The Government Debt Valuation Equation. An Appendix to "Understanding Policy."

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Abstract

This note derives the various forms of the government debt valuation equation.

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1 Introduction

This note establishes the basic present value forms of the government debt valuation equation.

2 Discrete time, one period debt

2.1 One period, discount debt

$B_{t-1}(t)$ is the face value of debt sold at the end of period $t-1$, repaid at period t. M_{t-1} is issued at period $t-1$ and held overnight to period t. In this case, we have

$$\frac{M_{t-1}+B_{t-1}(t)}{P_t} = E_t \sum_{j=0}^{\infty} m_{t,t+j} \left[(T_{t+j}-G_{t+j}) + \frac{M_{t+j}}{P_{t+j}} \frac{i_{t+j}}{1+i_{t+j}} \right]$$

Derivation: Start with the nominal flow budget constraint

$$M_{t-1}+B_{t-1}(t) = P_t\left(T_t - G_t\right) + M_t + Q_t^{(1)} B_t(t+1)$$

where $Q_t^{(1)}$ is the one-period nominal bond price,

$$Q_t^{(1)} = E_t\left(m_{t,t+1} \frac{P_t}{P_{t+1}} \right) = \frac{1}{1+i_t}.$$

Then substitute for $Q_t^{(1)}$,

$$\begin{aligned}
\frac{M_{t-1}+B_{t-1}(t)}{P_t} &= (T_t - G_t) + \frac{M_t}{P_t}\frac{i_t}{1+i_t} + \frac{M_t}{P_t} E_t\left(m_{t,t+1}\frac{P_t}{P_{t+1}}\right) + E_t\left(m_{t,t+1}\frac{B_t(t+1)}{P_{t+1}}\right) \\
\frac{M_{t-1}+B_{t-1}(t)}{P_t} &= (T_t - G_t) + \frac{M_t}{P_t}\frac{i_t}{1+i_t} + E_t\left(m_{t,t+1}\frac{M_t + B_t(t+1)}{P_{t+1}}\right)
\end{aligned}$$

and iterate forward.

2.2 Seignorage as a flow

An equivalent expression has only debt on the left, and counts seignorage as the flow proceeds from money creation rather than the interest spread,

$$\frac{B_{t-1}(t)}{P_t} = E_t \sum_{t=0}^{\infty} m_{t+j} \left[(T_{t+j}-G_{t+j}) + \frac{M_{t+j}-M_{t+j-1}}{P_{t+j}} \right].$$

Derivation: Again start with the flow constraint,

$$M_{t-1}+B_{t-1}(t) = P_t\left(T_t - G_t\right) + M_t + Q_t^{(1)} B_t(t+1)$$

$$Q_t^{(1)} = E_t\left(m_{t,t+1}\frac{P_t}{P_{t+1}}\right) = \frac{1}{1+i_t}.$$

$$\frac{B_{t-1}(t)}{P_t} = (T_t - G_t) + \frac{M_t - M_{t-1}}{P_t} + E_t\left(m_{t,t+1}\frac{B_t(t+1)}{P_{t+1}}\right)$$

and again iterate forward:

$$\frac{B_{t-1}(t)}{P_t} = (T_t - G_t) + \frac{M_t - M_{t-1}}{P_t} + E_t\left[m_{t,t+1}\left((T_{t+1} - G_{t+1}) + \frac{M_{t+1} - M_t}{P_{t+1}} + m_{t+1,t+2}\frac{B_{t+1}(t+2)}{P_{t+2}}\right)\right]$$

etc..

2.3 The discount rate

We can discount by a rate of return rather than the stochastic discount factor m.

$$\begin{aligned}
B_{t-1}(t) &= P_t s_t + \frac{1}{1+i_t}B_t(t+1) \\
\frac{B_{t-1}(t)}{P_t} &= s_t + \left(\frac{1}{1+i_t}\frac{P_{t+1}}{P_t}\right)\frac{B_t(t+1)}{P_{t+1}} \\
\frac{B_{t-1}(t)}{P_t} &= \sum_{j=0}^{\infty}\left[\prod_{k=0}^{j}\left(\frac{1}{1+i_{t+k}}\frac{P_{t+k}}{P_{t+k-1}}\right)\right]s_{t+j}
\end{aligned}$$

$$\frac{1}{1+i_t}\frac{P_{t+1}}{P_t} = \frac{1}{R_{t+1}}$$

is the ex-post real return on nominal bonds. So, we can "discount" by this return. It is not the real or nominal interest rate, however. The real interest and nominal interes rates are

$$\begin{aligned}
\frac{1}{1+i_t} &= E_t\left(m_{t+1}\frac{P_t}{P_{t+1}}\right) \\
\frac{1}{1+r_t} &= E_t(m_{t+1})
\end{aligned}$$

2.4 Real debt

Real debt $b_{t-1}(t)$ promises to pay P_t dollars. Equivalently, debt denominated in foreign currency or gold must be paid off with P_t dollars With real debt, the flow constraint is

$$\frac{M_{t-1}}{P_t} + b_{t-1}(t) = (T_t - G_t) + \frac{M_t}{P_t} + \frac{1}{1+r_t^f}b_t(t+1)$$

Iterating forward, the "present value" relations are

$$\begin{aligned} b_{t-1}(t) &= \sum_{j=0}^{\infty}\left[\prod_{k=0}^{j}\left(\frac{1}{1+r_{t+k}^{f}}\right)\right]\left[(T_{t+j}-G_{t+j})+\frac{M_{t+j}-M_{t+j-1}}{P_{t+j}}\right] \\ b_{t-1}(t) &= \sum_{j=0}^{\infty}E_t\left(m_{t,t+j}\right)\left[(T_{t+j}-G_{t+j})+\frac{M_{t+j}-M_{t+j-1}}{P_{t+j}}\right] \\ b_{t-1}(t) &= E_t\left\{\sum_{j=0}^{\infty}m_{t,t+j}\left[(T_{t+j}-G_{t+j})+\frac{M_{t+j}-M_{t+j-1}}{P_{t+j}}\right]\right\} \end{aligned}$$

These expressions hold *ex-post* as well as ex-ante. $b_{t-1}(t)$ places a constraint on subsequent actions, if the debt is not to default. Also, in this case we can use the real riskfree rate to discount.

2.5 Interest-paying debt

Suppose that bonds pay interest i_t at time $t+\Delta$. It's i_t because the nominal interest rate is fixed one period in advance. This case is useful also as the discrete-time counterpart to the continuous-time formulas. It also is more realistic to think of a steady stock of floating rate debt rather than continual rollover of explicit one-period discount debt, though the two arrangements are of course economically equivalent.

Debt B_t is outstanding at time t, and pays interest i_t at time $t+\Delta$. Thus, the time $t+\Delta$ flow constraint is

$$P_{t+\Delta}s_{t+\Delta}+(B_{t+\Delta}-B_t)-i_tB_t=0$$

I show in this case that the "present value" formula holds with slightly different timing.

$$\frac{B_t}{P_t}=E_t\sum_{j=1}^{\infty}\frac{\Lambda_{t+j\Delta}}{\Lambda_t}s_{t+j\Delta}.$$

Again, we can also discount at the ex-post real return on nominal bonds,

$$\frac{B_t}{P_t}=\sum_{j=1}^{\infty}\prod_{k=1}^{j}\left(\frac{1}{1+i_{(t+k-1)\Delta}}\frac{P_{t+k\Delta}}{P_{t+(k-1)\Delta}}\right)s_{t+j\Delta}=\sum_{j=1}^{\infty}\left(\prod_{k=1}^{j}\frac{1}{R_{(t+k-1)\Delta,(t+k)\Delta}}\right)s_{t+j\Delta}=\sum_{j=1}^{\infty}\frac{1}{R_{t,,(t+k)\Delta}}s_{t+j\Delta}$$

Derivations: Write the flow constraint as

$$s_{t+\Delta}+\frac{B_{t+\Delta}}{P_{t+\Delta}}-(1+i_t)\frac{P_t}{P_{t+\Delta}}\frac{B_t}{P_t}=0$$

$$\frac{1}{1+i_t}\frac{P_{t+\Delta}}{P_t}s_{t+\Delta}+\frac{1}{1+i_t}\frac{P_{t+\Delta}}{P_t}\frac{B_{t+\Delta}}{P_{t+\Delta}}=\frac{B_t}{P_t}$$

Iterating forward, we obtain

$$\frac{B_t}{P_t}=\sum_{j=1}^{\infty}\prod_{k=1}^{j}\left(\frac{1}{1+i_{(t+k-1)\Delta}}\frac{P_{t+k\Delta}}{P_{t+(k-1)\Delta}}\right)s_{t+j\Delta}$$

The ex-post real return on one period nominal bonds is of course

$$R_{t,t+\Delta}=(1+i_t)\frac{P_t}{P_{t+\Delta}}.$$

To arrive at the stochastic discount representation, multiply the flow constraint by $\Lambda_{t+\Delta}/\Lambda_t$ and take expectations,

$$E_t\left(\frac{\Lambda_{t+\Delta}}{\Lambda_t}s_{t+\Delta}\right)+E_t\left(\frac{\Lambda_{t+\Delta}}{\Lambda_t}\frac{B_{t+\Delta}}{P_{t+\Delta}}\right)-(1+i_t)\,E_t\left(\frac{\Lambda_{t+\Delta}}{\Lambda_t}\frac{P_t}{P_{t+\Delta}}\right)\frac{B_t}{P_t}=0.$$

But of course the price of one period bonds is

$$\frac{1}{1+i_t}=E_t\left(\frac{\Lambda_{t+\Delta}}{\Lambda_t}\frac{P_t}{P_{t+\Delta}}\right)$$

so the term in front of B_t/P_t vanishes, and, iterating forward we have again

$$\frac{B_t}{P_t}=E_t\sum_{j=1}^{\infty}\frac{\Lambda_{t+j\Delta}}{\Lambda_t}s_{t+j\Delta}.$$

3 Continuous time

3.1 Real debt

The flow constraint says that interest payments equal primary surplus plus new debt sales,

$$r_tb_tdt=s_tdt+db_t.$$

The flow constraint means that debt sales must also be of order dt, so we don't have to worry about db_t^2 terms. In this case the debt valuation equation takes the form

$$b_t=E_t\int_t^{\infty}\frac{\Lambda_\tau}{\Lambda_t}s_\tau d\tau.$$

We can also discount by the real interest rate, which is of course also the ex-post return on government debt.

$$b_t=\int_t^{T}\frac{V_t}{V_\tau}s_\tau d\tau$$

where V_t is the value process corrsponding to the real interest rate, i.e.

$$\frac{V_t}{V_0}=e^{\int_{\tau=0}^{t}r_\tau d\tau}.$$

This relation holds ex-post as well as ex-ante.

Derivations. Start with the flow constraint,

$$db_t - r_t b_t dt = -s_t dt.$$

Note

$$\begin{aligned} E_t\left[\frac{d(\Lambda_t b_t)}{\Lambda_t}\right] &= E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] b_t + E_t\left[db_t\right] \\ E_t\left[\frac{d(\Lambda_t b_t)}{\Lambda_t}\right] &= -r_t b_t dt + db_t \end{aligned}$$

Thus, we can write the flow constraint

$$E_t\left[d(\Lambda_t b_t)\right] = -\Lambda_t s_t dt.$$

Integrating,

$$E_t\left(\Lambda_T b_T\right) - \Lambda_t b_t = -E_t \int_t^T \Lambda_\tau s_\tau d\tau.$$

and imposing the transversaitly condition,

$$\Lambda_t b_t = E_t \int_t^\infty \Lambda_\tau s_\tau d\tau.$$

To express the present value relation discounted by bond returns, start from the same flow constraint,

$$r_t b_t dt = s_t dt + db_t.$$

Define the value corresponding the the return on government debt as

$$V_t = V_0 e^{\int_0^t r_t d\tau}.$$

Now,

$$d\left(\frac{b_t}{V_t}\right) = \frac{db_t}{V_t} - r_t \frac{b_t}{V_t} dt$$

Thus, divide the flow constraint by V_t and write

$$\begin{aligned} r_t \frac{b_t}{V_t} dt &= \frac{1}{V_t} s_t dt + \frac{db_t}{V_t} \\ \frac{db_t}{V_t} - r_t \frac{b_t}{V_t} dt &= -\frac{1}{V_t} s_t dt \\ d\left(\frac{b_t}{V_t}\right) &= -\frac{1}{V_t} s_t dt \\ \frac{b_T}{V_T} - \frac{b_t}{V_t} &= -\int_t^T \frac{1}{V_\tau} s_\tau d\tau \end{aligned}$$

and applying the transversality condition,

$$b_t = \int_t^T \frac{V_t}{V_\tau} s_\tau d\tau = \int_t^T \left[e^{-\int_{\sigma=t}^{\tau} r_\sigma d\sigma}\right] s_\tau d\tau.$$

3.2 Nominal debt and money

The nominal flow constraint is

$$P_t s_t dt + dB_t + dM_t - i_t B_t dt = 0. \tag{1}$$

The government gains money from primary surpluses, debt sales, money issue, and spends money on interest payments. Note here that $dB_t + dM_t$ must be of order dt, since both of the other terms are of order dt. To keep the analysis simple I will assume that each of dB and dM is of order dt rather than assume offsetting Ito terms or jumps.

I derive the following forms of the debt valuation equation. Let $s_t = T_t - G_t$ denote primary surpluses without seignorage. First, with seignorage counted as the flow from money creation,

$$\frac{B_t}{P_t} = E_t \int_{\tau=t}^{\infty} \frac{\Lambda_\tau}{\Lambda_t}\left(s_\tau + \frac{dM_\tau}{P_\tau}\right) d\tau.$$

Second, with seignorage counted as the interest savings on money and then with money on the left hand side,

$$\frac{M_t + B_t}{P_t} = E_t \int_{\tau=t}^{\infty} \frac{\Lambda_\tau}{\Lambda_t}\left(s_\tau + i_\tau \frac{M_\tau}{P_\tau}\right) d\tau.$$

We can also discount at the ex-post real return on nominal government debt, yielding

$$\frac{B_t}{P_t} = \int_{\tau=t}^{\infty} \frac{V_t}{V_\tau}\left(s_\tau d\tau + \frac{dM_\tau}{P_\tau}\right)$$

and

$$\frac{B_t + M_t}{P_t} = \int_{\tau=t}^{\infty} \frac{V_t}{V_\tau}\left(s_\tau + i_t \frac{M_\tau}{P_\tau}\right) d\tau$$

where V_t is the ex-post real cumulative value process from invesmtent in nominal government debt,

$$V_t = e^{\int_{\tau=0}^{t} i_\tau d\tau} \frac{P_0}{P_t}$$

i.e, it has a rate of return

$$\frac{dV_t}{V_t} = i_t dt - \frac{dP_t}{P_t}. \tag{2}$$

This is not the real interest rate.

Derivations Start with the flow constraint,

$$\begin{aligned} s_t dt + \frac{dB_t}{P_t} + \frac{dM_t}{P_t} - i_t \frac{B_t}{P_t} dt &= 0 \\ \Lambda_t s_t dt + \Lambda_t \frac{dM_t}{P_t} + \Lambda_t \frac{dB_t}{P_t} - \Lambda_t i_t \frac{B_t}{P_t} dt &= 0 \end{aligned}$$

Use the definition of the nominal interest rate

$$i_t dt = -E_t \left[d\left(\frac{\Lambda_t}{P_t}\right) / \left(\frac{\Lambda_t}{P_t}\right)\right]$$

to write

$$\begin{aligned}
\Lambda_t s_t dt + \Lambda_t \frac{dM_t}{P_t} + \Lambda_t \frac{dB_t}{P_t} + \Lambda_t E_t \left[d\left(\frac{\Lambda_t}{P_t}\right) / \left(\frac{\Lambda_t}{P_t}\right)\right] \frac{B_t}{P_t} &= 0 \\
\Lambda_t s_t dt + \Lambda_t \frac{dM_t}{P_t} + E_t \left[\frac{\Lambda_t}{P_t} dB_t + d\left(\frac{\Lambda_t}{P_t}\right) B_t\right] &= 0 \qquad (3)\\
\Lambda_t s_t dt + \Lambda_t \frac{dM_t}{P_t} + E_t \left[d\left(\frac{\Lambda_t}{P_t} B_t\right)\right] &= 0 \\
\Lambda_t s_t dt + \Lambda_t \frac{dM_t}{P_t} + E_t \left[d\left(\Lambda_t \frac{B_t}{P_t}\right)\right] &= 0
\end{aligned}$$

Now we can integrate, and impose the consumer transversality condition to obtain

$$\frac{B_t}{P_t} = E_t \int_{\tau=t}^{\infty} \frac{\Lambda_\tau}{\Lambda_t}\left(s_\tau d\tau + \frac{dM_\tau}{P_\tau}\right)$$

To express seignorage it in terms of interest cost, I proceed analogously. From (3), I add and subtract $E_t\left[d\left(\frac{\Lambda_t}{P_t}\right) M_t\right]$ and rearrange to

$$\begin{aligned}
\Lambda_t s_t dt - E_t\left[d\left(\frac{\Lambda_t}{P_t}\right) M_t\right] + E_t\left[\frac{\Lambda_t}{P_t} dB_t + \frac{\Lambda_t}{P_t} dM_t + d\left(\frac{\Lambda_t}{P_t}\right) B_t + d\left(\frac{\Lambda_t}{P_t}\right) M_t\right] &= 0 \\
\Lambda_t s_t dt + \frac{\Lambda_t}{P_t} i_t M_t dt + E_t\left[\frac{\Lambda_t}{P_t} d\left(B_t + M_t\right) + d\left(\frac{\Lambda_t}{P_t}\right)\left(B_t + M_t\right)\right] &= 0 \\
\Lambda_t s_t dt + \Lambda_t i_t \frac{M_t}{P_t} dt + E_t\left[d\left(\frac{\Lambda_t}{P_t}\left(B_t + M_t\right)\right)\right] &= 0 \\
\Lambda_t\left(s_t + i_t \frac{M_t}{P_t}\right) dt + E_t\left[d\left(\Lambda_t \frac{B_t + M_t}{P_t}\right)\right] &= 0
\end{aligned}$$

Integrating again, and imposing the consumer's transversality condition,

$$\frac{M_t + B_t}{P_t} = E_t \int_{\tau=t}^{\infty} \frac{\Lambda_\tau}{\Lambda_t}\left(s_\tau + i_\tau \frac{M_\tau}{P_\tau}\right) d\tau.$$

To discount with the ex-post return (2), start again with the nominal flow constraint (1),

$$\begin{aligned}
dB_t - i_t B_t dt &= -\left(P_t s_t dt + dM_t\right) \\
\frac{dB_t}{P_t} - \left(\frac{dP_t}{P_t} + i_t dt - \frac{dP_t}{P_t}\right)\frac{B_t}{P_t} &= -\left(s_t dt + \frac{dM_t}{P_t}\right) \\
\frac{1}{V_t}\frac{dB_t}{P_t} - \left(\frac{dP_t}{P_t} + \frac{dV_t}{V_t}\right)\frac{B_t}{V_t P_t} &= -\left(\frac{s_t}{V_t} dt + \frac{dM_t}{V_t P_t}\right) \\
\frac{1}{P_t V_t} dB_t + d\left(\frac{1}{V_t P_t}\right) B_t &= -\frac{1}{V_t}\left(s_t dt + \frac{dM_t}{P_t}\right) \\
d\left(\frac{1}{V_t}\frac{B_t}{P_t}\right) &= -\frac{1}{V_t}\left(s_t dt + \frac{dM_t}{P_t}\right) dt \\
\frac{B_t}{P_t} &= \int_{\tau}^{\infty} \frac{V_t}{V_\tau}\left(s_\tau d\tau + \frac{dM_\tau}{P_\tau}\right)
\end{aligned}$$

To do the same thing exressing seignorage as an interest expense,

$$\begin{aligned}
dB_t + dM_t - i_t B_t dt &= -P_t s_t dt \\
\frac{d\left(B_t + M_t\right)}{P_t} - i_t \frac{\left(B_t + M_t\right)}{P_t} dt &= -\left(s_t + i_t \frac{M_t}{P_t}\right) dt \\
\frac{d\left(B_t + M_t\right)}{P_t} - \left(\frac{dP_t}{P_t} + i_t dt - \frac{dP_t}{P_t}\right) \frac{\left(B_t + M_t\right)}{P_t} &= -\left(s_t + i_t \frac{M_t}{P_t}\right) dt \\
\frac{d\left(B_t + M_t\right)}{V_t P_t} - \left(\frac{dP_t}{P_t} + \frac{dV_t}{V_t}\right) \frac{\left(B_t + M_t\right)}{V_t P_t} &= -\frac{1}{V_t}\left(s_t + i_t \frac{M_t}{P_t}\right) dt \\
\frac{d\left(B_t + M_t\right)}{V_t P_t} + d\left(\frac{1}{V_t P_t}\right)\left(B_t + M_t\right) &= -\frac{1}{V_t}\left(s_t + i_t \frac{M_t}{P_t}\right) dt \\
d\left[\frac{1}{V_t P_t}\left(B_t + M_t\right)\right] &= -\frac{1}{V_t}\left(s_t + i_t \frac{M_t}{P_t}\right) dt \\
\frac{B_t + M_t}{P_t} &= \int_t^\infty \frac{V_t}{V_\tau}\left(s_\tau + i_t \frac{M_\tau}{P_\tau}\right) d\tau
\end{aligned}$$

Note that as a result of (2), the second order terms cancel in

$$V_t P_t d\left(\frac{1}{V_t P_t}\right) = -\frac{dV_t}{V_t} - \frac{dP_t}{P_t} + \frac{dV_t^2}{V_t^2} + \frac{dP_t^2}{P_t^2} + \frac{dV_t dP_t}{V_t P_t}.$$

4 Long term debt

Here I derive the present-value formula

$$\frac{\int_{j=0}^{\infty} Q_t^{(j)} B_t^{(j)} dj}{P_t} = E_t \int_{\tau=0}^{\infty} \frac{\Lambda_{t+\tau}}{\Lambda_t} s_{t+\tau} dt$$

from the flow budget constraint.

It helps to start with a discrete-time approach. $B_t^{(j)}$ is the stock of zero coupon bonds of j maturity outstanding at the beginning of t, and $Q_t^{(j)}$ is its price. $B_t^{(j)}$ is determined at the end of period $t - \Delta$ and is in in the $t - \Delta$ information set.

The flow constraint thus states that surplus, plus revenue from bond sales of all maturities, equals the cost of redeeming debt that comes due between time t and time $t + \Delta$,

$$P_t s_t \Delta + \int_{j=\Delta}^{\infty} Q_t^{(j)} (B_{t+\Delta}^{(j-\Delta)} - B_t^{(j)}) dj - \int_{j=0}^{\Delta} Q_t^{(j)} B_t^{(j)} dj = 0.$$

This constraint is equivalent to

$$P_t s_t \Delta + \int_{j=0}^{\infty} Q_t^{(j+\Delta)} B_{t+\Delta}^{(j)} dj - \int_{j=0}^{\infty} Q_t^{(j)} B_t^{(j)} dj = 0,$$

which states that surpluses, plus the revenue of selling new debt equals the cost of buying back and redeeming the outstanding stock of debt.

To iterate forward in this discretization we use the standard bond pricing formula, the value of a $j+\Delta$ period bond today is the discounted value of a j period bond tomorrow,

$$\Lambda_t \frac{Q_t^{(j+\Delta)}}{P_t} = E_t \left[\Lambda_{t+\Delta} \frac{Q_{t+\Delta}^{(j)}}{P_{t+\Delta}} \right].$$

Hence,

$$\Lambda_t s_t \Delta + E_t \int_{j=0}^{\infty} \Lambda_{t+\Delta} Q_{t+\Delta}^{(j)} \frac{B_{t+\Delta}^{(j)}}{P_{t+\Delta}} dj - \int_{j=0}^{\infty} \Lambda_t Q_t^{(j)} \frac{B_t^{(j)}}{P_t} dj = 0.$$

Iterating forward, we have

$$\frac{\int_{j=0}^{\infty} Q_t^{(j)} B_t^{(j)} dj}{P_t} = E_t \sum_{k=0}^{\infty} \frac{\Lambda_{t+k\Delta}}{\Lambda_t} s_{t+k\Delta} \Delta.$$

Now, we can take the limit,

$$\frac{\int_{j=0}^{\infty} Q_t^{(j)} B_t^{(j)} dj}{P_t} = E_t \int_{\tau=0}^{\infty} \frac{\Lambda_{t+\tau}}{\Lambda_t} s_{t+\tau} d\tau.$$

Doing the same thing directly in continuous time takes a bit more work. In continuous time, the flow constraint is

$$P_t s_t dt + \int_{j=0}^{\infty} \left[Q_t^{(j)} dB_t^{(j)} + B_t^{(j)} \frac{\partial Q_t^{(j)}}{\partial j} dt \right] dj = 0. \tag{4}$$

The $\partial Q_t^{(j)} / \partial j$ term compensates for the fact that $dB_t^{(j)} = \lim_{\Delta \to 0} B_{t+\Delta}^{(j)} - B_t^{(j)}$ is not the difference in quantity of the same bond, but is a bond maturing at $t+\Delta$ less a bond maturing at t. To derive this formula, break up the discrete time flow constraint as follows, and take the limit

$$\begin{aligned} P_t s_t \Delta + \int_{j=0}^{\infty} \left[Q_t^{(j+\Delta)} B_{t+\Delta}^{(j)} - Q_t^{(j+\Delta)} B_t^{(j)} + Q_t^{(j+\Delta)} B_t^{(j)} - Q_t^{(j)} B_t^{(j)} \right] dj &= 0 \\ P_t s_t \Delta + \int_{j=0}^{\infty} \left[Q_t^{(j+\Delta)} \left(B_{t+\Delta}^{(j)} - B_t^{(j)} \right) + \left(Q_t^{(j+\Delta)} - Q_t^{(j)} \right) B_t^{(j)} \right] dj &= 0 \end{aligned}$$

Next, in continous time the bond pricing relationship is

$$\frac{\Lambda_t}{P_t} \frac{\partial Q_t^{(j)}}{\partial j} dt = E_t \left[d \left(\Lambda_t \frac{Q_t^{(j)}}{P_t} \right) \right]. \tag{5}$$

Again, the derivative with respect to j appears because $Q_t^{(j)}$ is the value of $Q_{t+\Delta}^{(j-\Delta)}$ not of $Q_{t+\Delta}^{(j)}$. To show this, write

$$\begin{aligned} \frac{\Lambda_t}{P_t} \frac{\partial Q_t^{(j)}}{\partial j} \Delta &= \frac{\Lambda_t}{P_t} \left(Q_t^{(j+\Delta)} - Q_t^{(j)} \right) \\ &= E_t \left[\Lambda_{t+\Delta} \frac{Q_{t+\Delta}^{(j)}}{P_{t+\Delta}} \right] - \Lambda_t \frac{Q_t^{(j)}}{P_t} \end{aligned}$$

Now, using (5) in (4),

$$\Lambda_t s_t dt + \int_{j=0}^{\infty} \left[\Lambda_t \frac{Q_t^{(j)}}{P_t} dB_t^{(j)} + \Lambda_t \frac{B_t^{(j)}}{P_t} \frac{\partial Q_t^{(j)}}{\partial j} dt \right] dj = 0$$

$$\Lambda_t s_t dt + \int_{j=0}^{\infty} \left[\Lambda_t \frac{Q_t^{(j)}}{P_t} dB_t^{(j)} + B_t^{(j)} E_t \left[d \left(\Lambda_t \frac{Q_t^{(j)}}{P_t} \right) \right] \right] dj = 0$$

$$\Lambda_t s_t dt + E_t \int_{j=0}^{\infty} d \left[\Lambda_t \frac{Q_t^{(j)} B_t^{(j)}}{P_t} \right] dj = 0$$

(As before, I restrict $B_t^{(j)}$ not to have Ito terms). Integrating, we obtain the present value result

$$\frac{\int_{j=0}^{\infty} Q_t^{(j)} B_t^{(j)} dj}{P_t} = E_t \int_{\tau=0}^{\infty} \frac{\Lambda_{t+\tau}}{\Lambda_t} s_{t+\tau} dt$$