I. Options

1. We showed that the arbitrage bounds on option prices can be derived from the linear program,

\[
\min_{\pi^*(s)} \max_{\pi^*(S_T)} C = \frac{1}{R^T} \int \pi^*(S_T) \max(S_T - X, 0) dS_T \quad s.t \quad \pi^*(S_T) > 0
\]

\[
S = \frac{1}{R^T} \int \pi^*(s_T) S_T dS_T
\]

\[
1 = \int \pi^*(S_T) dS_T
\]

or with a discrete state space

\[
\min_{\pi^*(s)} \max_{\pi^*(s)} C = \frac{1}{R^T} \sum_s \pi^*(s) \max(S_T(s) - X, 0) \quad s.t \quad \pi^*(s) > 0
\]

\[
S = \frac{1}{R^T} \sum_s \pi^*(s) S_T(s)
\]

\[
1 = \sum_s \pi^*(s)
\]

(a) Construct the standard call option arbitrage bounds,

\[
S \geq C \geq \max(0, S - X/R^T)
\]

using this method. Draw a \(\pi^*(s)\) that achieves each bound. (The form of \(\pi^*(s)\) will vary with \(S\), so you may have to make more than two drawings.)

(b) Recalling that \(\pi(s)\) is given, so what you are choosing here are \(m(s)\), possible nonzero values of marginal utility as a function of \(S_T\). Are these “reasonable” \(m(s)\)?

(c) Use the same technique to find put option bounds. When you’re done, do the put option bounds and call option bounds obey put-call parity?

(d) Suppose the investor is allowed to trade once at \(T/2\). The arbitrage bound is

\[
\max_{m_{T/2}} \min_{m_T} C = \max_{\{m_{T/2}, m_T\}} \text{ or } \min E (m_{T/2} m_T x_{cT})
\]

Section 18.3 proves that this bound is recursive. Find the arbitrage bounds in this case. Are they tighter than the one-period bounds? Are arbitrage bounds converging to Black-Scholes as we let the rebalancing interval shrink?

2. A full set of call or put prices is equivalent to contingent claims. We can invert to find the contingent claim price from the second derivative of call price with respect to strike. To see this, consider a “butterfly” which is a payoff that is one if the stock has value \(X\), zero elsewhere
Total payoff of butterfly (area of triangle) = $\varepsilon^2$.
Price of 1 butterfly $= -[C(X - \varepsilon) - 2C(X) + C(X + \varepsilon)] = \varepsilon^2 \frac{\partial^2 C(X)}{\partial X^2}$
Buy $1/\varepsilon^2$ butterflies to get a payoff of 1 if $S_T = X$.
Price = value of $1/\varepsilon^2$ butterflies $= -\frac{\partial^2 C(X)}{\partial X^2}$
Thus, the contingent claims price is $\frac{\partial^2 C(X)}{\partial X^2}$. If we have probabilities $f(S_T)$, then the discount factor is $[\frac{\partial^2 C(X)}{\partial X^2}] / f(S_T)$

(a) Given a probability $f(S_T)$ a riskfree rate $R^f$ and the current call option price function $C(S, X, T)$, find the stochastic discount factor. Show how you’d value an arbitrary claim $x(S_T)$ using i) contingent claim prices ii) discount factor iii) risk neutral probability. Use put-call parity to figure out how to use puts instead of calls.

(b) Now, we’ll put this in to practice. Get the option data from the class website. These are June 30 2005, for 1 year to expiration S&P index puts and calls. First, plot the call and put prices as a function of strike/index (index=1191.33) which is “moneyness.” A call option with strike/index = 0.8 is “20% in the money”. Make sure this graph looks right to you before proceeding, and the puts and calls are correctly identified.

(c) Now plot the implied volatility (no need to recompute as it’s given in the dataset). If the Black Scholes formula is right, this should be the same for all strikes. Is it? This graph should also persuade you to use out of the money puts and calls to measure prices, not deeply in the money calls. (Look at the price graph and you’ll see why.) Are the actual prices of out of the money calls that large? If not, why is implied vol so large?

(d) In order to take a second derivative, we need to fit a smooth function to options data. I fit

$$p_i = e^{a + bx_i + cx_i^2 + dx_i^3 + ...}$$

where $p =$ price, and $x =$strike price, by running a regression of log p on the polynomial function of x. Experiment a bit until you find a good fit. You may have to do a little trimming. Plot your actual and fitted prices.

(e) Now, take second derivatives to find state-prices. To find discount factors, we’ll need a probability. To make a comparison with the Black-Scholes formula, use lognormal probabilities, $\mu = 0.09, \sigma = 0.145$ (this is the implied volatility at the money). Plot a) the contingent claim price and the probability and b) the log discount factor from the data, and the log discount factor from the Black-Scholes formula. Where are the state prices large or small? Where are probabilities large or small? Where is the discount factor large or small? (I used the more
reliable out of the money puts and calls to make a single line. I’d really like to redo this using a kernel density estimate of the stock distribution to evaluate fat tails under the real measure; this would change some of what looks like risk aversion to risk. Could it all go away? )

3. Problem 2, p. 326

**Bond Problems**

1. Problem 2, 4-7 p. 383.

2. Download the Fama-Bliss 1-5 year zero coupon bond price data from the class website. These are monthly observations of percent yields (not log) of 1-5 year zero coupon bonds, inferred from the treasury yield curve.

   (a) Let’s do a factor analysis of yields. Do an eigenvalue decomposition of the covariance matrix of yields, $QΛQ' = cov(y, y')$. Plot the columns of Q, and give the percent of variance explained by each factor and the standard deviations of the factors. (This is what I presented in lecture).

   (b) Reproduce the regression coefficients in the Fama Bliss regressions

   \[
   r_{t+1}^{(n)} - y_t^{(1)} = a + b \left( f_t^{(n-1-n)} - y_t^{(1)} \right) + \varepsilon_{t+1}
   \]

   \[
   y_{t+n-1}^{(1)} - y_t^{(1)} = a + b \left( f_t^{(n-1-n)} - y_t^{(1)} \right) + \varepsilon_{t+1}
   \]

   The sample is 1964:01-2006:9. For the problem set, you need only get the coefficients right. The standard errors in the table use olsgmm to correct for overlap. The point of the problem, of course, is to verify that you understand the regression. The most common source of trouble is not implementing the definitions of forward rate and return correctly. The first return on a two year bond is the Jan 1965 price of a one year bond divided by the Jan 1964 price of a two year bond. (And then take logs, or first find the log price then take the difference). The second return on a two year bond is the Feb 1965 price of a one year bond divided by the Feb 1964 price of a two year bond.

\[
\begin{array}{cccccc}
 n & a & b & σ(a) & σ(b) & R^2 \\
 2 & 0.02 & 0.91 & 0.27 & 0.26 & 0.14 & -0.02 & 0.09 & 0.27 & 0.26 & 0.00 \\
 3 & -0.19 & 1.21 & 0.50 & 0.34 & 0.15 & -0.33 & 0.45 & 0.59 & 0.30 & 0.04 \\
 4 & -0.43 & 1.42 & 0.69 & 0.44 & 0.16 & -0.71 & 0.71 & 0.70 & 0.21 & 0.10 \\
 5 & -0.16 & 1.11 & 0.94 & 0.51 & 0.07 & -0.88 & 0.87 & 0.79 & 0.19 & 0.14 \\
\end{array}
\]

forecasting one year returns on n-year bonds

forecasting one year rates n years from now