trillions of dollars from binding.

Figure 21 for Japan tells a similar story. Japanese interest rates declined swiftly in the early 1990s, and essentially hit zero in 1995. Again, there were widespread predictions of a deflation "spiral," sensible if you look at data up to 2001. But though Japan has

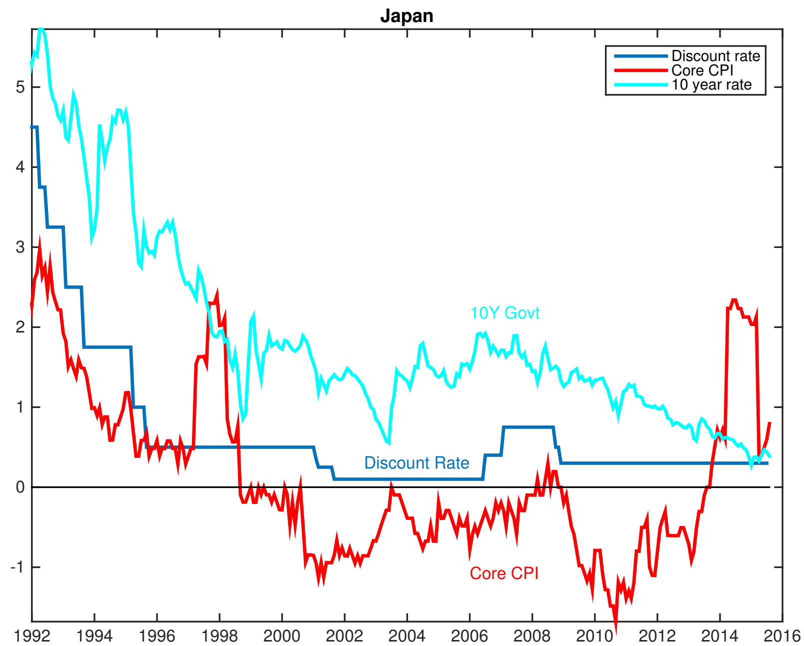


Figure 21: Discount rate, Core CPI, and 10 year Government Bond Yield in Japan.

had bouts of small deflation, deflation never spiraled. Despite large fiscal stimulus and quantitative easing operations, interest rates have stuck at zero with slight deflation, and Japan has lived 20 years of the optimum quantity of money Friedman (1969) (zero nominal rate, slight deflation).

The most recent Japanese data are revealing. In 2014, it seemed like the latest policies were finally going to bring back some inflation. But the 10 year government bond rate never budged from its steady downward trend. And the burst of inflation, largely due to a rise in the consumption tax, quickly reversed.

One can interpret data in many ways, of course. Perhaps hyperinflationary quantitative easing just offset a deflationary vortex, and the remarkable stability of the last 7 years in the US and 20 in Japan is the result of carefully calibrated policy. But a Fisherian interpretation that inflation is stable around an interest rate peg gives a very simple and straightforward account of these events.

The graphs and a theory that predicts stability under an interest rate peg should not be too reassuring. The theory says that a peg is only stable so long as fiscal policy remains solvent. And each further reduction in inflation requires larger surpluses to pay off the greater real value of nominal government debt. The large debt to GDP ratios in the US and Japan are not comforting. Woodford (2001) warns that the US peg of the 1940s and early 1950s fell apart from uncooperative fiscal policy, and that could happen again.

11.2. Model Solution

Here I derive the explicit solutions (3)-(4), for inflation and output given the equilibrium path of interest rates. The simple model (1)-(2) is

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t\pi_{t+1})$$
$$\pi_t = \beta E_t\pi_{t+1} + \kappa x_t.$$

The model with money generalizes the IS equation only, to (8)

$$x_t = E_t x_{t+1} + (\sigma - \xi)\left(\frac{m}{c}\right) E_t\left[\left(i_{t+1} - i^m_{t+1}\right) - \left(i_t - i^m_t\right)\right] - \sigma\left(i_t - E_t\pi_{t+1}\right).$$

We can treat the two cases simultaneously by defining

$$z_t \equiv i_t - \left(\frac{\sigma - \xi}{\sigma}\right)\left(\frac{m}{c}\right) E_t\left[\left(i_{t+1} - i^m_{t+1}\right) - \left(i_t - i^m_t\right)\right]$$

and writing the IS equation as

$$x_t = E_t x_{t+1} + E_t\pi_{t+1} - \sigma z_t.$$

One must be careful that lags of z_t are lags of expected interest rate changes, not lags of actual interest rate changes.

Expressing the model in lag operator notation,

$$E_t(1 - L^{-1})x_t = \sigma E_t L^{-1}\pi_t - \sigma z_t$$

$$E_t(1 - \beta L^{-1})\pi_t = \kappa x_t$$

Forward-differencing the second equation,

$$E_t(1-L^{-1})(1-\beta L^{-1})\pi_t = E_t(1-L^{-1})\kappa x_t$$

Then substituting,

$$E_t(1-L^{-1})\left(1-\beta L^{-1}\right)\pi_t = \kappa\sigma E_t L^{-1}\pi_t - \kappa\sigma z_t$$

$$E_t\left[(1-L^{-1})\left(1-\beta L^{-1}\right) - \kappa\sigma L^{-1}\right]\pi_t = -\kappa\sigma z_t$$

$$E_t\left[1-(1+\beta+\kappa\sigma)L^{-1} + \beta L^{-2}\right]\pi_t = -\kappa\sigma z_t.$$

Factor the lag polynomial

$$E_t(1-\lambda_1 L^{-1})(1-\lambda_2 L^{-1})\pi_t = -\kappa\sigma z_t$$

where

$$\lambda_i = \frac{(1+\beta+\kappa\sigma) \pm \sqrt{(1+\beta+\kappa\sigma)^2 - 4\beta}}{2}.$$

Since $\lambda_1 > 1$ and $\lambda_2 < 1$, reexpress the result as

$$E_t\left[(1-\lambda_1^{-1}L)(1-\lambda_2 L^{-1})\lambda_1 L^{-1}\pi_t\right] = \kappa\sigma z_t$$

$$E_t\left[(1-\lambda_1^{-1}L)(1-\lambda_2 L^{-1})\pi_{t+1}\right] = \kappa\sigma\lambda_1^{-1} z_t$$

The bounded solutions are

$$\pi_{t+1} = E_{t+1}\frac{\lambda_1^{-1}}{(1-\lambda_1^{-1}L)(1-\lambda_2 L^{-1})}\kappa\sigma z_t + \frac{1}{(1-\lambda_1^{-1}L)}\delta_{t+1}$$

where δ_{t+1} is a sequence of unpredictable random variables, $E_t\delta_{t+1} = 0$ and $\delta_{t+1} = \pi_{t+1} - E_t\pi_{t+1}$. I follow the usual practice and I rule out solutions that explode in the forward direction.

Using a partial fractions decomposition to break up the right hand side,

$$\frac{\lambda_1^{-1}}{\left(1-\lambda_1^{-1}L\right)\left(1-\lambda_2 L^{-1}\right)} = \frac{1}{\lambda_1-\lambda_2}\left(1+\frac{\lambda_1^{-1}L}{1-\lambda_1^{-1}L} + \frac{\lambda_2 L^{-1}}{1-\lambda_2 L^{-1}}\right).$$

So,

$$\pi_{t+1} = \frac{1}{\lambda_1 - \lambda_2} E_{t+1} \left(1 + \frac{\lambda_1^{-1} L}{1 - \lambda_1^{-1} L} + \frac{\lambda_2 L^{-1}}{1 - \lambda_2 L^{-1}} \right) \kappa\sigma z_t + \frac{1}{(1 - \lambda_1^{-1} L)} \delta_{t+1}$$

or in sum notation,

$$\pi_{t+1} = \kappa\sigma \frac{1}{\lambda_1 - \lambda_2} \left(z_t + \sum_{j=1}^{\infty} \lambda_1^{-j} z_{t-j} + \sum_{j=1}^{\infty} \lambda_2^j E_{t+1} z_{t+j} \right) + \sum_{j=0}^{\infty} \lambda_1^{-j} \delta_{t+1-j}.$$

We can show directly that the long-run impulse-response function is 1:

$$\frac{1}{(1 - \lambda_1^{-1})(1 - \lambda_2)} \frac{\kappa\sigma}{\lambda_1} = -\frac{\kappa\sigma}{(1 - \lambda_1)(1 - \lambda_2)}$$

$$= -\frac{\kappa\sigma}{(1 - (\lambda_1 + \lambda_2) + \lambda_1 \lambda_2)} = -\frac{\kappa\sigma}{(1 - (1 + \beta + \kappa\sigma) + \beta)} = 1.$$

Having found the path of π_t, we can find output by

$$\kappa x_t = \pi_t - \beta E_t \pi_{t+1}.$$

In lag operator notation, and shifting forward one period,

$$\kappa x_{t+1} = E_{t+1} \left[(1 - \beta L^{-1}) \pi_{t+1} \right]$$

$$\kappa x_{t+1} = \frac{\kappa\sigma}{\lambda_1 - \lambda_2} E_{t+1} \left[(1 - \beta L^{-1}) \left(1 + \frac{\lambda_1^{-1} L}{1 - \lambda_1^{-1} L} + \frac{\lambda_2 L^{-1}}{1 - \lambda_2 L^{-1}} \right) z_t \right] + E_{t+1} \frac{(1 - \beta L^{-1})}{(1 - \lambda_1^{-1} L)} \delta_{t+1}.$$

We can rewrite the polynomials to give

$$\kappa x_{t+1} = \frac{\kappa\sigma}{\lambda_1 - \lambda_2} E_{t+1} \left[\frac{1 - \beta\lambda_1^{-1}}{1 - \lambda_1^{-1} L} + \frac{(1 - \beta\lambda_2^{-1}) (\lambda_2 L^{-1})}{1 - \lambda_2 L^{-1}} \right] z_t + E_{t+1} \left[\frac{1 - \beta\lambda_1^{-1}}{1 - \lambda_1^{-1} L} \right] \delta_{t+1}.$$

(In the second term, I use $E_t \left[\beta L^{-1} \delta_{t+1} \right] = 0$) or, in sum notation,

$$\kappa x_{t+1} = \frac{\kappa\sigma}{\lambda_1 - \lambda_2} \left[(1 - \beta\lambda_1^{-1}) \sum_{j=0}^{\infty} \lambda_1^{-j} z_{t-j} + (1 - \beta\lambda_2^{-1}) \sum_{j=1}^{\infty} \lambda_2^j E_{t+1} z_{t+j} \right] +$$

$$+\left(1-\beta\lambda_1^{-1}\right)\sum_{j=0}^{\infty}\lambda_1^{-j}\delta_{t+1-j}.$$

12. Model and solutions in continuous time

It is convenient to work both in discrete and continuous time. To keep the math simple, I consider the perfect-foresight continuous-time specification, and treat the impulse response function as a once and for all unexpected shock. The continuous time version of the model is

$$\frac{dx_t}{dt}=\sigma\left(i_t-\pi_t\right) \tag{32}$$

$$\frac{d\pi_t}{dt}=\rho\pi_t-\kappa x_t \tag{33}$$

The solution is

$$\pi_t=\pi_0e^{-\lambda_2 t}-\frac{\kappa\sigma}{\rho}\left[\int_{s=0}^{t}e^{-\lambda_2(t-s)}i_s ds+\int_{s=t}^{\infty}e^{-\lambda_1(s-t)}i_s ds\right] \tag{34}$$

$$\kappa x_t=\lambda_1\pi_0e^{-\lambda_2 t}-\frac{\kappa\sigma}{\rho}\left[\lambda_1\int_{s=0}^{t}e^{-\lambda_2(t-s)}i_s ds-\lambda_2\int_{s=t}^{\infty}e^{-\lambda_1(s-t)}i_s ds\right] \tag{35}$$

where

$$\lambda_1=\frac{1}{2}\left(\sqrt{\rho^2+4\kappa\sigma}+\rho\right)$$

$$\lambda_2=\frac{1}{2}\left(\sqrt{\rho^2+4\kappa\sigma}-\rho\right).$$

To derive the solution we proceed as in discrete time. Difference (32),

$$\frac{d^2\pi_t}{dt^2}=\rho\frac{d\pi_t}{dt}+\kappa\sigma\pi_t-\kappa\sigma i_t$$

$$\left(\frac{d^2\pi_t}{dt^2}-\rho\frac{d\pi_t}{dt}-\kappa\sigma\pi_t\right)=-\kappa\sigma i_t.$$

We seek roots of the form

$$\left(\frac{d}{dt}-\lambda_1\right)\left(\frac{d}{dt}+\lambda_2\right)\pi_t=-\kappa\sigma i_t$$

in which case

$$\frac{d^2\pi_t}{dt^2} + (\lambda_2 - \lambda_1)\frac{d\pi_t}{dt} - \lambda_1\lambda_2\pi_t = -\kappa\sigma i_t. \tag{36}$$

Matching coefficients,

$$\lambda_1\lambda_2 = \kappa\sigma$$
$$\lambda_1 - \lambda_2 = \rho.$$

Solving,

$$\lambda_1 - \frac{\kappa\sigma}{\lambda_1} = \rho$$
$$\lambda_1^2 - \rho\lambda_1 - \kappa\sigma = 0,$$

and hence,

$$\lambda_1 = \frac{1}{2}\left(\sqrt{\rho^2 + 4\kappa\sigma} + \rho\right)$$
$$\lambda_2 = \lambda_1 - \rho = \frac{1}{2}\left(\sqrt{\rho^2 + 4\kappa\sigma} - \rho\right).$$

The solution to (36) is

$$\pi_t = \pi_0 e^{-\lambda_2 t} + \frac{\kappa\sigma}{\lambda_2 - \lambda_1}\left[\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right]$$

It's easier to check than to derive. The derivatives are

$$\frac{d\pi_t}{dt} = -\lambda_2\pi_0 e^{-\lambda_2 t} + \frac{\kappa\sigma}{\lambda_2 - \lambda_1}\left[-\lambda_2\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \lambda_1\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right]$$

$$\frac{d^2\pi_t}{dt^2} = -\lambda_2^2\pi_0 e^{-\lambda_2 t} + \frac{\kappa\sigma}{\lambda_2 - \lambda_1}\left[(\lambda_1 - \lambda_2)i_t - \lambda_2^2\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \lambda_1^2\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right]$$

Plugging these derivatives in to the differential equation (36), we have

$$
\begin{aligned}
&-\lambda_2^2\pi_0 e^{-\lambda_2 t} + \frac{\kappa\sigma}{\lambda_2-\lambda_1}\left[(\lambda_1-\lambda_2)i_t - \lambda_2^2\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \lambda_1^2\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right] \\
&+(\lambda_2-\lambda_1)\left(-\lambda_2\pi_0 e^{-\lambda_2 t} + \frac{\kappa\sigma}{\lambda_2-\lambda_1}\left[-\lambda_2\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \lambda_1\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right]\right) \\
&-\lambda_1\lambda_2\left(\pi_0 e^{-\lambda_2 t} + \frac{\kappa\sigma}{\lambda_2-\lambda_1}\left[\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right]\right) =? -\kappa\sigma i_t.
\end{aligned}
$$

Collecting terms

$$
-\lambda_2^2\pi_0 e^{-\lambda_2 t} + (\lambda_2-\lambda_1)\left(-\lambda_2\pi_0 e^{-\lambda_2 t}\right) - \lambda_1\lambda_2\pi_0 e^{-\lambda_2 t} = 0
$$

$$
\frac{\kappa\sigma}{\lambda_2-\lambda_1}\left[(\lambda_1-\lambda_2)i_t\right] + \kappa\sigma i_t = 0
$$

$$
\begin{aligned}
&\left[-\lambda_2^2 - (\lambda_2-\lambda_1)\lambda_2 - \lambda_1\lambda_2\right]\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \\
&+\left[\lambda_1^2 + (\lambda_2-\lambda_1)\lambda_1 - \lambda_1\lambda_2\right]\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds = 0
\end{aligned}
$$

We can find the output response from

$$
\begin{aligned}
\kappa x_t &= \rho\pi_t - \frac{d\pi_t}{dt} = \\
&\rho\pi_0 e^{-\lambda_2 t} - \kappa\sigma\left[\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right] \\
&-\pi_0\left(-\lambda_2\right)e^{-\lambda_2 t} + \frac{\kappa\sigma}{\rho}\left[-\lambda_2\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + \lambda_1\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right]
\end{aligned}
$$

$$
\lambda_1\pi_0 e^{-\lambda_2 t} - \frac{\kappa\sigma}{\rho}\left[(\rho+\lambda_2)\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds + (\rho-\lambda_1)\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right]
$$

or,

$$
\kappa x_t = \lambda_1\pi_0 e^{-\lambda_2 t} - \frac{\kappa\sigma}{\rho}\left[\lambda_1\int_{s=0}^{t} e^{-\lambda_2(t-s)} i_s ds - \lambda_2\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} i_s ds\right].
$$

12.1. Lagged inflation in the Phillips curve

The Phillips curve with lagged inflation is

$$\pi_t = \kappa \left(x_t + E_t \sum_{j=1}^{\infty} \phi^j x_{t+j} + \sum_{j=1}^{\infty} \rho^j x_{t-j} \right)$$
$$= E_t \left(1 + \frac{\phi L^{-1}}{1-\phi L^{-1}} + \frac{\rho L}{1-\rho L} \right) \kappa x_t$$

or, in autoregressive form

$$\pi_t = E_t \left(\frac{1-\rho\phi}{\left(1-\phi L^{-1}\right)\left(1-\rho L\right)} \right) \kappa x_t \tag{37}$$

$$E_t \left(1+\rho\phi - \phi L^{-1} - \rho L\right) \pi_t = (1-\rho\phi)\, \kappa x_t.$$

$$\pi_t = \frac{\phi}{1+\rho\phi} E_t \pi_{t+1} + \frac{\rho}{1+\rho\phi} \pi_{t-1} + \frac{(1-\phi\rho)}{(1+\phi\rho)} \kappa x_t.$$

To maintain the same steady state relationship between output and inflation, I constrain ρ and ϕ to satisfy

$$\frac{(1-\phi\rho)}{(1-\phi)(1-\rho)} = \frac{1}{(1-\beta)}.$$

So, for each choice of the weight ρ on past inflation, I use a weight ϕ on forward inflation given by

$$\phi = \frac{\beta - \rho}{1+\beta\rho - 2\rho}. \tag{38}$$

The case $\rho = \phi$ occurs where

$$\rho = \frac{\beta}{2-\beta}.$$

Now, to solve the model. To keep the algebra simple I find the perfect foresight solution and put the E_t back in at the end. The IS curve is

$$x_t = x_{t+1} + \sigma\pi_{t+1} - \sigma z_t$$

$$(1-L^{-1}) x_t = \sigma \left(L^{-1}\pi_t - z_t\right)$$

Forward-differencing (37) and substituting,

$$(1-L^{-1})\pi_t = \left(\frac{1-\phi\rho}{(1-\phi L^{-1})\,(1-\rho L)}\right)\left(1-L^{-1}\right)\kappa x_t$$

$$(1-L^{-1})\pi_t = \kappa\sigma\left(\frac{1-\phi\rho}{(1-\phi L^{-1})\,(1-\rho L)}\right)(L^{-1}\pi_t - z_t)$$

$$\left[(1-L^{-1})\left(1-\phi L^{-1}\right)(1-\rho L) - \kappa\sigma\,(1-\phi\rho)\,L^{-1}\right]\pi_t = -\kappa\sigma\,(1-\phi\rho)\,z_t$$

$$\left[-\rho L + (1+\rho\,(1+\phi)) - (1+\phi+\kappa\sigma+\phi\rho\,(1-\kappa\sigma))\,L^{-1} + \phi L^{-2}\right]\pi_t = -\kappa\sigma\,(1-\phi\rho)\,z_t$$

$$\left[-\frac{\rho}{\phi} + \frac{1+\rho\,(1+\phi)}{\phi}L^{-1} - \frac{1+\phi+\kappa\sigma+\phi\rho\,(1-\kappa\sigma)}{\phi}L^{-2} + L^{-3}\right]\phi L\pi_t = -\kappa\sigma\,(1-\phi\rho)\,z_t$$

Denoting the three roots of the lag polynomial λ_i^{-1},

$$E_t\left(L^{-1}-\lambda_1^{-1}\right)\left(L^{-1}-\lambda_2^{-1}\right)\left(L^{-1}-\lambda_3^{-1}\right)\phi L\pi_t = -\kappa\sigma\,(1-\phi\rho)\,z_t.$$

I find these roots numerically. Nonetheless we can characterize them somewhat. Evaluating the lag polynomial at $L=0$ we have

$$\lambda_1\lambda_2\lambda_3 = \frac{\phi}{\rho}. \tag{39}$$

while $L=1$ gives

$$\left(1-\lambda_1^{-1}\right)\left(1-\lambda_2^{-1}\right)\left(1-\lambda_3^{-1}\right) = \left(-\frac{\rho}{\phi} + \frac{1+\rho\,(1+\phi)}{\phi} - \frac{1+\phi+\kappa\sigma+\phi\rho\,(1-\kappa\sigma)}{\phi} + 1\right)$$

$$\left(1-\lambda_1^{-1}\right)\left(1-\lambda_2^{-1}\right)\left(1-\lambda_3^{-1}\right) = -\sigma\kappa\frac{(1-\phi\rho)}{\phi}$$

We will have $\lambda_1, \lambda_2 > 1$, $\lambda_3 < 1$ so it's convenient to write the result as

$$\lambda_3^{-1}\left[\left(1-\lambda_1^{-1}L\right)\left(1-\lambda_2^{-1}L\right)\left(1-\lambda_3 L^{-1}\right)\right]\phi L^{-1}\pi_t = \kappa\sigma\,(1-\phi\rho)\,z_t$$

$$\pi_{t+1} = \lambda_3\frac{\kappa\sigma}{\phi}\,(1-\phi\rho)\,\frac{1}{\left(1-\lambda_1^{-1}L\right)\left(1-\lambda_2^{-1}L\right)(1-\lambda_3 L^{-1})}z_t$$

I use the partial fractions identity

$$\frac{\lambda_3}{\left(1-\lambda_1^{-1}L\right)\left(1-\lambda_2^{-1}L\right)\left(1-\lambda_3L^{-1}\right)} == \frac{\lambda_3}{\left(1-\lambda_3\lambda_1^{-1}\right)\left(1-\lambda_3\lambda_2^{-1}\right)}\times$$

$$\times\left(1+\frac{\lambda_3L^{-1}}{\left(1-\lambda_3L^{-1}\right)}+\frac{\lambda_1^{-1}\left(1-\lambda_2^{-1}\lambda_3\right)}{\left(\lambda_1^{-1}-\lambda_2^{-1}\right)}\frac{\lambda_1^{-1}L}{1-\lambda_1^{-1}L}-\frac{\lambda_2^{-1}\left(1-\lambda_1^{-1}\lambda_3\right)}{\left(\lambda_1^{-1}-\lambda_2^{-1}\right)}\frac{\lambda_2^{-1}L}{1-\lambda_2^{-1}L}\right)$$

This takes pages of algebra to derive. It's easier just to check it. Thus, and reinserting the E_t

$$\pi_{t+1} = \kappa\sigma\frac{(1-\phi\rho)}{\phi}\frac{\lambda_3}{\left(1-\lambda_3\lambda_1^{-1}\right)\left(1-\lambda_3\lambda_2^{-1}\right)}\times$$

$$\times E_{t+1}\left[\left(1+\frac{\lambda_3L^{-1}}{\left(1-\lambda_3L^{-1}\right)}+\frac{\lambda_1^{-1}\left(1-\lambda_2^{-1}\lambda_3\right)}{\left(\lambda_1^{-1}-\lambda_2^{-1}\right)}\frac{\lambda_1^{-1}L}{1-\lambda_1^{-1}L}-\frac{\lambda_2^{-1}\left(1-\lambda_1^{-1}\lambda_3\right)}{\left(\lambda_1^{-1}-\lambda_2^{-1}\right)}\frac{\lambda_2^{-1}L}{1-\lambda_2^{-1}L}\right)E_tz_t\right] \tag{40}$$

The long-run response $(L=1)$ is

$$\begin{aligned}
\pi_{t+1} &= \lambda_3\frac{\kappa\sigma}{\phi}(1-\phi\rho)\frac{1}{\left(1-\lambda_1^{-1}\right)\left(1-\lambda_2^{-1}\right)\left(1-\lambda_3\right)}E_tz_t \\
&= -\frac{\kappa\sigma}{\phi}(1-\phi\rho)\frac{1}{\left(1-\lambda_1^{-1}\right)\left(1-\lambda_2^{-1}\right)\left(1-\lambda_3^{-1}\right)}E_tz_t \\
&= \frac{\kappa\sigma}{\phi}(1-\phi\rho)\frac{1}{\sigma\kappa}\frac{\phi}{(1-\phi\rho)}E_tz_t \\
&= 1E_tz_t
\end{aligned}$$

The case $\rho=0, \phi=\beta$ is the conventional forward looking curve. The case $\phi=0$, $\rho=\beta$, is a purely backward looking curve. In this case, the solution is

$$\left[\rho-(1+\rho)L^{-1}+(1+\kappa\sigma)L^{-2}\right]L\pi_t = \kappa\sigma z_t$$

$$\left(\frac{\rho}{(1+\kappa\sigma)}-\frac{(1+\rho)}{(1+\kappa\sigma)}L^{-1}+L^{-2}\right)(1+\kappa\sigma)L\pi_t = \kappa\sigma z_t$$

$$\left[\left(L^{-1}-\lambda_1^{-1}\right)\left(L^{-1}-\lambda_2^{-1}\right)\right](1+\kappa\sigma)L\pi_t = \kappa\sigma(1-\phi\rho)z_t$$

$$\lambda_1^{-1} = \frac{(1+\rho) \pm \sqrt{(1+\rho)^2 - 4\rho(1+\kappa\sigma)}}{2(1+\kappa\sigma)}$$
$$\lambda_2^{-1} = \frac{(1+\rho) \pm \sqrt{(1-\rho)^2 - 4\rho\kappa\sigma}}{2(1+\kappa\sigma)}$$
$$\lambda_3 = 0$$

$$\left[\left(1-\lambda_1^{-1}L\right)\left(1-\lambda_2^{-1}L\right)\right](1+\kappa\sigma)L^{-1}\pi_t = \kappa\sigma z_t$$
$$\pi_{t+1} = \frac{\kappa\sigma}{(1+\kappa\sigma)}\frac{1}{\left(1-\lambda_1^{-1}L\right)\left(1-\lambda_2^{-1}L\right)}z_t$$
$$\pi_{t+1} = \frac{\kappa\sigma}{(1+\kappa\sigma)}\left(1+\frac{1}{\lambda_1^{-1}-\lambda_2^{-1}}\left(\lambda_1^{-1}\frac{\lambda_1^{-1}L}{\left(1-\lambda_1^{-1}L\right)} - \lambda_2^{-1}\frac{\lambda_2^{-1}L}{\left(1-\lambda_2^{-1}L\right)}\right)\right)z_t$$

12.2. Linearized valuation equation

Here, I derive equation (15). We start with the government debt valuation formula, which says that the real value of nominal debt equals the present value of real primary surpluses,

$$\frac{B_{t-1}}{P_t} = E_t\left[\sum_{j=0}^{\infty}\beta^j\frac{u'(C_{t+j})}{u'(C_t)}S_{t+j}\right].$$

Start at the steady state (ignoring growth)

$$\frac{B}{P} = \sum_{j=0}^{\infty}\beta^j S = \frac{1}{1-\beta}S. \tag{41}$$

Write

$$\frac{B_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t}u'(C_t) = \sum_{j=0}^{\infty}\beta^j E_t\left[u'(C_{t+j})S_{t+j}\right].$$

Taking innovations,

$$\frac{B_{t-1}}{P_{t-1}}\Delta E_t\left[\frac{P_{t-1}}{P_t}u'(C_t)\right] = \sum_{j=0}^{\infty}\beta^j\Delta E_t\left[u'(C_{t+j})S_{t+j}\right].$$

where $\Delta E_t \equiv E_t - E_{t-1}$. To derive an approximate linear formula, write

$$u'(C_t) = C_t^{-\gamma} = e^{-\gamma c_t} \approx e^{-\gamma c - \gamma(c_t - c)} = e^{-\gamma c} e^{-\gamma x_t}.$$

We can characterize the fiscal policy needed to achieve any given equilibrium by the required permanent percent deviation from steady state Δs of primary surpluses. Surpluses are then $S_t = Se^{s_t} = Se^{\Delta s}$ where I use the latter notation to emphasize that we change fiscal policy forever. Then, we have

$$\frac{B}{P} \Delta E_t \left(e^{-\pi_t} e^{-\gamma c} e^{-\gamma x_t} \right) \approx \sum_{j=0}^{\infty} \beta^j \Delta E_t \left(e^{-\gamma c} e^{-\gamma x_{t+j}} S e^{\Delta s} \right)$$

Using (41)

$$\Delta E_t \left(e^{-\pi_t} e^{-\gamma x_t} \right) \approx (1 - \beta) \sum_{j=0}^{\infty} \beta^j \Delta E_t \left(e^{-\gamma x_{t+j}} e^{\Delta s} \right)$$

$$\Delta E_t \left(\pi_t + \gamma x_t \right) \approx (1 - \beta) \sum_{j=0}^{\infty} \beta^j \Delta E_t \left(\gamma x_{t+j} - \Delta s \right)$$

$$\Delta s \approx -\Delta E_t \left(\pi_t + \gamma x_t \right) + \gamma (1 - \beta) \sum_{j=0}^{\infty} \beta^j \Delta E_t \left(x_{t+j} \right)$$

$$\Delta s \approx -\Delta E_t \left(\pi_t \right) + \frac{(1 - \beta)}{\sigma} \sum_{j=0}^{\infty} \beta^j \Delta E_t \left(x_{t+j} - x_t \right).$$

In the last equation I use $\sigma \equiv 1/\gamma$.

12.3. The Model with Money

This section derives the model with money (8). The utility function is

$$\max E \int_{t=0}^{\infty} e^{-\delta t} u(c_t, M_t/P_t) dt.$$

The present-value budget constraint is

$$\frac{B_0 + M_0}{P_0} = \int_{t=0}^{\infty} e^{-\int_{s=0}^{t} r_s ds} \left[c_t - y_t + s_t + (i_t - i_t^m) \frac{M_t}{P_t} \right] dt$$

where

$$r_t = i_t - \frac{dP_t}{P_t}$$

and s denotes real net taxes paid, and thus the real government primary surplus. This budget constraint is the present value form of

$$d(B_t + M_t) = i_t B_t + i_t^m M_t + P_t(y_t - c_t - s_t).$$

Introducing a multiplier λ on the present value budget constraint, we have

$$\frac{\partial}{\partial c_t} : e^{-\delta t} u_c(t) = \lambda e^{-\int_{s=0}^{t} r_s ds}.$$

where (t) means $(c_t, M_t/P_t)$. Differentiating with respect to time,

$$-\delta e^{-\delta t} u_c(t) + e^{-\delta t} u_{cc}(t) \frac{dc_t}{dt} + e^{-\delta t} u_{cm}(t) \frac{dm_t}{dt} = -\lambda r_t \, e^{-\int_{s=0}^{t} r_s ds}$$

where $m_t \equiv M_t/P_t$. Dividing by $e^{-\delta t} u_c(t)$, we obtain the intertemporal first order condition:

$$-\frac{c_t u_{cc}(t)}{u_c(t)} \frac{dc_t}{c_t} - \frac{m_t u_{cm}(t)}{u_c(t)} \frac{dm_t}{m_t} = (r_t - \delta)\, dt. \tag{42}$$

The first-order condition with respect to M is

$$\begin{aligned}
\frac{\partial}{\partial M_t} : e^{-\delta t} u_m\,(t)\, \frac{1}{P_t} &= \lambda e^{-\int_{s=0}^{t} r_s ds}\, (i_t - i_t^m)\, \frac{1}{P_t} \\
e^{-\delta t} u_m\,(t) &= e^{-\delta t} u_c(t)\, (i_t - i_t^m) \\
\frac{u_m\,(t)}{u_c(t)} &= i_t - i_t^m. \qquad (43)
\end{aligned}$$

The last equation is the usual money demand curve.

Thus, an equilibrium $c_t = y_t$ satisfies

$$\begin{aligned}
\frac{-c_t u_{cc}(t)}{u_c(t)} \frac{dc_t}{c_t} - \frac{m_t u_{cm}(t)}{u_c(t)} \frac{dm_t}{m_t} &= -\delta dt + \left(i_t - \frac{dP_t}{P_t} \right) dt \qquad (44) \\
\frac{u_m\,(t)}{u_c(t)} &= i_t - i_t^m \qquad (45) \\
\frac{B_0 + M_0}{P_0} &= \int_{t=0}^{\infty} e^{-\int_{s=0}^{t} r_s ds} \left[s_t + (i_t - i_t^m)\, \frac{M_t}{P_t} \right] dt \qquad (46)
\end{aligned}$$

The last equation combines the consumer's budget constraint and equilibrium $c = y$. I call it the government debt valuation formula.

12.3.1. CES functional form

I use a standard money in the utility function specification,I specify a CES functional form,

$$u(c_t, m_t) = \frac{1}{1-\gamma}\left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}}.$$

I use the notation $m = M/P$, with capital letters for nominal and lowercase letters for real quantities.

This CES functional form nests three important special cases. Perfect substitutes is the case $\theta = 0$:

$$u(c_t, m_t) = \frac{1}{1-\gamma}\left[c_t + \alpha m_t\right]^{1-\gamma}.$$

The Cobb-Douglas case is $\theta \longrightarrow 1$:

$$u(c_t, m_t) \longrightarrow \frac{1}{1-\gamma}\left[c_t^{\frac{1}{1+\alpha}} m_t^{\frac{\alpha}{1+\alpha}}\right]^{1-\gamma}. \tag{47}$$

The monetarist limit is $\theta \to \infty$:

$$u(c_t, m_t) \to \frac{1}{1-\gamma}\left[\min\left(c_t, \alpha m_t\right)\right]^{1-\gamma}.$$

I call it the monetarist limit because money demand is then $M_t/P_t = c_t/\alpha$, i.e. $\alpha = 1/V$, and the interest elasticity is zero. The separable case is $\theta = \gamma$:

$$u(c_t, m_t) = \frac{1}{1-\gamma}\left[c_t^{1-\gamma} + \alpha m_t^{1-\gamma}\right].$$

In the separable case, u_c is independent of m, so money has no effect on the intertemporal substitution relation, and hence on inflation and output dynamics in a new-Keynesian model under an interest rate target. Terms in $(\theta - \gamma)$ or $(\sigma - \xi)$ with $\sigma = 1/\gamma$ and $\xi = 1/\theta$ will characterize deviations from the separable case, how much the marginal utility of consumption is affected by money.

With this functional form, the derivatives are

$$u_c = \left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]^{\frac{\theta-\gamma}{1-\theta}} c_t^{-\theta}$$

$$u_m = \left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]^{\frac{\theta-\gamma}{1-\theta}} \alpha m_t^{-\theta}.$$

Equilibrium condition (45) becomes

$$\frac{u_m(t)}{u_c(t)} = \alpha\left(\frac{m_t}{c_t}\right)^{-\theta} = i_t - i_t^m. \tag{48}$$

The second derivative with respect to consumption is

$$\frac{u_{cc}}{u_c} = (\theta-\gamma)\frac{1}{\left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]} c_t^{-\theta} - \theta c_t^{-1}$$

$$-\frac{cu_{cc}}{u_c} = -\frac{(\theta-\gamma)\, c_t^{1-\theta} - \theta\left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]}{\left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]}$$

$$-\frac{cu_{cc}}{u_c} = \frac{\gamma c_t^{1-\theta} + \theta\alpha m_t^{1-\theta}}{c_t^{1-\theta} + \alpha m_t^{1-\theta}}$$

$$-\frac{cu_{cc}}{u_c} = \gamma\frac{\left[1 + \frac{\theta}{\gamma}\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}\right]}{\left[1 + \alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}\right]}.$$

The cross derivative is

$$\begin{aligned}\frac{mu_{cm}}{u_c} &= (\theta-\gamma)\frac{\alpha m_t^{1-\theta}}{c_t^{1-\theta} + \alpha m_t^{1-\theta}}\\ &= (\theta-\gamma)\frac{\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}{1 + \alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}.\end{aligned}$$

or, using (48)

$$\frac{mu_{cm}}{u_c} = (\theta-\gamma)\frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1 + \left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}.$$

12.3.2. Money demand

Money demand (48) can be written

$$\frac{m_t}{c_t} = \left(\frac{1}{\alpha}\right)^{-\xi} (i_t - i_t^m)^{-\xi}. \tag{49}$$

where $\xi = 1/\theta$ becomes the interest elasticity of money demand, in log form, and α governs the overall level of money demand.

The steady state obeys

$$\frac{m}{c} = \left(\frac{1}{\alpha}\right)^{-\xi} (i - i^m)^{-\xi}. \tag{50}$$

so we can write money demand (49) in terms of steady state real money as

$$\frac{m_t}{c_t} = \left(\frac{m}{c}\right) \left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi}, \tag{51}$$

avoiding the parameter α. (Throughout, numbers without time subscripts denote steady state values.)

The product $\frac{m}{c}(i - i^m)$, the interest cost of holding money, appears in many subsequent expressions. It is

$$\frac{m}{c}(i - i^m) = \left(\frac{1}{\alpha}\right)^{-\xi} (i - i^m)^{1-\xi}.$$

With $\xi < 1$, as interest rates go to zero this interest cost goes to zero as well.

12.3.3. Intertemporal Substitution

The first order condition for the intertemporal allocation of consumption (44) is

$$-\frac{c_t u_{cc}(t)}{u_c(t)} \frac{dc_t}{c_t} - \frac{m_t u_{cm}(t)}{u_c(t)} \frac{dm_t}{m_t} = -\delta dt + (i_t - \pi_t)\, dt$$

where $\pi_t = dP_t/P_t$ is inflation. This equation shows us how, with nonseparable utility, monetary policy can distort the allocation of consumption over time, in a way not captured by the usual interest rate effect. That is the central goal here. In the case of complements, $u_{cm} > 0$ (more money raises the marginal utility of consumption), larger money growth makes it easier to consume in the future relative to the present, and acts like a higher interest rate, inducing higher consumption growth.

Substituting in the CES derivatives,

$$\gamma \frac{1+\frac{\theta}{\gamma}\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}{1+\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}\frac{dc_t}{c_t} - (\theta-\gamma)\frac{\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}{1+\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}\frac{dm_t}{m_t} = -\delta dt + (i_t - \pi_t)\,dt$$

and using (48) to eliminate α

$$\gamma \frac{1+\frac{\theta}{\gamma}\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1+\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}\frac{dc_t}{c_t} - (\theta-\gamma)\frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1+\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}\frac{dm_t}{m_t} = -\delta dt + (i_t - \pi_t)\,dt \tag{52}$$

We can make this expression prettier as

$$\gamma\frac{dc_t}{c_t} + (\theta-\gamma)\frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1+\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}\left(\frac{dc_t}{c_t} - \frac{dm_t}{m_t}\right) = -\delta dt + (i_t - \pi_t)\,dt$$

Rexpressing in terms of the intertemporal substitution elasticity $\sigma = 1/\gamma$ and interest elasticity of money demand $\xi = 1/\theta$, and multiplying by σ,

$$\frac{dc_t}{c_t} + \left(\frac{\sigma-\xi}{\xi}\right)\frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1+\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}\left(\frac{dc_t}{c_t} - \frac{dm_t}{m_t}\right) = -\delta\sigma dt + \sigma(i_t - \pi_t)\,dt. \tag{53}$$

We want to substitute interest rates for money. To that end, differentiate the money demand curve

$$\frac{m_t}{c_t} = \left(\frac{m}{c}\right)\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi}$$

$$\frac{m_t}{c_t}\left(\frac{dm_t}{m_t} - \frac{dc_t}{c_t}\right) = -\xi\left(\frac{m}{c}\right)\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi}\frac{d\left(i_t - i_t^m\right)}{i_t - i_t^m}$$

$$\left(\frac{dc_t}{c_t} - \frac{dm_t}{m_t}\right) = \xi\frac{\frac{m}{c}}{\frac{m_t}{c_t}}\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi}\frac{d\left(i_t - i_t^m\right)}{i_t - i_t^m}$$

Substituting,

$$\frac{dc_t}{c_t}+\left(\frac{\sigma-\xi}{\xi}\right)\frac{\left(\frac{m_t}{c_t}\right)\left(i_t-i_t^m\right)}{1+\left(\frac{m_t}{c_t}\right)\left(i_t-i_t^m\right)}\left(\xi\frac{\frac{m}{c}}{\frac{m_t}{c_t}}\left(\frac{i_t-i_t^m}{i-i^m}\right)^{-\xi}\frac{d\left(i_t-i_t^m\right)}{i_t-i_t^m}\right)=-\delta\sigma dt+\sigma\left(i_t-\pi_t\right)dt.$$

$$\frac{dc_t}{c_t}+\left(\sigma-\xi\right)\frac{m}{c}\frac{1}{1+\left(\frac{m_t}{c_t}\right)\left(i_t-i_t^m\right)}\left(\frac{i_t-i_t^m}{i-i^m}\right)^{-\xi}d\left(i_t-i_t^m\right)=-\delta\sigma dt+\sigma\left(i_t-\pi_t\right)dt.$$

With $x_t=\log c_t$, $dx_t=dc_t/c_t m$, approximating around a steady state, and approximating that the interest cost of holding money is small, $\left(\frac{m}{c}\right)\left(i-i^m\right) << 1$, we obtain the intertemporal substitution condition modified by interest costs,

$$\frac{dx_t}{dt}+\left(\sigma-\xi\right)\frac{m}{c}\frac{d\left(i_t-i_t^m\right)}{dt}=\sigma\left(i_t-\pi_t\right). \tag{54}$$

In discrete time,

$$E_t x_{t+1}-x_t+\left(\sigma-\xi\right)\left(\frac{m}{c}\right)\left[E_t\left(i_{t+1}-i_{t+1}^m\right)-\left(i_t-i_t^m\right)\right]=\sigma\left(i_t-E_t\pi_{t+1}\right).$$

For models with monetary control, one wants an IS curve expressed in terms of the monetary aggregate. From (53), with the same approximations and $\tilde{m}=\log(m)$,

$$\frac{dx_t}{dt}+\left(\frac{\sigma-\xi}{\xi}\right)\left(\frac{m}{c}\right)\left(i-i^m\right)\left(\frac{dx_t}{dt}-\frac{d\tilde{m}_t}{dt}\right)=\sigma\left(i_t-\pi_t\right)dt. \tag{55}$$

In discrete time,

$$\left(E_t x_{t+1}-x_t\right)+\left(\frac{\sigma-\xi}{\xi}\right)\left(\frac{m}{c}\right)\left(i-i^m\right)\left[\left(E_t x_{t+1}-x_t\right)-E_t\left(\tilde{m}_{t+1}-\tilde{m}_t\right)\right]=\sigma\left(i_t-\pi_t\right). \tag{56}$$

12.4. Money Demand

Figure 22 presents M1, M2, and reserves, each divided by GDP versus the three month treasury rate. The left side shows levels, the right side shows logs. Since the model is

$$\frac{m_t}{c_t}=\alpha(i_t-i_t^m)^{-\xi}=\frac{m}{c}\left(\frac{i_t-i_t^m}{i-i^m}\right)^{-\xi}$$

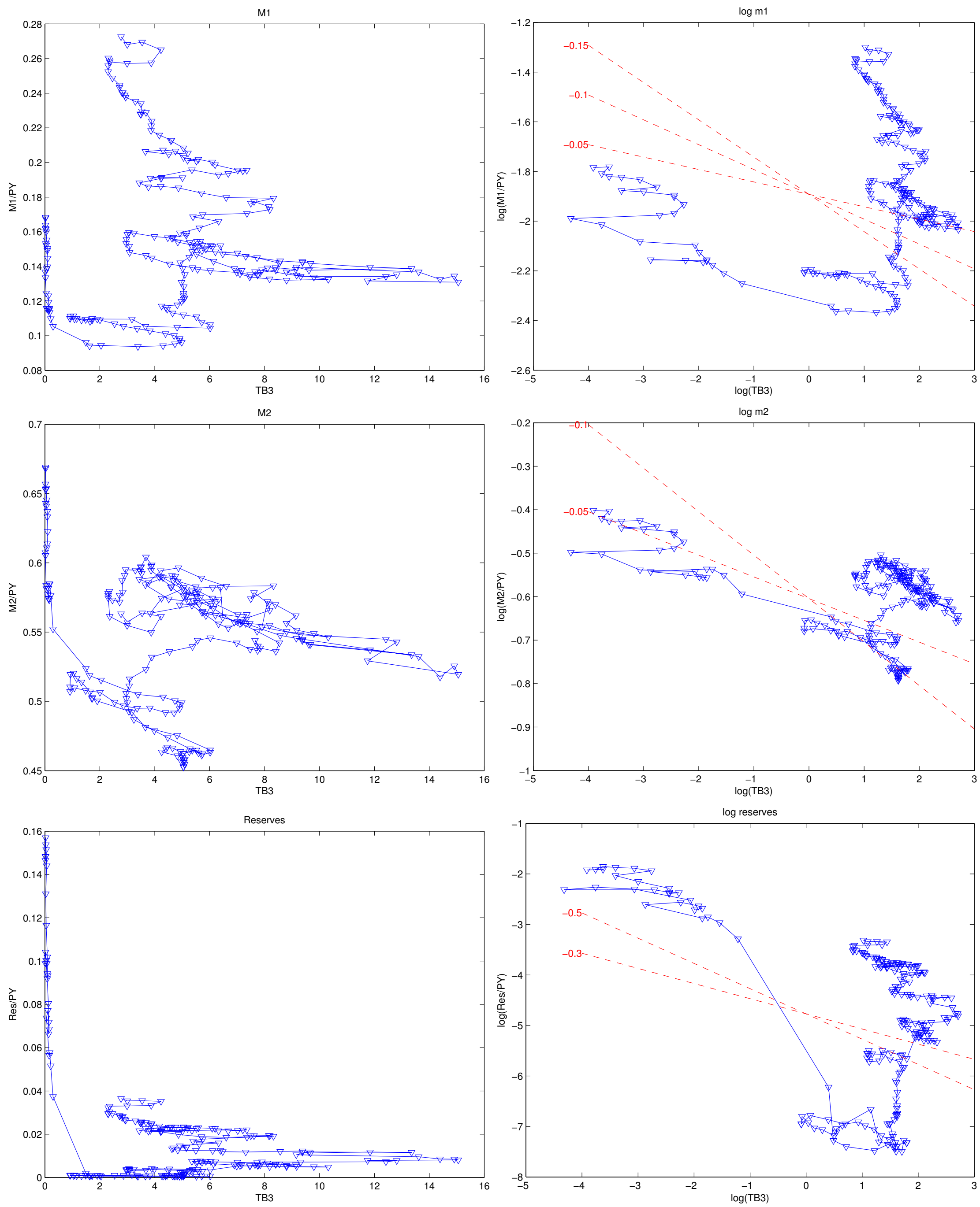


Figure 22: M1, M2, Reserves, vs three-month treasury rate

we want the log form of the elasticity, $\partial \log(m)/\partial \log(i - i^m)$.

The graphs include lines with slope $\xi = -0.05,\ -0.1$, and -0.15 so one can judge an interest elasticity by ocular regression.

In each case, on top of cyclical movements in money and interest rates, there was a clear downward trend in money holdings, conventionally attributed to improvements in financial technology. The cyclical variations however fit reasonably well with an interest elasticity around $\xi = 0.10$.

The era of zero interest rates following 2008 has led to dramatic increases in money holdings. The plots in levels on the left hand side suggest pure liquidity traps in which money and bonds are perfect substitutes. The log graphs on the right however suggest that even at these extremely low values some form of interest elasticity remains.

For the purposes of this paper, the most important point is to verify the general consensus (for example, Lucas (1988)) of an interest elasticity around $\xi = 0.1$, plausibly smaller than most estimates of the intertemporal substitution coefficient σ.

The overall level of money demand m/c depends on the definition of money. On the eve of the financial crisis, reserves were less than 1% of GDP, M1 was about 10% of GDP, and M2 was about 50% of GDP. However, much of M1 and M2 paid interest, so their $(i - i^m)$ was likely smaller.