A Continuous-Time Asset Pricing Model with Habits and Durability

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Abstract

I solve a continuous-time asset pricing economy with quadratic utility and complex temporal nonseparabilities.

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1 Introduction

I solve a linear-quadratic economy and find asset prices in continuous time. I then extend the model to include a habit/durability process. This problem is much easier in continuous time and using the Hansen-Sargent (1991) prediction formulas.

This problem applies the tools reviewed in Cochrane (2012). Those tools derive from Hansen and Sargent (1991), who extend the Hansen-Sargent (1980, 1981) methodology to continuous time. Heaton (1993) uses the same tools to solve the quantity dynamics of the models I present here.

2 Time-separable model

The warmup problem is the standard quadratic utility permanent income model, with an AR(1) income stream,

$$\max E_t \int_0^\infty e^{-r\tau}\left(-\frac{1}{2}\right)(c^* - c_{t+\tau})^2 d\tau \text{ s.t.} \tag{1}$$

$$dk_t = (rk_t + y_t - c_t)\,dt \tag{2}$$

$$dy_t = -\rho y_t dt + \sigma dB_t.$$

One of the conveniences of continuous time is apparent: you don't have to worry about timing, whether investment made at t joins the capital stock at t and generates a return at $t+1$, (finance timing) or whether investment made at t sits a period before contributing to capital stock at $t+1$ (macroeconoimc timing).

I find the equilibrium consumption process in level and differenced form,

$$c_t = rk_t + \frac{r}{r+\rho} y_t$$

$$dc_t = \frac{r}{r+\rho}\sigma dB_t.$$

Then, I find the price of the consumption stream,

$$p_t = E_t \int_{\tau=0}^\infty e^{-r\tau}\left(\frac{c^* - c_{t+\tau}}{c^* - c_t}\right) c_{t+\tau} d\tau.$$

The result is

$$p_t = \frac{c_t}{r} - \frac{1}{c^* - c_t}\frac{\sigma^2}{(r+\rho)^2},$$

or equivalently

$$p_t = E_t \int_{\tau=0}^\infty e^{-r\tau} c_{t+\tau} - \frac{1}{c^* - c_t}\frac{\sigma^2}{(r+\rho)^2}$$

The first term is the "risk neutral" term, as emphasized by its expansion in the last equality. The second term is a risk adjustment. The higher the variance of income, the lower the price. As c approaches c^* the quadratic utility investor gets more risk averse. At c^* he's at bliss and doesn't want more consumption, so you can't induce him to take risk by giving him more consumption. So as c rises to c^*, the price discount rises.

The price of the endowment stream has an identical risk correction,

$$\begin{aligned}
p_t &= E_t \int_{\tau=0}^{\infty} e^{-r\tau} \left(\frac{c^* - c_{t+\tau}}{c^* - c_t} \right) y_{t+\tau} d\tau \\
&= \frac{y_t}{\rho + r} - \frac{1}{c^* - c_t} \frac{\sigma^2}{(r+\rho)^2} \\
&= E_t \int_{\tau=0}^{\infty} e^{-r\tau} y_{t+\tau} d\tau - \frac{1}{c^* - c_t} \frac{\sigma^2}{(r+\rho)^2}
\end{aligned}$$

2.0.1 Model solution

First, I show that the "flow" constraint (2) (together with limits on how fast capital can grow) imply the "present value" constraint

$$E_t \int_0^{\infty} e^{-r\tau} c_{t+\tau} d\tau = E_t \int_0^{\infty} e^{-r\tau} y_{t+\tau} d\tau + k_t \tag{3}$$

Quotes on "present value" because $E_t \int_0^\infty e^{-r\tau} c_{t+\tau} d\tau$ is *not* the present value (price) of the risky consumption stream!

To show this equivalence, write the flow budget constraint (2) as

$$\begin{aligned}
(D - r)\, k_t &= y_t - c_t \\
k_t &= \frac{1}{(D-r)} (y_t - c_t)
\end{aligned}$$

writing it out,

$$k_t = \int_0^{\infty} e^{-r\tau} y_{t+\tau} d\tau - \int_0^{\infty} e^{-r\tau} c_{t+\tau} d\tau_t$$

Applying E_t to both sides, we obtain (3).

Second, since the rate of return r equals the discount rate, the basic asset pricing first order condition gives

$$u_t' = E_t u_{t+\tau}'$$
$$c^* - c_t = E_t (c^* - c_{t+\tau}).$$
$$c_t = E_t (c_{t+\tau})$$

and hence

$$E_t (dc_t) = 0.$$

Thus, we know that $dc_t = 0 \times dt + ()dB_t$, with the latter loading depending on the resource constraint.

Next, we substitute first order conditions $c_t = E_t c_{t+\tau}$ and the income forecast $E_t y_{t+\tau} = e^{-\rho\tau} y_t$ into the resource constraint (3) to find the actual consumption process,

$$\begin{aligned}
E_t \int_{\tau=0}^{\infty} e^{-r\tau} c_t d\tau &= E_t \int_{\tau=0}^{\infty} e^{-r\tau} e^{-\rho\tau} y_t d\tau + k_t \\
\frac{c_t}{r} &= \frac{y_t}{r+\rho} + k_t
\end{aligned}$$

$$c_t = rk_t + \frac{r}{r+\rho} y_t$$

This is the familiar permanent-income rule.

To derive the "random walk" rule, you can just take differences of the last equation,

$$\begin{aligned} dc_t &= rdk_t + \frac{r}{r+\rho} dy_t \\ dc_t &= r\left(rk_t + y_t - c_t\right) dt + \frac{r}{r+\rho}\left(-\rho y_t dt + \sigma dB_t\right) \\ &= r\left(rk_t + y_t - rk_t - \frac{r}{r+\rho} y_t\right) dt + \frac{r}{r+\rho}\left(-\rho y_t dt + \sigma dB_t\right) \end{aligned}$$

$$dc_t = \frac{r}{r+\rho} \sigma dB_t.$$

and hence

$$c_{t+\tau} = c_t + \frac{r}{r+\rho} \sigma \int_{s=0}^{\tau} dB_{t-s}. \tag{4}$$

Both of these operations are prettier as instances of the Hansen-Sargent prediction formulas of course.

The price of the consumption stream is

$$p_t = E_t \int_{\tau=0}^{\infty} e^{-r\tau} \left(\frac{c^* - c_{t+\tau}}{c^* - c_t}\right) c_{t+\tau} d\tau$$

I'll compute these by brute force, though one can also apply the Hansen-Sargent formulas. From (4),

$$\begin{aligned} E_t c_{t+\tau} &= c_t \\ E_t c_{t+\tau}^2 &= c_t^2 + \left(\frac{r\sigma}{r+\rho}\right)^2 \tau \end{aligned}$$

$$p_t = E_t \int_{\tau=0}^{\infty} e^{-r\tau} \frac{c^* c_t - c_t^2 - \left(\frac{r\sigma}{r+\rho}\right)^2 \tau}{c^* - c_t} d\tau$$

$$p_t = \frac{1}{r} \frac{c_t\left(c^* - c_t\right)}{c^* - c_t} - \frac{1}{c^* - c_t} \left(\frac{r\sigma}{r+\rho}\right)^2 \int_0^{\infty} \tau e^{-r\tau} d\tau$$

$$p_t = \frac{c_t}{r} - \frac{1}{c^* - c_t} \frac{\sigma^2}{(r+\rho)^2}.$$

Similarly, we can find the price of the income stream,

$$p_t = E_t \int_{\tau=0}^{\infty} e^{-r\tau} \left(\frac{c^* - c_{t+\tau}}{c^* - c_t}\right) y_{t+\tau} d\tau$$

Income follows

$$y_{t+\tau} = e^{-\rho\tau} y_t + \sigma \int_{s=0}^{\tau} e^{-\rho s} dB_{t-s},$$

so

$$
\begin{aligned}
E_t y_{t+\tau} &= e^{-\rho\tau} y_t \\
E_t\left(c_{t+\tau} y_{t+\tau}\right) &= e^{-\rho\tau} c_t y_t + \left(\frac{r\sigma^2}{r+\rho}\right) \int_{s=0}^{\tau} e^{-\rho s} ds = e^{-\rho\tau} c_t y_t + \left(\frac{r\sigma^2}{r+\rho}\right) \frac{1}{\rho}\left(1-e^{-\rho\tau}\right)
\end{aligned}
$$

$$
p_t = \int_{\tau=0}^{\infty} e^{-r\tau} \frac{c^* e^{-\rho\tau} y_t - e^{-\rho\tau} c_t y_t - \left(\frac{r\sigma^2}{r+\rho}\right)\frac{1}{\rho}\left(1-e^{-\rho\tau}\right)}{c^* - c_t} d\tau
$$

$$
p_t = y_t \int_{\tau=0}^{\infty} e^{-(r+\rho)\tau} d\tau - \frac{r\sigma^2}{(r+\rho)\,\rho} \frac{1}{c^* - c_t} \int_{\tau=0}^{\infty} e^{-r\tau}\left(1-e^{-\rho\tau}\right) d\tau
$$

$$
\begin{aligned}
p_t &= \frac{y_t}{\rho+r} - \frac{r\sigma^2}{(r+\rho)\,\rho}\frac{1}{c^*-c_t}\left(\frac{1}{r} - \frac{1}{r+\rho}\right) \\
p_t &= \frac{y_t}{\rho+r} - \frac{1}{c^*-c_t}\frac{\sigma^2}{(r+\rho)^2}
\end{aligned}
$$

3 A model with habits and durability

I generalize the linear-quadratic model to include a utility function with a rich temporal non-separability.

The quantity dynamics are solved by Heaton (1993), section 3.1, though I hope the notation here is a bit simpler. I extend Heaton's analysis to find asset prices. The consumer faces a linear capital accumulation technology and an AR(1) income stream as before.

The consumer's problem is

$$
\begin{aligned}
&\max_{\{c_t\}} -\frac{1}{2} E \int_{t=0}^{\infty} e^{-rt}\left(c^* - c_t - \int_{\tau=0}^{\infty} b(\tau)\, c_{t-\tau} d\tau\right)^2 \ s.t. \\
dk_t &= \left(r k_t + y_t - c_t\right) dt \\
dy_t &= -\rho y_t dt + \sigma dB_t
\end{aligned}
$$

We can write the objective in operator form,

$$
-\frac{1}{2} E \int_{t=0}^{\infty} e^{-rt}\left(c^* - \left[1 + \mathcal{L}_b(D)\right] c_t\right)^2 . \tag{5}
$$

The present-value form of the resource constraint is, as before,

$$
E_t \int_{\tau=0}^{\infty} e^{-r\tau} c_{t+\tau} d\tau = k_t + E_t \int_{\tau=0}^{\infty} e^{-r\tau} y_{t+\tau} d\tau = k_t + \frac{y_t}{r+\rho}. \tag{6}
$$

We normally think of the nonseparability $b(\tau)$ as generating "habit persistence" or "durability" in consumption. If $b(\tau) > 0$, then past consumption contributes positively to current utility, as past durable goods purchases do. If $b(\tau) < 0$, then past consumption raises current *marginal* utility, as habit-forming goods do.

We can also use nonseparability for a different purpose: The habit/durability term shifts the bliss point, which controls risk aversion. When current consumption c_t is closer to the composite bliss point, the consumer becomes more risk averse. Thus, temporal nonseparability allows us to control risk aversion and the cyclical behavior of asset prices.

Controlling risk aversion is useful to making the linear quadratic model vaguely reasonable. One of its biggest problems with a fixed bliss point is that higher consumption makes the consumer more risk averse, where in reality we think risk aversion is likely independent of wealth in the long run, and rises in consumption relative to the recent past may make people *less* risk averse, as in the Campbell-Cochrane (1999) model. By moving the bliss point, we can capture both ideas, and thus make the linear-quadratic model a more useful approximation.

As an example, I use a sum of two exponentials for the $b(\tau)$ function, so the problem is is

$$\max_{\{c_t\}} -\frac{1}{2}\int_{t=0}^{\infty} e^{-rt}\left(c^* - c_t - hx_t - kz_t\right)^2$$
$$x_t = \lambda \int_{\tau=0}^{\infty} e^{-\lambda\tau} c_{t-\tau} d\tau$$
$$z_t = \delta \int_{\tau=0}^{\infty} e^{-\delta\tau} c_{t-\tau} d\tau.$$

In the operator notation of (5),

$$\mathcal{L}_b(D) = \frac{h\lambda}{\lambda + D} + \frac{k\delta}{\delta + D}.$$

This formulation allows for both "habit" and "durable" effects. For example, consumption can be durable in the short run, but induce habits in the long run, with $h < 0$, $k > 0$, $\lambda < \delta$. Having included two exponentials, the generalization to an arbitrary sum of exponentials is clear.

Here are the major results: The consumption process follows

$$D\left[1 + \mathcal{L}_b(D)\right] c_t = \frac{r\sigma}{r+\rho}\left[1 + \mathcal{L}_b(r)\right] DB_t.$$

You can see the natural generalization from the time–separable case in which $\mathcal{L}_b(D) = 0$. In the double-exponential case,

$$dc_t = -\left[h\lambda\left(c_t - x_t\right) + k\delta(c_t - z_t)\right] dt + \frac{r\sigma}{\rho + r}\left(1 + \frac{h\lambda}{r+\lambda} + \frac{k\delta}{r+\delta}\right) dB_t \tag{7}$$

Without nonseparabilities, consumption follows the familiar random walk. With nonseparabilities, marginal utility is still a random walk, but consumption is not, and adjusts towards $(h, k > 0)$ or away $(h, k < 0)$ from its recent past.

The price of the consumption stream is

$$p_t = \frac{1}{\left[1 + \mathcal{L}_b(r)\right]}\left[1 + \frac{r\mathcal{L}_b(r) - D\mathcal{L}_b(D)}{(r - D)}\right]\frac{c_t}{r} - \frac{\left[1 + \mathcal{L}_b(r)\right]}{c^* - \left[1 + \mathcal{L}_b(D)\right] c_t}\left(\frac{\sigma}{\rho + r}\right)^2 \tag{8}$$

The first term is just the risk-neutral present value of the consumption stream, i.e.

$$E_t \int_{\tau=0}^{\infty} e^{-r\tau} c_{t+\tau} d\tau.$$

The second term in (8) is a discount for risk.

In the double-exponential case the price of the consumption stream is

$$p_t = \frac{1}{r}\frac{c_t + \frac{h\lambda}{r+\lambda}x_t + \frac{k\delta}{r+\delta}z_t}{1 + \frac{h\lambda}{r+\lambda} + \frac{k\delta}{r+\delta}} - \frac{\left(1 + \frac{h\lambda}{r+\lambda} + \frac{k\delta}{r+\delta}\right)}{c^* - c_t - hx_t - kz_t}\left(\frac{\sigma}{\rho + r}\right)^2$$

In either general or special formulas, the first term is now more complex because consumption dynamics are complex, and the first term includes expected increases or decreases in consumption itself. The denominator in the second term is the interesting component. This term is still current marginal utility. This term shows us how risk premiums evolve over time. When c_t is larger holding x_t and z_t constant, we still see risk aversion and the price discount rise. But now risk aversion is determined by c_t relative to the habit or durable stock x_t and z_t, which also vary over time. This generalization can allow the model to to produce more realistic time series.

The price of the endowment stream $\{y_t\}$ is similarly,

$$p_t = \frac{y_t}{r+\rho} - \frac{[1 + \mathcal{L}_b(r)]}{c^* - [1 + \mathcal{L}_b(D)]\, c_t}\frac{\sigma^2}{(r+\rho)^2}$$

In the double-exponential case,

$$p_t = \frac{y_t}{r+\rho} - \frac{\left[1 + \frac{h\lambda}{r+\lambda} + \frac{k\delta}{r+\delta}\right]}{c^* - [c_t + hx_t + kz_t]}\frac{\sigma^2}{(r+\rho)^2}$$

The first term is again the risk neutral value,

$$\frac{y_t}{r+\rho} = E_t \int_{\tau=0}^{\infty} e^{-r\tau} y_{t+\tau} d\tau.$$

The second term reflects the same time-varying risk aversion as before, due to changing marginal utility at time t.

3.0.2 Derivation

First order conditions and consumption drift.

The consumer's first-order conditions state that marginal utility is a martingale,

$$c_t + \int_{\tau=0}^{\infty} b(\tau) c_{t-\tau} d\tau = E_t\left[c_{t+\Delta} + \int_{\tau=0}^{\infty} b(\tau) c_{t+\Delta-\tau} d\tau\right] \quad (9)$$

Taking the limit, marginal utility follows a random walk

$$E_t\left[d\left(c_t + \int_{\tau=0}^{\infty} b(\tau) c_{t-\tau} d\tau\right)\right] = 0$$

I.e., we know that consumption follows a classic "autoregressive" process, which we can write either as

$$dc_t = -\left[b(0)c_t + \int_{\tau=0}^{\infty} b'(\tau) c_{t-\tau} d\tau\right] dt + \gamma dB_t$$

or

$$dc_t = -\left[\int_{\tau=0}^{\infty} b(\tau) dc_{t-\tau}\right] dt + \gamma dB_t.$$

We don't know what γ is, which the resource constraint will tell us.

In operator notation, marginal utility is

$$c^* - [1 + \mathcal{L}_b(D)]\, c_t$$

so the first order condition is

$$E_t \left\{ D\, [1 + \mathcal{L}_b(D)]\, c_t \right\} = 0.$$

The two "autoregressive" representations are potentially convenient rewritings of this condition

$$D\, [1 + \mathcal{L}_b(D)]\, c_t = [1 + \mathcal{L}_b(D)]\, Dc_t = [D + b(0) + \mathcal{L}_{b'}(D)]\, c_t = \gamma DB_t$$

Internal vs. external nonseparabilities

This first-order condition is the same whether the nonseparability is "internal" or "external." If external, these are directly the first order conditions, in equilibrium where individual = aggregate consumption. If internal, the marginal utility of consumption today includes its effects on future utility,

$$\frac{\partial U}{\partial c_t} = \left(c^* - c_t - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t-\tau} d\tau \right) + E_t \int_{s=0}^{\infty} \left(c^* - c_{t+s} - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t+s-\tau} d\tau \right) b(s) ds$$

The first order condition is now

$$\frac{\partial U}{\partial c_t} = E_t \left(\frac{\partial U}{\partial c_{t+\Delta}} \right)$$

$$\begin{aligned} &\left(c^* - c_t - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t-\tau} d\tau \right) + E_t \int_{s=0}^{\infty} \left(c^* - c_{t+s} - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t+s-\tau} d\tau \right) b(s) ds \\ = \;\; & E_t \left[\left(c^* - c_{t+\Delta} - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t+\Delta-\tau} d\tau \right) + \int_{s=0}^{\infty} \left(c^* - c_{t+\Delta+s} - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t+\Delta+s-\tau} d\tau \right) b(s) ds \right] \end{aligned}$$

Now you can see that (9) is still a solution. Substitute it in to the last equation,

$$\begin{aligned} &\left(c^* - c_t - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t-\tau} d\tau \right) + E_t \int_{s=0}^{\infty} \left(c^* - c_t - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t-\tau} d\tau \right) b(s) ds \\ = \;\; & E_t \left[\left(c^* - c_{t+\Delta} - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t+\Delta-\tau} d\tau \right) + \int_{s=0}^{\infty} \left(c^* - c_{t+\Delta} - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t+\Delta-\tau} d\tau \right) b(s) ds \right] \end{aligned}$$

And hence

$$\left(c^* - c_t - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t-\tau} d\tau \right) \left[1 + \int_{s=0}^{\infty} b(s) ds \right] = E_t \left(c^* - c_{t+\Delta} - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t+\Delta-\tau} d\tau \right) \left[1 + \int_{s=0}^{\infty} b(s) ds \right]$$

$$\left(c^* - c_t - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t-\tau} d\tau \right) = E_t \left(c^* - c_{t+\Delta} - \int_{\tau=0}^{\infty} b\,(\tau)\, c_{t+\Delta-\tau} d\tau \right)$$

Resource constraint and consumption shocks.

To find the response to shocks, I use the Hansen-Sargent prediction formulas. Hansen and Sargent (1991) show that if we express a process in moving-average form,

$$x_t = \int_{\tau=0}^{\infty} b(\tau)\sigma dB_{t-\tau} = \mathcal{L}_b(D)\sigma DB_t,$$

then we can find the moving average representation of the expected discounted value by

$$E_t \int_{\tau=0}^{\infty} e^{-r\tau} x_{t+\tau} d\tau = \left(\frac{\mathcal{L}_b(D) - \mathcal{L}_b(r)}{r - D} \right) \sigma DB_t. \tag{10}$$

The impact multiplier of the expected discounted value – how it responds to a shock – is

$$\lim_{D\to\infty} \left(D \frac{\mathcal{L}_b(D) - \mathcal{L}_b(r)}{r - D} \right) = \mathcal{L}_b(r). \tag{11}$$

(See Cochrane (2012) for an accessible derivation.)

Now, using the Hansen-Sargent response-to-shock formula (11), and the representation

$$\left[1 + \mathcal{L}_b(D)\right] Dc_t = \gamma DB_t$$

we have

$$d\left[E_t \int_{\tau=0}^{\infty} e^{-r\tau} c_{t+\tau} d\tau\right] = ()dt + \frac{1}{r}\frac{1}{1 + \mathcal{L}_b(r)}\gamma.$$

Differentiating the resource constraint (6), we obtain

$$d\left[E_t \int_{\tau=0}^{\infty} e^{-r\tau} c_{t+\tau} d\tau\right] = dk_t + \frac{dy_t}{r+\rho} = \left(rk_t - c_t + \frac{r}{r+\rho} y_t \right) dt + \frac{\sigma}{r+\rho} dB_t$$

Comparing the two expressions, the impact multiplier γ satisfies

$$\frac{1}{r}\frac{\gamma}{1 + \mathcal{L}_b(r)} = \frac{\sigma}{r+\rho}.$$

Therefore,

$$\gamma = \frac{r\sigma}{r+\rho}\left(1 + \mathcal{L}_b(r)\right)$$

In sum, then, consumption follows the process

$$dc_t = -\left[b(0)c_t + \int_{\tau=0}^{\infty} b'(\tau)c_{t-\tau}d\tau\right] dt + \frac{r\sigma}{r+\rho}\left(1 + \int_{\tau=0}^{\infty} e^{-r\tau} b(\tau)d\tau\right) dB_t$$

or,

$$dc_t = -\left[\int_{\tau=0}^{\infty} b(\tau)dc_{t-\tau}\right] dt + \frac{r\sigma}{r+\rho}\left(1 + \int_{\tau=0}^{\infty} e^{-r\tau} b(\tau)d\tau\right) dB_t$$

or, in operator notation,

$$\left[D + \mathcal{L}_b(D)D\right] c_t = \frac{r\sigma}{r+\rho}\left[1 + \mathcal{L}b(r)\right] DB_t.$$

We can write the same result by characterizing the marginal utility process,

$$d\left(c_t + \int_{\tau=0}^{\infty} b(\tau)c_{t-\tau}d\tau\right) = \frac{r\sigma}{r+\rho}\left(1 + \int_{\tau=0}^{\infty} e^{-r\tau} b(\tau)d\tau\right) dB_t \tag{12}$$

$$D\left[1+\mathcal{L}_b(D)\right]c_t = \frac{r\sigma}{r+\rho}\left[1+\mathcal{L}_b(r)\right]DB_t \tag{13}$$

Consumption process, exponential case

Expressing the state in terms of x_t and z_t is useful. Directly, (12) (13) are

$$d\left(c_t + hx_t + kz_t\right) = \frac{r\sigma}{r+\rho}\left[1+\mathcal{L}_b(r)\right]dB_t \tag{14}$$

If we want to study consumption growth, we can substitute from (13)

$$D\left[1+\frac{h\lambda}{\lambda+D}+\frac{k\delta}{\delta+D}\right]c_t = \frac{r\sigma}{r+\rho}\left[1+\frac{h\lambda}{\lambda+r}+\frac{k\delta}{\delta+r}\right]DB_t$$

$$\left[D+h\lambda\left(1-\frac{\lambda}{\lambda+D}\right)+k\delta\left(1-\frac{\delta}{\delta+D}\right)\right]c_t = \frac{r\sigma}{r+\rho}\left[1+\frac{h\lambda}{\lambda+r}+\frac{k\delta}{\delta+r}\right]DB_t.$$

Then, rewriting this in terms of the state variables x_t and z_t,

$$dc_t = -\left[h\lambda\left(c_t-x_t\right)+k\delta(c_t-z_t)\right]dt + \frac{r\sigma}{\rho+r}\left(1+\frac{h\lambda}{r+\lambda}+\frac{k\delta}{r+\delta}\right)dB_t \tag{15}$$

Price of the consumption stream

The price of the consumption stream is

$$p_t = E_t\int_{\tau=0}^{\infty}e^{-r\tau}\frac{c^*-\left[c_{t+\tau}+\int_{s=0}^{\infty}b(s)c_{t+\tau-s}ds\right]}{c^*-\left[c_t+\int_{s=0}^{\infty}b(s)c_{t-s}ds\right]}c_{t+\tau}d\tau$$

The formula simplifies if we recognize that marginal utility follows a random walk. Then

$$\left[c_{t+\tau}+\int_{s=0}^{\infty}b(s)c_{t+\tau-s}ds\right] = \left[c_t+\int_{s=0}^{\infty}b(s)c_{t-s}ds\right]+\frac{r\sigma}{r+\rho}\left(1+\mathcal{L}_b(r)\right)\int_{u=0}^{\tau}dB_{t+u}$$

Substituting this result in the pricing formula, we have

$$p_t = E_t\int_{\tau=0}^{\infty}e^{-r\tau}c_{t+\tau}d\tau - \frac{\frac{r\sigma}{r+\rho}\left[1+\mathcal{L}_b(r)\right]}{c^*-\left[1+\mathcal{L}_b(D)\right]c_t}E_t\int_{\tau=0}^{\infty}e^{-r\tau}\left(\int_{u=0}^{\tau}dB_{t+u}\right)c_{t+\tau}d\tau \tag{16}$$

Let's work on the first term. Using the Hansen-Sargent prediction formula and the operator expression for the consumption process,

$$\left[D+\mathcal{L}_b(D)D\right]c_t = \frac{r\sigma}{r+\rho}\left(1+\mathcal{L}_b(r)\right)DB_t,$$

we have

$$\begin{aligned}
E_t\int_{\tau=0}^{\infty}e^{-r\tau}c_{t+\tau}d\tau &= \frac{r\sigma}{r+\rho}\frac{\frac{(1+\mathcal{L}_b(r))}{D[1+\mathcal{L}_b(D)]}-\frac{(1+\mathcal{L}_b(r))}{r[1+\mathcal{L}_b(r)]}}{r-D}DB_t \\
&= \frac{\sigma}{r+\rho}\frac{r\left[1+\mathcal{L}_b(r)\right]-D\left[1+\mathcal{L}_b(D)\right]}{(r-D)\,D\left[1+\mathcal{L}_b(D)\right]}DB_t \\
&= \left[\frac{r\left[1+\mathcal{L}_b(r)\right]-D\left[1+\mathcal{L}_b(D)\right]}{(r-D)\left[1+\mathcal{L}_b(r)\right]}\right]\frac{c_t}{r} \\
&= \frac{1}{\left[1+\mathcal{L}_b(r)\right]}\left[1+\frac{r\mathcal{L}_b(r)-D\mathcal{L}_b(D)}{(r-D)}\right]\frac{c_t}{r}
\end{aligned}$$

I can't get further in general, so using the double-exponential functional form,

$$\begin{aligned}
E_t \int_{\tau=0}^{\infty} e^{-r\tau} c_{t+\tau} d\tau &= \frac{1}{\left[1+\frac{h\lambda}{r+\lambda}+\frac{k\delta}{r+\delta}\right]} \left[1+\frac{r\left(\frac{h\lambda}{\lambda+r}+\frac{k\delta}{\delta+r}\right)-D\left(\frac{h\lambda}{\lambda+D}+\frac{k\delta}{\delta+D}\right)}{(r-D)}\right] \frac{c_t}{r} \\
&= \frac{1}{\left[1+\frac{h\lambda}{r+\lambda}+\frac{k\delta}{r+\delta}\right]} \left[1+\frac{h\lambda^2}{(\lambda+r)(\lambda+D)}+\frac{k\delta^2}{(\delta+r)(\delta+D)}\right] \frac{c_t}{r} \\
&= \left[\frac{c_t+\frac{h}{(r+\lambda)}x_t+\frac{k}{(r+\delta)}z_t}{1+\frac{h\lambda}{r+\lambda}+\frac{k\delta}{r+\delta}}\right] \frac{1}{r}
\end{aligned}$$

Now for the second part. Start with the hard-looking part,

$$E_t \int_{\tau=0}^{\infty} e^{-r\tau} \left(\int_{u=0}^{\tau} dB_u\right) c_{t+\tau} d\tau.$$

Actually, this is easy, and the risk adjustment term is quite general. Start with any consumption process,

$$\begin{aligned}
c_t &= \int_{\tau=0}^{\infty} h(\tau) dB_{t-\tau} \\
E_t\left[c_{t+\tau} \int_{u=0}^{\tau} dB_{t+u}\right] &= \int_{s=0}^{\tau} h(s) ds \\
E_t \int_{\tau=0}^{\infty} e^{-r\tau} c_{t+\tau} \left(\int_{u=0}^{\tau} dB_{t+u}\right) d\tau &= \int_{\tau=0}^{\infty} e^{-r\tau} d\tau \int_{s=0}^{\tau} h(s) ds \\
&= \int_{s=0}^{\infty} h(s) \left(\int_{\tau=s}^{\infty} e^{-r\tau} d\tau\right) ds \\
&= \int_{s=0}^{\infty} h(s) \frac{1}{r} e^{-rs} d\tau \\
&= \frac{1}{r} \mathcal{L}_h(r).
\end{aligned}$$

Using our consumption process,

$$c_t = \frac{r\sigma}{r+\rho} \frac{[1+\mathcal{L}_b(r)]}{D[1+\mathcal{L}_b(D)]} DB_t = \frac{r\sigma}{r+\rho} \frac{\left[1+\frac{h\lambda}{r+\lambda}+\frac{k\delta}{r+\delta}\right]}{D\left[1+\frac{h\lambda}{D+\lambda}+\frac{k\delta}{D+\delta}\right]} DB_t,$$

we have

$$E_t \int_{\tau=0}^{\infty} e^{-r\tau} c_{t+\tau} B_{t+\tau} d\tau = \frac{\sigma}{r+\rho} \frac{\left[1+\frac{h\lambda}{r+\lambda}+\frac{k\delta}{r+\delta}\right]}{r\left[1+\frac{h\lambda}{r+\lambda}+\frac{k\delta}{r+\delta}\right]} = \frac{\sigma}{r+\rho} \frac{1}{r}$$

Now, adding back the other terms of (16),

$$\begin{aligned}
&\frac{\frac{r\sigma}{r+\rho}[1+\mathcal{L}_b(r)]}{c^*-[1+\mathcal{L}_b(D)]c_t} E_t \int_{\tau=0}^{\infty} e^{-r\tau} \left(\int_{u=0}^{\tau} dB_{t+u}\right) c_{t+\tau} d\tau \\
&= \frac{\frac{r\sigma}{r+\rho}[1+\mathcal{L}_b(r)]}{c^*-[1+\mathcal{L}_b(D)]c_t} \frac{\sigma}{r+\rho} \frac{1}{r} \\
&= \frac{[1+\mathcal{L}_b(r)]}{c^*-[1+\mathcal{L}_b(D)]c_t} \left(\frac{\sigma}{r+\rho}\right)^2
\end{aligned}$$

Price of the endowment stream

The price of the endowment stream is

$$p_t = E_t \int_{\tau=0}^{\infty} e^{-r\tau} \frac{c^* - [c_{t+\tau} + \int_{s=0}^{\infty} b(s) c_{t+\tau-s} ds]}{c^* - [c_t + \int_{s=0}^{\infty} b(s) c_{t-s} ds]} y_{t+\tau} d\tau$$

Working analogously, we have

$$\left[c_{t+\tau} + \int_{s=0}^{\infty} b(s) c_{t+\tau-s} ds\right] = \left[c_t + \int_{s=0}^{\infty} b(s) c_{t-s} ds\right] + \frac{r\sigma}{r+\rho}\left(1 + \mathcal{L}_b(r)\right) \int_{u=0}^{\tau} dB_u$$

$$p_t = E_t \int_{\tau=0}^{\infty} e^{-r\tau} y_{t+\tau} d\tau - \frac{\frac{r\sigma}{r+\rho}\left[1 + \mathcal{L}_b(r)\right]}{c^* - \left[1 + \mathcal{L}_b(D)\right] c_t} E_t \int_{\tau=0}^{\infty} e^{-r\tau} \left(\int_{u=0}^{\tau} dB_{t+u}\right) y_{t+\tau} d\tau$$

$$p_t = E_t \left(\int_{\tau=0}^{\infty} e^{-r\tau} e^{-\rho\tau} d\tau\right) y_t - \frac{\frac{r\sigma}{r+\rho}\left[1 + \mathcal{L}_b(r)\right]}{c^* - \left[1 + \mathcal{L}_b(D)\right] c_t} E_t \int_{\tau=0}^{\infty} e^{-r\tau} \left(\int_{u=0}^{\tau} dB_{t+\tau-u}\right) \left(\int_{u=0}^{\infty} e^{-\rho u} \sigma dB_{t+\tau-u}\right) d\tau$$

$$p_t = \frac{y_t}{r+\rho} - \frac{\frac{r\sigma^2}{r+\rho}\left[1 + \mathcal{L}_b(r)\right]}{c^* - \left[1 + \mathcal{L}_b(D)\right] c_t} \int_{\tau=0}^{\infty} e^{-r\tau} \left(\int_{u=0}^{\tau} e^{-\rho u}\right) d\tau$$

$$p_t = \frac{y_t}{r+\rho} - \frac{\frac{r\sigma^2}{r+\rho}\left[1 + \mathcal{L}_b(r)\right]}{c^* - \left[1 + \mathcal{L}_b(D)\right] c_t} \frac{1}{\rho}\left(\frac{1}{r} - \frac{1}{r+\rho}\right)$$

$$p_t = \frac{y_t}{r+\rho} - \frac{\left[1 + \mathcal{L}_b(r)\right]}{c^* - \left[1 + \mathcal{L}_b(D)\right] c_t} \frac{\sigma^2}{(r+\rho)^2}$$

In the double-exponential case,

$$p_t = \frac{y_t}{r+\rho} - \frac{\left[1 + \frac{h\lambda}{r+\lambda} + \frac{k\delta}{r+\delta}\right]}{c^* - [c_t + h x_t + k z_t]} \frac{\sigma^2}{(r+\rho)^2}$$

4 References

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