Problem Set 2
Due in class, week 3

In this problem, you’ll explore VAR representations, such as Figure 20.4 and 20.5 in Asset Pricing, and their connection to univariate representations.

Rethinking it, the move to expressing in terms of dividend growth may not have been such a good idea. It’s simpler for the base case, but the intuition is really about the x, r relationship.

1. Start by replicating the VAR I presented in class:

   (a) Fit a restricted VAR to your value-weighted return data, i.e.

   \[
   \Delta d_{t+1} = 0 \times (d_t - p_t) + \varepsilon_{t+1}^d \\
   r_{t+1} = b_r \times (d_t - p_t) + \varepsilon_{t+1}^r \\
   d_{t+1} - p_{t+1} = \phi \times (d_t - p_t) + \varepsilon_{t+1}^{dp} \\
   \text{cov}(\varepsilon^d, \varepsilon^{dp}) = 0.
   \]

   Make sure your numbers for \(b_r, \phi, \rho, \) and the covariance matrix of \(\varepsilon\) obey identities that flow from \(r_{t+1} = -\rho \times (d_{t+1} - p_{t+1}) + (d_t - p_t) + \Delta d_{t+1}!\) Display your \(b_r, \phi, \rho, \) and standard deviation-correlation matrix of \(\varepsilon.\) (We’re imposing restrictions now because I want to do some simulations, “what if the world looks like this,” and to contrast it with an unrestricted VAR below.)

   (b) Plot responses to an “expected return” shock and to an “expected cashflow” shock. Include responses of \(p_t, d_t, r_t, \sum r_t, dp_t\) as in class.

2. We saw one strongly mean-reverting component of prices in the impulse-response functions, and one “permanent” shock. Surely, one would think, a return shock alone would combine the two and show some mean reversion? Let’s ask about the univariate return representation, i.e. what do you learn about the future if you only observe a big return, but you don’t know what happened separately to prices and dividends. (This is, I hope, a much clearer and easier version of the section of the book that contrasts univariate and multivariate mean-reversion.)

   First, consider the general question: suppose returns follow the “latent variable” process

   \[
   x_t = \phi x_{t-1} + \varepsilon_t^x \\
   r_{t+1} = x_t + \varepsilon_{t+1}^r
   \] (1)

   (a) Show that the univariate return representation implied by (1) is an ARMA(1,1),

   \[
   (1 - \phi L) r_{t+1} = (1 - \theta L) v_{t+1} \\
   v_{t+1} = r_{t+1} - E(r_{t+1} | r_t, r_{t-1}...)
   \]

   You do not have to do any spectral stuff as in Asset Pricing. Just apply \((1 - \phi L)\) to \(r_{t+1}.\) (i.e. make a lucky guess that the AR root is the same in univariate and multivariate representations.) You’ll then be able to show that an ARMA(1,1) representation exists. Hint: The defining feature of a time-series process is the autocovariance function. Thus, if you can write down a univariate representation that captures the autocovariance function generated from a multivariate representation, you have the univariate Wold representation in hand.
(b) Sketch the response of returns $r_{t+j}$ and cumulative returns $\sum r_{t+j}$ to a shock $v_t$ for the case $\phi > \theta, \phi = \theta$, and $\phi < \theta$. What is the limiting value of this response function? (The point here is to gain some familiarity with the ARMA(1,1) process, which crops up all over the place in macro and finance.)

Note: The latent variable model (1) is very common in finance. We often think that agents see $x_t$, but we don’t and have to infer $x_t$ from other data. Here, we’ve asked what the implications for $r_t$ alone are.

The corresponding univariate ARMA(1,1) process $(1 - \phi L)r_t = (1 - \theta L)v_t$ with $\phi \approx \theta$ is very important in macroeconomics and finance. It is a good example in which “short run” dynamics are quite different from “long-run” dynamics, so it’s a good example for the dangers of inferring long run properties of a series with short-run methods.

As you’ve seen, dividend yields allow you to see $x_t$ directly with a huge difference in what we as econometricians can predict.

3. Obviously, we have a case of this sort, with $x_t = b_t(d_t - p_t)$. Find $\theta$ for the VAR of question 1. To do this, you have to match the variance and autocovariance of $(1 - \phi L)r_t$ across multivariate and univariate representations. You’ll end up with a pretty formula that should suggest $\theta$ is near $\phi$, of the form $1 + \phi^2 + \text{(small number)} = \theta + \theta^2$ but solving for $\theta$ means solving an ugly quadratic which you’ll have to do numerically.

(a) Find $\theta$ in the form

$$\frac{\phi + \text{(small number)}}{1 + \phi^2 + \text{(small number)}} = \frac{\theta}{1 + \theta^2}$$

(If you can find a prettier expression, let me know! This is the best I could do.) Hint: I found this easiest by substituting $\varepsilon^r = -\rho \varepsilon^{dp} + \varepsilon^d$ before taking variances and imposing $\sigma(\varepsilon^{dp}, \varepsilon^d) = 0$ rather than substituting later for a general case allowing correlation $\sigma(\varepsilon^r, \varepsilon^{dp})$ or $\sigma(\varepsilon^d, \varepsilon^{dp})$. You’re welcome to explore the general case allowing correlation, but it’s not as pretty and not required here.

(b) Using numerical values from the VAR above, find $\theta$ numerically.

(c) Plot the response of returns and cumulative returns to the univariate return shock, and compare the results to the responses of the two components of the multivariate return shock you just plotted.

(d) Is it possible for the univariate representation of the VAR to be a pure random walk? Verify the limit $\theta = \phi$ when $\sigma_{dp}^2 = 0$. This isn’t very useful; it just says that when the variance of expected returns is zero, returns are a random walk. Show that we also have $\theta = \phi$ when $\rho = \phi$. Is this plausible? Is it unlikely that our data is drawn from a world in which $\phi = \rho = 0.96$ or greater?

(e) We seem to have a puzzle. In the VAR, the response to a positive return shock $\varepsilon^r = 1, \varepsilon^d = 1, \varepsilon^{dp} = 0$ gives rise to a purely transitory response, while the response to a positive return shock $\varepsilon^r = 1, \varepsilon^d = 0, \varepsilon^{dp} = -1/\rho$ gives rise to a mean-reverting response. Yet here, the response to a positive return shock all on its own, which averages across these two possibilities, gives rise to a purely transitory response. Worse, if $\phi > \rho$, a large return forecasts positive future returns, which neither underlying shock had. (Check yourself if you want, but not necessary.) How is this possible? (No math needed, just point out the central mechanism/logical issue.)

(f) Related, how is it possible that we do not see the wide swings in expected return by looking at the history of returns? Surely when $dp_t$ takes one of its generation-long positive excursions raising $E_t(r_{t+1})$ we should see a string of positive actual returns, so the history
of returns alone should say something. Yet, in this example, though $E(r_{t+1}|dp_t)$ varies over time, $E(r_{t+1}|r_t, r_{t-1}, ...)$ (the ARMA(1,1) forecast) is always exactly the same – we cannot see that we’re in a period of high expected returns, even one that lasts decades, by noting high ex-post returns.