The Fragile Benefits of Endowment Destruction

John Y. Campbell* and John H. Cochrane†‡

March 18, 2015

Abstract

The benefits of endowment destruction documented by Ljungqvist and Uhlig (2014), and the related possibility that consumption can lower habits, are fragile. Both issues result from a particular way of discretely approximating the underlying continuous-time model, or of adapting it to jumps. Other ways of calculating the discrete-time approximation or extending the model to jumps easily overturn the results, while making no difference to the model's description of asset prices and quantities. This analysis shows how to extend models so that jumps give the same result as a jump limit of continuous movements.

*Harvard University and NBER.

†University of Chicago Booth School of Business, Hoover Institution, and NBER.

‡We thank Darrell Duffie, Lars Ljungqvist, Stavros Panageas, Monika Piazzesi, and Philip Protter for helpful comments and discussion.

1 Beneficial Endowment Destruction?

If a consumer has preferences with habits, lowering consumption today can lower future habits, and potentially raise utility overall. Is habit persistence this strong in the Campbell-Cochrane (1999, 2000) model?

Suppose a Campbell-Cochrane consumer at time $t = 0$ is at the steady-state log surplus consumption ratio $s_0 = \bar{s}$. Suppose log endowment grows steadily at the rate g for periods 0, 1, 2, i.e. $y_0 = -g$, $y_1 = 0$, $y_2 = g$. Suppose that the government destroys some of the time-1 endowment, so that log time-1 consumption $c_1 = \psi < 0$. Thereafter the log endowment follows the usual process

$$y_{t+1} = g + y_t + v_{t+1},\ t > 2, v_{t+1} \sim \mathcal{N}(0, \sigma_v^2) \tag{1}$$

and $c_t = y_t$. We simulate this endowment process for a variety of ψ, and we evaluate the utility function by averaging over a large number of simulations.

The solid line in Figure 1 presents the consumer's utility. We include $\psi > 0$, transfers from abroad or manna from heaven, as well as endowment destruction $\psi < 0$.

Near $\psi = 0$, utility rises with ψ. Despite habit formation, endowment destruction hurts and transfers help. However, the relationship is u-shaped so that destroying a discrete amount of the endowment raises utility. This is Ljungqvist and Uhlig's (2014) main point.

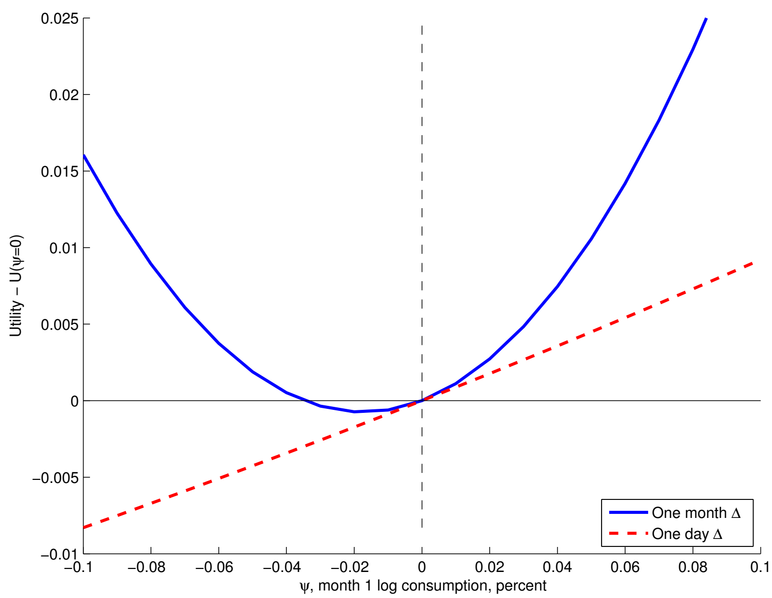

Figure 1: Effect of endowment destruction. At time $t = 1$ an amount ψ is added or subtracted to log consumption. The figure plots achieved utility as a function of ψ. The solid line uses a monthly time interval, and perturbs consumption at month $t = 1$ only. The dashed line uses a daily time interval, modifying consumption in a V shaped pattern for two months.

The solid line uses a monthly time interval, as we did in Campbell and Cochrane (1999). The dashed line of Figure 1 presents instead the same endowment destruction episode, with the model simulated at a daily time interval. Starting at time $t = 0$ we add (or subtract, when $\psi < 0$) $\psi/30$ from log consumption each day for 30 days. We then restore consumption the same way, producing a V-shaped daily consumption pattern that bottoms out at the same ψ value on the 30th day. We simulate the model forward as before.

In this daily simulation, the Ljungqvist and Uhlig pattern disappears. Output destruction is always harmful, and transfers are always welcome. Still smaller time intervals lead to visually indistinguishable results.

To produce Figure (1), we simulate the monthly or daily version of (1), using $\sigma_v = 0.015\Delta$, where $\Delta = 1/12$ or $\Delta = 1/360$ is the simulation interval. We then recursively calculate the surplus consumption ratio,

$$s_{t+\Delta} = (1-\phi^{\Delta})\bar{s} + \phi^{\Delta} s_t + \lambda(s_t)(c_{t+\Delta} - c_t - g\Delta), \tag{2}$$

where

$$\lambda(s) \equiv \begin{cases} \frac{1}{\bar{S}}\sqrt{1-2(s-\bar{s})} - 1; & s \le s_{\max} \\ 0; & s \ge s_{\max} \end{cases}, \tag{3}$$

$$s_t = \log(S_t) = \log\left(\frac{C_t - X_t}{C_t}\right),$$

$c_t = \log(C_t)$ is log consumption, X_t is habit, and $\bar{s} = \log \bar{S} = \log(0.057)$, $\phi = 0.87$, $g = 0.0189$ and $s_{\max} = \bar{s} + 1/2(1-\bar{S}^2)$ are parameters. We then evaluate the utility function

$$U = \frac{1}{1-\gamma} E \sum_{t=0,\Delta,2\Delta,\dots}^{\infty} \delta^t (C_t - X_t)^{1-\gamma} = \frac{1}{1-\gamma} E \sum_t \delta^t e^{(1-\gamma)(c_t + s_t)}$$

with $\delta = 0.89$, $\gamma = 2.00$ by averaging over simulations. This is the Campbell-Cochrane (1999) model and parameters.

Why are the results of the daily simulation so different from those of the monthly simulation? Examine (2) closely. During the daily simulation $\Delta = 1/360$, the surplus consumption ratio responds to each little bit of consumption each day, and a new $\lambda(s_t)$ is recomputed each day. During the monthly simulation the same $\lambda(s_0)$ applies to all the daily changes from $t = 0$ to $t = 1/12$, and the same $\lambda(s_{1/12})$ applies to all the daily changes from $t = 1/12$ to $t = 2/12$. A daily simulation that uses the beginning-of-the-month value of $\lambda(s_t)$ rather than the continuously evolving one would generate Ljungqvist and Uhlig's result.

Simulation interval per se is not a key difference. The key difference is whether the surplus consumption ratio can adjust within the endowment-destruction period to the evolving consumption decline.

Whether habits adjust contemporaneously to a consumption change is also not a key difference. The specification (2) already allows habits to change contemporaneously with changes in consumption, even with s_t predetermined in $\lambda(s_t)$. (This feature ensures that consumption cannot fall below habit in this discrete-time version of the model.) The key difference is the nature of habit adjustment when s_t is held fixed for a month in $\lambda(s_t)$, vs the nature of habit adjustment when s_t changes during the month in $\lambda(s_t)$.

2 Continuous Time

Ljungqvist and Uhlig might respond, let the government destroy endowment for one day only, returning the next day, and the effect is restored. And we might respond, let the surplus consumption ratio respond each hour as consumption is being destroyed during the day, and the effect disappears.

These issues are best understood by writing the underlying continuous time version of our model. The endowment follows a geometric Brownian motion

$$dc_t = gdt + \sigma dz_t, \tag{4}$$

the surplus consumption ratio responds to consumption via

$$ds_t = (1-\phi)(\bar{s} - s_t)\, dt + \lambda(s_t)(dc_t - gdt), \tag{5}$$

and expected utility is

$$W = E\int_{t=0}^{\infty} e^{-\delta t}\frac{(C_t - X_t)^{1-\gamma}}{1-\gamma}dt = E\int_{t=0}^{\infty}\frac{e^{-\delta t}e^{(1-\gamma)(c_t+s_t)}}{1-\gamma}dt.$$

This continuous-time diffusion model does not produce benefits of output destruction. In order to produce Ljungqvist and Uhlig's result, one must extend the model to consider endowment jumps,

$$dc_t = gdt + \sigma dz_t + dJ_t. \tag{6}$$

(For this purpose, one doesn't really need jumps in the underlying consumption process, i.e. to specify a stochastic process for dJ_t including its frequency and distribution. What matters is that the government can induce a downward jump in consumption and its reversal, which can be completely unexpected. If one specifies dJ_t as a process with nonzero mean, one should adjust the gdt drift term as usual.)

Equation (5) on its own cannot handle jumps, as it is not clear whether s_t in $\lambda(s_t)$ refers to the right limit, the left limit, or some intermediate value. So one must generalize (5). To produce the endowment destruction result, one generalizes (5) by specifying that $\lambda(s_{t\_})$ applies to the entire jump episode,

$$ds_t = (1-\phi)(\bar{s} - s_t)\, dt + \lambda(s_{t\_})(dc_t - gdt), \tag{7}$$

where $s_{t\_}$ denotes the left-hand limit, the value of s_t just before the jump.

One then specifies a downward jump $dc_0 = \psi < 0$, followed immediately by an opposite upward jump; i. e. followed by an upward jump at at time ε later, and take the limit as $\varepsilon \to 0$. (The endowment destruction effect can appear for $\varepsilon > 0$, but this limit gives an elegant and simple version of the result because dt terms drop out.)

In the downward jump, the surplus consumption ratio then changes by

$$s_0 - s_{0\_} = \lambda(s_{0\_})\psi. \tag{8}$$

On the way back up, however,

$$s_\varepsilon - s_{\varepsilon\_} = \lambda(s_{\varepsilon\_})(-\psi). \tag{9}$$

For small ε, so $s_0 = s_{\varepsilon\_}$, the round trip produces a net change in the surplus consumption ratio equal to

$$s_\varepsilon - s_{0\_} = \left[\lambda(s_{0\_}) - \lambda(s_0)\right]\psi.$$

The slope of the $\lambda(s)$ function ($\lambda'(s) < 0$) means that the surplus consumption ratio rises at time ε by more than it declined at 0, in response to $\psi < 0$, producing a net rise in the surplus consumption ratio, $s_\varepsilon > s_{0\_}$ and thus a decline in the habit. As $\varepsilon \to 0$, the endowment destruction episode has less and less impact on the flow utility from consumption.

In the limit, then, an instantaneous jump-valued endowment destruction and reversal generates a downward reset of habits, free of any direct utility cost, simply and costlessly raising future utility. This is a powerful version of Ljungqvist and Uhlig's result.

However, suppose we produce the same decline and rise in consumption by a rapid but continuous sample path, such as a V shape, that takes place in time ε. Return to the diffusion model (4)-(5), and let us produce the ψ decline by a linear path $z_t = 2(\psi/\varepsilon)t,\ t \in (0, \varepsilon/2)$, and a contrary linear recovery between time $\varepsilon/2$ and ε. As $\varepsilon \to 0$ the dt terms of (5) drop out, so the surplus consumption ratios at times $\varepsilon/2$ and ε solve the differential equations

$$\int_{s_0}^{s_{\varepsilon/2}} \frac{1}{\lambda(s)} ds = \sigma \int_{t=0}^{\varepsilon/2} dz_t = c_{\varepsilon/2} - c_0 = \psi \tag{10}$$

$$\int_{s_{\varepsilon/2}}^{s_\varepsilon} \frac{1}{\lambda(s)} ds = \sigma \int_{t=\varepsilon/2}^{\varepsilon} dz_t = c_\varepsilon - c_{\varepsilon/2} = -\psi. \tag{11}$$

But $\int_a^b 1/\lambda(s)ds = -\int_b^a 1/\lambda(s)ds$, so $s_\varepsilon = s_0$. The second differential equation (11) exactly retraces the steps of the first one (10). This continuous-sample-path endowment-destruction operation produces no change in surplus consumption ratio at all, and thus no change in overall utility.

The difference of the two approaches is clearest to see if we unite (8), written as

$$\frac{1}{\lambda(s_{0_-})}(s_0 - s_{0_-}) = \psi.$$

with its counterpart (10), taking the $\varepsilon \to 0$ limit,

$$\int_{s_{0_-}}^{s_0} \frac{1}{\lambda(s)} ds = \psi$$

In the first case, $\lambda(s_{0_-})$ applies to the entire jump, while in the second case, $\lambda(s)$ adapts continuously as the jump adapts.

These are curious results. A jump in consumption produces a different result than the jump limit of a continuous change in consumption. The consumption jump outpaces surplus-consumption-ratio adjustment, but a femtosecond continuous consumption change does not. An instantaneously reversed jump produces a change in the surplus consumption ratio, but a continuous V-shaped movement arbitrarily close to the jump produces no change at all in the surplus consumption ratio. Two jumps in a row produce a different response than a single jump twice the size. And in any discretely sampled data there is no way to tell the difference between a fast continuous change and a jump.

These results are not, however, necessary features of an extension of the habit model to jump processes. One can easily extend the specification of the surplus-consumption-ratio adjustment function (5) in a different way from the left-limit approach of (7), so that jumps in consumption produce the same surplus-consumption-ratio change as their continuous-sample-path limits produce, as follows.

Write the solutions to the differential equation

$$\int_{s_-}^{s} \frac{1}{\lambda(\xi)} d\xi = c - c_- \tag{12}$$

as

$$s - s_- = f(c - c_-, s_-). \tag{13}$$

Then, write the generalization of (5) to handle jumps, as

$$\begin{aligned} ds_t &= (1-\phi)\left(\bar{s}-s_{t_-}\right)dt + f(dc_t - gdt, s_{t_-}) \\ &= (1-\phi)\left(\bar{s}-s_{t_-}\right)dt + \lambda(s_{t_-})dz_t + f(dJ_t, s_{t_-}) \end{aligned} \tag{14}$$

rather than (5).

Since, by differentiating (12),

$$\left.\frac{\partial f(c-c_-, s_-)}{\partial(c-c_-)}\right|_{c-c_-=0} = \lambda(s_-),$$

(14) is the same as the original specification (5) for diffusion changes, dc_t of order dt or dz. We could have written the original specification (5) in this form. Thus, the form (14) is also a generalization of the original (5) to handle jumps, not a modification of that specification. And by construction, the specification (14) produces the same result for a jump dc_t as for the jump-limit of continuous sample paths.

If we generalize the model to jumps via (14) rather than (7), then, the results of a jump are the same as those of an arbitrarily fast continuous approximation to the jump. And instantaneous output destruction has no effect on habits, surplus consumption ratio, or utility.

In our case, with $\lambda(s)$ given by (3), $\lambda(s) = \frac{1}{\bar{S}}\left(\sqrt{1-2(s-\bar{s})}-1\right)$, we can perform the integral on the left of (12) yielding[1]

$$\bar{S}\left[\log(\lambda(s)) - \log(\lambda(s_-))\right] + \bar{S}^2\left[\lambda(s)-\lambda(s_-)\right] = c - c_-.$$

This expression defines $f(\cdot)$ implicitly. However, we cannot solve this expression for a closed-form representation of $f(\cdot)$ in (13) or (14).

To clarify the idea with an explicit example, then, suppose that the surplus consumption ratio follows consumption by

$$dc_t = gdt + \sigma dz_t, \tag{15}$$

$$ds_t = s_t\mu dt + s_t dc_t. \tag{16}$$

The solutions to (15)-(16) are

$$s_{t+\Delta} = s_t e^{(c_{t+\Delta}-c_t)+(\mu-\frac{1}{2}\sigma^2)\Delta}. \tag{17}$$

Taking the jump limit, $\Delta \to 0$ holding a fixed change in consumption, the jump model that produces the same response as its continuous-sample-path limit is

$$dc_t = gdt + \sigma dz_t + dJ_t \tag{18}$$

$$\begin{aligned} ds_t &= s_{t_-}(e^{dc_t}-1) \equiv f(dc_t, s_{t_-}) \\ &= s_{t_-}(gdt + \sigma dz_t + e^{dJ_t}). \end{aligned} \tag{19}$$

[1] The algebra:

$$d\lambda(s) = \frac{-1}{\bar{S}\sqrt{1-2(s-\bar{s})}}ds = \frac{-1}{\bar{S}\left[1+\bar{S}\lambda(s)\right]}ds$$

$$\int \frac{1}{\lambda(s)}ds = \bar{S}\int\left(\frac{1}{\lambda}+\bar{S}\right)d\lambda = \bar{S}\log(\lambda) + \bar{S}^2.$$

The second equality in (19) relates this example to the notation of (14). When there is no jump in c, if dc_t is of order dz or dt, formula (19) reduces to the original (16). (One needs second-order terms for $d(e^c)$ but not for e^{dc}.) Equation (19) is thus a generalization not a modification of (16).

(To derive equation (12), we approximated the jump by differentiable functions of time. In equation (19), we get the same result if we approximate the jump as a limit of fast diffusion realizations as well. Though (19) is a stochastic differential equation, the jump function is still defined by the ordinary differential equation (12).)

The usual left limit,

$$ds_t = s_{t\_} dc_t \tag{20}$$

also reduces to (16) when there are no jumps, so it too is a valid generalization. This generalization gives a different answer from (19) – the limit point of a jump occasions a different response from limiting continuous changes, and two half-jumps produce a different answer than a full jump. For example, (20) allows s_t to jump to negative values, where (19) forces s_t to remain positive for any jump.

A nonlinear function of a jump such as in (19) is a standard idea in the time-series literature. For example, see Bass (2004) p. 5 and p. 6 equation (3.7). Kurtz, Pardoux, and Protter (1995), section 6 p. 365 ff., describe jump processes as a limit of continuous approximants.

Nonlinear functions are not that common in economics and finance applications because, when observing only one series, one can change instead the distribution of the jump variable. The issue comes up when one wants to link two observed variables that depend on the same jump, as in consumption and the surplus consumption ratio, asset prices and wealth, or stock price and option price.

3 Which Model is Correct?

The point is not that one or the other method of extending a model to jumps is right or wrong. The point is that there *is* a way (19) to extend a diffusion model to include jumps, an alternative to the method (20) of just substituting left limits for state variables, so that the result of a jump is the same as the result of infinitely fast continuous movement, and that multiple small jumps have the same result as a large jump. The continuous-time version of our model does not produce benefits of endowment destruction, and there is a way to extend that model to jumps which also does not produce benefits of endowment destruction.

Which extension to jumps is correct? The answer depends on the economic situation.

For example, consider models with bankruptcy constraints. Agents who can continuously adjust their investments may always avoid bankruptcy in a diffusion setting. If we extend such a model to jumps as in (19), implicitly preserving the investor's ability to trade as fast as asset prices change even in the jump limit, we will preserve bankruptcy avoidance in face of a jump in prices. However, if we model portfolio adjustment to jumps with the left-limit generalization as in (20), agents may be forced in to bankruptcy for price jumps.

Sometimes, one introduces jumps precisely to model a situation in which prices can move faster than agents can adjust their portfolios, so agents may be forced to bankruptcy. Then the left-limit generalization is correct. But if one wants to extend a model to jumps for other reasons, while avoiding bankruptcy, negative consumption, negative marginal utility (consumption below habits in some habit specifications), violations of budget constraints, feasibility conditions, borrowing constraints, and so forth, then one should choose a generalization such as (19) in which the jump gives the same result as the continuous limit.

Similarly, when extending option pricing models to jumps, one may want to model the jump in such a way that investors cannot adjust portfolios fast enough. Then the left-limit extension is appropriate, and investors must hold the jump risk. But one may wish to accommodate jumps in asset prices to better fit asset price dynamics while maintaining investor's ability to dynamically hedge. Then the nonlinear extension is appropriate, maintaining the equivalence between jumps and the limiting diffusion.

Which is the right way to generalize our habit model to jumps? In our view, the continuous-limit specification is a more sensible economic model. While asset prices may move faster than investors can trade, it is not obvious that one should modify our model so that consumption can move faster than the surplus consumption ratio can adjust. Already, our model specifies that habits themselves move contemporaneously with consumption when consumption changes, even when the surplus consumption ratio stays constant, in order to avoid consumption falling below habit.

In fact, this is the important lesson we draw from Ljungqvist and Uhlig's result. Like them, we regard increases in utility from habit destruction as a pathological result. The continuous-time version of this result is even more pathological: The government can make us all better off by shutting down electric power for a millisecond, resetting our habits as it makes all the clocks flash 12:00. Yes, sensible specifications of habit-persistence utility should not produce such results. Yes, therefore, write the model in continuous time and extend the model to jumps, if one wishes to do so, in a way in which jumps have the same effects as arbitrarily close continuous paths. Let the surplus consumption ratio adapt as quickly as consumption can change, as habits themselves already do in our specification.

4 Habits That Move the Wrong Way

Figure 2 presents the relationship between monthly log consumption growth and log habit. The evolution of surplus consumption ratio described by (2) or (5) is really just a means to this end, a description of how habits x adapt to consumption c. The figure calculates $s_{t+\Delta}$, from (2) and then unwinds the definition of $s_{t+\Delta}$ to the implied habit $x_{t+\Delta}$.

The solid line verifies Ljungqvist and Uhlig's second paradoxical result: Discrete increases in consumption can lead to a contemporaneous decline in habits. This result is behind their benefits of endowment destruction. It is not so much the decline in consumption pushing habits down that does the work. Rather it is the further decline in habits when consumption takes the second discrete upward jump that leads to the benefits of endowment destruction.

The dashed line in Figure 2 again subdivides the monthly consumption change into 30 increments. This line is also visually identical for any finer time interval, and to the continuous time result for arbitrarily fast but continuous consumption changes. We see that in this version, habit is a nondecreasing function of consumption throughout.

The negative relation between consumption and habit is also artifact of introducing jumps in such a way that the jump produces a different result from its continuous limit.

(Figure 2 presents the calculation when the surplus consumption ratio is at its steady state $\bar{s}$, which is the hardest case. The derivative $dx_t/dc_t = 0$ at the expected value $\Delta c_t = g$ in this case, represented by the slope of the solid line where it intersects the vertical line at $\Delta c_t = g$. For other

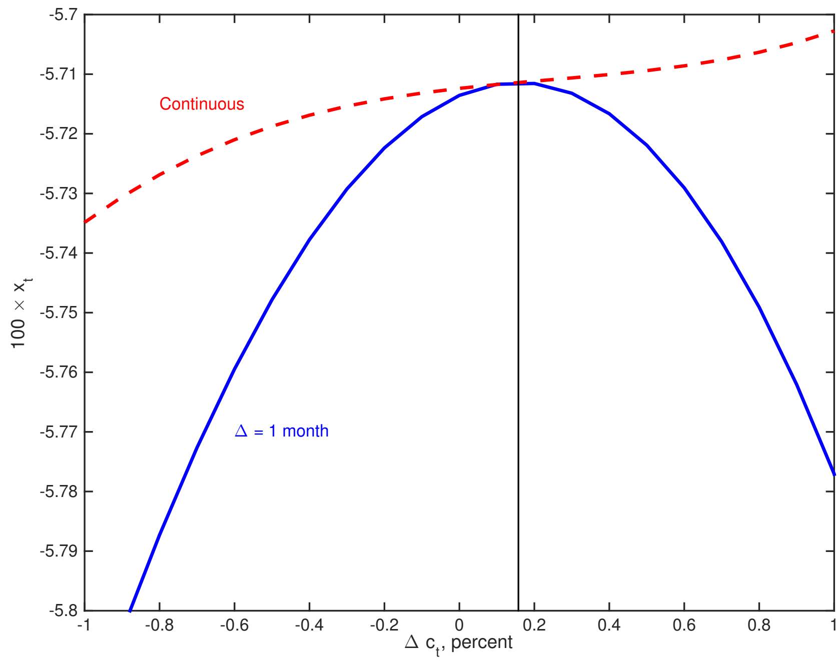

Figure 2: The effect of log consumption growth Δc_{t+1} on contemporaneous log habit x_{t+1}, when $s_t = \bar{s}$. The solid line uses a monthy time interval. The dashed line subdivides the consumption change into 30 steps. The vertical line indicates the value of consumption $c_{t+1} - c_t = g$ at which the term multiplying $\lambda(s_t)$ is zero in the surplus consumption ratio transition equation.

values of the initial surplus consumption ratio, we have $dx_t/dc_t > 0$ at $\Delta c_t = g$; the solid curve moves to the right leaving a positive derivative in the middle. This leaves a larger area with a positive derivative, and requires a larger positive consumption realization to see a decline in habit.)

5 Conclusion

We verify and generalize Ljungqvist and Uhlig's (2014) results: In the Campbell and Cochrane (1999) model, specified in discrete time at a monthly frequency, a one-month discrete endowment destruction can raise utility, and a discrete increase in consumption can lower habits.

However, simulating the model at a daily frequency, these results are overturned. And the underlying continuous-time diffusion model shows neither result. Extending the model to jumps in one way, in which consumption jumps produce a different effect on habits from the jump-limit of continuous consumption movements, produces beneficial endowment destruction and habits that go the wrong way. Extending the model to jumps in another way, in which consumption jumps have the same effects as the jump-limits of continuous movements, again removes the benefits of endowment destruction and contrary movement of habits.

None of these variations affect the model's description of the joint movements of asset prices and consumption for which the model was designed, and for which we chose the simple monthly discretization and simulation interval.

More directly, in our model one can add to the utility function any function of aggregate consumption $v(c_t^a)$ and thereby change the welfare implications of endowment addition or destruction completely, while not changing at all the individual's first order conditions and therefore asset pricing and quantity predictions. Just add a u-shaped $v(c_1^a)$ function to Figure 1 matching the u shape shown there, and utility becomes a smoothly rising function of time 1 consumption ψ, even in the monthly simulation.

Though models with temporally-dependent preferences often do imply interesting optimal dynamics, we agree that beneficial output destruction would be an unwelcome prediction of our model. However, one wishes for a fundamental, robust critique of a model, either of its predictions for data or of its policy implications, that the critique is not easily resolved by small changes in specification or numerical approximation procedure that have no effect on the points for which the model was created.

For this reason, we take as the important lesson of Ljungqvist and Uhlig's (2014) result that if one wishes to extend our habit model to continuous time and consumption jump processes, one should choose the extension that does not produce benefits of endowment destruction, and in which jumps have the same effect as their continuous-sample-path limits. We also are reminded that evaluating simplified models, taken literally, based on policy prescriptions, or features of the data far beyond the range of phenomena for which they were developed, can often lead to fragile results.

6 References

Bass, Richard F. 2004. "Stochastic Differential Equations with Jumps." *Probability Surveys* 1, 1-19. DOI: 10.1214/154957804100000015.

Campbell, John Y., and John H. Cochrane. 1999. "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior." *Journal of Political Economy* 107, 205-251.

Campbell, John Y., and John H. Cochrane. 2000. "Explaining the Poor Performance of Consumption-Based Asset Pricing Models." *Journal of Finance* 55, 2863-2878.

Kurtz, Thomas G., Étienne Pardoux and Phillip Protter. 1995. "Strantovich stochastic differential equations driven by general semimartingales." *Annales de l'I.H.P., section B,* tome 31, n.2, 351-366. `http://www.numdam.org/item?id=AIHPB_1995__31_2_351_0`.

Ljungqvist, Lars, and Harald Uhlig. 2014. "Comment on the Campbell-Cochrane Habit Model." *Journal of Political Economy*, this issue.