1. In class, we did an eigenvalue decomposition of bond *yields*. Here we’ll do the eigenvalue factor decomposition for bond *excess returns*.

   (a) Form the excess log returns. (Same data as last week. Excess here means return - one year rate. There will be four excess returns, for 2-5 year bonds). Find their covariance matrix. Do the eigenvalue decomposition of the covariance matrix. Plot Q and give the standard deviations of the factors. You will see that the shape of the dominant “level” factor is different here than it was for yields. How does it make sense – why does the level factor here seem to rise with maturity, while the level factor in yields was flat or slightly declining in maturity? (Hint: what happens to bond returns when there is a parallel shift to *yields*?)

   (b) As you will see, the first two factors account for nearly all the variance of returns. Let’s try these factors in a factor pricing model, i.e. just like the FF factors. Create the factors \( x_t = Q' y_t \). Run time series regressions of the 4 bond excess returns on the two biggest factors. Compare the betas to the columns of Q. Do they relate in a sensible way? Make a table of mean excess returns, alphas, and betas, and make a plot of actual mean excess returns \( y \) against the predictions of the model \( (\beta_1 E(\text{factor 1}) + \beta_2 E(\text{factor 2})) \) Does the two factor model do a good job of explaining expected excess bond returns?

Try to summarize what you’ve learned. What is the nature of the risk for which you earn a return by holding long term bonds? I.e. “You earn a large expected excess (log) return if you have a large beta on ....” (The point here is to treat bonds exactly as Fama and French treat stocks.)

(Note: This exercise just looks at the *unconditional* risk premium – \( E(R^e) \). Fama Bliss and Cochrane Piazzesi are about *conditional* risk premia, \( E_t(R^e_{t+1}) \) that vary over time. We can explain them in similar ways, but you need to think about time-varying betas and time varying lambdas to do it. We’re not going to do it here. This exercise just addresses the long term average premium. But that’s a start!)

2. Now let’s try the same thing for the Fama French 25 portfolios.

   (a) Load up the Fama French data, and subtract off Rf to make them excess returns. Use only the 1947-now subsample (247:end). (It doesn’t really matter but it comes out prettier that way without having to do any fancy tricks.) Form the covariance matrix of the FF 25 portfolios, and take the eigenvalue decomposition. Plot the standard deviation of the factors (square root of \( \Lambda \) diagonals), then look at numbers. How many factors seem important?

   (b) Now, let’s look at the four factors with largest standard deviations. First we want to look at the columns of Q corresponding to the largest factors. I found 5 x 5 bar plots most revealing. For example, here’s the first factor. Since everyone is positive, it is a sort of level factor, or an equal-weighted market factor. (rmrf would be a *value* weighted market factor which would load almost entirely on the large size portfolios.) How can you interpret the remaining three of the first four factors? (Yes 4 – let’s go one past FF and see what happens.)
(c) As with the bonds, take the first three and then the first four factors, and run time series regressions of the returns against these factors. How do the regression coefficients compare with the columns of $Q$? How well do these models work as factor models? Are the alphas small? (No need to repeat GRS tests, etc. Just plot or make a table so you can see how big they are.) How do these models compare to the Fama French regressions?

Note: It is not true that factor models attack average returns or alphas one by one. The factor model finds in order the largest common movements in the covariance matrix of returns. There is no mathematical reason that this should relate at all to the mean returns or alphas left over from the last factor model. In fact, you see a pattern that each factor model attacks patterns seen in mean returns of the last factor model. This is finance, not math, it’s a result that could have come out differently; it’s a sign of the basic idea in finance working. The central idea in finance is that risk premia will attach only to common, undiversifiable movements in asset returns. And, lo and behold, it does: every dimension of expected return turns out to correspond to a common factor in variance, and many dimensions (not all) of the variance of returns turn out to correspond to a pattern in mean returns.

3. If that was “what Fama and French should have done”, this is “What Cochrane and Piazzesi should have done.” Let’s do an eigenvalue factor decomposition of expected excess bond returns. Get the yield data, and start in 1964 (observation 140) as Fama-Bliss and Cochrane-Piazzesi do.

(a) Start by running the four bond excess returns (2,3,4,5 year maturity) on the one year yield and the four forward rates. i.e. run unrestricted regressions

$$rx_{t+1}^{(n)} = a + b_{n,1}y_{t}^{(1)} + b_{n,2}f_{t}^{(1-2)} + b_{n,3}f_{t}^{(2-3)} + b_{n,4}f_{t}^{(3-4)} + b_{n,5}f_{t}^{(4-5)} + \varepsilon_{t+1}; \ n = 2, 3, 4, 5$$

Plot the $b$ coefficients, and check that the plot looks something like the top panel of CP figure 1.

(b) Now, create the $T \times 4$ matrix of expected returns, given by the fitted values of the four regressions, i.e.

$$Et\left(rx_{t+1}^{(n)}\right) = a + b_{n,1}y_{t}^{(1)} + b_{n,2}f_{t}^{(1-2)} + b_{n,3}f_{t}^{(2-3)} + b_{n,4}f_{t}^{(3-4)} + b_{n,5}f_{t}^{(4-5)}$$
We’re going to do an eigenvalue decomposition of these 4 expected returns, exactly as we did for the stock and bond returns above.

(c) Find the eigenvalue decomposition of the $4 \times 4$ covariance matrix of expected returns, and display the square roots of the eigenvalues (standard deviations of the factors) and the loadings (columns of $Q$) as you did above. Do you see the same or different patterns in expected returns that you saw in returns or yields?

(d) You will see that the data strongly suggest a one factor model. Make a time-series graph comparing the fit of a one-factor model with the actual expected returns. (You won’t see much in the overall sample. My plot goes from 1990 to 2004).

4. Let’s modify the basic term structure model, and see if we can account for Fama-Bliss regressions. All I do is add one extra term, so the model is

$$x_{t+1} - \delta = \rho(x_t - \delta) + \varepsilon_{t+1}$$

$$\log m_{t+1} = -x_t - \frac{1}{2} (\lambda_0 + \lambda_1 x_t)^2 \sigma^2_t - (\lambda_0 + \lambda_1 x_t) \varepsilon_{t+1}$$

(a) Find $p_{t}^{(1)}, p_{t}^{(2)}$, hence $y_{t}^{(1)}, f_{t}^{(2)}, rx_{t+1}^{(2)}, E_t rx_{t+1}^{(2)}$ in this model. Hint: they are still linear functions (stuff)$+ (stuff) x_t$! (Yes, you could simply find these as instances of general cases in the notes, but derive them on your own, as you might have to do on a test.)

(b) Find the predicted value of the Fama-Bliss coefficients, i.e. write $E_t rx_{t+1}^{(2)} = (\cdot) + (\cdot)f_{t}^{(2)} - y_{t}^{(1)}$. Forget the mess in the constant. Can we find $\lambda_0, \lambda_1$ so that this model captures the Fama-Bliss slope coefficient of approximately 1?

(c) Does this seem like a satisfactory model to capture Fama-Bliss’ regressions, or would you want to modify it in some way?

5. You can invest in a stock which currently has price $100. It will either go up to $130 or down to $90, with probability 1/2 of each event. (Call the two states $u$ and $d$.) You can also invest in a bond, which pays zero interest—A $100 investment gives $100 for sure.

(a) Find a discount factor $m_{t+1}$ that prices stock and bond.

(b) A one-period investor with log utility $u(W_{t+1}) = \ln(W_{t+1})$ has initial wealth $100. Find this investor’s optimal allocation to the stock and bond. Hint: first find optimal wealth in the two states tomorrow. Then figure out how to obtain this optimal wealth by investing in the $h$ shares of stock and $k$ bonds.

6. Last week, you found the actual discount factor / contingent claim price vector from a set of option data. This week, find and plot a power utility investor’s optimal payoff as a function of index value, using these empirical contingent claim prices. I’m looking for plots of payoff vs. index as presented in lecture. I plotted the result for $\gamma = 1$ and $\gamma = 5$. Include the optimal payoff assuming the Black-Scholes discount factor and reasonable coefficients. This should be an easy problem. That’s the point – I can’t begin to think of how to solve it using a portfolio approach.

7. Suppose a quadratic utility investor with a one-year horizon (no intermediate consumption) wakes up in the lognormal iid world.

(a) Mirroring what we did with power utility, find the return on his optimal portfolio in terms of a discount factor $m$ and its moments. Using the definition $\frac{\sigma^2}{2m^2} - 1 = \frac{1}{\gamma}$, you can express
this return \( \hat{R} = \frac{\hat{x}}{W} \) in terms of the local risk aversion at initial wealth, without \( c_b \) or \( W \). We are looking here for the analogue to

\[
\hat{R} = \frac{\hat{x}}{W} = \frac{m^{-\frac{1}{\gamma}}}{E(m^{1-\frac{1}{\gamma}})}
\]

(b) Adding the lognormal iid assumption, i.e. \( dS/S = \mu dt + \sigma dz \), \( dB/B = r dt \), find the return on the investor’s portfolio as a function of the stock return. We are looking here for the analogue to

\[
\hat{R} = e^{(1-\alpha)(r + \frac{1}{2}\sigma^2)} R_T^\alpha
\]

where \( R_T = S_T/S_0 \) denotes the stock return, and

\[
\alpha \equiv \frac{1}{\gamma} \frac{\mu - r}{\sigma^2}.
\]

(c) For \( \gamma = 1, 3.125, 5, 20 \) and \( \mu = 0.09, \sigma = 0.16, r = 0.01 \), make a plot to compare the function \( \hat{R} = \ldots R_T \) in the log and power utility cases. How good is the quadratic as an approximation to power in the range \( R_T = 1 \pm 2\sigma \), where the stock is most likely to end up? How good is the quadratic as an approximation to power in the full range, and in particular for describing demands for out of the money options?

(d) Now, solve the portfolio weight problem for the quadratic utility investor by dynamic programming, mirroring what we did with power utility.

(e) To solve the Bellman equation, you can either guess a quadratic form \( V(W, t) = -\frac{1}{2}e^{2\eta(T-t)} (e^{-r(T-t)} c_b - W)^2 \) and solve for \( \eta \). There is a better way however. Since you have solved for the optimal payoffs above, you know the distribution of \( W_T = \hat{x}_T \), so you can find the value function directly. Thus, first find the value function from the above solution, then use this as a guess, i.e. verify that it solves the Bellman equation. (This is a great idea, now that I think of it, and offers a constructive way to find value functions for difficult portfolio problems.)

(f) Simulate the stock process using a daily interval. For each value of risk aversion, plot the resulting wealth process for the power and quadratic utility investor, starting at \( W_0 = 1 \). For each value of risk aversion, also plot the optimal weight in the stock over time for the quadratic and power utility investor. (The power utility investor puts a constant weight in the risky asset, so that one is a horizontal line, but the quadratic utility investor’s allocation to the risky asset varies over time.) Again, comment on the dimensions for which the quadratic and power solutions seem similar for a given initial risk aversion, and the dimensions for which the solutions seem quite different.