The New-Keynesian Liquidity Trap

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Abstract

In standard solutions, the new-Keynesian model produces a deep recession with deflation in a liquidity trap. The model also makes unusual policy predictions: Useless government spending, technical regress, and capital destruction have large multipliers. These predictions become larger as prices become *less* sticky. I show that both data and policy predictions are strongly affected by equilibrium selection. For the same interest-rate path, different choices of equilibria – either by the researcher's direct selection or the researcher's specification of expected Federal Reserve policy – can overturn all these results. A set of "local-to-frictionless" equilibria predicts mild inflation, no output reduction and negative multipliers during the liquidity trap, and its predictions approach the frictionless model smoothly, all for the same interest rate path.

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1 Introduction

Standard solutions of new-Keynesian models predict deep recessions and deflation in a liquidity trap, when the "natural" rate of interest is negative and the nominal rate is stuck at zero. Those solutions also produce unusual policy predictions. "Open-mouth operations," "forward guidance," and promises to keep interest rates low in the far future, with no current action, can strongly stimulate the current level of output. Fully-expected future inflation raises output. Deliberate capital destruction or productivity reduction can raise output. Government spending, even if financed by current taxation, and even if completely wasted, can have large output multipliers. Even more puzzling, as prices become *more* flexible, standard solutions of new-Keynesian models predict that deflation and depression during the trap get *worse*, and these paradoxical policy prescriptions become *stronger*, so that infinitesimal price stickiness produces enormous results. Thus, although price stickiness is the central friction keeping the economy from achieving its optimal output, policies that reduce price or wage stickiness would make deflation and depression worse. The flexible-price limit *points* are well-defined however, and these solutions are discontinuous at the flexible-price limit.

For a given path of expected interest rates, new-Keynesian models allow multiple equilibrium paths for inflation and output. Thus, to produce a prediction from a New-Keynesian model, a researcher must choose an equilibrium as well as a path for expected future interest rates. Most often, the researcher does this implicitly, choosing an equilibrium by making assumptions about the form of a Taylor rule.

I show that these liquidity-trap predictions are quite sensitive to a particular equilibrium choice. Choosing different equilibria, either directly or by different specifications of Fed policy, despite exactly the same path of interest-rate expectations, the new-Keynesian model can predict gentle inflation matching the negative natural rate, small output gaps, and normal signs and magnitudes of policies. These equilibria and policy predictions smoothly approach the frictionless limit.

I do not, in this paper, argue for a specific equilibrium choice, though I analyze the issues involved and speculate that the standard equilibrium choice will be found wanting, and one of the equilibria displaying mild inflation will eventually be preferred. I emphasize that equilibrium selection can be an empirical issue, and one needs a realistic model before going to the data.

You may be reading this article after interest rates are no longer zero, and zero lower bound controversies are making their way to economic history rather than current policy advice and analysis. This fact does not make the analysis irrelevant. First, another time will come. Economists saw and analyzed the zero bound in the Great Depression, we saw it and analyzed it again when Japan hit the zero bound in the 1990s, and we saw it and analyzed it a third time in the US starting in 2008. We will likely see it again. Understanding what policies are appropriate in a zero bound, especially when the claims for various policy measures are so extreme, will be useful. Second, understanding this episode is a particularly useful window into understanding the general structure of macroeconomic models, which will remain with us as interest rates recover. The equilibrium selection issues made stark at the zero bound remain at higher rates.

This paper is not a critique of new-Keynesian DSGE models. It does more, in my view, to save than to criticize those models. Forward-looking consumers who optimize intertemporally and microfounded price stickiness are sensible ingredients of a macroeconomic model. This paper shows that those same ingredients can produce economic equilibria with reasonable signs and magnitudes of policy intervention – no paradoxes, no enormous multipliers to wasted government spending, no strong effects of promises about the far-off future, no explosive limits as price stickiness is reduced, and no discontinuity at the frictionless limit point – if we simply choose a wide class of equilibria, different from the standard choice. That is all good news for serious economic modeling of nominal distortions, and policy options, maintaining the basic ingredients of the model.

1.1 Literature

This paper owes an obvious debt to Werning (2012) whose structure I adopt. It is not a critique of Werning. Werning acknowledges multiple equilibria, I explore their nature.

Kiley (2013) and Wieland (2014) nicely summarize the puzzling predictions of new-Keynesian zero-bound analyses. Kiley contrasts sticky-price and sticky-information models' behavior with the policy paradoxes, and finds that sticky-information models also resolve some paradoxes.

Eggertsson and Woodford (2003) study monetary policy in a liquidity trap, emphasizing the dangers of the trap and the idea that the Fed commit to keep interest rates low after the trap ends, which is the centerpiece of Werning (2012).

Christiano, Eichenbaum and Rebelo (2011), Eggertsson (2011), Woodford (2011), and Carlstrom, Fuerst, and Paustian (2012) all find large fiscal multipliers, and multipliers that increase with the duration of fiscal expansion, in new-Keynesian zero-bound models.

Eggertsson (2010) and Wieland (2014) analyze the "paradox of toil" that negative productivity can be expansionary. Wieland shows that several cases of endowment destruction did not seem to have the predicted stimulative effects. Eggertsson, Ferrero, and Raffo (2014) argue that European structural reforms will be contractionary at the zero bound. See also the excellent discussion in Fernández-Villaverde (2013).

As I emphasize, many of the seemingly paradoxical results of the New-Keynesian model, such as "more growth is bad," come from the assumption that the economy always returns to trend. Fernández-Villaverde, Guerrón-Quintana, and Rubio-Ramírez (2014) point out that "supply-side" policies can raise the future level of output above the preexisting trend. This wealth or permanent income effect, ruled out by the assumption of a return to trend, also overturns the results that productivity improvements lower output today.

Braun, Körber, and Waki (2012) find that the paradoxical properties of new-Keynesian models are artifacts of linearization. Christiano and Eichenbaum (2012) argue that the criticism is not "E-learnable" and does not apply for small enough deviations. I use the linearized system entirely, so this controversy is not relevant to my points.

Cochrane (2011a) studies the plausibility of explosive threats underlying Taylor rules in new-Keynesian models. That is not the central issue here. Here, I play by standard rules of the new-Keynesian game and simply rule out equilibria that explode as time goes forward.

Benhabib, Schmitt-Grohé and Uribe (2001, 2002) point out local indeterminacy induced by the fact that the Taylor rule can't operate at the zero bound, and the possibility of sunspot equilibria. This paper is not about that source of indeterminacy either. I ignore the possibility of sunspots.

Aruoba, Cuba-Borda, and Schorfheide (2013), Mertens and Ravn (2014) point out that the same values of output and inflation can be seen when the economy is in the zero bound equilibrium and the positive inflation equilibrium of the Benhabib, Schmitt-Grohé and Uribe model subject to shocks, and quite different fiscal multipliers will be seen in each, a point related to the large difference in multipliers I see across equilibria. However, all my equilibria escape the zero bound.

2 Model

I follow Werning (2012) and study the continuous-time specification of the standard new-Keynesian model. Werning's brilliantly simple framework clarifies the issues relative to potentially more realistic but more complex environments. The model is

$$\frac{dx_t}{dt} = \sigma^{-1}\left(i_t - r_t - \pi_t\right) \tag{1}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t. \tag{2}$$

Here, x_t is the output gap, i_t is the nominal rate of interest, r_t is the "natural" real rate of interest, and π_t is inflation. I specialize to $\sigma = 1$ in what follows as variation in this parameter plays no role in the analysis.

Equation (1) is the "IS" curve, which ought to be renamed the "intertemporal substitution" curve. It is derived from the first-order condition for allocation of consumption over time, and consumption equals output with no capital. Equation (2) is the new-Keynesian forward-looking Phillips curve. Solving it forwards, it expresses inflation in terms of expected future output gaps,

$$\pi_t = \kappa \int_{s=0}^{\infty} e^{-\rho s} x_{t+s} ds.$$

Like Werning, I suppose that the economy suffers from a temporarily negative natural rate $r_t = -r < 0$, which lasts until time T before returning to a positive value.

I complete the model by specifying directly the path of equilibrium interest rates. In this perfect-foresight model these are also expected interest rates, and it is useful to think of the path of expected future interest rates and other variables determining events at time t. I specify that the equilibrium nominal interest rate is zero, the lower bound, up to period T. After time T, people expect equilibrium interest rates to rise to equal the natural rate $i_t = r_t > 0$. Thus, people expect interest rates to follow the path

$$\begin{aligned} t < T &: i_t = 0 \\ t \geq T &: i_t = r_t \geq 0. \end{aligned} \tag{3}$$

I use $r = 5\%$, $T = 5$, $\rho = 0.05$ and $\kappa = 1$ for all calculations.

Perfect foresight of a trap end date is unrealistic, of course. As usual, however, this specification provides useful intuition for how models behave driven by expectations of a stochastically ending trap and other shocks, or for how models behave in which the expected natural rate and interest rate follow a slowly mean-reverting process like an AR(1). The fixed end date makes the algebra simpler and the mechanics clearer.

Next, I find the set of output $\{x_t\}$ and inflation $\{\pi_t\}$ paths that, via (1) and (2), are consistent with this path of equilibrium expected interest rates, and do not explode as time increases. It will turn out that there are many such paths.

2.1 A brief Taylor-rule digression and preview

Typically, the new-Keynesian literature completes the model by an assumption about expected central bank behavior rather than by directly specifying the path of equilibrium expected interest rates.

Most typically, one assumes that the central bank follows a Taylor-type rule of the form (or algebraically equivalent to the form)

$$i_t = i_t^* + \phi(\pi_t - \pi_t^*) \tag{4}$$

where i_t^* and π_t^* are the equilibrium values of inflation and output that the central bank wishes to produce, among the set of possible equilibrium paths defined in the last paragraph. The assumption $\phi > 1$ and a rule against explosive equilibria then produce $\{i_t^*,\ \pi_t^*\}$ as the unique locally-bounded equilibrium.

By writing the Taylor rule this way, however, one can see quickly that the researcher can produce any equilibrium that results from the first method – specify the equilibrium interest rate path directly, find equilibrium inflation and output gap – by the second method – specify this form of a Taylor rule, and derive the unique locally-bounded equilibrium.

Thus, by specifying directly the equilibrium interest rate path, I am *not* assuming that the central bank follows a peg, and the multiplicity of equilibria which we will see are *not* a repetition of the familiar point that with an interest rate peg and passive fiscal policy, this class of models has multiple equilibria. The range of equilibria I investigate are the range of equilibria that a researcher may produce as unique equilibria under an active Taylor rule, by writing different forms of that rule.

Starting with the equilibrium interest rate path and then constructing a Taylor rule, however, leads the multiple equilibria open to more general interpretations, and represent more general work in this area. For example, they could be selected by an actual peg and an active-fiscal policy specification, by a peg with some other equilibrium-selection criterion, by the "sophisticated" equilibrium-selection policies of Atkeson, Chari and Kehoe (2010), or by a variety of other selection methods used in the literature.

Most of all, proceeding in this way is quicker. I will prove below, as part of a larger discussion of the equilibrium-selection issue, that one can select via a Taylor rule any of the equilibria I derive by specifying the interest rate path, as if that path were a peg. If the reader will trust that proof is coming, then we can very quickly get to computing equilibria and evaluating their economic properties.

Werning (2012) innovated this simplifying strategy. He displayed one equilibrium consistent with the specified interest rate path, and asserted, correctly, in a footnote, that a wide variety of Taylor-rule or other equilibrium-selection strategies could be invoked to deliver that equilibrium as the unique equilibrium.

2.2 The flexible-price case

Again, our goal is to find the set of non-explosive inflation and output gap paths, $\{\pi_t, x_t\}$, consistent with the path of expected interest rates, $i_t = 0$ before T and following the natural rate thereafter $i_t = r_t$, $t > T$. I start with the flexible-price case.

To understand the flexible-price limit point, it is best to write the Phillips curve (2) as

$$x_t = \frac{1}{\kappa}\left(\rho\pi_t - \frac{d\pi_t}{dt}\right).$$

As κ rises, the output gap x_t for any inflation path π_t becomes smaller and smaller. The flexible-price limit point is the case in which inflation can do whatever it wants, and the output gap is zero. Thus, the flexible-price limit point is $\kappa = \infty$, $x_t = 0$.

Turning to the IS curve (1), if $x_t = 0$ then $dx_t/dt = 0$ and we must have

$$i_t - r_t = \pi_t.$$

This is just the linearized Fisher relationship. When prices are flexible, inflation and nominal rates must move together, and the real rate must equal the natural rate.

Thus, the unique flexible-price solution to our liquidity-trap scenario (3) is

$$t < T : \pi_t = r,\ x_t = 0$$
$$t \geq T : \pi_t = 0,\ x_t = 0.$$

During the liquidity trap, inflation in the frictionless world will rise to exactly equal to the negative natural rate, all on its own without extra prodding by the Fed, and there will be no output gap.

In the frictionless version of this model, the Fisher effect is king. If the Fed had not lowered the interest rate to zero, there would have been even more inflation.

2.3 Equilibria with price stickiness

Now, I solve the sticky-price case $\kappa < \infty$. The model (1)-(2) is

$$\frac{d}{dt}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} ir_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -\kappa & \rho \end{bmatrix}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \tag{5}$$

where $ir_t = i_t - r_t$, with $ir_t = r > 0$ for $t < T$ and $ir_t = 0$ for $t \geq T$.

The model is a standard first-order matrix differential equation. I solve it by finding the eigenvalue decomposition of the square matrix. I first find the solution for $t > T$, and then paste the $t < T$ solution to it. I present the algebra in the Appendix. The answer is

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} ir_t - \frac{1}{\lambda - \delta}\begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix}\begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} ir_t + \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_T e^{\delta(t-T)}. \tag{6}$$

where

$$\lambda \equiv \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4\kappa}\right) > 0 \tag{7}$$
$$\delta \equiv \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4\kappa}\right) < 0.$$

The first term of (6) is the steady state. The second term describes mixed exponential dynamics. The sum of first and second terms vanish as $t \to T$. (You need the identity $\rho = \lambda_- + \lambda$

to see that fact.) Because $ir_t = 0$, $t > T$, the first and second terms also vanish for $t > T$, leaving only the third term, which consists of exponentially decaying dynamics.

There are, in general, two free constants. To arrive at (6), I impose the standard new-Keynesian assumption that equilibria must not explode as $t \to \infty$ to eliminate one constant, thus setting to zero terms multiplying the forward-explosive eigenvalue λ in the $t > T$ region. The remaining equilibria tie output and inflation paths together in a one-dimensional family. I use the parameter π_T in the third term of (6), inflation at the end of the liquidity trap, to index all the possible multiple equilibria that are locally bounded as $t \to \infty$.

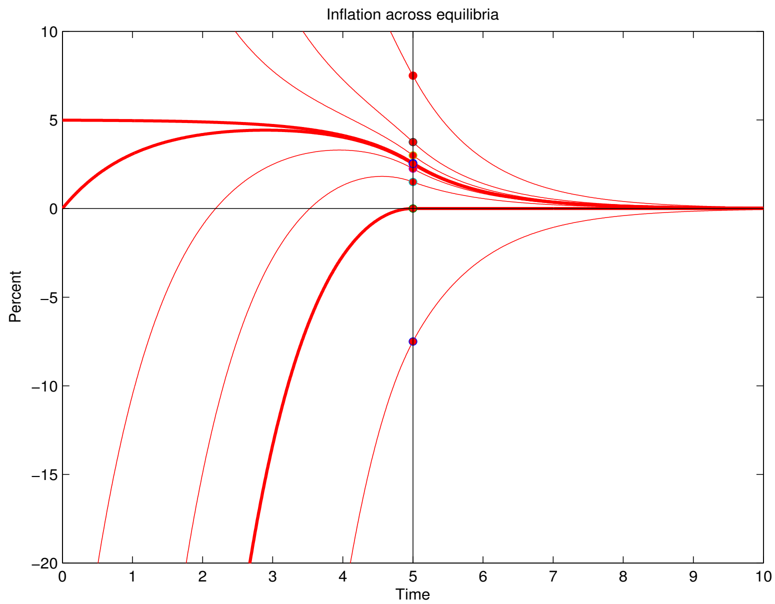

Figure 1: Inflation in all equilibria. Equilibria are indexed by the expected value of inflation π_T at $T = 5$, shown by the circles. The thicker lines show the standard deflation equilibrium, the no-inflation-jump equilibrium, and the backward-stable equilibrum. Thinner lines show a range of equilibria indexed by different choices for π_T. The price-stickiness parameter $\kappa = 1$ and the discount rate $\rho = 0.05$.

Figure 1 and Figure 2 show inflation and output gaps in a range of equilibria. These are all equilibria of the *same* model, with the *same* expected interest rate path.

The large dots at $T = 5$ in the figures index equilibria by different choices for π_T, expected inflation just as the trap ends, mirroring the form of Equation (6). One could equivalently index the alternative equilibria by expectations about the output gap at T, x_T, but I follow tradition in specifying them as a function of inflation expectations. I also follow the tradition of describing a causal chain from expectations of future inflation to the determination of current inflation and to output, though the equations do not specify a causal ordering. One can also index the multiple equilibria of this perfect-foresight model by one of inflation or output at any date. In particular, in some contexts it is interesting to index multiple equilibria by the value of inflation or output at time zero, π_0 or x_0, which represent the contemporaneous responses of inflation and output to the natural-rate shock.

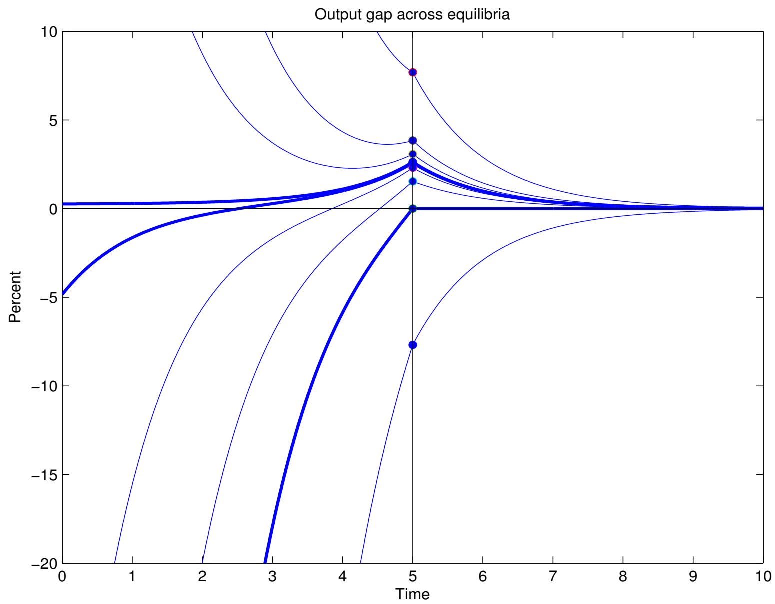

Figure 2: Output gaps in all equilibria. Alternative equilibria are indexed by the value of the output gap at $T = 5$, shown by the circles. The thicker lines show the standard deflation equilibrium, the no-inflation-jump equilibrium, and the backward-stable equilibrium. Thinner lines show a range of equilibria.

2.4 The standard equilibrium

Now, let's look at a few equilibria in detail.

Werning (2012), like the rest of the literature, chooses the equilibrium $\pi_T = 0$. Plugging in to the general solution, (6), we verify that the economy is in the steady state as soon as the liquidity trap ends,

$$t \geq T: \ x_t = \pi_t = 0. \tag{8}$$

Before T, during the liquidity trap episode, (6) becomes

$$t \leq T: \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} ir - \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} ir \tag{9}$$

Figure 3 presents this standard equilibrium choice, (8) and (9). Its output and inflation paths are also plotted as thicker lines in Figure 1 and Figure 2, for comparison with other equilibria.

This equilibrium shows a dramatic deflation and large output gaps during the liquidity-trap period. We also see strong dynamics – deflation steadily improves, and expected output *growth* is strong. This model does not produce a "slump," or "secular stagnation," a long period of a steady output gap and steady inflation or even slight deflation. The intertemporal first order condition (1) says that the level of consumption can only be below potential if consumption is expected to grow back to potential. The new-Keynesian Phillips curve (2) links the output gap to the change in inflation, producing a gap when inflation is lower today than in the future. If inflation is to end up at zero, and if one wants a large output gap, the Phillips curve requires swiftly decreasing, and therefore substantial, deflation.

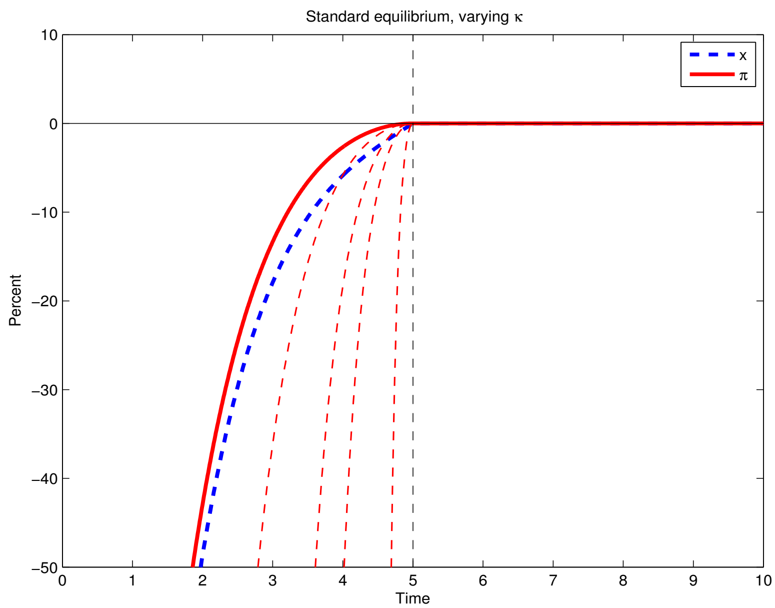

Figure 3: Output and inflation in the standard equilibrium. The thick lines show $\kappa = 1$. The thin dashed lines give inflation as the price-stickiness parameter κ varies from 1, 2, 5, 10, 100.

This equilibrium explodes *backward* in time, and deviations grow arbitrarily large as the period of the liquidity trap expands. That observation turns out to be key to understanding many properties of its predictions. Technically, in order to arrive at $\pi_T = 0$, the the $t < T$ solution loads on both eigenvalues, including the stable forward, but unstable backward $\delta < 0$.

The dashed lines in Figure 3 show how solutions behave as we reduce price frictions. Deflation and (not shown) output gaps become *larger* as price stickiness is *reduced*. Despite this infinite limit, the limit *point* of the frictionless equilibrium is well-behaved at $\pi_t = 5\%$, $x_t = 0$, $t < T$. Thus, the model with this equilibrium selection makes the strange prediction that for prices very near frictionless, there will be huge deflation and output gaps, while for the frictionless limit point, it instead predicts steady inflation and no output gap. This behavior makes technical if not economic sense: As pricing frictions decrease and κ increases, both eigenvalues λ and δ increase in absolute value. Dynamics, forward and backward, happen faster and faster as prices become more flexible.

2.5 The backward-stable equilibrium

Another interesting choice of equilibrium is to require that the economy approaches the steady state as t goes *backwards* in time. This is a natural counterpart to the criterion we used in the $t > T$ region, that the economy should converge to the steady state as t goes *forward* in time. I do not argue that it is the right one or the correct one, just that this equilibrium choice is possible and it makes a particularly interesting contrast to the standard equilibrium choice.

To impose this backward-stable criterion, we pick π_T so that the loading of the $t \le T$ solution on the backwards-explosive eigenvalue δ is zero. Writing (6) for $t < T$ in the form

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} ir_t + \left(\begin{bmatrix} \lambda & 0 \\ 1 & 0 \end{bmatrix} \pi_T - \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} ir \right) \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix}, \quad (10)$$

we see that the choice

$$\pi_T = \frac{\lambda}{\lambda - \delta} ir$$

puts no loading on the backwards-explosive root δ for $t < T$. Imposing that equilibrium choice, the solution (6) and (10) becomes

$$t < T : \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} ir + \frac{\delta}{\lambda - \delta} \begin{bmatrix} \delta \\ 1 \end{bmatrix} e^{\lambda(t-T)} ir, \tag{11}$$

$$t > T : \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{\lambda}{\lambda - \delta} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta(t-T)} ir. \tag{12}$$

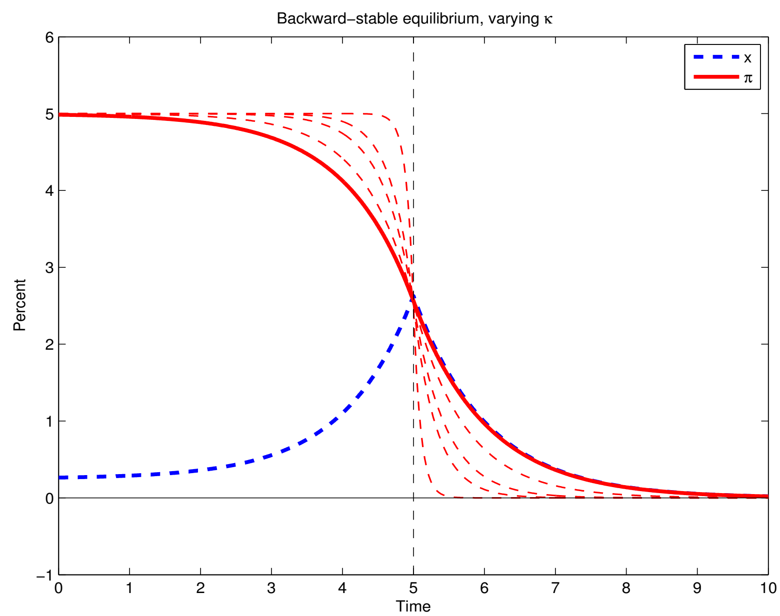

Figure 4: Output and inflation with the backward-stable equilibrium choice. Solid lines show $\kappa = 1$. Thin dashed lines show inflation as the price-stickiness parameter κ takes values 1, 2, 5, 10, 100.

Figure 4 presents the backward-stable equilibrium (11)-(12). Output and inflation are, by construction, well-behaved in both directions. The economy starts near a steady state with 5% inflation and a small permanent output gap. As the end of the liquidity trap, inflation comes down slowly, following a "glidepath," in Fed parlance, to zero.

There is a small *increase* in output towards the end of the liquidity trap. This behavior is a standard and much-criticized feature of the usual new-Keynesian Phillips curve (2), that output is high when inflation is declining, i.e. when inflation today is higher than expected future inflation. The major message of the Figure, however, is that the output gap is small, as inflation/deflation and its dynamics are small, certainly in comparison with the standard equilibrium choice. These output and inflation paths are included in Figure 1 and Figure 2, for comparison with other equilibria.

The backwards-stable equilibrium of Figure 4 and the standard equilibrium of Figure 3 are both equilibria of the *same* model with the *same* parameters, and the *same interest rate path.* They do not differ at all by expectations of what the Fed will actually *do*. I will exhibit a Taylor rule that underlies each of them, so a researcher can produce either as the "prediction of the new-Keynesian model," even with Taylor rules.

The two equilibrium choices have drastically different implications for how the economy behaves in a liquidity trap. Figure 4 suggests that a liquidity trap is associated with a mild inflation, which will emerge on its own without any additional policy, and a small increase in output relative to "potential" as the trap ends. Figure 3 suggests that a liquidity trap is an economic calamity, with large output gap and rampant deflation, though with strong expected output growth, and rapidly-decreasing deflation.

At a minimum, one cannot argue that expectations of future interest rates alone are central, and the choice of equilibrium (or, equivalently, but less transparently, the corresponding choice of Taylor-rule specification) is an innocuous technical detail, to be swept into footnotes! Equilibrium choice is *central* to the model's economic predictions.

The comparison of the two equilibria concurs with Werning's (2012) central message. The real problem of the liquidity trap is not so much the lower bound on contemporaneous interest rates as in static Keynesian models, but expectations of what will happen once the trap passes. The only difference between a gentle inflation and a disastrous recession is people's expectations of inflation in the immediate aftermath of the liquidity trap. If people expect inflation can tranquilly approach zero in a smooth glidepath, we get the backward-stable equilibrium. If people think inflation will be zero the minute the liquidity trap ends, then we have the disastrous standard equilibrium. Unlike Werning, however, who modifies the interest-rate path to obtain a more benign equilibrium, the backward-stable equilibrium choice displays higher expected inflation at the end of the trap without changing the path of expected interest rates at all.

Figure 4 also shows how inflation in the backward-stable equilibrium behaves as we increase the price-stickiness parameter κ, lowering the pricing friction, in the sequence of thin dashed lines. The backward-stable equilibrium smoothly approaches the frictionless limit as price frictions are reduced. Output gaps also smoothly approach the frictionless solution $x_t = 0$.

I will call equilibria with this well-behaved limit "local-to-frictionless." For price stickiness very near frictionless, inflation and output gaps deviate very little from frictionless values.

Examining the limiting behavior in Figure 4 illuminates why the standard equilibrium choice has a discontinuous frictionless limit. The perfect-foresight new-Keynesian Phillips curve (2) does not allow expected inflation jumps with a finite output gap. It does allow unexpected inflation jumps, such as at time 0. This backward-stable solution looks a lot like the frictionless equilibrium, but smooths naturally around the end of the trap with inflation following an S shape. But some of that smoothing must occur in the $t > T$ period. Insistence that inflation is zero *immediately* at $t = T$, for any value of price stickiness, is the crucial specification that drives the economic dislocation and puzzling limiting behavior of the standard solution.

2.6 The no-inflation-jump equilibrium

In both the standard and backward-stable equilibria, inflation jumps the moment that the (unexpected) liquidity trap begins, at time 0. In the standard solution, the extreme deflation breaks out instantly at its worst value and then recovers. In the backward-stable solution, inflation jumps up to match the natural-rate shock.

Another interesting equilibrium choice results by requiring that inflation does not jump on news, or in our case, $\pi_0 = 0$. To characterize this equilibrium, I use the general solution (6), and find π_T so that $\pi_0 = 0$. The equation is not particularly interesting and is given in the Appendix.

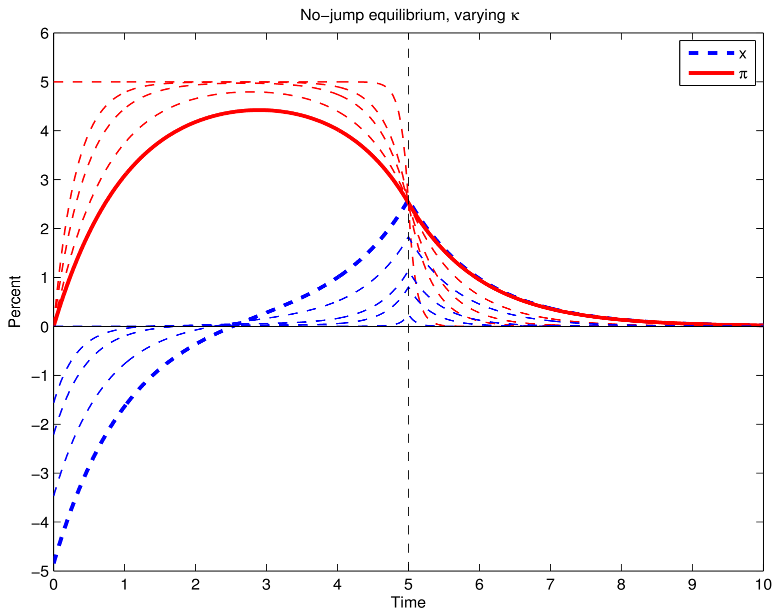

Figure 5: Output and inflation in the no-inflation-jump equilibrium. The thin lines present progressively less price stickiness with greater values of κ, for both output gap and inflation.

Figure 5 presents output and inflation in the no-inflation-jump equilibrium. Inflation ramps up towards the negative of the natural rate. Through the standard forward-looking Phillips curve mechanism, this equilibrium produces a mild recession early in the trap while inflation is increasing (inflation less than expected future inflation), and then a mild expansion late in the trap as inflation decreases.

As pricing frictions decrease, this no-inflation-jump equilibrium also approaches the frictionless equilibrium. Thus, this equilibrium is also "local-to-frictionless."

The local-to-frictionless property is thus not unique. In fact, as one can guess from the graph, *any* equilibrium choice rule which bounds the initial jump in inflation π_0 or output gap x_0, is local-to-frictionless. Such an equilibrium choice must put lower and lower weight on the backwards-explosive eigenvalue as price stickiness diminishes. The standard equilibrium choice allows arbitrarily large jumps in inflation and output gap as pricing fictions decrease.

A no-output-jump equilibrium choice $x_0 = 0$ is also attractive. Comparing Figure 5 and Figure 4 we see that the equilibrium will lie between the backward-stable and no-inflation-jump equilibrium, and is also local-to-frictionless. I spare the reader another set of graphs. In what follows I present the backward-stable equilibrium, but as Figure 5 suggests, results for either no-jump equilibrium are quite similar.

Again, I am displaying a variety of possibilities without arguing that any of these equilibria is the "right" choice. The point is the variability – for the same interest rate path, all sorts of equilibria are possible. The equilibria vary sharply in their description of the data, and, as we shall see, in their predictions about the impact of policies. The point in this paper is that equilibrium selection matters a great deal, fixing interest rate expectations, not that one equilibrium selection is right.

3 Magical multipliers and paradoxical policies

To analyze the fiscal multipliers and paradoxical policies, I add a shifter variable to the Phillips curve

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa\left(x_t + g_t\right). \tag{13}$$

The variable g_t can represent totally useless (since it does not affect utility and does not enter the IS curve) government spending – hiring people to dig ditches and fill them up, or constructing defenses against imaginary alien invasions, to use two classic examples. It also can represent destruction of capital or technological regress – throwing away ATM machines to employ bank tellers, idling bulldozers to employ people with shovels, or even spoons; breaking glass or welcoming hurricanes, to use classic colorful examples.

These policy steps matter in this model because they increase inflation π_t for a given output gap, and thus they reduce the real interest rate and consumption growth.

The new-Keynesian model is utterly different from static Keynesian intuition in this regard. The static Keynesian model posits that consumption depends on income, and income is the model's key equilibrating variable. High real interest rates lower investment demand, which lowers output, which lowers consumption. The fiscal multiplier works because more income generates more consumption which generates more income.

By contrast, this basic new-Keynesian model posits that consumption depends on expected future consumption and the real interest rate. Solving (1)forward,

$$x_t = \int_{s=0}^{\infty} \left(i_{t+s} - r_{t+s} - \pi_{t+s}\right) ds,$$

the level of consumption is the integral of all future growth rates, and thus all future real interest rates. More growth is bad, because it is only achievable by lowering today's level. Permanent income, the trend, does not change, so the marginal propensity to consume out of income is *zero*. The interest rate, not income, is the key equilibrating variable, by reallocating consumption over time.

In a new-Keynesian model, the fiscal multiplier works entirely by creating expected inflation, which lowers real interest rates, lowers consumption growth, and thus raises today's consumption level. It has nothing to do with the standard income-multiplication mechanism. Capital destruction or technical regress similarly create inflation, thus lower interest rates, lower growth and raise consumption and output levels. And since, by assumption, we always return to an unchanged trend no matter how much capital or technology we destroy, wealth destruction is irrelevant to today's output and consumption[1].

Modeling fiscal stimulus by an increase in a Philips-curve shifter g is admittedly simplistic. More realistic and microfounded treatments, such as that in Werning (2012) and Wieland (2014),

[1]Dupor and Li (2013) find that stimulus spending was not associated with a rise in expected inflation, casting doubt on the new-Keynesian multiplier. Weiland (2014) similarly finds no inflation associated with negative supply shocks. These papers nicely show how imposing a model's mechanism –expected inflation in this case – can add a lot to identification beyond the usual VAR approach tying government spending to output and employment such as Ramey (2011).

recognize the potential value of government spending in the utility function, or treat its effects on potential output, or on the natural rate. But the point of this paper is to be as simple as possible to focus on equilibrium selection.

I assume that the government sets $g_t = g$ for $t < T$ and $g_t = 0$ thereafter, and I examine how increasing g in this way affects equilibrium output and employment. I characterize the effects of this policy by the private-consumption multiplier $\partial x_t/\partial g$. If g represents government spending, conventional multipliers would add g itself. I present $\partial x_t/\partial g$ multipliers since g can represent other Phillips curve shocks. Additionally, when the equations typically represent log-linearized deviations rather than deviations of levels, $y = x+g$ does not hold. The versions of this equation in Werning (2012) and Wieland (2014) (see his equation (2)) have additional scaling factors in front of x and g. The point here however is not quantitative assessment of one policy. The point is to see how the multipliers change across equilibrium choices.

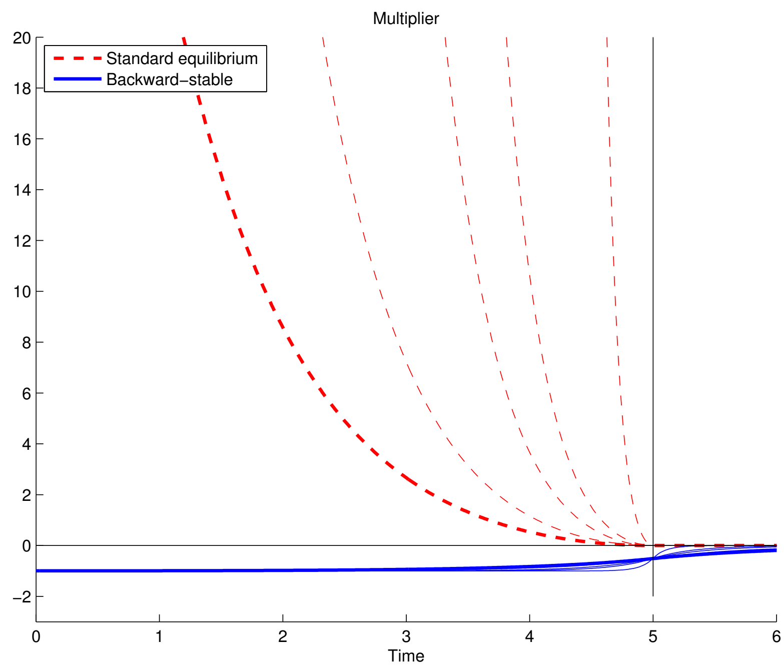

Figure 6: Private-consumption multipliers. I modify the Phillips curve to $d\pi_t/dt = \rho\pi_t - \kappa(x_t + g_t)$. The graph plots the multiplier $\partial x_t/\partial g$ for an increase in g that lasts from the indicated time until the end of the liquidity trap at $T = 5$. The dashed lines present the standard solution, the solid lines present the backward-stable solution. The thin lines show what happens as price stickiness is steadily reduced. Parameters are $\kappa = 1$, $\rho = 0.05$, $r = -5\%$ and the thin lines present $\kappa = 2$, 5, 10, 100.

Figure 6 presents these private-consumption multipliers. The calculation is in the Appendix. The thick red dashed line presents the multipliers in the standard equilibrium choice. The graph shows very large multipliers. The figure also shows how multipliers increase as the time of the liquidity trap and g policy increases, moving to the left. The thin dashed red lines present multipliers from the standard equilibrium choice, for steadily less price stickiness.

Multipliers paradoxically *increase* as price stickiness is *reduced.* In the limit that price stickiness goes to zero, the multiplier goes to infinity. Very small amounts of price stickiness generate very large multipliers. The multiplier is -1 at the limit *point*, however, so the multiplier displays the same discontinuous behavior as the equilibrium quantities display.

This increase in multipliers presents an interesting paradox for advocates of such policies using this equilibrium choice. Microeconomic efforts to reduce price stickiness – the central friction in the economy, in this view – might be resisted, since they make the depression worse, according to Figure 3. But such efforts would make multipliers larger, greatly increasing the effectiveness of fiscal or broken-window stimulus.

The thick solid blue line is the multiplier in the backward-stable solution. The multiplier is negative throughout and it asymptotes smoothly to the frictionless limit -1, and also as price stickiness is reduced. The multiplier in the no-jump equilibrium is similar.

The intuition for this contrast is fairly straightforward. The Phillips curve (13) shows that the private-consumption multiplier is -1 unless some change occurs in inflation π_t or its derivative $d\pi_t/dt$. In a steady state, with constant inflation, an increase in g is matched by a decrease in x, no change in inflation, and no change in the terms of the IS curve. Thus, to obtain a large positive multiplier, we need an equilibrium that already features large changes in inflation $d\pi_t/dt$ in the Phillips curve. The standard equilibrium has strong inflation dynamics, and hence a strong multiplier. The backward-stable equilibrium, like the no-jump equilibrium, has small inflation dynamics, and hence a small multiplier. That observation suggests that repairing the model to eliminate the counterfactual strong inflation dynamics will also eliminate the large multipliers.

In sum, magical multiplier predictions are direct results of *equilibrium choice*, holding the model and the path of interest rates constant. The backward-stable equilibrium produces a multiplier that is, if anything, lower than conventional wisdom, and more in line with the complete crowding-out results of some equilibrium models.

3.1 Forward guidance and commitments

Many authors have advocated "forward guidance" and "commitment," "open-mouth" policies, or promises to temporarily raise future inflation targets, as policies to ameliorate a liquidity trap. If the Fed could only commit itself to keep rates low for some time after the negative natural rate passes, we would get a little bit of inflation at that time, and the discounted effect of lower future real rates would stimulate output today. Werning's (2012) optimal policy is of this sort.

To address the power of this sort of policy, I now assume that people expect that the interest rate will remain at zero $i_t = 0$ for a time interval τ after T, when the natural rate rises to a new constant and positive value r_T. In the previous simulations, people expected the Fed to raise rates to $i_t = r_T$ immediately at $t = T$, so $\tau = 0$.

I solve the model as before. I pick non-explosive solutions for $t > T + \tau$, eliminating the explosive eigenvalue in that region. To work backwards, I now paste twice, once at $t = T + \tau$ when interest rates rise from $i_t = 0$ to $i_t = r_T$ and again at $t = T$ when the natural rate switches from $-r$ to $+r_T$. The standard equilibrium chooses $\pi_{T+\tau} = 0$. The backwards-stable equilibrium chooses the equilibrium that is bounded as time goes backwards, though this allows some inflation to continue past $\pi_{T+\tau}$. The equations are in the Appendix.

Figure 7 presents the predictions of the standard equilibrium for a variety of time intervals τ. In the top left, I present the previous solution with $\tau = 0$, which reminds us of the deep recession and deflation baseline.

The top right panel supposes that people expect the interest-rate rise to be delayed for six months. (However, since the parameters are not calibrated, the value of the time interval is not meaningful.) This delay allows a little inflation to emerge between $t = T$ and $t = T + \tau$. The solutions are the same for $t < T$ given π_T, so the little bit of inflation $\pi_T > 0$ improves the situation for $t < T$ by switching to one of the $\pi_T > 0$ equilibria. The solutions for $t < T$ depend only on π_T, so there is a delay policy of this sort that creates the backward-stable and no-jump equilibria exactly. Then, multiplying small changes in terminal conditions by backward-explosive eigenvalues has large effects. The vertical difference between the top left and top right equilibria at $t = 2$, for example, is quite large, and larger still at $t = 1$. Thus, this equilibrium choice displays the paradoxical prediction that promises further in the future have *larger* effects today.

Moving to the bottom left and bottom right panels, allowing one year and then two years of delayed interest rate rises has dramatic effects. Now the deflation and depression during the liquidity trap is turned into an inflation with a boom. Not shown, as pricing frictions decrease, and the backward-explosive eigenvalue increases, all of these effects become larger.

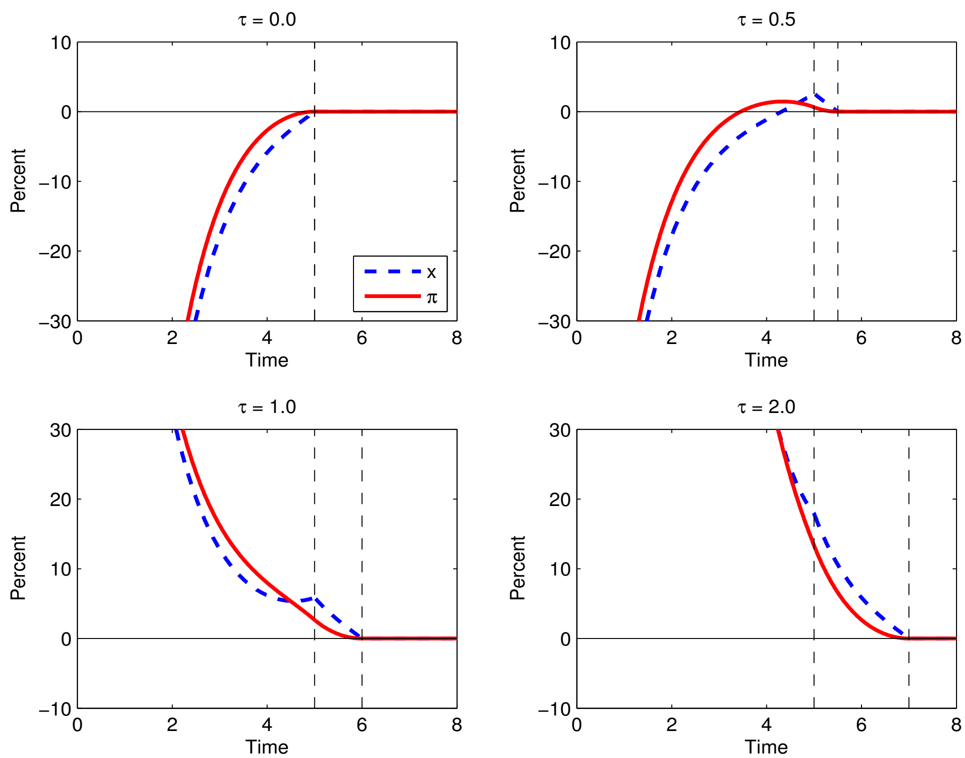

Figure 7: Equilibrium values of output gap x and inflation π in the standard equilibrium choice, when the interest-rate rise is delayed. At $t = T = 5$, marked by the left dashed line, the natural rate changes from -5% to +5%. At $t = T + \tau$ people expect the nominal interest rate to rise from $i = 0$ to $i = 5\%$, for τ as indicated. The standard selecton criterion is $\pi_{T+\tau} = 0$.

Figure 8 presents the backward-stable equilibrium choice in the same situation. As we might have expected, the backward-stable equilibrium shows very little current $(t < T)$ effect of the delayed-interest-rate-rise policy. Moreover, it displays the normal economic prediction that promises about the further-off future have *less* effect today.

The jumpy dynamics around T and $T + \tau$ reflect the usual local-to-frictionless behavior of

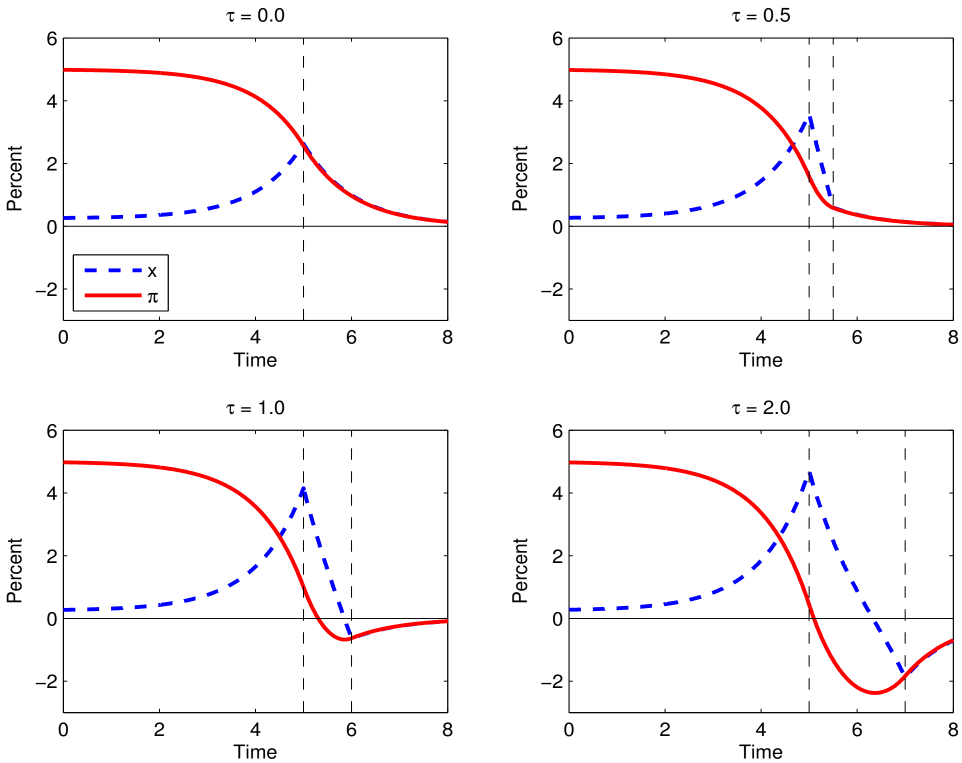

Figure 8: Equilibrium values of output gap x and inflation π in the backward-stable equilibrium choice, when the interest-rate rise is delayed. At $t = T = 5$, marked by the left dashed line, the natural rate changes from -5% to +5%. At $t = T + \tau$ people expect the nominal interest rate from $i = 0$ to $i = 5\%$, for τ as indicated.

this equilibrium. The completely frictionless solution (not shown) requires $x_t = 0$ at all times. Thus, we will see inflation $\pi_t = r$ for $t < T$, a period of *de*flation $\pi_t = -r_T$ during the period $T < t < T + \tau$ of zero nominal rate and positive natural rate $r_T > 0$, and then no inflation again when $i_t = r_T$ for $t > T + \tau$.

The main point remains: equilibrium choice is vitally important to analyzing predictions of this model, and all the dramatic properties disappear in local-to-frictionless equilibrium choices.

3.2 Equations

The standard equilibrium shown in Figure 7 is, for $t < T$,

$$\left[\begin{array}{c} \kappa x_t \\ \pi_t \end{array}\right] = \left[\begin{array}{c} \rho \\ 1 \end{array}\right] r - \frac{1}{\lambda - \delta}\left[\begin{array}{cc} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{array}\right]\left\{\left[\begin{array}{c} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{array}\right] r + \left[\begin{array}{c} e^{\delta(t-T)}\left(1 - e^{-\delta\tau}\right) \\ e^{\lambda(t-T)}\left(1 - e^{-\lambda\tau}\right) \end{array}\right] r_T\right\}.$$

The Appendix gives the algebra. We recognize the solution (9) from before, plus the r_T term. The r_T term thus captures the effects of the interest-rate-rise postponement policy. The $e^{\delta(t-T)}$ terms explode backward so are the most important. Thus, the $\left[e^{\delta(t-T)}\right] r$ multiplied by λ and λ^2 are the terms that drive the negative explosion going back in time in the standard solution. Now, $e^{-\delta\tau} > 1$, so the top right term is negative, offsetting the $\left[e^{\delta(t-T)}\right] r$ term. For larger enough

τ, the total effect multiplying the $e^{-\delta(t-T)}$ root becomes positive, which gives the inflationary solutions shown in Figure 7 for $\tau = 1$ and $\tau = 2$.

Between T and $T + \tau$, the standard solution is

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} - 1 \\ e^{\lambda(t-(T+\tau))} - 1 \end{bmatrix} r_T$$

Both terms are positive, so inflation and output gap are positive during this period. Beyond $t > T + \tau$, both inflation and output gap are zero.

The backward-stable solution is, for $t < T$

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \frac{1}{\lambda - \delta} \begin{bmatrix} \delta^2 \\ \delta \end{bmatrix} \left[r + \left(1 - e^{-\lambda\tau}\right) r_T \right] e^{\lambda(t-T)}$$

We see that the backwards-stable root $e^{\lambda(t-T)}$ is the only one left, and the system approaches the steady state going back in time. For $t > T + \tau$ this solution leaves some slowly decaying inflation and output gap, now controlled by the forward-stable root $e^{\delta(t-T)}$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 \\ \lambda \end{bmatrix} \left[r + r_T \left(1 - e^{-\delta\tau}\right) \right] e^{\delta(t-T)}.$$

The behavior between T and $T + \tau$ is not very interesting (well, I couldn't make it pretty), but for completeness it is

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} - 1 \\ e^{\lambda(t-(T+\tau))} - 1 \end{bmatrix} r_T \\ &\quad + \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 \\ \lambda \end{bmatrix} \left[r + r_T \left(1 - e^{-\delta\tau}\right) \right] e^{\delta(t-T)}. \end{aligned}$$

4 Taylor rules

How do we choose the equilibrium? In the standard new-Keynesian approach, the researcher specifies that the Fed chooses the equilibrium. The Fed chooses the desired equilibrium interest rate path $\{i_t^*\}$. It then *also and additionally* conducts an equilibrium-selection policy to select which of the many possible equilibria $\{\pi_t\}$ and $\{x_t\}$ consistent with that $\{i_t^*\}$ will emerge as the equilibrium $\{\pi_t^*\}$ and $\{x_t^*\}$. Finally, people know about all this, as it is their expectations of Fed equilibrium-selection policy in the future that determines which equilibrium emerges today.

To be specific, after choosing the desired equilibrium interest rate path $\{i_t^*\}$, the Fed selects the equilibrium π_t^* from the set $\{\pi_t\}$ consistent with $\{i_t^*\}$ (the set graphed in Figure 1) by following for $t > T$ a Taylor-rule inspired policy of the form

$$i_t = i_t^* + \phi(\pi_t - \pi_t^*). \tag{14}$$

This policy de-stabilizes the economy, by making both eigenvalues explosive. With $\|\phi\| > 1$ in this model, all the equilibria $\{\pi_t, x_t\}$ other than $\{\pi_t^*, x_t^*\}$ now explode as $t \to \infty$. The new-Keynesian tradition adopts as an equilibrium-selection principle that the economy will not choose non-locally bounded equilibria, and thus predicts that π_t^* is the unique observed equilibrium.

Equation (14) shows instantly that interest rate policy, $\{i_t^*\}$ and equilibrium-selection policy $\{\pi_t^*\}$ are completely separate parts of monetary policy. It also shows constructively how the researcher can specify that any of the equilibria studied so far are derived as the unique locally-bounded equilibrium of a Taylor-rule new-Keynesian model. Just choose $\{\pi_t^*\}$ to be whichever equilibrium you want the model to produce.

The form (14), due to King (2000), may seem unusual. The interest rate rule is often parameterized as a simple Taylor rule

$$i_t = \bar{\imath}_t + \phi \pi_t, \tag{15}$$

where $\bar{\imath}_t$ is a potentially time-varying intercept or policy shock, in which the Fed responds directly to other events or shocks in the economy. Woodford (2004) advocates such an intercept as "Wicksellian policy" and shows its desirable, indeed optimal, properties. Svensson and Woodford (2005) similarly show how the interest target should respond directly to economic shocks, such as the natural rate shock in this economy, to produce optimal monetary policy. The Fed should, in Woodford's model, and does, in practice, follow a rule with a disturbance term ($R^2 < 1$), or equivalently a time-varying intercept. The Fed explains deviations from "normal" policy as responses to shocks in the economy. Including the possibility of such an intercept is not unnatural – assuming it away would be unnatural.

The stochastic-intercept formulation in (15) is equivalent to the equilibrium-selection formulation in (14). In equilibrium, (15) is

$$i_t^* = \bar{\imath}_t + \phi \pi_t^*.$$

Then, taking the difference of (15) from its equilibrium, we can rewrite (15) in the form (14)

$$i_t = (i_t^* - \phi \pi_t^*) + \phi \pi_t = i_t^* + \phi\left(\pi_t - \pi_t^*\right).$$

So, my initial claim that new-Keynesian Taylor rules have two, distinct parts, an interest rate path $\{i_t^*\}$ and a separate equilibrium-selection policy $\{\pi_t^*\}$ may have seemed unfamiliar. And this fact is not obvious in staring at (15). But since (15) is algebraically equivalent to (14), the claim is true. A Wicksellian stochastic intercept is the same thing as a time-varying (more generally, state-contingent) inflation target which is the same thing as an equilibrium selection policy that acts distinctly and on top of interest-rate policy. And all the Fed has to do in order to produce, say, the zero-jump equilibrium, in this model, given the expected path of interest rates $\{i_t^*\}$, is to adopt the right equilibrium-selection policy and get people to believe it.

The rule (14) is similar to off-equilibrium threats in game theory. If the private sector deviates from the desired path π_t^*, the Fed will deviate from the desired path i_t^*, and the threat of subsequent explosion is enough to persuade the private sector not to deviate in the first place. In equilibrium, we only see $\{i_t^*, \pi_t^*\}$. Following this view, one can imagine many other actions the Fed could take to enforce its inflation target π_t^*. Atkeson, Chari and Kehoe (2010) discuss general "sophisticated" equilibrium-selection strategies. But Taylor-inspired rules dominate applied analysis, so I will stick to that formulation here.

Werning (2012) sums up equilibrium selection compactly. He also specifies the interest-rate path, and he specifies which equilibrium he wants to select, with no output gap or inflation after T. He then writes "I assume that the central bank can guarantee.. $\pi(t), x(t) = (0, 0)$ for $t \geq T$," and that this assumption "presumes that the central bank somehow overcomes the indeterminacy of equilibria that plagues these models. A few ideas have been advanced to accomplish this, such as adhering to a Taylor rule with appropriate coefficients, or the fiscal theory of the price level." This is correct. But the previous figures emphasize that this assumption is not one to be relegated to a footnote. People's expectation of the Fed's equilibrium-selection policy completely determines whether the liquidity trap will be benign or disastrous. And here I point out that the Fed could just as easily select any of the other equilibria.

In sum, though we often focus on the current and expected future path of interest rates $\{i_t^*\}$ in evaluating monetary policy, in this class of models, assumptions about "equilibrium selection policy," the specification of $\{\pi_t^*\}$ and the reaction $\phi(\pi_t - \pi_t^*)$ in (14), matter enormously to the nature of the model's predictions.

A researcher can, rather than directly pick equilibria, assume that the Fed picks any equilibrium in the set studied so far. So the mere assumption of a Taylor-rule selection mechanism does nothing to prune equilibria, or to select the standard choice in particular. The prediction of a bad outcome in a liquidity trap, the standard outcome rather than the no-jump or backward-stable outcome, comes entirely from the researcher's assumption about people's expectations of inappropriate "equilibrium selection policy."

4.1 A closer look at Taylor rules

To consider these ideas, it is better to make them concrete. Here I display expectations of Fed behavior that underlie the selection of one vs. another equilibrium.

I trade a little complexity for a rule that does not seem to stack the deck against reasonable results. Following Sims (2004) and Fernández-Villaverde, Posch, and Rubio-Ramírez (2012), I write a continuous-time Taylor rule in partial-adjustment form, which is also how it is typically estimated,

$$i_t - i_t^* = \phi\theta \int_{s=0}^{\infty} e^{-\theta s}(\pi_{t-s} - \pi_{t-s}^*)ds.$$

In differential form,

$$\frac{d\,(i_t - i_t^*)}{dt} = \theta\left[\phi\,(\pi_t - \pi_t^*) - (i_t - i_t^*)\right]. \tag{16}$$

The parameter ϕ is the usual Taylor parameter, measuring the eventual response of interest rates to inflation. The parameter θ controls speed of adjustment. I use $\theta = 1$ or a half-life of one year. A rule $i_t - i_t^* = \phi\,(\pi_t - \pi_t^*)$ may seem simpler, but has some undesirable properties in continuous time, set out in the Appendix. Empirical Taylor rules include output responses, $i_t - i_t^* = \phi_\pi\,(\pi_t - \pi_t^*) + \phi_x\,(x_t - x_t^*)$, but this complexity is not important here as the inflation response is crucial for equilibrium selection and determinacy issues.

I also modify (16) to respect the zero bound. We then have for $t > T$,

$$\frac{di_t}{dt} = \left\{ \begin{array}{cc} \theta\left[\phi\,(\pi_t - \pi_t^*) - (i_t - i_t^*)\right] & \text{if } i_t > 0 \\ \max\left\{\theta\left[\phi\,(\pi_t - \pi_t^*) - (i_t - i_t^*)\right], 0\right\} & \text{if } i_t = 0 \end{array} \right\}, \tag{17}$$

and in this simple scenario $i_t^* = r$.

Some of the equilibria in the $t < T$ region have positive inflation, and so might result in non-zero interest rates if one imagined a Taylor rule always operating. I will leave the Taylor rule off for $t < T$, however. Adding it just adds conceptual complications, and doesn't address any issues in the literature – this is still a critique paper. One could imagine that the Fed adds such a large stochastic intercept (large π_t^*) in a financial crisis that interest rates are zero in any of the plotted equilibria for $t < T$.

There is no change to any equilibria in $t < T$. The difference is that each equilibrium is now pasted to different expectations about what will happen for $t > T$. And those different $t > T$ paths can rule out equilibria other than $\{\pi_t^*, x_t^*\}$.

Figure 9 presents the resulting set of equilibrium-selection expectations underlying the assumption that the Fed selects $\pi_t^* = 0$ for $t > T$, the standard solution. The Appendix details the calculations. This is an example of how Werning (2012) might have filled out the above-quoted footnote. Compare it to Figure 1 – the figures are the same for $t < T$, but now we see the effect of the Taylor-rule equilibrium selection efforts for $t > T$. Figure 10 graphs the output gap x_t. Compare it to Figure 2. Finally, Figure 11 presents the interest rate.

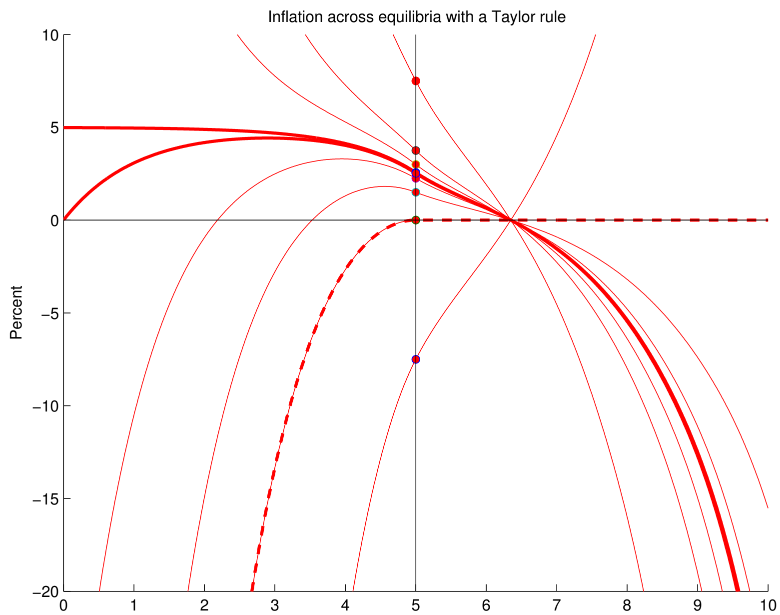

Figure 9: Path of inflation in all equilibria with a Taylor rule selecting $\pi_t^* = 0$. At $t = T = 5$ people expect the Fed to follow a Taylor rule to enforce the $\pi^* = x^* = 0$ equilibrium. The Taylor parameter is $\phi = 1.1$ and the half-life of adjustment is $\theta = 1$. The thick dashed line is the standard equilibrium.

The comparison between the sets of figures verify that the Taylor rule is doing what it's supposed to do: $\phi > 1$ induces explosive dynamics in all variables except for the unique locally bounded equilibrium $x_t^* = 0, \pi_t^* = 0$, $t > T$, where before there were multiple locally-bounded equilibria for $t > T$. If we rule out all but locally-bounded equilibria, we choose the $x_t^* = 0$, $\pi_t^* = 0$ equilibrium.

If the economy were to follow the backwards-stable equilibrium, for example, highlighted in bold in the figure, people believe that after $t > T$, the Fed would not tolerate the "glide path" to zero inflation shown in Figure 4. The Fed would instead start aggressively raising interest rates above the natural rate. But the problem is *not* that people expect the Fed to push the glide path too quickly to zero inflation. This explosion is central: If people believe the Fed takes any stabilizing action, the alternative path converges to zero and is not ruled out as an equilibrium.

Figure 11 shows that interest rates do hit the zero bound for many equilibria. At that bound, the eigenvalues of the system are again $\lambda > 0$, and $\delta < 0$. Nonetheless, all of the plotted equilibria still explode and are ruled out. The reason is that during the period of non-

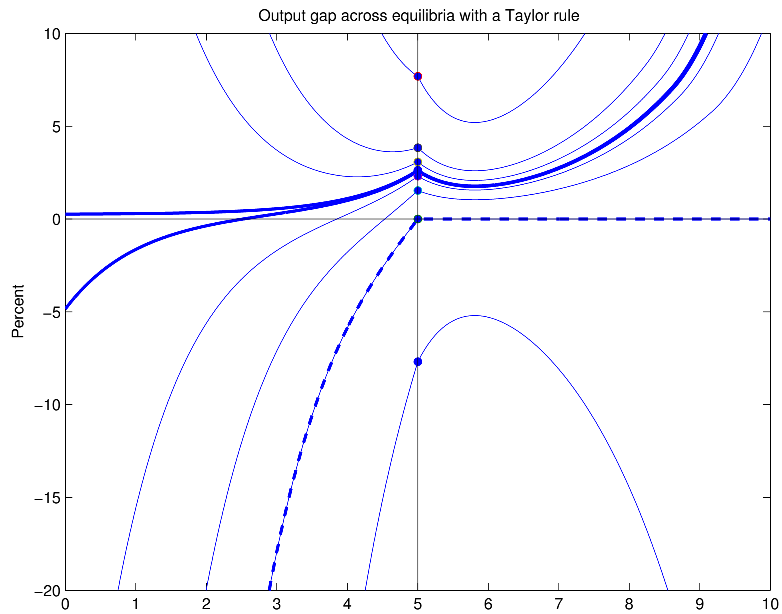

Figure 10: Path of output gaps in all equilibria with a Taylor rule selecting $\pi_t^* = 0$ for $t > T$. The thick dashed line gives the standard equilibrium.

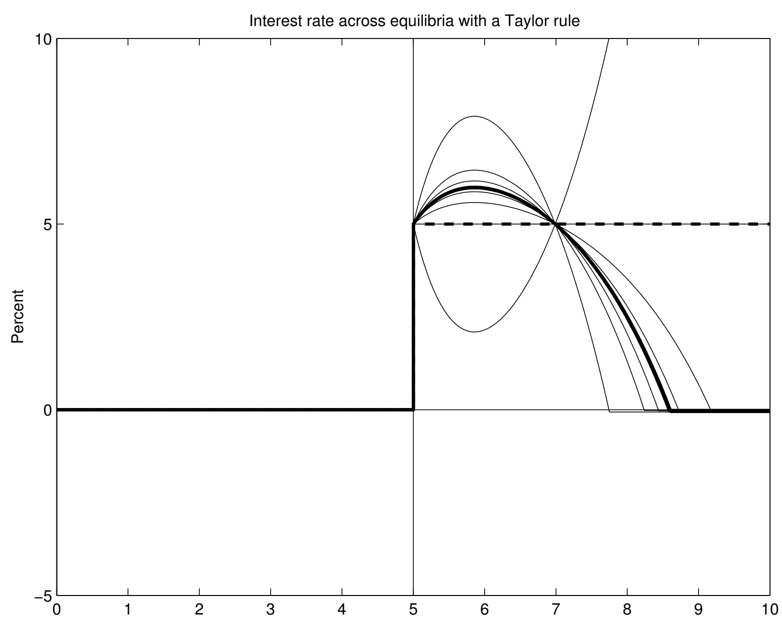

Figure 11: Path of interest rates in all equilibria with a Taylor rule selecting $\pi_t^* = 0$. The Taylor parameter is $\phi = 1.1$ and the half-life of adjustment is $\theta = 1$. The thick dashed line gives the standard equilibrium. The natural rate is 5% for $t > T$.

zero interest rates, these equilibria left the linear combination of (x_t, π_t) that loads only on the stable eigenvalue δ. However, the usual indeterminacies at a zero bound now remain, in that we cannot rule out "sunspot" equilibria, not shown, in which $\{x_t, \pi_t\}$ unexpectedly jump to a linear combination that does load only on δ and thus returns stably to zero. (Benhabib, Schmitt-Grohé and Uribe 2001, 2002.) In turn, appending such sunspots may mean that even the apparently explosive equilibria here are eventually absorbed by the zero bound. However, I follow the standard rules of the game and ignore this possibility.

There is, however, nothing special about the $\pi_t^* = 0$ equilibrium choice in a Taylor rule. That is a main point of this section. A Fed able to follow an "equilibrium selection policy" can select any of the other equilibria just as well, and the researcher may produce any of the equilibria studied so far by assuming the Fed selects it. To emphasize this point, Figures 12 and 13 show how the Fed would select the backward-stable equilibrium. The Figures show the range of equilibria when the Fed follows the same Taylor rule (17) but now selecting π_t^* to be the glide-path for $t > T$ that leads to the backward-stable equilibrium for $t < T$. Now all the equilibria other than the backward-stable one explode, including, in particular, the standard equilibrium with $\pi_t^* = 0$.

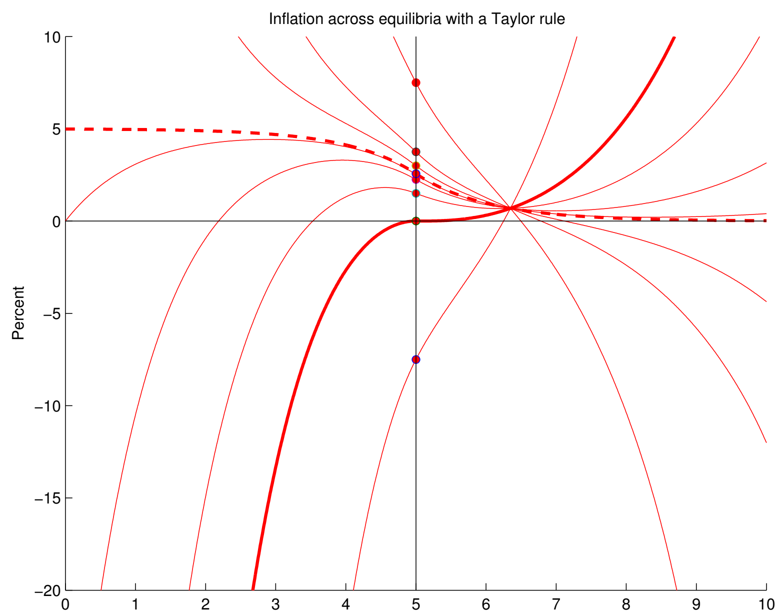

Figure 12: Path of inflation in all equilibria with a Taylor rule, when the Fed selects the backward-stable equilibrium.

4.2 Doubts about Taylor-rule equilibrium selection

With the assumed expectations before us, we can now ask, do these figures represent at all a reasonable specification of how people think the Fed would react to inflation above its target? The Fed does announce what it would like inflation to do, which we may interpret as $\{\pi_t^*\}$. And it talks a lot about interest rates, which we may interpret as $\{i_t^*\}$. But do people really believe that if inflation comes out to something different, the Fed will deliberately send the economy to a "non-locally-bounded" *deflationary boom* equilibrium? It would be an interesting exercise for survey researchers to find out how many people's expectations of what the Fed would do if its inflation target were violated correspond to these figures.

More deeply, does the Fed even have, and do people expect, an "equilibrium-selection" policy, that *de*stabilizes the economy for inflation not equal to its target, distinct from its "interest rate policy?" You will be hard pressed to find mention of this concept on the Fed's website. The

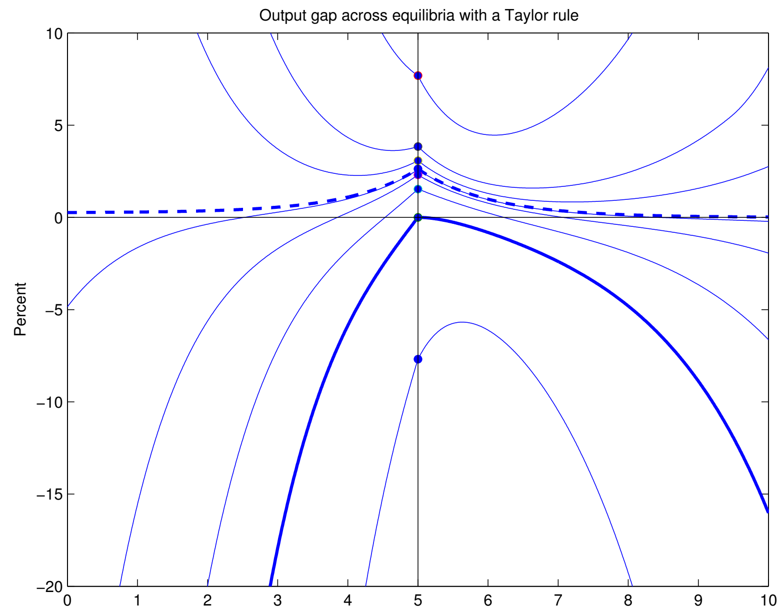

Figure 13: Path of output gap in all equilibria with a Taylor rule for $t > T$, when the Fed selects the backward-stable equilibrium.

Fed resolutely describes its behavior as stabilizing, reacting to unexpected inflation in a way to bring inflation back down again.

Yes, people believe the Fed will react to inflation and output, and historically observed interest rates i_t^* correlate with equilibrium output x_t^* and inflation π_t^*. But these observations are irrelevant to the question. The first question is, how does the equilibrium interest rate target – the interest rate we will see, i_t^*– vary over time, in ways correlated with observed equilibrium inflation π_t^* and output gap x_t^*? The second question is, how do conjectured deviations from that equilibrium $i_t - i_t^*$, which we never see, vary with conjectured deviations from equilibrium inflation $\pi_t - \pi_t^*$ and output $x_t - x_t^*$, which we also never see?

Cochrane (2011a) criticizes equilibrium selection by this specification that people expect the Fed to explode the economy in more theoretical detail. The novelty here is to look at the pictures for this example, and to ponder whether it makes sense in this scenario to assume that people have "equilibrium-selection" expectations such as those displayed here.

4.3 What equilibrium will the Fed choose?

Suppose the Fed has, and people expect, an equilibrium-selection policy, either of the Taylor rule form, or perhaps one of the more "sophisticated" forms suggested by Atkeson, Chari, and Kehoe (2010). Why should people expect the Fed to select the $\pi_T^* = 0$ equilibrium, with its drastic consequences for inflation and output before T, when the Fed could just as easily select the backward-stable equilibrium, by selecting a "glide path" of inflation back to zero starting at time T? Why should a researcher make this assumption and select the disastrous standard equilibrium as the model's prediction?

Werning (2012) offers an answer. People expect that the Fed will do ex-post what looks best going forward no matter what last year's "forward guidance" was. Werning finds the optimal *discretionary* policy. His Fed maximizes a standard loss function

$$L = \frac{1}{2}\int_0^\infty e^{-\rho t}\left(x_t^2 + \lambda \pi_t^2\right) dt, \tag{18}$$

choosing i_t and π_t at each date in a completely forward-looking manner, subject to the model dynamics (1)-(2). At time T, $\pi_t^* = x_t^* = 0$ $t \geq T$ is optimal. So the Fed will choose this equilibrium at T, people know that, and we are stuck on the disastrous standard equilibrium for $t < T$.

Since there are multiple equilibria given $\{i_t^*\}$, Werning's monetary policy consists of *both* the specification of i_t^* and of π_t^*, with an unstated equilibrium-selection device for the latter. (See Werning's footnote 7, p. 9, quoted above.) We have learned, relative to Werning's analysis, that the equilibrium-selection footnote is key: The Fed could achieve a benign outcome very close to Werning's fully-optimal policy with commitment, without changing interest-rate policy at all from its discretionary path, if the Fed could only change the equilibrium-selection policy to the backward-stable or no-jump equilibrium.

That the Fed – and certainly our Fed, as it operated in 2008-2014 – cannot and is not expected by people to precommit, is a sensible assumption, especially regarding interest rate policy. Nobody expects the Fed chair to show up in Congress in 2016 and say "it would now be appropriate to raise rates. But we promised a long period of zero rates to fight the crisis, so now we're going to make good on that promise."

However, there is a deep flaw – a form of subgame imperfection – in this specification. In Figures 9 and 10, all equilibria except the selected x_t^*, π_t^* one are disastrous going forward, for $t > T$, from the Fed's quadratic loss objective. Thus, to select at time T the $x_t^* = 0, \pi_t^* = 0$, $t > T$ equilibrium, the Fed must *completely* precommit to follow an equilibrium-selection threat which, ex-post, is disastrous for its objectives.

So, we must imagine the Fed is incapable of precommitment in selecting its interest rate i_t^* and inflation π_t^* targets, not even allowing a gentle glide-path, but then capable of a monstrous precommitment in its equilibrium-selection policy. And, again, that people know this and believe it ahead of time.

A Fed that cannot precommit at all might choose $x_t^* = 0$, $\pi_t^* = 0$ for $t > T$. But then, if the benign backwards stable equilibrium emerges, a Fed that cannot precommit would be far better off ex-post leaving $i = i^*$ and tolerating the glidepath rather than making good on the selection threat and blowing up the economy. If it cannot precommit to its inflation target, surely it also cannot precommit to its equilibrium-selection policy. And then the alternative equilibria are free to show up.

The point here is not to defend a different equilibrium as the "right" one. The point is to undermine the view that equilibrium-selection policy by a Fed unable to pre-commit gives the conventional answer.

4.4 Inflation targets

One might view the equations as validating calls that the Fed adopt an explicit inflation target separately from its interest rate policy announcements, and temporarily raise that target. After all, the local-to-frictionless solution differs only from the disastrous standard solution in that the Fed changes its inflation target, changing π^* in $i_t = i_t^* + \phi(\pi_t - \pi_t^*)$ while leaving i^* alone.

This is true, but the whole enterprise hinges on the question, what does the Fed do if the world doesn't follow its announcements of what inflation it would like to see? What is the stick corresponding to the "speak loudly" advice? In this model, the answer is the Taylor-inspired rule, in its off-equilibrium threat form. People are assumed to expect that if the wrong equilibrium emerges, the Fed will drive the economy off a cliff.

Many people who advocate a higher inflation target, such as Blanchard, Dell'Ariccia, and Mauro (2010), do not have such equilibrium-selection policy in mind as the point of an inflation target. They have in mind a simpler old-Keynesian view that the central bank will keep *equilibrium* interest rates low for a while as inflation picks up, not a different equilibrium-selection policy for the *same* equilibrium interest rate path. In these models, as in Taylor (1993, 1999), the Taylor rule introduces stable eigenvalues, not unstable ones, and the interest rate target is simply where the Fed will slowly bring the economy back to after any disturbance. They do not mean "and if the desired inflation does not occur, the Fed should precommit to drive the economy off a cliff."

4.5 Taylor-rule summary

We started with multiple equilibria $\{\pi_t^*, x_t^*\}$ corresponding to a given set of interest-rate expectations, $\{i_t^*\}$. Some, such as the standard choice, produced disastrous predictions in the liquidity trap, and unusual and paradoxical policy predictions. Others, and especially the local-to-frictionless backward-stable and zero-jump equilibria, produced mild inflation, little output gap, and normal policy predictions. Our question is, how should we pick from these equilibria?

In this section, I pursued the answer that the Fed picks the equilibrium – that the researcher specifies that people expect the Fed to pick the equilibrium, to be precise – via an "equilibrium selection policy" distinct from "interest rate policy," as in the Taylor–inspired-rule specification of Fed policy in New-Keynesian models. Alas, we find that the researcher can assume that the Fed chooses any equilibrium the researcher likes via a Taylor rule. So the simple fact of joining the $t < T$ equilibria to a return to Taylor rules when the trap fades does nothing to help us to select equilibria.

I criticized the general idea that the Fed has, or people expect, an "equilibrium-selection policy" of the new-Keynesian Taylor rule type. But even if it does, there seems little reason to specify that the Fed selects the standard equilibrium, when it could just as easily select the no-jump or backward-stable equilibrium. The Fed has already loudly said that its policies in the trap are a departure from "normal" and that it plans a gradual "normalization" of policy. Does this not sound exactly like a glide-path for $\{\pi_t^*\}$, a gentle removal of the stochastic intercept, as in the backward-stable equilibrium?

Werning (2012) gives a principled reason to select the disastrous standard equilibrium: That the Fed cannot precommit, so will follow a forward-looking optimal policy, which means setting $\pi_T^* = 0$ immediately when the trap ends and not allowing a glidepath. But I have shown that enforcing that equilibrium-selection policy requires the Fed to have a perfect precommitment ability to blow up the economy should alternative equilibria emerge anyway. A Fed that cannot precommit cannot follow a new-Keynesian Taylor-rule equilibrium-selection policy.

One can assume that the Fed can fully precommit to both interest-rate policy and equilibrium-selection policy. This criterion generates Werning's (2012) optimal policy with commitment solution. It features a slight delay in raising equilibrium interest rates, and, more importantly, an equilibrium selection policy that selects gentle inflation during the liquidity trap that gently subsides afterwards, very much like the backward-stable or no-jump equilibria presented here. If that's the answer, then both as a positive and normative matter, a liquidity trap is simply not much of a problem.

So, we are back where we started. The model leaves many equilibria, and no obvious reason to pick the disastrous one rather than one of the more benign equilibria as the model's prediction for a liquidity trap and associated policies.

5 Equilibrium choice

This discussion should leave us hungry for better principles on which to select equilibria. I do not propose a resolution to this question in this paper. But it is useful to outlines the possibilities that one might follow. Fixing the path of expected interest rates $\{i_t^*\}$, how can we pick the right path $\{\pi_t^*\}$?

5.1 Philosophy and equilibrium

In models with multiple equilibria, a wide range of principles have been advocated to select equilibria. Examples include the "minimum state variable" criterion, "E-stability" and "learnability." See Evans and Hohkapohja (2001), McCallum (2003), (2009), a rejoinder in Cochrane (2009), and Christiano and Eichenbaum (2012). In a sense, these principles are really philosophical, as they lie beyond the standard definition of equilibrium. That's an observation, not a criticism: Adopting such a principle, in essence expanding the definition of equilibrium, is a way to pick equilibria in a coherent way.

The basic new-Keynesian procedure has a philosophical element as well. It rests primarily on the principle that expectations should "coordinate" on "locally bounded" equilibria. As Woodford (2004, p.128) explains, "The equilibrium ..[π^*].. is nonetheless *locally* unique, which may be enough to allow expectations to coordinate upon that equilibrium rather than on one of the others." See also King (2000, p. 58-59). Efforts to turn locally-bounded into a completely economic criterion in passive-fiscal models have, as far as I was able to ascertain in an extensive survey (Cochrane 2011a), failed, and that includes the standard citation to Obstfeld and Rogoff (1983). (Sims 2013 reads like a counterexample, but it isn't. See Cochrane 2015.) And this approach still relies on the Fed to induce a second unstable eigenvalue.

One can make similar philosophical cases for equilibrium-selection in this environment. The local-to-frictionless property is attractive – don't pick equilibria whose predictions are discontinuous at the frictionless limit. That principle does not pick a unique equilibrium, as any equilibrium that limits the initial response is local-to-frictionless, including the backward-stable and no-jump equilibria here. But that principle can serve to rule out equilibria, and the standard equilibrium choice in particular.

The local-to-frictionless criterion can also be read as a slight extension of the standard new-Keynesian criterion that equilibria should be locally bounded. The local-to-frictionless criterion specifies that equilibria remain locally bounded as price stickiness goes to zero[2].

One could extend the principle, perhaps, and choose the minimal perturbation of the underly-

[2] I thank an anonymous referee for this suggestion.

ing real equilibrium to account for price stickiness – finding the local-to-frictionless equilibrium that minimizes the sum of squared deviations from the frictionless equilibrium. Again, this criterion would rule out the standard choice.

The standard new-Keynesian Taylor-rule philosophy rules out forward explosions. One could argue similarly for the backwards-stable equilibrium, which is locally bounded in *both* time directions. Of course, we don't see full backward explosions; at date zero even the standard solution jumps only so much, and in stochastic models every day is a new date zero.

However, a deeper, related criterion could be that news about further-off future events should have smaller effects today. That criterion limits the loading on backward-explosive eigenvalues, so central to the dramatic conclusions of the standard equilibrium choice.

The no-inflation-jump or no-output-jump equilibria are also attractive, as large jumps in output or inflation seem counterintuitive. One might elevate the lack of such contemporaneous jumps to an equilibrium-selection principle. However, such jumps are legitimate in this specific model. The intuition that there should not be jumps wants to build a more complex model with costs to changing inflation as well as prices, or costs such as capital adjustment or habits to changing output. New-Keynesian models which add such frictions then still have jumps in other state variables, such as expected future output.

Optimal policy calculations such as Werning's typically produce a single best equilibrium, which we could advance as the correct choice, after we agree on the optimization problem and its commitment constraints. But that leaves open the question whether the equilibrium is unique given the tools at the central bank's disposal. If the bank can only choose the interest rate path, there is no guarantee that the economy will settle on the optimal inflation equilibrium.

I leave these as possibilities. A good equilibrium-selection principle should work in a wide range of models, and stochastic ones in particular. It should not just be an excuse to pick a nice-looking equilibrium of this very stylized model. The end of a paper built narrowly around other people's models is not the time to start innovating equilibrium-selection philosophies. Finally, I think there is a better way to select equilibria, considered next.

5.2 Fiscal theory

The fiscal theory of the price level neatly solves nominal indeterminacies under interest rate targets, using the completely standard Walrasian definition of equilibrium. It thus poses, I think, a more promising path.

Each of the equilibria I have plotted has different fiscal implications. The large deflation predicted by the standard equilibrium choice would have implied an unexpected bonanza to the holders of long-term U.S. government debt, requiring large new taxes or spending reductions. The small inflation rise of the backward-stable equilibrium implies a small devaluation of bond-holders' claims on the Treasury, allowing a bit more primary deficit than otherwise expected. Each equilibrium, together with the maturity structure of outstanding debt at time 0, requires a different value of the innovation to the present value of real primary surpluses, and can be indexed by the change in such surpluses it implies.

The standard new-Keynesian solution method throws away this ability to select equilibria. It *produces* indeterminacy in a model which would otherwise have a single equilibrium path, by assuming that fiscal policy is "passive," meaning that the Treasury is assumed to raise whatever revenue is required by any equilibrium inflation or deflation path.

Certainly, as a first step to equilibrium selection, even under the "passive" fiscal assumption, it is worth examining the fiscal backing required of any equilibrium, and deciding if that backing is reasonable. The big deflation predicted by the standard equilibrium, if it had occurred, would have put the "passive" assumption to the test. A large increase in surpluses, to fund an unexpected deflation-induced transfer from taxpayers to typical Treasury bondholders, was not exactly high on the 2009 political agenda, even if it had been economically possible. That consideration suggests, I think, that equilibria with deep deflations are not likely to be validated by fiscal policy. People know this, so equilibria with deep deflations didn't happen. There is a silver lining to a thunder-cloud of nominal debt.

The fiscal theory also helps to remove strange limiting properties of the standard equilibrium choice. As one reduces pricing frictions, the initial deflation jump is larger and larger. The implied increase in surpluses to pay off bondholders is larger and larger. The Laffer curve puts an upper bound on the present value of surpluses that can ever be raised – 100% of GDP – and, thus, a bound on how much deflation may jump. At some point we *must* abandon the standard solution.

Going beyond that qualitative consideration, to using the fiscal theory to pick a particular equilibrium, we need to think hard about what fiscal policy does in response to a "natural rate" shock. It's straightforward to find the equilibrium choice that has no fiscal implications, one in which the Treasury does not make any adjustments to surpluses. But that calculation is hardly realistic. Fiscal and monetary policy are coordinated, and fiscal policy responds to economic shocks as well. The right answer pairs a fiscal and monetary response to the shock, and results in a unique following equilibrium. (Cochrane 2014 includes impulse-responses in a sticky-price model with fiscal price determination and different assumptions about fiscal and monetary responses to shocks.)

A realistic fiscal policy specification is not easy either. Deficits increased in the great recession, yes, but what counts for the fiscal theory is the present value of all future surpluses. If, as usual, current debts imply larger future taxes, then there is no inflationary effect. And the discount rates of future surplus matter as much as their expected values. If a shock lowers the discount rate applied to the same stream of surpluses, that shock is deflationary without any change in fiscal expectations. So, while applying the fiscal theory to choose equilibria is simple in principle – choose the innovation in present value of future surpluses that accompanies the liquidity trap and monetary policy response to that trap, and this picks the equilibrium – doing a serious job is an extended exercise beyond the scope of this paper.

5.3 Empirical equilibrium selection

Equilibrium selection can, and should, in my view, be an *empirical* project as well as a theoretical one. The equilibrium choice rather obviously and centrally matters to how the model fits the data. Of course, to do a serious job of that empirical project requires a more fleshed-out model. Still, we can think of how one would go about the empirical project, and I can point to the big-picture fit of the model at hand to speculate constructively about what the empirical project might produce in the context of a more realistic model.

First, we can ask which equilibrium choice produces a better fit with the data, and the effects of policies in recent history. The US economy 2009-2014 features steady but slow growth, a level of output stuck about 6-7% below the previous trendline, a stagnant employment-population ratio, and steady positive 2% or so inflation.

The backward-stable and no-inflation-jump equilibria as shown in Figures 4 and 5 can produce this steady outcome. However, they do not produce a big output gap. Thus, they only account for the disappointing level and growth rate of output if one thinks that current output is about equal to potential, i.e. that the problem is "supply" rather than "demand," and that the CBO and other calculations of "potential" or non-inflationary output and employment are optimistic, as they were in the 1970s. The substantial ex-post downward revision in potential output calculations in hindsight supports this view.

Inflation did not rise, either. But there is no independent measurement of the natural rate shock, so fitting a model with this equilibrium would likely produce a negative 2% natural rate, 2% inflation, and 0% interest rate.

The standard equilibrium choice as shown in Figure 3 cannot produce stagnation. Here and in more general models, the standard equilibrium choice counterfactually predicts large and time-varying deflation (Hall (2011), forcefully, Ball and Mazumder (2011), King and Watson (2013), Coibion and Gorodnichenko (2013)), which simply did not happen, and it counterfactually predicts strong growth.

The problem in generating steady stagnation is central to the new-Keynesian model. The "IS" curve and the assumption that we return to trend means that we can only have a low *level* of output and consumption if we expect strong *growth*. The Phillips curve says that to have a large output gap, we must have inflation today much below expected inflation tomorrow and, thus, strongly declining deflation. One would have to imagine a steady stream of unexpected negative shocks – that each year, the expected duration of the negative natural rate increases unexpectedly by one more year – to rescue the model. Some models produce a slump with a Poisson emergence from the trap, which is exactly this specification. But five tails in a row is still pretty unlikely. Authors who modify the standard model to produce a slump, such as Del Negro, Giannoni and Schorfheide (2013) and, especially, Eggertsson and Mehrotra (2014), undertake fundamental modifications. This observation strengthens the case that this model doesn't produce a slump, so within the context of this model the data would choose something like the no-jump equilibrium.

Furthermore, the analysis revealed that all of the magical multipliers and paradoxical policies depended crucially on strong inflation dynamics, and a loading on a backwards-explosive eigenvalue. Produce a steady slump, and the multipliers evaporate.

Again, the very stylized model used here is too simplistic for a serious comparison to data. The main point here is just that given a model, the equilibrium selection is a measurable quantity, producing better or worse fit with the data. Considerations like these can be used to think about equilibrium selection rules in more elaborated models that are fit to data.

Second, we might consider which equilibrium choice is correct by examining data and policies from longer periods in history. Zero-bound predictions emerge in these models any time interest rates do not respond to changes in equilibrium output and inflation. Thus, they should occur in periods such as the Great Depression, at the zero bound, in the late 1940s and early 1950s, when interest rates were explicitly fixed, and in the 1970s, when new-Keynesian thought (Clarida, Galí and Gertler (2000)) claims that interest rates did not respond enough to inflation. At a casual level, attempts to deliberately inflate, output destruction, technical regress, useless government spending, and promises all seem to have been tried in those periods without the large effects claimed for them now. That too, however, is a suggestion, not a claim, and one that requires much more careful documentation. (Dupor and Li (2013) do this exercise, finding no multiplier and no inflation channel during the 1970s.) Again, the point here is method rather than result.

Third, and perhaps most concretely and constructively, recall that we can index equilibria by the jump of inflation or output at time zero, in response to the shock, π_0 and x_0. Thus, we can measure equilibrium selection by the contemporaneous impulse-response function. In my fiscal interpretation, we can measure the fiscal response to a typical shock as we measure the monetary response. That is, in stochastic models, the whole point of equilibrium selection: The model ties down $E_t x_{t+1}$ for many variables x, but $x_{t+1} - E_t x_{t+1}$ indexes multiple equilibria. In my fiscal indexation of equilibrium choices, the contemporaneous impulse-response function measures the fiscal policy response to shocks, which selects equilibria.

That exercise is still tricky. These are contemporaneous responses, usually orthogonalized by assumption. And we need to identify structural shocks – a natural rate shock in this case. And the zero bound may mean that responses are different when they cross the zero bound than when they don't. In addition, of course, the impulse-response prediction depends on the specific model one adopts, and which state variables are allowed to jump.

Still, whether by matching data, by matching policy experience, or by matching impulse-response functions, an equilibrium selection rule is an identifiable and measurable part of a model. It is not an issue that must perpetually remain a philosophical controversy, or a controversy about observationally equivalent interpretations of the same equations.

6 Concluding comments

I examine a standard new-Keynesian analysis of the zero bound, following Werning (2012): A negative natural rate will last until time T, and the nominal rate is stuck at zero. After that, policy returns to normal, meaning that people expect interest rates to rise again and thereafter follow the natural rate. I calculate the model equilibria in this circumstance. I find there are many locally-bounded (nonexplosive as $t \to \infty$) equilibria, which we can index by the value of expected inflation at the end of the trap, π_T, or by the jump at time 0, π_0. These equilibria all share the *same* interest rate path.

The new-Keynesian literature analyzes one equilibrium: it features a deep recession with deflation. It also features strong expected output growth, which is why the level of output is so low, and rapidly declining deflation. It predicts large multipliers to wasted government spending, and wealth or productivity destruction. It predicts large effects of announcements about far-off future policy. These predictions grow larger the longer the period of the liquidity trap, and as the degree of price stickiness is *reduced.* An ε price stickiness produces stone-age output levels and nearly infinite multipliers. Though price stickiness is the central friction causing the economy to deviate from efficient output levels, reducing price frictions would lower output further. Lowering price stickiness would, however, raise the effectiveness of wasted-spending and broken-window multipliers, again to arbitrarily high levels.

But there are many equilibria of this model. The "backward stable" and "no-inflation-jump" equilibria of the same model, with the same interest rate path, instead predict mild *in*flation during the liquidity trap, little if any reduction in output relative to potential, small negative multipliers, and little effects of promises of far-off policies. Their predictions, like any other equilibrium with bounded initial jumps, smoothly approach the frictionless limit as pricing frictions are reduced.

The equilibrium we will observe during the trap depends on people's expectations of what will happen following the trap. If people think we must have exactly zero inflation (deviation from long run trend) as soon as the trap ends, then we will experience the new-Keynesian recession and its paradoxical policy implications. If people expect that we can retain a mild inflation – about half the negative "natural rate" in the trap – and then a smoothly declining inflation "glide path" as graphed in Figure 4, then we will experience a benign period during the liquidity trap and policies will not have magical or paradoxical results.

At a minimum, this analysis shows that equilibrium selection, rather than just interest rate policy, is vitally important for understanding these models' predictions for a liquidity trap and the effectiveness of stimulative policies. In usual interpretations of new-Keynesian model results, authors feel that interest rate policy is central, and equilibrium-selection policy by the Fed or by the author are details relegated to technical footnotes (as in Werning 2012), game-theoretic foundations, or philosophical debates, which can all safely be ignored in applied research.

The standard new-Keynesian approach specifies that expectations about Fed behavior select equilibria. In that analysis, the Fed has *two* separate and important policy tools – interest rate policy, which specifies the equilibrium interest rate path $\{i_t^*\}$ (which may vary systematically with equilibrium output x_t^*, inflation π_t^*, and other shocks to the model), and an equilibrium-selection policy, which selects one particular equilibrium path $\{\pi_t^*\}$ from the set consistent with $\{i_t^*\}$. Equilibrium selection is accomplished by a Taylor-inspired rule, $i_t = i_t^* + \phi\left(\pi_t - \pi_t^*\right)$, $\phi > 1$. People believe that the Fed selects equilibria by this method, and the Fed will explode the economy should undesired equilibria emerge.

I show that adding such a Taylor rule does not restrict the range of equilibria. The researcher can produce any of the equilibria by changing his or her assumption about equilibrium selection policy, the inflation target $\{\pi_t^*\}$.

And that discussion still leaves unanswered, why do people expect the Fed to choose an equilibrium with such disastrous consequences, when the same sort of threats could support benign equilibria. All it takes is an expectation that the Fed will gradually "normalize" policy after a trap, which is exactly what the Fed says it plans to do. I analyzed Werning's (2012) claim that optimal discretionary policy selects the standard disastrous equilibrium, but found it requires full precommitment to an ex-post disastrous off-equilibrium threat.

You may say, "That is not reasonable. The Fed doesn't have an equilibrium-selection policy, and does not threaten to explode the economy for equilibria it doesn't like." You may object that none of this has any relation to the Fed's clear statements of how it would *stabilize* the economy if inflation came out higher than the Fed's target, not the opposite. You may object that nobody has written any op-eds or policy essays castigating the Fed for its equilibrium-selection policies. You may look at my graphs of the expectations for alternative equilibria required to support a given equilibrium choice, and feel they are nutty assumptions to make about what people expect. If so, you conclude that the Fed really doesn't have an "equilibrium selection" policy, and this class of models needs a different equilibrium-selection mechanism in order to provide a definite prediction for the data and reliable policy prescriptions.

I have throughout added what may have seemed unnecessary words to emphasize that equilibria are selected not by what the Fed will do, could do, or promises to do, but by what people *expect* to happen. The logic of these models is strong – the expectation of future inflation (or output) paths is what selects equilibria today. In evaluating which equilibrium emerges today, it is not just desirable but necessary to think about what people actually expect to happen when the liquidity trap ends.

I have not advocated a specific alternative equilibrium selection criterion, or a specific choice of equilibrium. The backward-stable and no-jump equilibria have some points to commend them: they produce roughly normal policy predictions, they have a smooth limit as price stickiness is reduced, they do not presume an enormous fiscal support for deflation, and they do not predict that events further in the future have larger effects today, by limiting loading on a backward-explosive eigenvalue. But these observations are not yet economic proof that either is the "right" equilibrium choice. I have emphasized that, within the context of a particular model, equilibrium selection has testable content. Since each equilibrium has a distinct fiscal backing, I suggested that both theoretical and empirical analysis of fiscal-monetary policy coordination is the most productive way to answer the equilibrium-selection question in the future.

I emphasize, these are constructive suggestions. The point of this paper is not that disciplined policy conclusions are impossible. The point is that the results depend a great deal on how the researcher selects equilibria, or how the researcher models the Federal Reserve to select equilibria, how the researcher models the economy to select equilibria. That fact means that equilibrium selection discipline is more important, not less important. The fact that I stop short of advocating one solution to this problem does not mean that such solutions are unimportant or irrelevant. Indeed, it means they are important but hard.

I close with a few kinds words for the new-Keynesian model. This paper is really an argument to *save* the core of the new-Keynesian model – proper, forward-looking intertemporal, budget-constrained behavior in its "IS" and price-setting equations – rather than to attack it. Inaccurate predictions for data (deflation, depression, strong growth), puzzling policy predictions, a paradoxical limit as price stickiness declines, and explosive off-equilibrium expectations are *not* essential results of the model's core ingredients. A model with the core ingredients can give a very conventional view of the world, if one only picks a local-to-frictionless equilibrium. A model with the same core ingredients can also avoid all the theoretical troubles of explosive self-destructive equilibrium-selection threats by simply adding explicit fiscal-monetary policy coordination, recognizing that the effort to overcome indeterminacy with interest rate targets otherwise has failed.

Such a model will build neatly on a stochastic growth model, represented here in part by the forward-looking IS equation, and adding changes in potential output. Its price stickiness will modify dynamics in small but sensible ways and allow a description of the effects of monetary policy. In such a model, we will not likely to be able to chalk all our economic troubles up to one big simple story, a "negative natural rate" (whatever that means, since it is not independently measured) facing a lower bound on short term nominal rates, and we will not likely to be able to remedy those troubles with remarkable policies that change the signs on all the dismal parts of our dismal science. Technical regress, wasted government spending, and deliberate capital destruction are not likely to work. That outcome is bad news for those who found magical policies an intoxicating possibility, but good news for a realistic and sober macroeconomics, for the basic structure and usefulness of the core of the new-Keynesian model, and for the general program of incorporating nominal pricing frictions into well-posed equilibrium models.

7 References

Atkeson, Andrew, Chari, Varadarajan V. and Patrick J. Kehoe, 2010. "Sophisticated Monetary Policies." *Quarterly Journal of Economics* 125, 47–89.

Aruoba, S. Borağan, Pablo Cuba-Borda, and Frank Schorfheide, 2014. "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries." NBER Working Paper 19248, updated draft `http://sites.sas.upenn.edu/schorf/files/acs-zlb.pdf`

Ball, Laurence M. and Sandeep Mazumder, 2011. "Inflation Dynamics and the Great Recession," *Brookings Papers on Economic Activity*, Spring, 337-402.

Benhabib, Jess, Stephanie Schmitt-Grohé and Martin Uribe, 2001. "The Perils of Taylor Rules." *Journal of Economic Theory* 96, 40-69.

Benhabib, Jess, Stephanie Schmitt-Grohé and Martin Uribe, 2002. "Avoiding Liquidity Traps." *Journal of Political Economy* 110, 535-563.

Blanchard, Olivier, Giovanni Dell'Ariccia, and Paolo Mauro, 2010, "Rethinking Macroeconomic Policy." *Journal of Money, Credit and Banking* 42 , 199–215.

Bianchi, Francesco and Leonardo Melosi, 2014, "Escaping the Great Recession? " Manuscript, `https://bianchi.economics.cornell.edu/docs/BianchiMelosi_Escaping.pdf`.

Braun, R. Anton, Lena Mareen Körber, and Yuichiro Waki, 2012. "Some Unpleasant Properties of Log-Linearized Solutions when the Nominal Rate is Zero." Manuscript.

Campbell, Jeffrey R., Charles Evans, Jonas D. M. Fisher, and Alejandro Justiniano, 2012. "Macroeconomic effects of Federal Reserve forward guidance." Working Paper Series WP-2012-03, Federal Reserve Bank of Chicago.

Carlstrom, Charles T., Timothy S. Fuerst, and Matthias Paustian, 2012. "Fiscal multipliers under an interest rate peg of deterministic vs. stochastic duration." Working Paper 1235, Federal Reserve Bank of Cleveland.

Christiano, Lawrence J., Martin Eichenbaum, and Sergio Rebelo, 2011. "When is the Government Spending Multiplier Large?" *Journal of Political Economy* 119, 78-121.

Christiano, Lawrence J. and Martin Eichenbaum, 2012. "Notes on Linear Approximations, Equilibrium Multiplicity and E-learnability in the Analysis of the Zero Lower Bound." Manuscript, Northwestern University.

Clarida, Richard, Jordi Galí and Mark Gertler, 2000. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *The Quarterly Journal of Economics* 115, 147-180. doi: 10.1162/003355300554692

Cochrane, John H., 2009. "Can Learnability Save New-Keynesian Models?" *Journal of Monetary Economics* 56, 1109–1113.

Cochrane, John H., 2011a. "Determinacy and Identification with Taylor Rules." *Journal of Political Economy* 119, 565-615.

Cochrane, John H., 2011b. "Online Appendix B to Determinacy and Identification with Taylor Rules." *Journal of Political Economy* 119, 565-615.

Cochrane, John H., 2014. "Monetary Policy with Interest on Reserves." *Journal of Economic Dynamics and Control* 49, 74-108.

Cochrane, John H., 2015. "A Response to Sims (2013)." Manuscript, `http://faculty.chicagobooth.edu/john.cochrane/research/papers/sims_response.pdf`

Coibion, Olivier and Yuriy Gorodnichenko, 2013' "The Missing Deflation," Manuscript.

Correia, Isabel, Emmanuel Farhi, Juan Pablo Nicolini, and Pedro Teles, 2013. "Unconventional Fiscal Policy at the Zero Bound," *American Economic Review* 103, 1172-1211. `http://dx.doi.org/10.1257/aer.103.4.1172`.

Del Negro, Marco, Marc Giannoni, and Christina Patterson, 2012. "The Forward Guidance Puzzle." Staff Reports 574, Federal Reserve Bank of New York.

Del Negro, Marco, Marc P. Giannoni, Frank Schorfheide, 2013. "Inflation in the Great Recession and New Keynesian Models." Federal Reserve Bank of New York Staff Report #618.

Dupor, Bill, and Rong Li, 2013. "The 2009 Recovery Act and the Expected Inflation Channel of Government Spending." Manuscript.

Eggertsson, Gauti B., 2010. "The Paradox of Toil." Federal Reserve Bank of New York Staff Report N. 433.

Eggertsson, Gauti B., 2011. "What Fiscal Policy is Effective at Zero Interest Rates?" In Daron Acemoglu and Michael Woodford, eds., *NBER Macroeconomics Annual 2010*, 59-112. Chicago: University of Chicago Press.

Eggertsson, Gauti B. and Michael Woodford, 2003. "The Zero Bound on Interest Rates and Optimal Monetary Policy."*Brookings Papers on Economic Activity* 34, 139-235.

Eggertsson, Gaudi and Neil Mehrotra, 2014. "A Model of Secular Stagnation." Manuscript

Eggertsson, Gauti, Andrea Ferrero, and Andrea Raffo, 2013. "Can Structural Reforms Help Europe?" *Journal of Monetary Economics* 61, 2-22.

Evans, George W. and Seppo Honkapohja, 2001. *Learning and Expectations in Macroeconomics.* Princeton: Princeton University Press.

Fernández-Villaverde, Jewsús , 2013. "Discussion of 'Can Structural Reforms Help Europe?' by Gauti Eggertsson, Andrea Ferrero, and Andrea Raffo." *Journal of Monetary Economics,* 61, 23-31. `http://dx.doi.org/10.1016/j.jmoneco.2013.11.005i`

Fernández-Villaverde, Jesús, Olaf Posch, and Juan F. Rubio-Ramírez, 2012. "Solving the New Keynesian Model in Continuous Time." Manuscript.

Fernández-Villaverde, Jesús , Pablo Guerrón-Quintana, Juan F. Rubio-Ramírez 2014. "Supply-Side Policies and the Zero Lower Bound." Manuscript.

Hall, Robert E., 2011. "The Long Slump." *American Economic Review* 101, 431-469.

Kiley, Michael T. 2013. "Policy Paradoxes in the New Keynesian Model." Manuscript.

King, Robert G., 2000. "The New IS-LM Model: Language, Logic, and Limits." Federal Reserve Bank of Richmond *Economic Quarterly* 86, 45-103.

King, Robert G. and Mark W. Watson, 2012. "Inflation and Unit Labor Cost." Manuscript.

Levin, Andrew, David Lopez-Salido, Edward Nelson, and Tack Yun, 2010. "Limitations on the Effectiveness of Forward Guidance at the Zero Lower Bound." *International Journal of Central Banking* 6, 143-189.

Mertens, Karel and Morten O. Ravn, 2014. "Fiscal Policy in an Expectations Driven Liquidity Trap." *Review of Economic Studies*, Forthcoming, 10.1093/restud/rdu016.

McCallum, Bennett T., 2003. "Multiple-Solution Indeterminacies in Monetary Policy Analysis." *Journal of Monetary Economics* 50, 1153-75.

McCallum, Bennett T., 2009. "Inflation Determination With Taylor Rules: Is the new-Keynesian Analysis Critically Flawed? *Journal of Monetary Economics* 56, 1101-1108.

Obstfeld, Maurice and Kenneth Rogoff, 1983. "Speculative Hyperinflations in Maximizing Models: Can we Rule them Out?" *Journal of Political Economy* 91, 675-687.

Obstfeld, Maurice and Kenneth Rogoff, 1986. "Ruling out Divergent Speculative Bubbles." *Journal of Monetary Economics* 17, 349-362.

Ramey, Valerie A. 2011. "Can Government Purchases Stimulate the Economy?" *Journal of Economic Literature* 49, 673-85, doi:10.1257/jel.49.3.673.

Sims, Christopher, 2004. "Limits to Inflation Targeting." In Bernanke, Ben S., and Michael Woodford, eds., *The Inflation-Targeting Debate*, NBER Studies in Business Cycles Volume 32, Ch. 7, p. 283-310. Chicago: University of Chicago Press. `http://sims.princeton.edu/yftp/Targeting/TargetingFiscalPaper.pdf`.

Sims, Christopher, 2013. "Paper Money." *American Economic Review* 103, 563-584. `http://dx.doi.org/10.1257/aer.103.2.563`

Svensson, Lars E. O., and Michael Woodford, 2005, "Implementing Optimal Policy through Inflation-Forecast Targeting." In Bernanke, Ben S., and Michael Woodford, eds. (2005), *The Inflation-Targeting Debate*, Chicago: University of Chicago Press, 19-83. `http://larseosvensson.se/files/papers/swift.pdf`.

Taylor, John B., 1993. "Discretion Versus Policy Rules in Practice." *Carnegie-Rochester Series on Public Policy* 39, 195-214.

Taylor, John B., 1999. "The Robustness and Efficiency of Monetary Policy Rules as Guidelines for Interest Rate Setting by the European Central Bank." *Journal of Monetary Economics* 43, 655-79.

Werning, Ivan, 2012. "Managing a Liquidity Trap: Monetary and Fiscal Policy." Manuscript.

Wieland, Johannes, 2014. "Are Negative Supply Shocks Expansionary at the Zero Lower Bound?" Manuscript.

Woodford, Michael, 2003. *Interest and Prices,* Princeton: Princeton University Press.

Woodford, Michael, 2011. "Simple Analytics of the Government Expenditure Multiplier." *American Economic Journal: Macroeconomics* 3, 1-35.

8 Appendix

8.1 Derivation of the model solution

The model is

$$\begin{aligned}\frac{dx_t}{dt} &= ir_t - \pi_t \\ \frac{d\pi_t}{dt} &= \rho\pi_t - \kappa\left(x_t + g_t\right).\end{aligned}$$

where $ir_t = ir$ for $t < T$ and $ir = 0$ for $t > T$, and g_t is a constant g for $t < T$ and $g_t = 0$ for $t > T$. It is convenient to scale x_t by κ, yielding

$$\frac{d}{dt}\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix} = \kappa\begin{bmatrix}ir_t \\ -g_t\end{bmatrix} + \begin{bmatrix}0 & -\kappa \\ -1 & \rho\end{bmatrix}\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix}. \tag{19}$$

The steady state of (19) is

$$\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix} = \begin{bmatrix}\rho ir - \kappa g \\ ir\end{bmatrix}$$

Thus, we find solutions to the homogenous part – without the first term in (19) – and then add back this steady state.

Eigenvalue decomposing the transition matrix, we have

$$\begin{bmatrix}0 & -\kappa \\ -1 & \rho\end{bmatrix} = \begin{bmatrix}\lambda & \delta \\ 1 & 1\end{bmatrix}\begin{bmatrix}\delta & 0 \\ 0 & \lambda\end{bmatrix}\begin{bmatrix}\lambda & \delta \\ 1 & 1\end{bmatrix}^{-1}$$

where

$$\begin{aligned}\lambda &= \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4\kappa}\right) \geq 0 \\ \delta &= \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4\kappa}\right) \leq 0.\end{aligned}$$

For simplifying algebra it is convenient to note

$$\begin{aligned}\lambda - \delta &= \sqrt{\rho^2 + 4\kappa} \\ \delta + \lambda &= \rho \\ \delta\lambda &= -\kappa.\end{aligned} \tag{20}$$

Define

$$\begin{bmatrix}z_t \\ w_t\end{bmatrix} = \begin{bmatrix}\lambda & \delta \\ 1 & 1\end{bmatrix}^{-1}\left(\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix} - \begin{bmatrix}ir - \kappa g \\ ir\end{bmatrix}\right)$$

and thus

$$\begin{bmatrix}x_t \\ \pi_t\end{bmatrix} = \begin{bmatrix}\rho ir - \kappa g \\ ir\end{bmatrix} + \begin{bmatrix}\lambda & \delta \\ 1 & 1\end{bmatrix}\begin{bmatrix}z_t \\ w_t\end{bmatrix}.$$

Then (19) reduces to

$$\frac{d}{dt}\begin{bmatrix}z_t \\ w_t\end{bmatrix} = \begin{bmatrix}\delta & 0 \\ 0 & \lambda\end{bmatrix}\begin{bmatrix}z_t \\ w_t\end{bmatrix}$$

with solutions

$$\begin{bmatrix} z_t \\ w_t \end{bmatrix} = \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \begin{bmatrix} z_T \\ w_T \end{bmatrix}$$

where and z_T and w_T are constants indexing multiple equilibria.

Transforming back to x and π, the general solution to (19) when g and ir are constant is therefore

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} + \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \begin{bmatrix} z_T \\ w_T \end{bmatrix} \tag{21}$$

For $t \geq T$, I follow the standard procedure by requiring that x_t and π_t do not explode as $t \to \infty$. This requirement means that the constant w_T multiplying the explosive eigenvalue λ must be zero. We also have $ir = 0$ and $g = 0$ for $t \geq T$, and so we have

$$t \geq T: \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} z_T e^{\delta(t-T)}.$$

The single parameter z_T controls both x_t and π_t equilibria, and $z_T = \pi_T$ so I use that notation.

$$t \geq T: \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_T e^{\delta(t-T)}.$$

For $t \leq T$, we pick a different set of constants z_T^* and w_T^*. We pick these constants so that the solution $(\kappa x_t, \pi_t)$ pastes in to the $t \geq T$ solution at $t = T$. This consideration does not rule out solutions that load on the positive eigenvalue λ. The pasting condition is

$$\begin{bmatrix} \kappa x_T \\ \pi_T \end{bmatrix} = \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} + \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_T \tag{22}$$

We can solve (22) for $\begin{bmatrix} z_T^* & w_T^* \end{bmatrix}'$ and substitute into (21).

$$\begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix} = \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \lambda \\ 1 \end{bmatrix} z_T - \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} \right)$$

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} \\ &+ \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_T - \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} \right) \end{aligned}$$

Using (20) we can write each of the rightmost terms as

$$\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_T = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta(t-T)} \pi_T,$$

$$\begin{aligned} &\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \delta + \lambda \\ 1 \end{bmatrix} ir \\ &= \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} ir \end{aligned}$$

and

$$\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \kappa g = \frac{1}{\lambda-\delta} \begin{bmatrix} \lambda & -\delta \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} \kappa g$$

Putting it all back together,

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho i r_t - \kappa g_t \\ i r_t \end{bmatrix} \\ &- \frac{1}{\lambda-\delta}\left(\begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} i r_t - \begin{bmatrix} \lambda & -\delta \\ 1 & -1 \end{bmatrix} \kappa g_t \right) \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} \\ &+ \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_T e^{\delta(t-T)} \end{aligned} \tag{23}$$

To find the no-inflation-jump equilibrium, we need the value of π_T such that $\pi_0 = 0$. That value is, from (6),

$$\pi_0 = 0 = \left(1 - \frac{\lambda e^{-\delta T} - \lambda_- e^{-\lambda T}}{\lambda - \delta}\right) i r + \pi_T e^{-\delta T}.$$

$$\pi_T = e^{\delta T} \left(\frac{\lambda\left(e^{-\lambda_- T} - 1\right) - \delta\left(e^{-\lambda T} - 1\right)}{\lambda_+ - \delta} \right) i r$$

Simply plugging in to (6) now gives an equation for the solution in this case.

8.2 Multiplier calculation

Starting from the general solution (23), the standard equilibrium choice with g present and $\pi_T = 0$ is

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho i r_t - \kappa g_t \\ i r_t \end{bmatrix} - \frac{1}{\lambda-\delta}\left(\begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} i r_t - \begin{bmatrix} \lambda & -\delta \\ 1 & -1 \end{bmatrix} \kappa g_t \right) \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix}. \tag{24}$$

The backward-stable solution picks

$$\pi_T = \frac{\lambda i r - \kappa g}{\lambda - \delta} \tag{25}$$

and is therefore, for $t < T$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho i r - \kappa g \\ i r \end{bmatrix} + \frac{1}{\lambda-\delta}\left(\begin{bmatrix} \delta^2 \\ \delta \end{bmatrix} i r - \begin{bmatrix} \delta \\ 1 \end{bmatrix} \kappa g \right) e^{\lambda(t-T)} \tag{26}$$

and for $t > T$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{\lambda i r - \kappa g}{\lambda - \delta} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta(t-T)}.$$

(The algebra:

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho i r - \kappa g \\ i r \end{bmatrix} - \frac{1}{\lambda^p - \lambda^m}\left(\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} i r - \begin{bmatrix} \lambda^p & -\lambda^m \\ 1 & -1 \end{bmatrix} \kappa g \right) \begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} \\ &+ \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \frac{\lambda^p i r - \kappa g}{\lambda^p - \lambda^m} e^{\lambda^m(t-T)} \end{aligned}$$

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} - \frac{1}{\lambda^p - \lambda^m} \times$$
$$\times \left(\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} ir - \begin{bmatrix} \lambda^p & -\lambda^m \\ 1 & -1 \end{bmatrix} \kappa g - (\lambda^p ir - \kappa g) \begin{bmatrix} \lambda^p & 0 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} e^{\lambda^m (t-T)} \\ e^{\lambda^p (t-T)} \end{bmatrix}$$

and simplify.)

Now we can take derivatives. In the standard solution (24),

$$t < T : \frac{\partial}{\partial g} \begin{bmatrix} x_t \\ \frac{1}{\kappa}\pi_t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \frac{1}{\lambda - \delta} \left(\begin{bmatrix} \lambda & -\delta \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix}$$
$$t > T : \frac{\partial}{\partial g} \begin{bmatrix} x_t \\ \frac{1}{\kappa}\pi_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

while in the backward-stable solution, (26)

$$t < T : \frac{\partial}{\partial g} \begin{bmatrix} x_t \\ \frac{1}{\kappa}\pi_t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \frac{1}{\lambda - \delta} \begin{bmatrix} \delta \\ 1 \end{bmatrix} e^{\lambda(t-T)}$$
$$t > T : \frac{\partial}{\partial g} \begin{bmatrix} x_t \\ \frac{1}{\kappa}\pi_t \end{bmatrix} = \frac{-1}{\lambda - \delta} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta(t-T)}.$$

Recall that the root $\delta < 0$ leads to $e^{\delta(t-T)}$ that explodes as we go backward in time. Hence, the standard-solution multiplier depends on the length of time the liquidity trap and g policy are expected to last, and can be arbitrarily large. Since the eigenvalue δ increases in absolute value as price stickiness declines, $\kappa \to \infty$, we see analytically that the multiplier grows larger as frictions disappear and the multiplier approaches infinity in the frictionless limit.

By contrast, the root $\lambda > 0$ is the only one that appears in (26), and $e^{\lambda(t-T)}$ converges to zero as we go backward in time. Hence, the backward-stable multiplier is at its largest in absolute value just before the liquidity trap ends, and is lower the earlier it is applied. Furthermore, the backward-stable multiplier is negative throughout. And its multiplier approaches the frictionless limit -1 smoothly as we approach the frictionless limit.

8.3 Solution with a postponed interest rate rise

Here I consider a policy in which the Fed keeps the interest rate at zero from period T to period $T+\tau$, though the natural rate has risen from $-r$ to $+r$. The model is

$$\frac{dx_t}{dt} = ir_t - \pi_t$$
$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t.$$

where $ir_t = r$ for $t < T$; $ir = -r_T$ for $T < t < t+\tau$ and and $ir = 0$ for $t > T$

The general solution to (19) when ir is constant is given in (21). We just paste twice.

For $t \geq T+\tau$, we again set to zero the explosive root,

$$t \geq T+\tau : \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta[t-(T+\tau)]}\pi_{T+\tau}.$$

For $T < t < T+\tau$, since $ir = -r_T$, we use the pasting condition at $t = T+\tau$

$$\begin{bmatrix} \kappa x_{T+\tau} \\ \pi_{T+\tau} \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} (-r_T) + \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta\tau} & 0 \\ 0 & e^{\lambda\tau} \end{bmatrix} \begin{bmatrix} z_T^{**} \\ w_T^{**} \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_{T+\tau}$$

where r_T is the positive natural rate that holds between time T and time $T+\tau$. We solve this equation for $\begin{bmatrix} z_{T+\tau}^{**} & w_{T+\tau}^{**} \end{bmatrix}'$ and substitute into (21).

$$\begin{bmatrix} z_{T+\tau}^{**} \\ w_{T+\tau}^{**} \end{bmatrix} = \begin{bmatrix} e^{-\delta\tau} & 0 \\ 0 & e^{-\lambda\tau} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_{T+\tau} + \begin{bmatrix} \rho \\ 1 \end{bmatrix} r_T \right)$$

so between T and $T+\tau$,

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_T + \\ &\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} & 0 \\ 0 & e^{\lambda(t-(T+\tau))} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_{T+\tau} + \begin{bmatrix} \rho \\ 1 \end{bmatrix} r_T \right) \end{aligned}$$

Using (20) we can write each of the rightmost terms as

$$\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} & 0 \\ 0 & e^{\lambda(t-(T+\tau))} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_{T+\tau} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \pi_{T+\tau} e^{\delta(t-(T+\tau))},$$

$$\begin{aligned} &\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} & 0 \\ 0 & e^{\lambda(t-(T+\tau))} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \delta+\lambda \\ 1 \end{bmatrix} r_T \\ &= \frac{1}{\lambda-\delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} \\ e^{\lambda(t-(T+\tau))} \end{bmatrix} r_T \end{aligned}$$

Putting it all back together, for $T < t < T+\tau$

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_T + \frac{1}{\lambda-\delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} \\ e^{\lambda(t-(T+\tau))} \end{bmatrix} r_T + \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta(t-(T+\tau))} \pi_{T+\tau}. \quad (27)$$

Now, we paste again at $t = T$, with the general solution for $t < T$ matching this solution for $T < t < T+\tau$ at $t = T$,

$$\begin{aligned} \begin{bmatrix} \kappa x_T \\ \pi_T \end{bmatrix} &= \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix} \\ &= -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_T + \frac{1}{\lambda-\delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{-\delta\tau} \\ e^{-\lambda\tau} \end{bmatrix} r_T + \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{-\delta\tau} \pi_{T+\tau} \end{aligned}$$

Solving for z_T^* and w_T^*,

$$\begin{aligned} \begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix} &= -\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho \\ 1 \end{bmatrix} (r_T + r) \\ &+ \frac{1}{\lambda-\delta} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{-\delta\tau} \\ e^{-\lambda\tau} \end{bmatrix} r_T \\ &+ \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{-\delta\tau} \pi_{T+\tau} \end{aligned}$$

We substitute in (21) once again, repeated here and specialized,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix}.$$

The terms are, first using $\lambda + \delta = \rho$

$$\begin{aligned} &\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \left(- \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho \\ 1 \end{bmatrix} (r_T + r) \right) \\ &= \frac{1}{\lambda - \delta} \begin{bmatrix} -\lambda^2 & \delta^2 \\ -\lambda & \delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} (r_T + r)\,; \end{aligned}$$

next,

$$\begin{aligned} &\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{-\delta\tau} \\ e^{-\lambda\tau} \end{bmatrix} r_T \\ &= \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} \\ e^{\lambda(t-(T+\tau))} \end{bmatrix} r_T \end{aligned}$$

and finally

$$\begin{aligned} &\begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} & 0 \\ 0 & e^{\lambda(t-T)} \end{bmatrix} \begin{bmatrix} \lambda & \delta \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{-\delta\tau} \pi_{T+\tau} \\ &= \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta(t-(T+\tau))} \pi_{T+\tau}. \end{aligned}$$

Putting it all together, for $t < T$,

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = &\begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \frac{1}{\lambda - \delta} \begin{bmatrix} -\lambda^2 & \delta^2 \\ -\lambda & \delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} (r_T + r) \\ &+ \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \begin{bmatrix} e^{\delta(t-(T+\tau))} \\ e^{\lambda(t-(T+\tau))} \end{bmatrix} r_T + \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta(t-(T+\tau))} \pi_{T+\tau} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = &\begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \\ &+ \frac{1}{\lambda - \delta} \begin{bmatrix} -\lambda^2 & \delta^2 \\ -\lambda & \delta \end{bmatrix} \left\{ \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} r + \begin{bmatrix} e^{\delta(t-T)} \left[1 - e^{-\delta\tau}\right] \\ e^{\lambda(t-T)} \left[1 - e^{-\lambda\tau}\right] \end{bmatrix} r_T \right\} + \\ &+ \begin{bmatrix} \lambda \\ 1 \end{bmatrix} e^{\delta(t-(T+\tau))} \pi_{T+\tau} \end{aligned} \tag{28}$$

8.3.1 Standard solution

The standard solution just sets $\pi_{T+\tau} = 0$ as usual. For $\tau < T$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} r - \frac{1}{\lambda - \delta} \begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix} \left\{ \begin{bmatrix} e^{\delta(t-T)} \\ e^{\lambda(t-T)} \end{bmatrix} r + \begin{bmatrix} e^{\delta(t-T)} \left[1 - e^{-\delta\tau}\right] \\ e^{\lambda(t-T)} \left[1 - e^{-\lambda\tau}\right] \end{bmatrix} r_T \right\}$$

while for $T < t < T + \tau$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_T + \frac{1}{\lambda - \delta}\begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix}\begin{bmatrix} e^{\delta(t-(T+\tau))} \\ e^{\lambda(t-(T+\tau))} \end{bmatrix} r_T$$

and with $\rho = \lambda + \delta$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{1}{\lambda - \delta}\begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix}\begin{bmatrix} e^{\delta(t-(T+\tau))} - 1 \\ e^{\lambda(t-(T+\tau))} - 1 \end{bmatrix} r_T$$

and of course $\pi_t = 0, x_t = 0$ for $t > T + \tau$.

8.3.2 Backward-stable solution

To make the loading on $e^{\delta(t-T)}$ equal to zero in the $t < T$ period, we need to choose $\pi_{T+\tau}$ by

$$e^{-\delta\tau}\pi_{T+\tau} = \frac{\lambda}{\lambda_+ - \delta}\left[r + r_T\left(1 - e^{-\delta\tau}\right)\right].$$

So, for $t > T + \tau$ we have

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{1}{\lambda - \delta}\begin{bmatrix} \lambda^2 \\ \lambda \end{bmatrix}\left[r + r_T\left(1 - e^{-\delta\tau}\right)\right] e^{\delta(t-T)}$$

Substituting in (27) we have for $T < t < T + \tau$,

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = & -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_T + \\ & + \frac{1}{\lambda - \delta}\begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix}\begin{bmatrix} e^{\delta(t-(T+\tau))} \\ e^{\lambda(t-(T+\tau))} \end{bmatrix} r_T \\ & + \begin{bmatrix} \lambda \\ 1 \end{bmatrix}\frac{\lambda}{\lambda - \delta}\left[r + r_T\left(1 - e^{-\delta\tau}\right)\right] e^{\delta(t-T)} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = & \frac{1}{\lambda - \delta}\begin{bmatrix} \lambda^2 & -\delta^2 \\ \lambda & -\delta \end{bmatrix}\begin{bmatrix} e^{\delta(t-(T+\tau))} - 1 \\ e^{\lambda(t-(T+\tau))} - 1 \end{bmatrix} r_T + \\ & + \frac{1}{\lambda - \delta}\begin{bmatrix} \lambda^2 \\ \lambda \end{bmatrix}\left[r + r_T\left(1 - e^{-\delta\tau}\right)\right] e^{\delta(t-T)} \end{aligned}$$

8.4 Solutions with a Taylor rule

As explained in the text, I specify a partial - adjustment Taylor rule with a zero bound.

$$\frac{di_t}{dt} = \begin{cases} \theta\phi\left(\pi_t - \pi_t^*\right) - \theta\left(i_t - r\right) & i_t > 0 \\ \max\left[\theta\phi\left(\pi_t - \pi_t^*\right) - \theta\left(i_t - r\right), 0\right] & i_t = 0 \end{cases} \tag{29}$$

In the bottom equation, when the interest rate is zero it can rise above zero, but cannot fall further.

The model is now, for $t > T$,

$$\frac{dx_t}{dt} = ir_t - \pi_t$$
$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t.$$

along with (29)where $ir_t \equiv i_t - r$.

Putting it all together, when $i_t > 0$, or $i_t = 0$ and $\phi(\pi_t - \pi_t^*) > ir_t$ we have

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t \\ \pi_t \\ ir_t \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & \theta\phi & -\theta \end{bmatrix}\begin{bmatrix} \kappa x_t \\ \pi_t \\ ir_t \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \phi\theta \end{bmatrix}\pi_t^*. \tag{30}$$

Now, π_t^* and x_t^* obey the standard model with $ir_t = ir_t^* = 0$ for $t > T$,

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \end{bmatrix} = \begin{bmatrix} 0 & -\kappa \\ -1 & \rho \end{bmatrix}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \end{bmatrix},$$

which I can write in the form

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & \theta\phi & -\theta \end{bmatrix}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \phi\theta \end{bmatrix}\pi_t^*. \tag{31}$$

Subtracting (31) from (30), we can describe deviations from the desired equilibrium,

$$\frac{d}{dt}\begin{bmatrix} \kappa(x_t - x_t^*) \\ \pi_t - \pi_t^* \\ ir_t \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & \theta\phi & -\theta \end{bmatrix}\begin{bmatrix} \kappa(x_t - x_t^*) \\ \pi_t - \pi_t^* \\ ir_t \end{bmatrix} \tag{32}$$

When $i_t = 0$ and $\phi(\pi_t - \pi_t^*) < ir_t$ we have instead

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t \\ \pi_t \\ ir_t \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \kappa x_t \\ \pi_t \\ ir_t \end{bmatrix}.$$

In this case, writing the dynamics of the $*$ equilibrium as

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \\ 0 \end{bmatrix},$$

we can subtract again and obtain

$$\frac{d}{dt}\begin{bmatrix} \kappa(x_t - x_t^*) \\ \pi_t - \pi_t^* \\ ir_t \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \kappa(x_t - x_t^*) \\ \pi_t - \pi_t^* \\ ir_t \end{bmatrix} \tag{33}$$

We know the equilibrium values. The point is to look at the off equilibrium behavior. Thus, I simulate *forward*, from the known $\{\pi_T, x_T\}$ of various equilibria, to see what would happen to alternative multiple equilibria. At each point, I find whether the zero bound restrictions bite, and then simulate either (32) or (33) forward as necessary.

For greater accuracy, I simulate forward the solutions. Each of (32) or (33) is of the form

$$\frac{d}{dt}X_t = AX_t = Q\Lambda Q^{-1}X_t$$

with solution

$$X_t = Qe^{\Lambda t}Q^{-1}X_0$$

so long as it is valued. Thus, I simulate forward starting at time T,

$$X_{t+\Delta} = Qe^{\Lambda\Delta}Q^{-1}X_t$$

at each date t.

8.4.1 Taylor rules without partial adjustment

The simpler Taylor rule without partial adjustment, the limit $\theta \to \infty$, also works in this model to generate a unique locally-bounded equilibrium. However, it cannot be used in the frictionless model or frictionless limit. And, in the simple three-equation model, it quickly generates complex dynamics. For these reasons, it is better to use a model with partial adjustment in continuous time.

In a discrete-time frictionless perfect foresight model with $r = 0$, the IS curve and Taylor rule are

$$i_t = \pi_{t+\Delta}$$
$$i_t = \phi\pi_t$$

leading to equilibrium dynamics

$$\pi_{t+\Delta} = \phi\pi_t,$$

where Δ is the time interval. The Taylor coefficient ϕ is the system eigenvalue, so the full class of solutions is

$$\pi_t = \phi^{t/\Delta}\pi_0.$$

A coefficient $\phi > 1$ gives a unique locally bounded solution, $\pi_t = 0$ in this case, to avoid the other solution's explosions at the rate $\phi^{1/\Delta}$.

If we take this example directly to continuous time, $\Delta \to 0$, however, we have

$$i_t = \pi_t$$
$$i_t = \phi\pi_t$$

and now the corresponding explosion happens infinitely quickly, $\phi^{t/\Delta} \to \infty$. In discrete time there is a one-period lag between the inflation that the interest rate controls $-E_t\pi_{t+1}-$ and the inflation that the Fed responds to $-\ \pi_t\ -$ which vanishes in continuous time.

One could say, fine, the infinitely-reactive Taylor rule still leads to a unique locally bounded solution. But the infinite speed of the dynamics is hard to take seriously. To keep reasonable dynamics in the continuous time limit, we have to reintroduce that lag somehow. The partial adjustment form does so.

In the frictionless model,

$$
\begin{aligned}
i_t &= \pi_t \\
\frac{di_t}{dt} &= -\theta\left(i_t - \phi\pi_t\right)
\end{aligned}
$$

leads to equilibria

$$
\begin{aligned}
\frac{d\pi_t}{dt} &= \theta\left(\phi - 1\right)\pi_t \\
\pi_t &= \pi_0 e^{\theta(\phi-1)t}
\end{aligned}
$$

which all explode, but with finite speed, if $\phi > 1$.

Now consider the simple new-Keynesian model with $i^* = r$,

$$
\begin{aligned}
\frac{dx_t}{dt} &= i_t - r - \pi_t \\
\frac{d\pi_t}{dt} &= \rho\pi_t - \kappa x_t.
\end{aligned}
$$

If we add a Taylor rule,

$$
i_t = r + \phi(\pi_t - \pi_t^*)
$$

then deviations from an equilibrium satisfy

$$
\begin{aligned}
\frac{d\left(x_t - x_t^*\right)}{dt} &= \left(\phi - 1\right)\left(\pi_t - \pi_t^*\right) \\
\frac{d\left(\pi_t - \pi_t^*\right)}{dt} &= \rho\left(\pi_t - \pi_t^*\right) - \kappa\left(x_t - x_t^*\right)
\end{aligned}
$$

or

$$
\frac{d}{dt}\begin{bmatrix} \kappa x_t - \kappa x_t^* \\ \pi_t - \pi_t^* \end{bmatrix} = \begin{bmatrix} 0 & \kappa\left(\phi - 1\right) \\ -1 & \rho \end{bmatrix}\begin{bmatrix} \kappa x_t - \kappa x_t^* \\ \pi_t - \pi_t^* \end{bmatrix}
$$

We have

$$
\begin{bmatrix} 0 & \kappa\left(\phi - 1\right) \\ -1 & \rho \end{bmatrix} = \begin{bmatrix} \delta_+ & \delta_- \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \delta_- & 0 \\ 0 & \delta_+ \end{bmatrix}\begin{bmatrix} \delta_+ & \delta_- \\ 1 & 1 \end{bmatrix}^{-1}.
$$

$$
\delta_\pm \equiv \frac{1}{2}\left[\rho \pm \sqrt{\rho^2 - 4\kappa\left(\phi - 1\right)}\right]
$$

$\phi > 1$ guarantees that the real part of both eigenvalues is positive, so x_t^*, π_t^* is the unique locally bounded solution.

However, since ρ is a discount rate, of order $\rho = 0.05$, these eigenvalues will be complex, leading to slowly-explosive oscillatory paths, for ϕ even slightly greater than one.

8.5 State space

Werning (2012) presents equilibria in state space rather than graph variables over time. In homage to Werning, and to connect my additional equilibria to the ones he derives, Figure 14 presents the standard equilibrium and the backwards-stable equilibrium in Werning's state space.

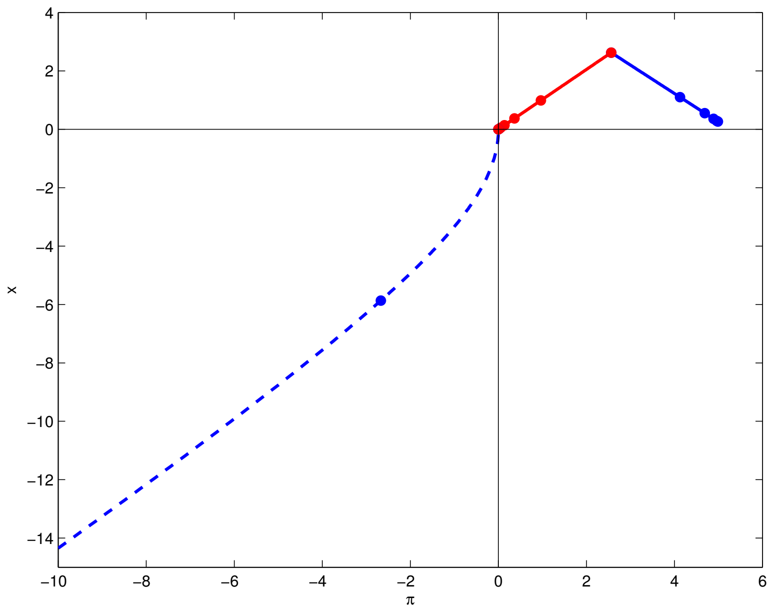

Figure 14: The backward-stable and standard equlibrium choices compared in state space. The dashed blue line gives the standard equilibrium for $t < T$. For $t > T$, the standard solution stops at the central dot. The solid lines give the backward-stable equilibrium. Dots indicate each year. The blue line is for $t < T$ and the red line is for $t > T$. This equilibrium starts on the right and proceeds to the left

The blue dashed line is the standard equilibrium choice, as in Werning (2012) Figure 1. Inflation and output gap approach from the bottom left, the region of deflation and depression. Dots indicate years. At $t = T$ Werning's standard solution attains the central red dot and stays there.

The solid lines display the backwards-stable equilibrium. The red part rising from the origin is $t \geq T$, and the economy actually approaches the origin from the north-east here. Once we eliminate the explosive solution or $t > T$, there is a whole range of non-explosive solutions that converge to the origin along the red ray. Equilibrium choice comes down to where we specify that the $t < T$ solution will join this path. The standard choice picks the origin itself. The backward-stable choice merges at a point to the north-east of the origin at $t = T$, at just the right place so that the blue line is non- explosive going backward in time, i.e. to the bottom right in the graph. Going forward in time, this solution starts at the right end of the blue line and works left, hitting the output gap peak at $t = T$ and then converging back to the steady state at the origin.