The New-Keynesian Liquidity Trap

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Abstract

Many new-Keynesian models produce a deep recession with deflation at the zero lower bound. These models also make unusual policy predictions: Useless government spending, technical regress, capital destruction, and forward guidance can raise output. These predictions are larger as prices become less sticky and as changes are expected further in the future. These predictions are strongly affected by equilibrium selection. For the same interest-rate path, equilibria that bound initial jumps predict mild inflation, small output variation, negative multipliers, small effects of far-off expectations and a smooth frictionless limit. Fiscal policy considerations suggest the latter equilibria.

Keywords: Zero bound, multiplier, multiple equilibria, fiscal theory

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1. Introduction

Many models in the new-Keynesian tradition predict a deep recession with deflation when the "natural" rate of interest is negative and the nominal rate is stuck at zero. Those models also produce unusual policy predictions. Forward guidance about central bank actions can strongly stimulate the current level of output. Fully-expected future inflation can raise output. Deliberate capital destruction or productivity reduction can raise output. Government spending, even if financed by current taxation, and even if completely, can have large output multipliers. A given promise or expectation further in the future has larger effects

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today. As prices become more flexible, deflation and depression get worse and unusual policy prescriptions become stronger. Tiny price stickiness has unboundedly large effects.

For a given path of expected interest rates, new-Keynesian models allow multiple stable equilibrium paths for inflation and output. Thus, to produce a prediction, a researcher must choose an equilibrium as well as a path for expected interest rates.

I show that these liquidity-trap predictions are sensitive to equilibrium choice. Choosing different equilibria, either directly or by different specifications of Fed policy, despite exactly the same path of interest-rate expectations, the same model can predict gentle inflation matching the negative natural rate, small output gaps, and normal signs and magnitudes of policies. Inflation, output and policy predictions are smaller for events expected further in the future, and smoothly approach the frictionless limit.

In the most general terms, the standard models choose equilibria by thinking about expectations of output and inflation when the economy exits the zero bound, and then working backwards. The alternative equilibria I study limit how much inflation and output can jump on the day that the economy learns of the natural rate shock. A variety of criteria suggest such a limitation, especially fiscal policy considerations. Since a sharp deflation raises the value of government bonds, a limitation on the government's ability or willingness to raise taxes limits initial deflation, and consequently all effects of the zero bound.

1.1. Literature

Werning (2012) shows clearly the predictions for a depression and deflation at the zero bound, and some policy paradoxes. I adopt his simple modeling framework. This paper is not a critique of Werning. Werning studies the properties of one equilibrium. He acknowledges multiple equilibria. I explore their nature.

Kiley (2016) and Wieland (2014) nicely summarize the puzzling predictions of new-Keynesian zero-bound analyses. Christiano, Eichenbaum and Rebelo (2011), Eggertsson (2011), Woodford (2011), and Carlstrom, Fuerst and Paustian (2014) all find large fiscal multipliers, and multipliers that increase with the duration of fiscal expansion. Eggertsson (2010) and Wieland (2014) analyze the "paradox of toil" that negative productivity can be expansionary. Eggertsson, Ferrero and Raffo (2013) argue that structural reforms are contractionary. (See also the discussion in Fernández-Villaverde (2013).)

Werning's (2012) main point, as that of Eggertsson and Woodford (2003) and Woodford (2012), is to study optimal policy. These authors find a path of inflation, output, and interest rates that maximizes a planner's objective. This path typically involves keeping interest rates low for some time after the natural-rate shock ends. They then advocate "forward guidance," that Federal Reserve officials announce and somehow commit to such policies.

This paper makes no optimal policy calculations. I study outcomes for a variety of given policies, as in the above-cited literature. Some of those policies resemble optimal policies. For example, I study postponed rises in interest rates. I focus on the "implementation" problem: To achieve optimal results, it is not enough for the Fed to specify the path of interest rates. The Fed must take some other action to select among multiple equilibria consistent with the optimal interest rate path. Looking at those equilibria, I find that this selection is far more important to the results than is the path of equilibrium interest rates.

2. Model

I use Werning's (2012) simple continuous-time specification of the standard new-Keynesian model:

$$\frac{dx_t}{dt} = \sigma \left(i_t - r_t - \pi_t \right) \tag{1}$$
$$\frac{d\pi_t}{dt} = \rho \pi_t - \kappa (x_t + g_t). \tag{2}$$

Here, x_t is the output gap, i_t is the nominal rate of interest, r_t is the "natural" real rate of interest, π_t is inflation, and g_t is a Phillips curve disturbance discussed below. I abstract from constants, so these are all deviations from steady state values.

Equation (1) is the "IS" curve, which ought to be renamed "intertemporal substitution." It results from the first-order condition for allocation of consumption over time, and consumption equals output. Equation (2) is the new-Keynesian Phillips curve. Solved forwards, it expresses inflation in terms of expected future output gaps.

Like Werning, I suppose that starting at $t = 0$, the economy suffers from a negative natural rate $r_t = r = -2\%$, which lasts until time $t = T = 5$ before returning to a positive value. Also following Werning, I complete the model by specifying that the path of equilibrium nominal interest rates is zero up to period T, and then rises back to the natural rate $i_t = r_t \geq 0$, $t \geq T$. I use $\rho = 0.05$, $\sigma = 1$ and $\kappa = 1$.

Perfect foresight of a trap end date is unrealistic. However, it is simple and clear, and it provides a useful guide to the behavior of models with a stochastically ending trap or a slowly mean-reverting processes like an AR(1).

Then, I find the set of output $\{x_t\}$ and inflation $\{\pi_t\}$ paths that, via (1) and (2), are consistent with this path of interest rates, and do not explode as time increases. It will turn out that there are many such paths.

Specifying directly the equilibrium path of interest rates does not mean that I assume a peg, that interest rates are exogenous, or that I ignore Taylor rules. Typically, one assumes that after the trap ends, the central bank follows a rule of the form

$$i_t = i_t^* + \phi(\pi_t - \pi_t^*) \tag{3}$$

or equivalently,

$$i_t = \hat{\imath}_t + \phi\pi_t. \tag{4}$$

Here, i_t^* and π_t^* are the Fed's desired equilibrium values of interest rate and inflation, consistent with (1) and (2). The assumption $\phi > 1$ then produces $\{i_t^*,\ \pi_t^*\}$ as the unique locally-bounded equilibrium, which is the new-Keynesian selection rule. We see immediately though, that if there are multiple stable $\{\pi_t^*, x_t^*\}$ consistent with a given $\{i_t^*\}$–and there are–then any of them can be implemented by the Taylor rule (3). The Fed just chooses the desired $\{\pi_t^*\}$. By specifying interest rates, we find the set of equilibria, each implemented by a Taylor rule, and each unique given that Taylor rule, that result in the same equilibrium interest rate path. Equation (4) is algebraically equivalent to (3); choosing $\hat{\imath}_t = i_t^* - \phi\pi_t^*$ gives the same range of possibilities as choosing $\{i_t^*, \pi_t^*\}$ subject to (1) and (2). So, the same point applies.

Closing the model with a Taylor rule *per se* does nothing to prune multiple stable equilibria. To use Taylor rules to pick equilibria we have to ask *which* Taylor rule – which $\{\pi_t^*\}$ or $\{\hat{\imath}_t\}$–people expect the Fed to follow, which I consider below.

2.1. The flexible-price case

The flexible-price case is an important benchmark, and it is easiest to compute directly rather than by taking limits of the general solution. Prices become more flexible as κ increases, and $\kappa = \infty$ is the flexible-price case. To see this point, write the Phillips curve (2) (simplifying to $g_t = 0$) as

$$x_t = \frac{1}{\kappa}\left(\rho\pi_t - \frac{d\pi_t}{dt}\right).$$

As κ rises, the output gap x_t becomes smaller for any inflation path π_t. The flexible-price limit point is the case in which the output gap is zero for any inflation. If $\kappa = \infty$, $x_t = 0$.

Turning to (1), if $x_t = 0$ then $dx_t/dt = 0$ and we must have

$$i_t - r_t = \pi_t.$$

This is just the linearized Fisher relationship, which is the entire flexible-price model.

Thus, the flexible-price solution to our liquidity-trap scenario is $\pi_t = -r = 2\%,\ x_t = 0$ for $0 < t < T$, and $\pi_t = 0,\ x_t = 0$ for $t > T$. Inflation in the frictionless world rises to exactly equal to the negative natural rate, all on its own without extra prodding by the Fed, and there is no output gap.

In a perfect foresight model, this equilibrium is unique, up to the overall price level. If the natural rate change is unexpected, then there can be a price-level jump at the moment of that shock. In discrete time, $i_t = r_t + E_t\pi_{t+1}$ allows multiple equilibria $\pi_{t+1} - E_t\pi_{t+1}$ consistent with a given equilibrium nominal rate i_t. But then the path $\{E_t\pi_{t+j}\}$ is then unique. A price-level jump at 0 is the continuous-time counterpart.

2.2. Equilibria with price stickiness

Equations (1) and (2) are a pair of linear differential equations. Their solutions are

$$\pi_t = Ce^{-\lambda^b t} + \frac{1}{\lambda^f + \lambda^b}\left[\int_{s=-\infty}^{t} e^{-\lambda^b(t-s)} z_s ds + \int_{s=t}^{\infty} e^{-\lambda^f(s-t)} z_s ds\right] \tag{5}$$

$$\kappa x_t = -\kappa g_t + \lambda^f Ce^{-\lambda^b t} + \frac{1}{\lambda^f + \lambda^b}\left[\lambda^f \int_{s=-\infty}^{t} e^{-\lambda^b(t-s)} z_s ds - \lambda^b \int_{s=t}^{\infty} e^{-\lambda^f(s-t)} z_s ds\right], \tag{6}$$

where

$$\lambda^f \equiv \frac{1}{2}\left(\sqrt{\rho^2 + 4\kappa\sigma} + \rho\right);\ \lambda^b \equiv \frac{1}{2}\left(\sqrt{\rho^2 + 4\kappa\sigma} - \rho\right);\ z_t \equiv \kappa\sigma(i_t - r_t) + \kappa g_t. \tag{7}$$

Inflation and output are each two-sided moving averages of the driving processes. Inflation is a positive function of the driving disturbance in both directions. Output is a negative function of future disturbances, but a positive function of the past. Since ρ is a small number, the forward λ^f and backward λ^b eigenvalues are nearly, but not quite the same, and the forward-looking weights λ^f are a bit smaller.

Following standard procedure, I set to zero the corresponding forward-explosive equilibria $C_2 e^{\lambda^f t}$. All of these equilibria are stable going forward. There remain multiple forward-stable equilibria indexed by the free constant C.

These formulas are perfect foresight solutions. As such, they capture the impulse-response function, and the path of expected values in a stochastic model. In the case of an unexpected shock, the economy jumps to these solutions on the date that the shock is known.

2.3. Inflation and output paths

From (5)-(6), it is straightforward to calculate the paths of inflation and output for the forcing variable $z_t = \kappa\sigma(i_t - r_t)$ that starts at zero, rises to $i_t - r_t = 2\%$ for $0 < t < T$, and then falls to zero again. The algebra follows.

Figure 1 shows inflation in a range of such equilibria, generated by a range of values for the free constant C. These are all equilibria of the *same* model, with the *same* interest rate and natural rate path. The equilibria are all forward-stable, following the usual rules. They all use forward-looking solutions of the unstable eigenvalue. This paper is *not* about multiple explosive equilibria.

2.4. Algebra for the solution

One can solve the model (1)-(2) with the continuous-time version of standard lag operator techniques. Differentiate (2), and substitute from (1) for dx_t/dt to obtain

$$\frac{d^2\pi_t}{dt^2} - \rho\frac{d\pi_t}{dt} - \kappa\sigma\pi_t = -z_t,$$

and write this differential equation in the operator form

$$\left(\frac{d}{dt} - \lambda^f\right)\left(\frac{d}{dt} + \lambda^b\right)\pi_t = -z_t \tag{8}$$

where λ^f, λ^b, z are defined in (7). To invert the differential operator (8), note

$$\left(\frac{d}{dt} - \lambda^f\right)\pi_t = y_t$$

has solution

$$\pi_t = Ce^{\lambda^f t} - \int_{s=t}^{\infty} e^{-\lambda^f (s-t)} y_s ds,$$

while

$$\left(\frac{d}{dt} + \lambda^b\right)\pi_t = y_t$$

has solution

$$\pi_t = Ce^{-\lambda^b t} + \int_{s=-\infty}^{t} e^{-\lambda^b (t-s)} y_s ds.$$

These expressions allow us an interpretation of the inverse operators.

Now, write (8) as

$$\pi_t = \frac{-1}{\left(\frac{d}{dt} - \lambda^f\right)\left(\frac{d}{dt} + \lambda^b\right)} z_t = \frac{1}{\lambda^f + \lambda^b}\left[\frac{1}{\frac{d}{dt} + \lambda^b} - \frac{1}{\frac{d}{dt} - \lambda^f}\right] z_t,$$

set to zero the forward-explosive solutions $Ce^{\lambda^f t}$, and we immediately have the solution (5).

We can find the solutions for x_t similarly, or more easily by evaluating (2)

$$\kappa x_t = -\kappa g_t + \rho \pi_t - \frac{d\pi_t}{dt}$$

leading directly to (6).

These multiple forward-stable equilibria do not occur in many similar familiar models. For example, the standard asset pricing model with dividend x_t and constant return r, $dp_t/p_t + x_t/p_t dt = rdt$, has general solution

$$p_t = \int_{s=t}^{\infty} e^{-r(s-t)} x_s ds + Ce^{rt}$$

The unique forward stable solution, imposed by the consumer's transversality condition, sets $C = 0$. There is no additional family of forward-stable equilibria to pick from.

3. Three equilibria

I examine here the properties of three specific equilibria. Section 5 discuss principles for choosing among equilibria later. The latter are not decisive, and the main message is that results are sensitive to equilibrium choice. Also, knowing the consequences of equilibrium choice helps to make that choice.

3.1. The standard equilibrium

Werning (2012) chooses the equilibrium with zero inflation on the date that the trap ends, $\pi_T = 0$. To calculate this solution, I find the value of C in (5) that yields $\pi_T = 0$.

Figure 2 presents inflation and output in this standard equilibrium choice. It shows a large deflation and large output gaps during the liquidity-trap period $0 < t < T$.

We also see strong dynamics – deflation steadily improves, and expected output *growth* is strong. The intertemporal first order condition (1) says that the level of consumption can only be below potential if consumption is expected to grow back to potential. The forward-looking Phillips curve (2) produces a large gap when inflation is lower today than in the future. If inflation is to end up at zero, it produces substantial, but improving, deflation.

This equilibrium does not show an unstable deflation "spiral," in which a small deflation grows bigger over time. Such a spiral is a feature of models with adaptive expectations. In this forward-looking model, inflation is stable, even in a liquidity trap. It also does not produce a "slump," a large but steady output gap and steady but low inflation.

This equilibrium explodes *backward* in time. That observation helps to understand many of its predictions. In order to arrive at $\pi_T = 0$, the $t < T$ solution includes a nonzero $Ce^{-\lambda^b t}$ term, which explodes backwards. The backward explosion implies a large downward jump in inflation and output when the natural rate shock is observed, at $t = 0$ here.

A backward explosion also means that inflation and output deviations grow arbitrarily large as the period of the liquidity trap expands.

The dashed lines in figure 2 show how solutions with $\pi_T = 0$ behave as we reduce price frictions. Deflation and (not shown) output gaps become *larger* as price stickiness is *reduced*. As pricing frictions decrease, dynamics happen faster, and both eigenvalues λ^f and λ^b increase in absolute value. Faster backward explosions, tethered to $\pi_T = 0$, imply lower and lower inflation and output. It seems that structural reform to reduce price stickiness would only make matters worse.

Despite this infinite limit, the limit *point* of the frictionless equilibrium is well-behaved at two percent inflation and no output gap. The model with $\pi_T = 0$ equilibrium selection thus displays a large discontinuity, and tiny price stickiness has huge effects.

3.2. The backward-stable equilibrium

Figure 3 presents the equilibrium with $C = 0$ and no extra $Ce^{-\lambda^b t}$ term. Now inflation and output are each two-sided moving averages of the driving shocks. The thick lines show the standard experiment, that the natural rate shock at $t = 0$ is unexpected.

In this equilibrium, inflation jumps up by 1% on the onset of the trap, rather than jump down by 132% as in the standard equilibrium. The small variation in inflation corresponds to small variation in output, with output low when inflation is low relative to the future and vice versa. At $t = T = 5$, inflation comes down smoothly to zero.

Figure 3 also shows how inflation behaves as we reduce the pricing friction. The $C = 0$ equilibrium smoothly approaches the frictionless limit. I will call equilibria with this well-behaved limit "local-to-frictionless." With this property, small amounts of price stickiness gives inflation and output gaps close to frictionless values.

As the length of the trap episode widens, inflation just takes a similar hat-shaped path, approaching the negative of the natural rate $\pi_t = 2\%$ in the middle of the trap, and the output gap spends more time at zero. Unlike the standard solution, that gets exponentially worse for longer traps, this equilibrium is insensitive to that length.

Figure 3 also shows what happens if the trap is expected ahead of time, in the thin lines marked "Expected" for $t < 0$. The solution is a two-sided moving average, so inflation and output gap smoothly move ahead of the trap. News of a trap further and further in the future has smaller and smaller impacts on inflation and output before the trap.

A natural description of this property is that the solutions are "backward-stable." I'll use this property to give the equilibrium a more memorable name than "$C = 0$ equilibrium."

With a large $Ce^{-\lambda^b t}$ term, the standard $\pi_T = 0$ equilibrium shown in figure 2 explodes (even more) as we move back in time, if people know that the trap is coming ahead of time. (The backward explosion continues to the left of $t = 0$, not shown in figure 2.) Similarly, news of a trap further in the future has larger effects on inflation and output today. In a linear system, we read these perfect-foresight solutions as expected values. Thus, unless the probability of a future trap starting at date $t + s$ also declines at the rate $e^{-\lambda^b s}$, even a small probability of a trap further s in the future has a larger effect on inflation and output today. A constant small possibility of a trap at any date in the future produces an *infinitely* negative inflation and output today.

In sum, the backward-stable $C = 0$ equilibrium of figure 3 suggests that a negative natural rate and the zero bound is a mild event, associated with a mild inflation, which will emerge on its own without any additional policy, and little output variation. It suggests that longer traps are if anything less of a problem, because prices have more time to adjust, and that expectations of far off events have smaller and smaller effects today. By contrast the standard $\pi_T = 0$ equilibrium of figure 2 suggests that this liquidity trap produces a large output gap and deflation, that longer-lasting traps are exponentially (literally) worse, and that expectations of low-probability traps in the far-off future are worse still. At a minimum, we learn that the choice of equilibrium is not an innocuous technical detail, and instead that equilibrium choice is central to the model's economic predictions.

Figure 3 illuminates why the standard equilibrium choice has a discontinuous frictionless limit. The new-Keynesian Phillips curve (2) does not allow expected inflation jumps with a finite output gap. It does allow unexpected inflation jumps, such as at time 0. This backward-stable solution smooths the frictionless case naturally around the end of the trap with inflation following an S shape. But some of that smoothing must occur in the $t > T$ period. Insistence that inflation is zero *immediately* at $t = T$, for any value of price stickiness drives the economic dislocation and puzzling limiting behavior of the standard solution.

3.3. The no-inflation-jump equilibrium

We can also index equilibria by their jump at time 0, π_0. The standard $\pi_T = 0$ equilibrium choice shown in figure 2 features a large downward jump in inflation and output when the trap is announced. The backward-stable solution in figure 3 has a much smaller upward jump. An interesting case is the equilibrium in which inflation does not jump on news, $\pi_0 = 0$. To characterize this equilibrium, I find C such that $\pi_0 = 0$ in (5).

Figure 4 presents output and inflation in the no-inflation-jump $\pi_0 = 0$ equilibrium. The results are not terribly different from those of the backward-stable equilibrium, figure 3.

This equilibrium is also local-to-frictionless, in that the $\kappa \to \infty$ limit smoothly approaches 2% inflation for $0 < t < 5$ as in figure 3. The local-to-frictionless property is not unique.

In this $\pi_0 = 0$ equilibrium output still jumps at $t = 0$. A no-output-jump equilibrium choice $x_0 = 0$ gives a small jump in inflation and otherwise looks similar.

This equilibrium selection concept sets $\pi_t = 0$ on the date t that news of the trap arrives. As one moves the date of news of the trap back to $t = -1, t = -2$, etc., one must find a new C each time to ensure that $\pi_{-1} = 0$, $\pi_{-2} = 0$, etc. As a result, though individual choices of C here are not backward-stable, the equilibrium concept "pick $\pi_t = 0$ on the date people learn about the trap" is also backward-stable. News about traps in the further future have no effect on inflation, and lower effects on output.

3.4. Algebra for step function impulses

For $z_t = z$, $T_l < t < T_h$ and zero otherwise, evaluating the integrals in (5) and (6) yields

$$\pi_t = Ce^{-\lambda^b t} + \frac{z}{\lambda^f + \lambda^b} w_t$$

$$\kappa x_t = -\kappa g_t + \lambda^f C e^{-\lambda^b t} + \frac{z}{\lambda^f + \lambda^b} v_t$$

where

$$t < T_l : w_t = \frac{1}{\lambda^f}\left(e^{-\lambda^f (T_l - t)} - e^{-\lambda^f (T_h - t)}\right) \tag{9}$$

$$T_l < t < T_h : w_t = \frac{1}{\lambda^b}\left(1 - e^{-\lambda^b (t - T_l)}\right) + \frac{1}{\lambda^f}\left(1 - e^{-\lambda^f (T_h - t)}\right)$$

$$t > T_h : w_t = \frac{1}{\lambda^b}\left(e^{-\lambda^b (t - T_h)} - e^{-\lambda^b (t - T_l)}\right); \tag{10}$$

$$t < T_l : v_t = -\frac{\lambda^b}{\lambda^f}\left(e^{-\lambda^f (T_l - t)} - e^{-\lambda^f (T_h - t)}\right) \tag{11}$$

$$T_l < t < T_h : v_t = \frac{\lambda^f}{\lambda^b}\left(1 - e^{-\lambda^b (t - T_l)}\right) - \frac{\lambda^b}{\lambda^f}\left(1 - e^{-\lambda^f (T_h - t)}\right)$$

$$t > T_h : v_t = \frac{\lambda^f}{\lambda^b}\left(e^{-\lambda^b (t - T_h)} - e^{-\lambda^b (t - T_l)}\right). \tag{12}$$

Figures 1 through 3 plot the case $T_l = 0$, $T_h = T$. To select equilibria with $\pi_0 = 0$ or by $\pi_T = 0$, we solve for the corresponding C, from

$$0 = \pi_0 = C + \frac{z}{\lambda^f + \lambda^b}\frac{1}{\lambda^f}\left(1 - e^{-\lambda^f T}\right) \tag{13}$$

$$0 = \pi_T = Ce^{-\lambda^b T} + \frac{z}{\lambda^f + \lambda^b}\frac{1}{\lambda^b}\left(1 - e^{-\lambda^b T}\right). \tag{14}$$

To plot equilibria, one can simply use these values in (9)-(10). To find the explicit formula for the standard equilibrium $\pi_T = 0$, (useful later) substitute for C from (14) in (9)-(10), giving

$$t<0:\pi_t = \frac{z}{\lambda^f+\lambda^b}\left[\frac{1}{\lambda^b}\left(e^{-\lambda^b t}-e^{-\lambda^b (t-T)}\right)+\frac{1}{\lambda^f}\left(e^{\lambda^f t}-e^{\lambda^f (t-T)}\right)\right] \tag{15}$$

$$0<t<T:\pi_t = \frac{z}{\lambda^f+\lambda^b}\left[\frac{1}{\lambda^b}\left(1-e^{\lambda^b (T-t)}\right)+\frac{1}{\lambda^f}\left(1-e^{-\lambda^f (T-t)}\right)\right]$$

$$t>T:\pi_t = 0 \tag{16}$$

$$t<0:\kappa x_t = -\kappa g_t + \frac{z}{\lambda^f+\lambda^b}\left[\frac{\lambda^f}{\lambda^b}\left(e^{-\lambda^b t}-e^{-\lambda^b (t-T)}\right)-\frac{\lambda^b}{\lambda^f}\left(e^{\lambda^f t}-e^{\lambda^f (t-T)}\right)\right] \tag{17}$$

$$0<t<T:\kappa x_t = -\kappa g_t + \frac{z}{\lambda^f+\lambda^b}\left[\frac{\lambda^f}{\lambda^b}\left(1-e^{\lambda^b (T-t)}\right)-\frac{\lambda^b}{\lambda^f}\left(1-e^{-\lambda^f (T-t)}\right)\right]$$

$$t>T:\kappa x_t = -\kappa g_t. \tag{18}$$

To find the explicit formula for the no-jump equilibrium $\pi_0 = 0$, substitute for C from (13) in (9)-(10), giving

$$t<0:\pi_t = \frac{z}{\lambda^f+\lambda^b}\frac{1}{\lambda^f}\left(1-e^{-\lambda^f T}\right)\left(e^{\lambda^f t}-e^{-\lambda^b t}\right) \tag{19}$$

$$0<t<T:\pi_t = \frac{z}{\lambda^f+\lambda^b}\left[\frac{1}{\lambda^b}\left(1-e^{-\lambda^b t}\right)+\frac{1}{\lambda^f}\left(1-e^{-\lambda^f (T-t)}\right)-\frac{1}{\lambda^f}\left(1-e^{-\lambda^f T}\right)e^{-\lambda^b t}\right]$$

$$t>T:\pi_t = \frac{z}{\lambda^f+\lambda^b}\left[\frac{1}{\lambda^b}\left(1-e^{-\lambda^b T}\right)e^{-\lambda^b (t-T)}-\frac{1}{\lambda^f}\left(1-e^{-\lambda^f T}\right)e^{-\lambda^b t}\right] \tag{20}$$

$$t<0:\kappa x_t = -\kappa g_t - \frac{z}{\lambda^f+\lambda^b}\left(1-e^{-\lambda^f T}\right)\left(\frac{\lambda^b}{\lambda^f}e^{\lambda^f t}+e^{-\lambda^b t}\right) \tag{21}$$

$$0<t<T:\kappa x_t = -\kappa g_t + \frac{z}{\lambda^f+\lambda^b}\left[\frac{\lambda^f}{\lambda^b}\left(1-e^{-\lambda^b t}\right)-\frac{\lambda^b}{\lambda^f}\left(1-e^{-\lambda^f (T-t)}\right)-\left(1-e^{-\lambda^f T}\right)e^{-\lambda^b t}\right]$$

$$t>T:\kappa x_t = -\kappa g_t + \frac{z}{\lambda^f+\lambda^b}\left[\frac{\lambda^f}{\lambda^b}\left(1-e^{-\lambda^b T}\right)e^{-\lambda^b (t-T)}-\left(1-e^{-\lambda^f T}\right)e^{-\lambda^b t}\right]. \tag{22}$$

4. Large multipliers and paradoxical policies

I add a disturbance g_t to the Phillips curve,

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa\left(x_t+g_t\right), \tag{23}$$

following Werning (2012) and Wieland (2014). The variable g_t can represent useless (since it does not affect utility) government spending. It also can represent destruction of capital or technological regress. These policies increase inflation π_t for a given output gap, and thus they reduce the real interest rate and consumption growth. Assuming a return to trend,[1] reducing the consumption growth rate increases the current level of consumption: Solving (1) forward,

$$x_t = \int_{s=0}^{\infty} \frac{dx_{t+s}}{ds} ds = \int_{s=0}^{\infty} \sigma \left(i_{t+s} - r_{t+s} - \pi_{t+s} \right) ds.$$

Expected future inflation is the key for stimulus in this model, not current inflation, or unexpected current inflation. Similarly, since output is demand-determined, wealth or capital destruction does not directly affect output or consumption.

The new-Keynesian multiplier is utterly different from static Keynesian intuition. The static Keynesian multiplier results because more income generates more consumption which generates more income. $Y = C + I + G$, and $C = \bar{c} + mY$ imply $Y = (\bar{c} + I + G)/(1 - m)$ and thus a multiplier $1/(1 - m)$. In this new-Keynesian model, the marginal propensity to consume is effectively zero, as the consumer is intertemporally unconstrained and there are no permanent changes in the level of consumption. Fiscal policy acts entirely by creating future inflation, affecting the intertemporal allocation of consumption.

Modeling fiscal stimulus, capital destruction, or technical regress by a Phillips-curve shifter g is admittedly simplistic. More realistic and microfounded treatments, such as that in Werning (2012) and Wieland (2014), recognize the potential value of government spending in the utility function, or treat its effects on potential output, or on the natural rate. But the point of this paper is to examine equilibrium selection in the most transparent model, not to give a realistic quantitative evaluation of policies.

I specify that $g_t = g$ during the trap, for $0 < t < T$, and $g_t = 0$ thereafter. I examine how increasing g affects equilibrium output and employment by the private-consumption multiplier $\partial x_t / \partial g$ evaluated at $g = 0$. If g represents government spending, conventional multipliers would add g itself. I present $\partial x_t / \partial g$ multipliers since g can represent other Phillips

[1] Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2014) challenge this common assumption. They find that supply-side policies can raise potential enough that productivity and growth raise output.

curve shocks. Additionally, since the equations are log-linearized, $y = x + g$ does not hold. The versions of this equation in Werning (2012) and Wieland (2014) (see his equation (2)) have additional scaling factors in front of x and g. Again, the point here is not quantitative assessment of a policy, it is to see how the multipliers change across equilibrium choices.

Taking the derivative with respect to g of (11)-(12) using $C = 0$, of (21)-(22), and of (17)-(18), evaluated at $g = 0$, we obtain the same multiplier formula in each case:

$$t < 0;\ t > T : \left.\frac{\partial x_t}{\partial g}\right|_{g=0} = -\frac{x_t}{\sigma r};\ \ 0 < t < T : \left.\frac{\partial x_t}{\partial g}\right|_{g=0} = -\frac{x_t}{\sigma r} - 1. \tag{24}$$

Dividing by r means the multiplier is about 50 times the output gap. These are not the same multipliers, as the x_t are different in each case. Equilibria with large output gaps have large multipliers, and vice versa.

Figure 5 presents these multipliers. Multipliers are large for the standard $\pi_T = 0$ equilibrium – ten at year two, for example. The multipliers increase exponentially as the length of the liquidity trap increases, moving to the left. Multipliers *increase* as price stickiness is *reduced*. In the limit that price stickiness goes to zero, the multiplier goes to infinity. Very small amounts of price stickiness generate very large multipliers. The multiplier is -1 at the limit *point*, however.

This increase in multipliers presents an interesting policy paradox. Microeconomic efforts to reduce price stickiness make the depression worse, according to figure 2. But such efforts make multipliers larger, increasing the effectiveness of fiscal or broken-window stimulus.

By contrast, the multiplier in the no-jump $\pi_0 = 0$ equilibrium is small, and clustered around the frictionless value -1, as its output gaps are small. As price-stickiness is reduced or the period of the trap lengthens the no-jump equilibrium multipliers converge smoothly to -1. The multipliers in the backward-stable equilibrium, not shown, are similar.

In sum, large multiplier predictions are direct results of equilibrium choice. The no-jump or backward-stable equilibria produce fiscal or productivity-reduction multipliers that are, if anything, lower than conventional wisdom, and more in line with the complete crowding-out or supply-limited results of equilibrium models.

4.1. Forward guidance

Many authors have advocated forward guidance policies to ameliorate a liquidity trap, in which the Fed announces a commitment to keep rates low for some time after the negative natural rate passes. The optimal policy in Werning (2012) and Woodford (2012) takes this form. A temporarily higher inflation target has a similar effect.

To address forward guidance, I assume that the interest rate remains zero for some time τ after T, when the natural rate rises.

Figure 6 presents the standard equilibrium, selected by $\pi_{T+\tau} = 0$ for a variety of time intervals τ. The top left presents the previous solution with $\tau = 0$, which reminds us of the deep recession and deflation baseline. The remaining panels suppose that people expect the interest-rate rise to be delayed for $\tau = 0.6$, 0.703, and 0.8 years. This delay allows a little inflation to emerge between $t = T$ and $t = T + \tau$. Then, allowing small changes in the π_T terminal condition has large effects on inflation and output during the trap.

An 0.6 year delay, in the top right panel, raises inflation and output substantially. A 0.703 year delay in raising interest rates, bottom left panel, produces the benign results of the no-jump equilibrium. While not exactly the optimal policy of Werning (2012) and Woodford (2003), this choice carries their central message: by committing to a delay in raising rates after the trap is over, the central bank can dramatically improve an otherwise dismal outcome, even if it enforces the $\pi_{T+\tau} = 0$ equilibrium. An 0.8 year delay raises inflation π_T even further, and produces an upward jump at time $t = 0$, an inflationary boom.

This exercise is paradoxical in several ways, however. First, the vertical difference between the $\tau = 0$ and $\tau > 0$ solutions in each panel is larger as one moves back in time. Promises further in the future have larger effects today.

Second, I do not show the solutions for $t < 0$, but the backward-explosive eigenvalue continues to operate. Thus, a promise to hold rates low for half a year after a future trap ends has larger effects on output today, the further in the future that trap and promise occur.

Third, all the dynamics happen faster as prices become less sticky (also not shown). Forward guidance has larger effects for less sticky prices, and infinitely large effects in the flexible price limit – and then no effect at all at the flexible price limit point.

Fourth, the graphs reveal a strong sensitivity of forward guidance predictions to the length τ of the delayed rate rise. The $\tau = 0.703$ year delay produces a benign result. But get it just a little wrong – promise 0.6 years, or 0.8 years – and the economy still shows strong deflationary recession or a strong inflationary boom.

Figure 7 presents the no-jump equilibrium in the same situation. The no-jump equilibrium shows very little effect of the delayed-interest-rate-rise policy. It displays the normal economic prediction that promises about the further-off future have less effect today. Not shown, greater price-stickiness just brings the inflation and output paths closer to their frictionless values. The delayed rise's main effect here is to bring inflation down more quickly after then end of the trap than would occur otherwise.

The main point: equilibrium choice is centrally important to analyzing predictions of this model. The interest rate path makes almost no difference compared to the choice of equilibrium. For example, the benign $\tau = 0.703$ delay with the standard $\pi_{T+\tau} = 0$ equilibrium choice (bottom left, figure 6) is almost identical to the the no-jump equilibrium with no delay $\tau = 0$ (top left of figure 7). Within the no-jump equilibria of figure 7, the interest rate delay makes almost no difference. As far as improving outcomes during the trap, the $\tau = 0.703$ delay of figure 6 is just a way to raise inflation π_T and thus to choose the no-jump equilibrium for $0 < t < T$.

Why then do Werning (2012) and Woodford (2012) find that delay is an optimal policy? Since in optimal policy exercises the Fed can choose any of these equilibria, why not just choose the no-jump equilibrium, by a suitable inflation target π_T? The graphs reveal the answer: Once one chooses a (nearly) optimal equilibrium, either by choosing π_T or by choosing π_0, outcomes during the trap are basically unaffected by delay or no delay. But this model is Fisherian: inflation is a positive function of nominal interest rates (see (5), a two-sided moving average of interest rates with positive weights). So keeping interest rates low for a while after the trap brings inflation down faster than it otherwise would fall, and that slightly improves the Fed's objective in the post-trap world.

4.2. Formulas for forward guidance

The postponed interest rate rise solution comes from adding up two cases of (9)-(12), $T_l = 0$, $T_h = T$ with $z_1 = \kappa\sigma(i - r) = 2\%$ and $T_l = T$, $T_h = T + \tau$ using $z_2 = -2\%$. We obtain:

$$\pi_t = Ce^{-\lambda^b t} + \frac{w_t}{\lambda^f + \lambda^b}$$
$$\kappa x_t = -\kappa g_t + \lambda^f Ce^{-\lambda^b t} + \frac{v_t}{\lambda^f + \lambda^b}$$

where

$$t < 0 : w_t = \frac{z_1}{\lambda^f}\left(1 - e^{-\lambda^f T}\right)e^{\lambda^f t} + \frac{z_2}{\lambda^f}\left(1 - e^{-\lambda^f \tau}\right)$$
$$0 < t < T : w_t = \frac{z_1}{\lambda^b}\left(1 - e^{-\lambda^b t}\right) + \frac{z_1}{\lambda^f}\left(1 - e^{-\lambda^f (T-t)}\right) + \frac{z_2}{\lambda^f}\left(1 - e^{-\lambda^f \tau}\right)e^{\lambda^f (t-T)}$$
$$T < t < T+\tau : w_t = \frac{z_1}{\lambda^b}\left(e^{\lambda^b T} - 1\right)e^{-\lambda^b t} + \frac{z_2}{\lambda^b}\left(1 - e^{-\lambda^b (t-T)}\right) + \frac{z_2}{\lambda^f}\left(1 - e^{-\lambda^f (T+\tau-t)}\right)$$
$$t > T+\tau : w_t = \frac{z_1}{\lambda^b}\left(e^{\lambda^b T} - 1\right)e^{-\lambda^b t} + \frac{z_2}{\lambda^b}\left(e^{\lambda^b \tau} - 1\right)e^{-\lambda^b (t-T)}$$

$$t < 0 : v_t = -\frac{\lambda^b z_1}{\lambda^f}\left(1 - e^{-\lambda^f T}\right)e^{\lambda^f t} - \frac{\lambda^b z_2}{\lambda^f}\left(1 - e^{-\lambda^f \tau}\right)e^{\lambda^f (t-T)}$$
$$0 < t < T : v_t = \frac{\lambda^f z_1}{\lambda^b}\left(1 - e^{-\lambda^b t}\right) - \frac{\lambda^b z_1}{\lambda^f}\left(1 - e^{-\lambda^f (T-t)}\right) - \frac{\lambda^b z_2}{\lambda^f}\left(1 - e^{-\lambda^f \tau}\right)e^{\lambda^f (t-T)}$$
$$T < t < T+\tau : v_t = \frac{\lambda^f z_1}{\lambda^b}\left(e^{\lambda^b T} - 1\right)e^{-\lambda^b t} + \frac{\lambda^f z_2}{\lambda^b}\left(1 - e^{-\lambda^b (t-T)}\right) - \frac{\lambda^b z_2}{\lambda^f}\left(1 - e^{-\lambda^f (T+\tau-t)}\right)$$
$$t > T+\tau : v_t = \frac{\lambda^f z_1}{\lambda^b}\left(e^{\lambda^b T} - 1\right)e^{-\lambda^b t} + \frac{\lambda^f z_2}{\lambda^b}\left(e^{\lambda^b \tau} - 1\right)e^{-\lambda^b (t-T)}$$

I then pick $C = 0$, the C that delivers $\pi_{T+\tau} = 0$ and the C that delivers $\pi_0 = 0$.

4.3. Jumps and limits

The paradoxical policies – large multipliers, large effects of far-off promises, less price stickiness makes matters worse – all have their roots in backward-explosive solutions, and thus large jumps at time 0.

Therefore, reversing these predictions is not unique to the specific no-jump $\pi_0 = 0$ and $C = 0$ backward-stable equilibria. Any limit on the size of the initial jump produces a local-to-frictionless and backward-stable result, and declining effects of expectations.

Specifically, consider equilibria in which the initial jump is limited, $\|\pi_t\| < \Pi$ where people learn the shock at t. In this set of equilibria, 1) Expectations of future events have smaller and smaller effects the farther in the future the event lies; 2) As price stickiness decreases $\kappa \to \infty$, inflation and output smoothly approach the frictionless limit point; 3) The Phillips-disturbance multiplier $\partial x/\partial g$ is bounded by the output jump x_t per (24), and smoothly approaches -1 as price stickiness declines.

5. Choosing equilibria

With an understanding of the effects of equilibrium choices, we can now consider how we ought to make that choice.

5.1. Taylor rules

In the standard new-Keynesian approach, the Fed chooses the desired equilibrium interest rate path $\{i_t^*\}$. It then *also and additionally* conducts an equilibrium-selection or implementation policy to select which of the many possible equilibria $\{\pi_t\}$ and $\{x_t\}$ consistent with that $\{i_t^*\}$ will emerge as the equilibrium $\{\pi_t^*\}$ and $\{x_t^*\}$. Finally, people know about all this, as it is their expectations of Fed equilibrium-selection policy in the future that determines which equilibrium emerges today.

To be specific, after choosing the equilibrium interest rate path $\{i_t^*\}$, the Fed selects the equilibrium π_t^* from the set $\{\pi_t\}$ consistent with $\{i_t^*\}$ (for example, the set graphed in figure 1) by following for $t > T$ a Taylor-rule inspired policy of the form

$$i_t = i_t^* + \phi(\pi_t - \pi_t^*) = \hat{\imath}_t + \phi\pi_t. \tag{25}$$

This policy de-stabilizes the economy. With $\|\phi\| > 1$, all the equilibria $\{i_t, \pi_t, x_t\}$ other than $\{i_t^*, \pi_t^*, x_t^*\}$ now explode forward[2] as $t \to \infty$. The new-Keynesian tradition adopts as an equilibrium-selection principle that the economy will not choose non-locally-bounded

[2] In continuous time, one must specify a rule with some persistence, such as

$$i_t - i_t^* = \phi\theta \int_{s=0}^{\infty} e^{-\theta s}(\pi_{t-s} - \pi_{t-s}^*)ds$$

or in equivalent differential form

$$\frac{d(i_t - i_t^*)}{dt} = \theta\left[\phi(\pi_t - \pi_t^*) - (i_t - i_t^*)\right].$$

One can also append a zero bound to such rules, e.g. the last equation when $i_t > 0$ but

$$\frac{d(i_t - i_t^*)}{dt} = \max\left\{\theta\left[\phi(\pi_t - \pi_t^*) - (i_t - i_t^*)\right], 0\right\}$$

when $i_t = 0$. See Sims (2004) and Fernández-Villaverde, Posch and Rubio-Ramírez (2012). Cochrane (2013) contains simulations of these rules.

equilibria, and thus predicts that π_t^* is the unique observed equilibrium[3].

For example, Werning (2012) writes "I assume that the central bank can guarantee... $\pi(t), x(t) = (0, 0)$ for $t \geq T$," and this "presumes that the central bank somehow overcomes the indeterminacy of equilibria that plagues these models. A few ideas have been advanced to accomplish this, such as adhering to a Taylor rule with appropriate coefficients..."

(I ignore an additional strand of multiplicity described by Benhabib, Schmitt-Grohé and Uribe (2001). Since the Taylor rule must respect the zero bound even after the natural rate shock passes, there really are still multiple equilibria. Mertens and Ravn (2014) argue for such equilibria in which the bound and policies have mild and standard effects.)

As in section 2, such a Taylor rule by itself does nothing to select equilibria. Equation (25) shows how to construct$\{\pi_t^*\}$ or $\{\hat{\imath}_t\}$ that deliver any equilibrium shown in figure 1. To use Taylor rules for equilibrium selection, we have to think about *which* Taylor rule, why the Fed might insist on $\pi_t^* = 0$ or $\hat{\imath}_t = 0$ for $t > T$ – and why people expect this choice.

Many papers just assume a rule with $\pi_T = 0$, $\pi_t^* = 0$, or $\hat{\imath}_t = 0$ for $t > T$. But given large historical deviations from Taylor rules ($R^2 < 1$); given the strong persistence in empirical Taylor rules (lagged interest rate terms); given much Fed talk of temporary deviations from "normal" policy, and "glidepath" and "soft landing" inflation goals; given that optimal policy recommends intercepts $\hat{\imath}_t$ that vary with shocks (Woodford (2003), Svensson and Woodford (2005)), and given the advantages for the Fed not to insist on $\pi_T = 0$, it is a weak assumption.

Werning (2012) offers a principled reason for people to expect $\pi_T = 0$: People expect that the Fed is fully discretionary. It will do ex-post what looks best going forward no matter what last year's forward guidance was. At time T, $\pi_t^* = x_t^* = 0$ $t \geq T$ is forward-looking optimal. The delayed rise in Werning and Woodford (2012) proposals requires pre-commitment as well as guidance, which both authors emphasize.

That the Fed is not expected by people to pre-commit to things it will regret ex-post, is a sensible assumption, buttressed by fairly explicit statements from Fed officials and FOMC.

But there is a deep contradiction in this view about what the Fed can and cannot commit

[3]Atkeson, Chari, and Kehoe (2010) discuss more general "sophisticated" equilibrium-selection strategies. Taylor rules dominate applied analysis, so I stick to that formulation here.

to. Under an "active" $\phi > 1$ policy, all equilibria except the selected x_t^*, π_t^* are disastrous for the Fed's objective – output and inflation explode. So people must believe that the Fed cannot commit at all to interest rate and inflation targets, $\{\pi_t^*, i_t^*\}$, but the same Fed *completely* pre-commits to a doomsday-machine equilibrium-selection threat which, ex-post, is disastrous for its objectives.

The whole idea of Taylor-rule-inspired equilibrium selection is not unassailable. Taylor (for example Taylor (1993)) advanced the $\phi > 1$ rule in the context of an adaptive-expectations, backward-looking model. In that case, $\phi > 1$ brings *stability* to an economy that is *unstable* under an interest rate peg. If inflation rises, interest rates rise more, real interest rates rise, demand decreases and expected future inflation decreases.

But this conventional intuition does not apply to forward-looking new-Keynesian models, such as (1)-(2). Here, the economy is already stable under an interest rate peg; by $\phi > 1$ the Fed *destabilizes* the economy, in order to select from multiple equilibria. If inflation rises, interest rates rise more, but this leads to *more* subsequent inflation, spiraling off to infinity, so inflation had better not rise in the first place.

Does the Fed really have, and do people believe that it has, an "equilibrium-selection" policy, that *de*stabilizes the economy for inflation not equal to its target, distinct from its "interest rate policy?" The Fed resolutely describes its behavior as stabilizing, reacting to unexpected inflation in a way to bring inflation back down again. Furthermore, the $\phi > 1$ reaction is unobservable and hence unlearnable from time series. If the model is right, we only see the equilibrium $\pi_t = \pi_t^*$, and hence can never learn the value of ϕ or the existence of equilibrium selection policy. (For more on these doubts, see Cochrane (2011))

The point: Equilibrium selection by Taylor rules may not be as rock-solid as it appears. We can at least contemplate other equilibria and other ways of choosing equilibria.

5.2. Fiscal theory

The fiscal theory of the price level neatly solves nominal indeterminacies under interest rate targets, and thus offers help for equilibrium selection.

I present here the simplest case, one-period nominal government debt in discrete time. Then, the proposition that the real value of nominal government debt equals the present value of primary surpluses reads

$$\frac{B_{t-1}}{P_t} = E_t \left[\sum_{j=0}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} s_{t+j} \right], \tag{26}$$

where B_{t-1} is the face value of debt sold at period $t-1$ and due at t, P_t is the price level, and s_t is the real primary surplus. This is a valuation equation, like price = present value of dividends; it is a consequence of equilibrium and the consumer's transversality condition not a budget constraint; and it does assume surpluses are "exogenous" any more than the standard asset pricing equation assumes that dividends are "exogenous." (For response to standard objections see Cochrane (2005). For an explicit integration of fiscal theory with a sticky-price model and interest rate target, see Cochrane (2014).)

Multiply and divide by P_{t-1} and take innovations, yielding

$$\frac{B_{t-1}}{P_{t-1}} \left(E_t - E_{t-1} \right) \left(\frac{P_{t-1}}{P_t} \right) = \left(E_t - E_{t-1} \right) \left[\sum_{j=0}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} s_{t+j} \right]. \tag{27}$$

This equation tells us that unexpected inflation – the jump at $t = 0$ – corresponds entirely to innovations in the present value of future surpluses.

Multiplying by marginal utility and taking expected values,

$$\frac{B_{t-1}}{P_{t-1}} E_{t-1} \left(\beta \frac{u'(c_t)}{u'(c_{t-1})} \frac{P_{t-1}}{P_t} \right) = \frac{B_{t-1}}{P_{t-1}} \frac{1}{1+i_{t-1}} = E_{t-1} \left[\sum_{j=0}^{\infty} \beta^{j+1} \frac{u'(c_{t+j})}{u'(c_{t-1})} s_{t+j} \right]. \tag{28}$$

This equation shows how the government can follow an interest rate target. By fixing the interest rate on government debt i_{t-1} with constant surpluses, it describes the number of bonds B_{t-1} that will be sold, and verifies that number is determined, positive, and finite.

Together, (26)-(28) challenge standard monetary doctrines: The government can follow an interest rate peg (static or time-varying), or a passive $\phi < 1$ interest rate rule. In either case, the price level and inflation rate are both stable and uniquely globally determined.

In the continuous-time perfect-foresight simplified setup of this paper, with a single unexpected jump at time 0, (26) reads

$$\frac{B_t}{P_t} = \int_{j=t}^{\infty} e^{-\int_{\tau=t}^{j} r_\tau d\tau} s_j dj \tag{29}$$

and the continuous time version of (27) describes a price level jump,

$$\frac{B_0}{P_{0-}} \left(P_{0-} \left(d\frac{1}{P_0} \right) \right) = d \left(\int_{j=0}^{\infty} e^{-\int_{\tau=0}^{j} r_\tau d\tau} s_j dj \right). \tag{30}$$

Here B_0 is predetermined and can't jump, P_{0-} is the value just before a jump, and d is the forward differential operator.

Equations (27) or (30) apply immediately to our equilibrium-selection problem. They tells us to pick equilibria by inflation π_0 at time $t = 0$, or when people learn of the negative natural rate shock, not by expectations of inflation at time T. They tell us to pick equilibria by understanding *fiscal* policy responses to the natural rate shock, rather than expectations of Fed equilibrium-selection policy after the shock ends. In this simple framework, fiscal considerations have no other effect than to choose the initial jump, and thus the equilibrium.

Even if one does not wish to use the fiscal theory to select equilibria, it is useful to index equilibria by their fiscal consequences. These equations are present in all models. In the standard new-Keynesian model, one assumes that (27) describes the behavior of the Congress and the Treasury: they adjust taxes and spending "passively" ex-post to validate any price level. If, for example, the price level falls by half, then the Government will double fiscal surpluses to pay off an unexpected windfall to bond holders. Even with this view, it remains useful to index equilibria by the time-zero jump, and examine the magnitude and plausibility of the consequent "passive" fiscal policy reaction.

The no-jump equilibrium $\pi_0 = 0$ occurs if there is no change in present value of future surpluses coincident with or in response to the negative natural rate shock. This is not an obvious choice. Given the large deficits and fiscal stimulus in the 2008-2009 recession and beyond, the assumption of looser fiscal policy seems initially more plausible. That line of thought pushes us to a positive inflation jump at time zero, such as the backward-stable equilibrium figure 3 or even more. And that line of thought suggests that the zero bound is even less of a problem than the no-jump $\pi_0 = 0$ equilibrium suggests.

However, equation (27) directs us to examine the innovation to the *present value* of *all* future surpluses. If the government reacts to the negative natural rate shock with large deficits during the trap, $s_t < 0$ for $0 < t < T$, but also credibly promises to pay back the resulting debt by future tax increases or spending cuts, $s_t > 0$ for $t > T$, stimulus now but austerity later, then there is no innovation to the present value of future deficits. This is a plausible assumption. Increases in debt usually convey expectations that the debt will be paid back, as governments finance wars with current deficits but future surpluses. Even in the middle of the stimulus debates of 2009, the Administration promised to follow current stimulus with future debt reduction, not default via inflation.

Furthermore, discount rates matter to present values. In 2008, real interest rates on government bonds dropped suddenly. A plausible way therefore to make sense of the small but sharp disinflation in 2008-9 via (27), is that the larger value of government debt, corresponded to sharply lower real interest rates, not a tightening of current or expected future surpluses. This is the "flight to quality." (Again, (27) holds in every model, and as an identity using ex-post returns. Thus the question is *how* it holds, not *if* it holds.)

By contrast, the $\pi_T = 0$ standard solution graphed in figure 2 includes a -132% deflation at $t = 0$, corresponding to a jump of the price level down to $100 \times e^{-1.32} = 27\%$ of its initial level, and $100 \times e^{1.32} = 376\%$ increase in the value of government debt. Raising taxes or cutting spending that much would surely strain the "passive" assumption. Large unexpected deflations require large ex-post taxes. In the fiscal theory, this is why large unexpected deflations don't happen.

The point here is not to advocate a particular fiscal assumption as the right one. The point is that we can think about equilibrium selection this way. The jump at π_0, or when people learn of the trap, which indexes equilibria, corresponds to expectations about fiscal policy and discount rates. To figure out which is the right equilibrium, we have to think as hard about fiscal policy and discount rates as we think hard about monetary policy, expected interest rate paths, equilibrium selection policies, pre-commitment, and so forth.

But even without picking a specific value, fiscal considerations at least suggest that one place a limit on the allowable jumps in inflation at time 0. Per section 4.3, such a limit cures the strange limiting behavior of the model and its policy predictions. Conversely, the paradoxical limits resulting from the standard equilibrium choice require that "passive" fiscal policy validate *unbounded* increases in the value of government debt.

5.3. Other equilibrium-selection principles

In models with multiple equilibria, a wide range of principles outside of the standard definition of equilibrium have been advocated to select equilibria.

The basic new-Keynesian selection procedure has such an element as well. It rests primarily on the principle that expectations should "coordinate" on particular equilibria. (See Woodford (2003) p.128 and King (2000) p. 58-59.) A long list of efforts to uniquely select equilibria using completely economic criteria in passive-fiscal models fail on closer examination (See Cochrane (2011), and Cochrane (2015) response to Sims (2013).)

One could make similar cases for equilibrium selection here, by turning properties of various equilibria into criteria for their selection. Rules that eliminate sets of equilibria are also useful. If we can bound initial jumps, we resolve most of the issues.

The local-to-frictionless property is attractive – pick equilibria in which small frictions have small effects. That principle does not pick a unique equilibrium here, as any equilibrium that limits the initial response is local-to-frictionless. But that principle can serve to rule out equilibria, and the standard equilibrium choice in particular.

The property that news about further-off events should have smaller effects today, or that equilibria should not explode backward, are properties that one could use for equilibrium selection. They have some of the same flavor as the views in Woodford (2003) and King (2000) about sensible expectations and coordination mechanisms. They also bound initial jumps and thus solve most of the issues.

One could also bound initial jumps directly as an equilibrium-selection device.

5.4. Empirical equilibrium selection

Equilibrium selection can be an empirical project as well as a theoretical one. The equilibrium choice centrally matters to how the model fits the data, just like preferences and technology. So, one can ask the data which equilibrium choice fits best. The present model is not rich enough, nor have I calibrated or estimated parameters and shocks, to do a serious job of such estimation. But I can point to the general issues.

First, we can ask which equilibrium choice produces a better fit with the data. In this simple model, the stability of zero bound experience would be a key observation. The US economy 2009-2014 featured steady slow growth, a level of output stuck about 7% below the previous trendline, and steady positive 2% or so inflation. European and Japanese experience has been similar.

The backward-stable and no-inflation-jump equilibria shown in figures 3 and 4 can produce this steady outcome. However, they do not produce a big output gap. Thus, they only account for disappointing output if one thinks that growth has been limited by "supply" rather than "demand," that calculations of potential output are optimistic. Substantial ex-post downward revision in potential output calculations lends support to this view.

The standard equilibrium choice as in figure 2 cannot produce stagnation. Here and in more general models, the standard equilibrium choice counterfactually predicts large and time-varying deflation (Hall (2011), Ball and Mazumder (2011), King and Watson (2012), Coibion and Gorodnichenko (2013)), which did not happen, and it counterfactually predicts strong growth. To generate stagnation, one has to imagine a stream of unexpected negative shocks. (Failure of a Poisson exit shock to appear is a negative shock relative to expectations.)

Alternatively, one can fundamentally modify the model itself, as do Del Negro, Giannoni and Schorfheide (2015) and Eggertsson and Mehrotra (2014). But the latter course strengthens the case that *this* model doesn't produce a slump, so within the context of this model the data are likely to choose something like the no-jump equilibrium.

Second, for an exercise such as the one in this paper, one could condition on the observed downward jump in inflation to select equilibrium. The standard equilibrium produced a sharp -132% downward jump in the price level. In the data, core inflation decreased from about 2.5% in mid 2008 to just a bit below 1% in 2011 before rebounding. We could pick the equilibrium with (say) -1.5% deflation at $\pi_0 = 0$.

Third, for model simulations, one could measure the typical downward jump in inflation and output in response to shocks. We can measure equilibrium selection by the correlation of shocks in the impulse-response function. Yes, identifying shocks is hard, but this is regular empirical macroeconomics, not a special task that must be relegated to theory alone.

Finally, the equilibrium choice, along with the rest of the model, can be evaluated by its policy predictions and the historical record. We have the recent past and the Great Depression, at the zero bound, and zero-bound predictions emerge in this model when interest rates respond less than one for one $\phi < 1$ to inflation, so early postwar interest rate targets and the 1970s apply. At a casual level, deliberate inflation, output destruction, technical regress, more price stickiness (price controls), useless government spending, and central banker promises do not seem to have had in those periods the large effects claimed for them now. As real examples, Dupor and Li (2013) find that stimulus spending was not associated with a rise in expected inflation and thus no multiplier by this mechanism; Wieland (2014) shows that several cases of endowment destruction and adverse supply shocks did not induce inflation or stimulus at the zero bound; and Del Negro, Giannoni and Patterson (2015) measure the effects of forward guidance, finding that "standard medium-scale DSGE models tend to grossly overestimate the impact of forward guidance."

The point here is not to settle the case, but to outline the methodological possibility. Whether by matching data directly, by conditioning on an observation, by matching impulse-response functions, or by matching policy experience, equilibrium selection rules are identifiable and measurable parts of a model. They do not have perpetually to remain a theoretical or philosophical controversy.

6. Concluding comments

I examine a standard new-Keynesian analysis of the zero bound, following Werning's (2012) elegantly simple example: A negative natural rate lasts from time 0 to time T, and the nominal rate is stuck at zero. I find there are many equilibria, each bounded, forward-stable, and nonexplosive as $t \to \infty$.

The conclusion that the zero bound is a big economic problem, and that counterintuitive policies can have dramatic curative effects, follows from selecting equilibria by expected inflation at the end of the trap, $\pi_T = 0$. This equilibrium features a deep recession with deflation. It also features strong expected output growth, which is why the level of output is so low, and rapidly declining deflation. It predicts large multipliers to wasted government spending, and wealth or productivity destruction. It predicts that announcements about far-off future policies have large effects. These predictions grow larger the longer the period of the liquidity trap, and as price stickiness is reduced. It has a large jump in inflation and output at time 0, when people learn of the negative natural rate shock.

Indexing equilibria by the initial jump in inflation π_0, and limiting such jumps overturns all of these results. In particular, the "backward stable" and "no-inflation-jump" equilibria of the same model, with the same interest rate path, instead predict mild *in*flation during the liquidity trap, little if any reduction in output relative to potential, small negative multipliers, and little effects of promises of far-off policies. Their predictions smoothly approach the frictionless limit as pricing frictions are reduced.

At a minimum, this analysis shows that equilibrium selection, rather than just the path of expected interest rates, is vitally important for understanding these models' predictions. In usual interpretations of new-Keynesian model results, authors feel that interest rate policy is central, and equilibrium-selection policy by the Fed or by the author are "implementation" details relegated to technical footnotes (as in Werning (2012)), game-theoretic foundations, or philosophical debates, which can all safely be ignored in applied research.

My most concrete suggestion for addressing multiple equilibria is to marry new-Keynesian models with the fiscal theory of the price level. Every model includes the valuation equation for government debt, which can determine the price level. The standard approach, codified in Woodford (2003), throws that equation out, by assuming a "passive" fiscal policy. Restoring that equation, global determinacy is restored. Operationally, each value of the initial jump in inflation corresponds to a jump in the present value of future fiscal surpluses. Just what that jump is requires us to think hard about fiscal policy responses to shocks, and endogenous discount-rate responses.

The new-Keynesian structure plus fiscal theory – or with any other limitation on the size of jumps in endogenous state variables in response to shocks – produces an attractive and tractable model of nominal stickiness and interest rate targets. But it eliminates the puzzle, or the promise, depending on your reaction to earlier work, of some new-Keynesian models' diagnoses and their policy prescriptions for the zero bound.

7. References

Atkeson, Andrew, Chari, Varadarajan V. and Patrick J. Kehoe, 2010. Sophisticated Monetary Policies. Quarterly Journal of Economics 125, 47-89.

Ball, Laurence M. and Sandeep Mazumder, 2011. Inflation Dynamics and the Great Recession. Brookings Papers on Economic Activity, Spring, 337-402.

Benhabib, Jess, Stephanie Schmitt-Grohé and Martin Uribe, 2001. The Perils of Taylor Rules. Journal of Economic Theory 96, 40-69.

Carlstrom, Charles T., Timothy S. Fuerst, and Matthias Paustian, 2014. Fiscal Multipliers under an Interest Rate Peg of Deterministic versus Stochastic Duration. Journal of Money, Credit and Banking 46 1293-1312.

Christiano, Lawrence J., Martin Eichenbaum, and Sergio Rebelo, 2011. When is the Government Spending Multiplier Large? Journal of Political Economy 119, 78-121.

Cochrane, John H., 2005. Money as Stock. Journal of Monetary Economics 52, 501-528.

Cochrane, John H., 2011. Determinacy and Identification with Taylor Rules. Journal of Political Economy 119, 565-615.

Cochrane, John H., 2014. Monetary Policy with Interest on Reserves. Journal of Economic Dynamics and Control 49, 74-108.

Cochrane, John H., 2013. The New-Keynesian Liquidity Trap. NBER Working Paper 19476.

Cochrane, John H., 2015. A Response to Sims (2013). Manuscript.

Coibion, Olivier and Yuriy Gorodnichenko, 2013. Is The Phillips Curve Alive and Well After All? Inflation Expectations and the Missing Disinflation. American Economic Journal: Macroeconomics 7, 197-232.

Del Negro, Marco, Marc Giannoni, and Christina Patterson, 2015. The Forward Guidance Puzzle. Staff Report 574, Federal Reserve Bank of New York.

Del Negro, Marco, Marc P. Giannoni, and Frank Schorfheide, 2015. Inflation in the Great Recession and New Keynesian Models. American Economic Journal: Macroeconomics 7, 168-96.

Dupor, Bill, and Rong Li, 2013. The 2009 Recovery Act and the Expected Inflation Channel of Government Spending. Manuscript.

Eggertsson, Gauti B., 2010. The Paradox of Toil. Federal Reserve Bank of New York Staff Report N. 433.

Eggertsson, Gauti B., 2011. What Fiscal Policy is Effective at Zero Interest Rates? In Daron Acemoglu and Michael Woodford (Eds.), NBER Macroeconomics Annual 2010, 59-112. University of Chicago Press, Chicago.

Eggertsson, Gauti B. and Michael Woodford, 2003. The Zero Bound on Interest Rates and Optimal Monetary Policy. Brookings Papers on Economic Activity 34, 139-235.

Eggertsson, Gaudi and Neil Mehrotra, 2014. A Model of Secular Stagnation. Manuscript.

Eggertsson, Gauti, Andrea Ferrero, and Andrea Raffo, 2013. Can Structural Reforms Help Europe? Journal of Monetary Economics 61, 2-22.

Fernández-Villaverde, Jesús, 2013. Discussion of "Can Structural Reforms Help Europe?" Journal of Monetary Economics 61, 23-31.

Fernández-Villaverde, Jesús, Olaf Posch, and Juan F. Rubio-Ramírez, 2012. Solving the New Keynesian Model in Continuous Time. Manuscript.

Fernández-Villaverde, Jesús, Pablo Guerrón-Quintana, Juan F. Rubio-Ramírez, 2014. Supply-Side Policies and the Zero Lower Bound. Manuscript.

Hall, Robert E., 2011. The Long Slump. American Economic Review 101, 431-469.

Kiley, Michael T., 2016. Policy Paradoxes in the New Keynesian Model. Review of Economic Dynamics 21,1-16.

King, Robert G., 2000. The New IS-LM Model: Language, Logic, and Limits. Federal Reserve Bank of Richmond *Economic Quarterly* 86, 45-103.

King, Robert G. and Mark W. Watson, 2012. Inflation and Unit Labor Cost. Manuscript.

Mertens, Karel and Morten O. Ravn, 2014. Fiscal Policy in an Expectations Driven Liquidity Trap. The Review of Economic Studies 81, 1637-1667.

Sims, Christopher, 2004. Limits to Inflation Targeting. In Ben S. Bernanke and Michael Woodford (Eds.), The Inflation-Targeting Debate. University of Chicago Press, Chicago, pp. 283-310.

Sims, Christopher, 2013. Paper Money. American Economic Review 103, 563-584.

Svensson, Lars E. O., and Michael Woodford, 2005. Implementing Optimal Policy through Inflation-Forecast Targeting. In Ben S. Bernanke and Michael Woodford (Eds.), The Inflation-Targeting Debate. University of Chicago Press, Chicago, pp. 19-83.

Taylor, John B., 1993. Discretion Versus Policy Rules in Practice. Carnegie-Rochester Series on Public Policy 39, 195-214.

Werning, Ivan, 2012. Managing a Liquidity Trap: Monetary and Fiscal Policy. Manuscript.

Wieland, Johannes, 2014. Are Negative Supply Shocks Expansionary at the Zero Lower Bound? Manuscript.

Woodford, Michael, 2003. Interest and Prices. Princeton University Press, Princeton.

Woodford, Michael, 2011. Simple Analytics of the Government Expenditure Multiplier. American Economic Journal: Macroeconomics 3, 1-35.

Woodford, Michael, 2012. Methods of Policy Accommodation at the Interest-Rate Lower Bound. In Federal Reserve Bank of Kansas City, The Changing Policy Landscape. pp. 185-288.

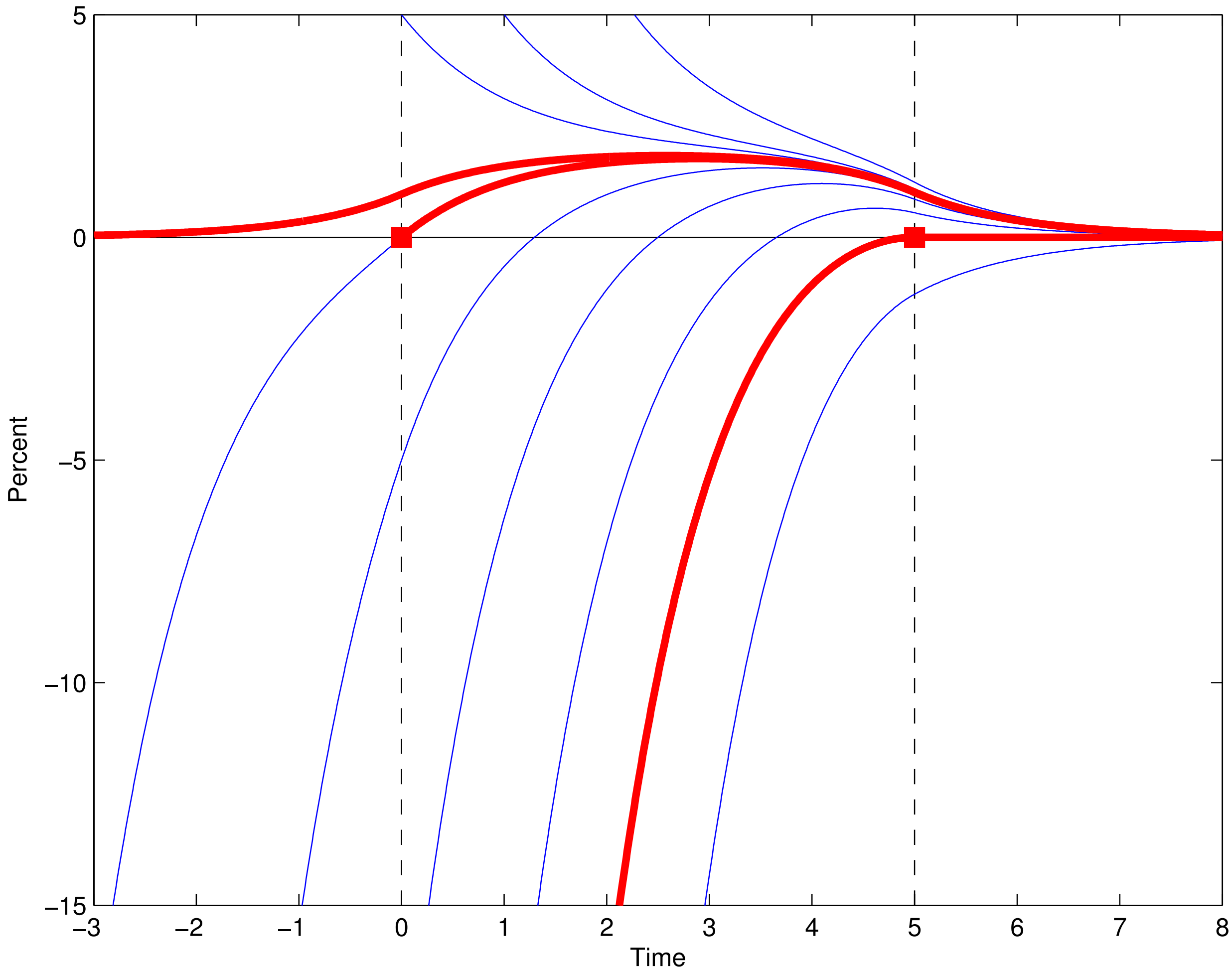

Figure 1: Inflation in a range of multiple equilibria. $i_t - r_t = -2\%$ between $t = 0$ and $t = 5$, shown by vertical dashed lines, and $i_t = r_t$ otherwise. The thick lines show the backward-stable equilibrium, the no-jump equilibrium, and the standard equilibrium discussed below. Thinner lines show a range of additional possible equilibria.

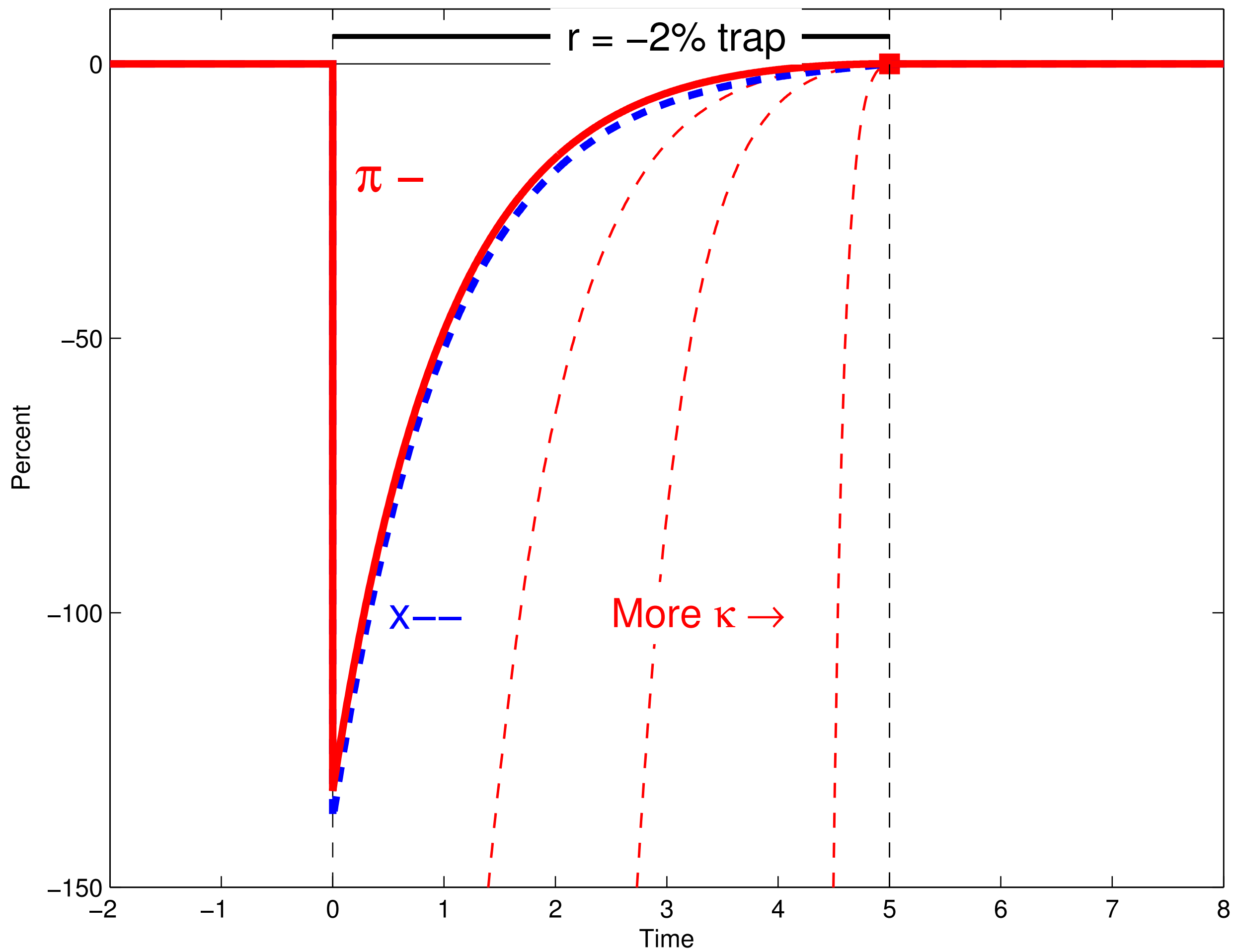

Figure 2: Output and inflation in the standard equilibrium. The thick lines show $\kappa = 1$. The thin dashed lines give inflation as the price-stickiness parameter κ increases from to 2, 5, and 100. I assume that the natural rate shock is unexpected at time $t = 0$, and then lasts until $t = 5$. The square at $t = 5$ indicates the selection assumption $\pi_5 = 0$.

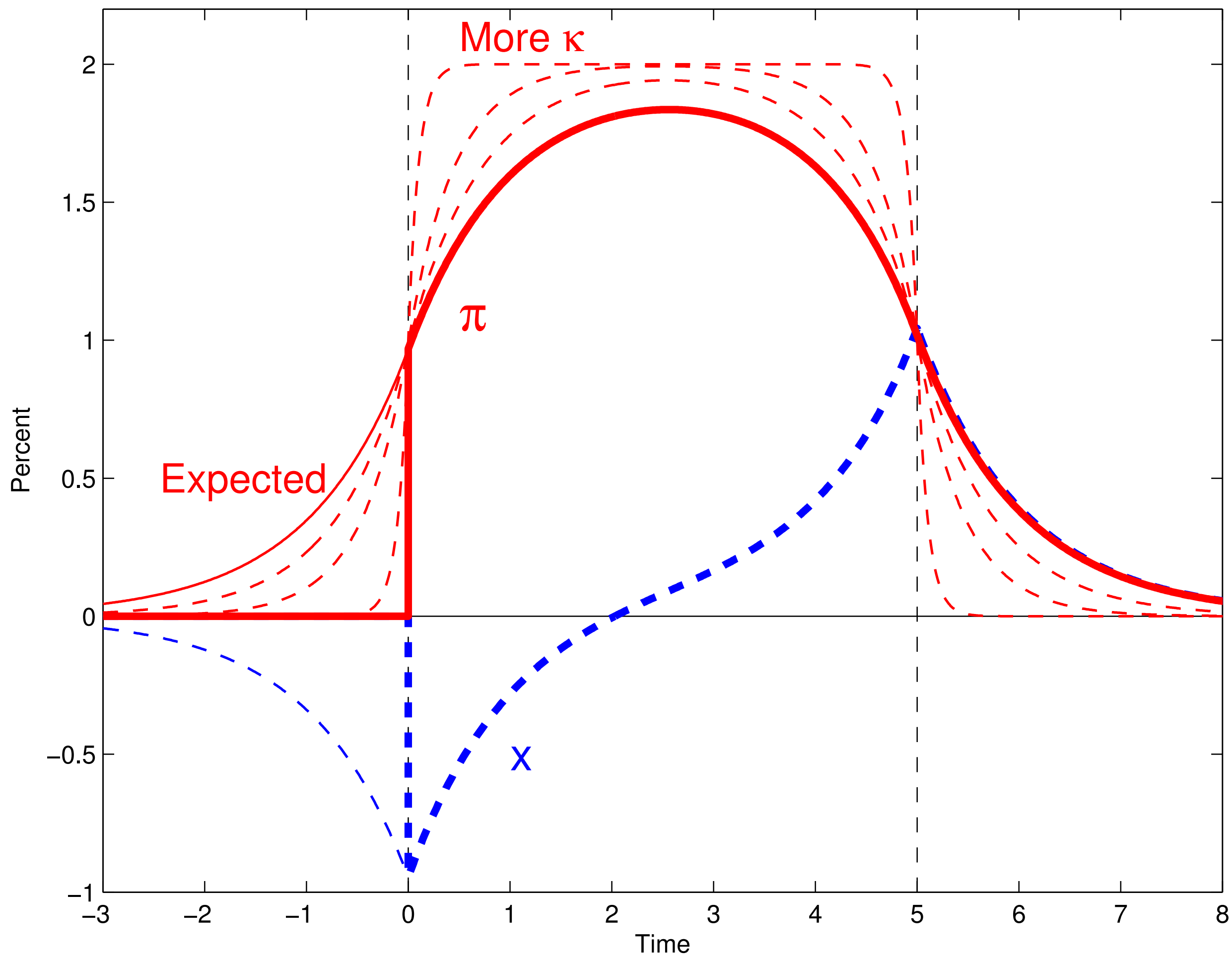

Figure 3: Output and inflation in the backward-stable $C = 0$ equilibrium, $i_t = 0$, $r_t = -2\%$ between $t = 0$ and $t = 5$. Thick lines show inflation and output when the trap is unexpected at $t = 0$. Lines to the left of $t = 0$ show inflation and output when the event is expected. Thin dashed lines show inflation as price-stickiness diminishes from κ =1, to 5, 10, 100.

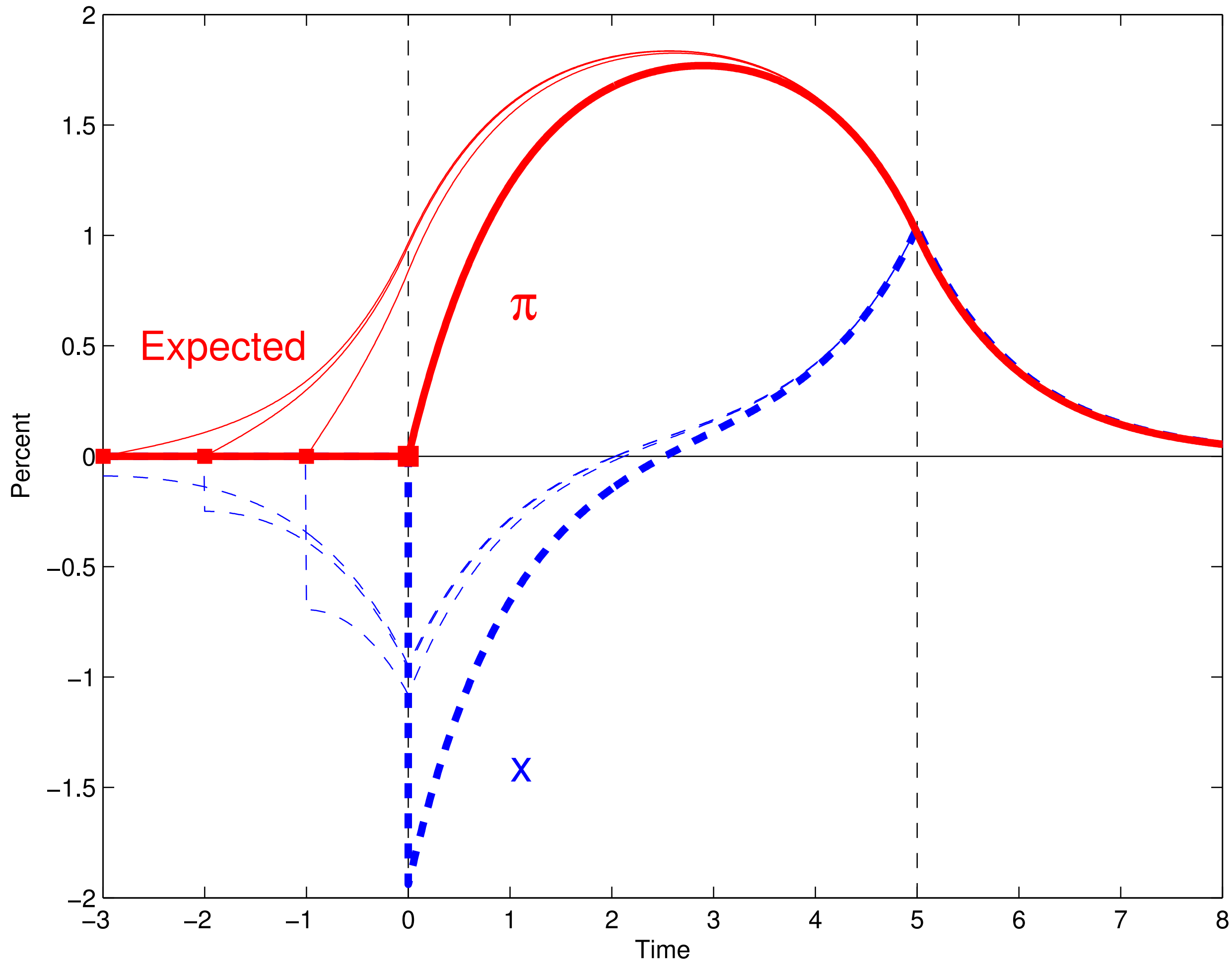

Figure 4: Output and inflation in the no-inflation-jump $\pi_0 = 0$ equilibrium. The thin lines give equilibria with no inflation jump at time $t = -1, -2$, and -3, corresponding to news of the trap arriving on those dates. $\kappa = 1$ throughout. The solid square reminds us visually of the equilibrium selection by $\pi_t = 0$.

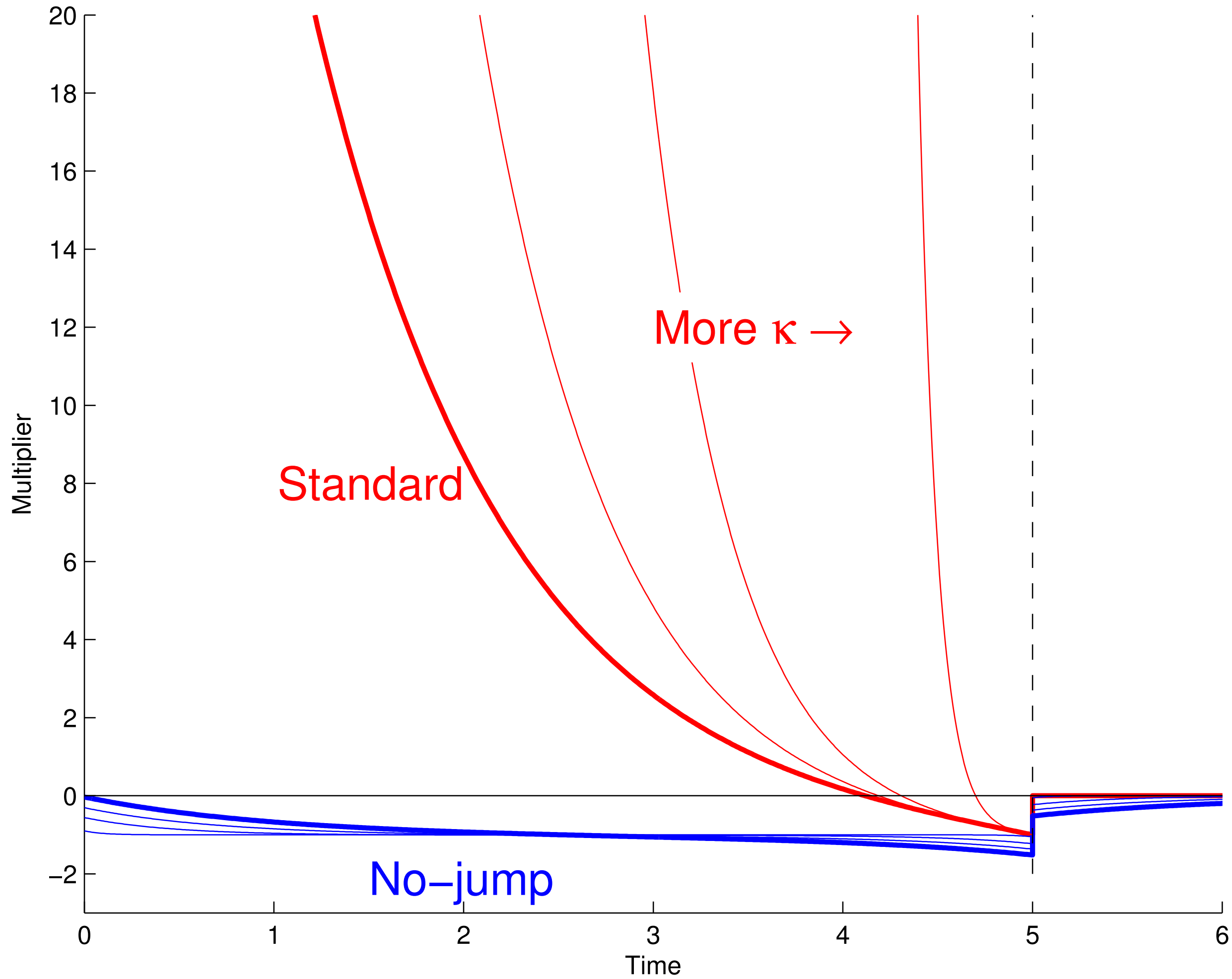

Figure 5: Multipliers with respect to a Phillips curve disturbance. I modify the Phillips curve to $d\pi_t/dt = \rho\pi_t - \kappa(x_t + g_t)$. The graph plots the multiplier $\partial x_t/\partial g$ for an increase in g through the trap episode from $t = 0$ to $t = T = 5$. The thin lines show what happens as price stickiness is steadily reduced to $\kappa = 2$, 5, 100.

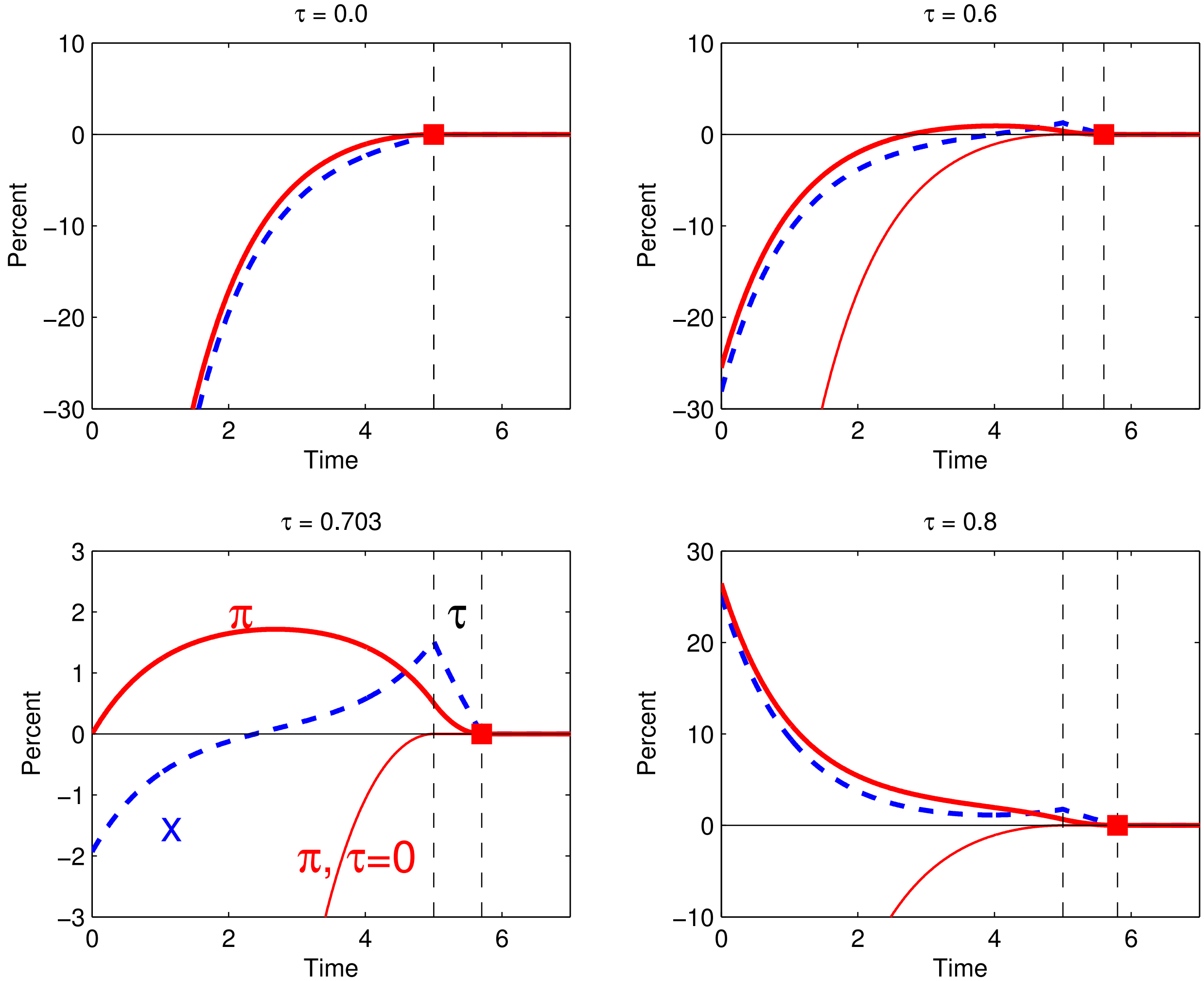

Figure 6: Output gap x and inflation π in the standard equilibrium choice $\pi_{T+\tau} = 0$, when the interest-rate rise is delayed. At $t = T = 5$, marked by the left dashed line, the natural rate changes from -2% to +2%. At $t = T + \tau$, marked by the right dashed line, people expect the nominal interest rate to rise from $i = 0$ to $i = 2\%$, for τ as indicated. The thin line marked "π, $\tau = 0$" repeats the $\tau = 0$ inflation line for comparison.

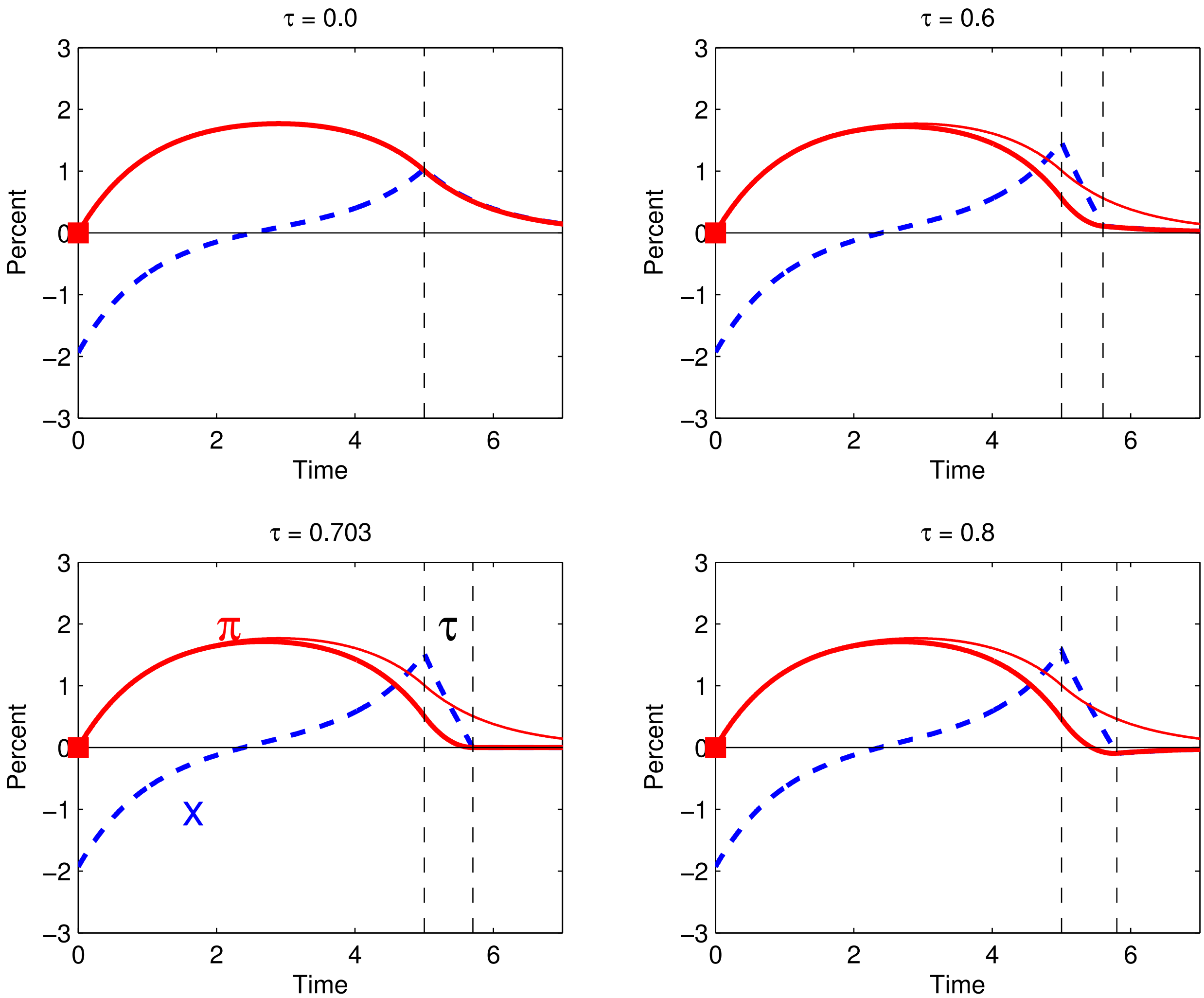

Figure 7: Output gap x and inflation π in the no-jump equilibrium choice $\pi_0 = 0$, when the interest-rate rise is delayed. At $t = T = 5$, marked by the left dashed line, the natural rate changes from -2% to +2%. At $t = T + \tau$, marked by the right dashed line, people expect the nominal interest rate from $i = 0$ to $i = 2\%$, for τ as indicated. The thin line presents the $\tau = 0$ inflation value for comparison.