Assignment 1 answers

1. a) In the SW regime, we have real debt and a money demand equation. We consider the usual perfect foresight model with a one time unanticipated shock at time 1. Thus the equations are

\[ M_t v = P_t y \]

\[ v_t = \sum_{j=0}^{\infty} \frac{1}{R^j} \left( s_{t+j} + \frac{M_{t+j} - M_{t+j-1}}{P_{t+j}} \right) \]

The Fed sets the price level each period, and so seems to have great monetary control. But it must do so respecting the present value budget constraint (it is a constraint, since this is real debt, and we’re ruling out default.) We can substitute out for \( \alpha \) and look directly at prices,

\[ b_t = \sum_{j=0}^{\infty} \frac{1}{R^j} \left( s_{t+j} + \frac{P_{t+j} - P_{t+j-1} y}{P_{t+j} v} \right) \]

We start and return to a steady state. I calibrate my economy to \( R = 1.05, b = 0.5, P = 1, M = 1/6, v = 6, y = 1 \) With no steady state growth, that means \( b = \frac{1}{1-R}s = \frac{R}{R-1}s \) and hence \( s = \frac{R-1}{R}b = \frac{0.05}{1.05}0.5 = 0.0238 \).

At time \( t \) we unexpectedly get bad news that \( s \) will be lower by 10%. The left hand side does not change after the shock, so, taking the difference from steady state, the right hand side terms must offset. The seigniorage terms must increase (from zero) to

\[ \sum_{j=0}^{\infty} \frac{1}{R^j} \left( \frac{P_{t+j} - P_{t+j-1} y}{P_{t+j} v} \right) = 0.1 \frac{R}{R-1}s \]

The Fed can choose any history of prices with this property. I suggested a one time increase at time 1 and a one time increase at time 10. You can try others! If at time 1, the price rise must be enough to raise enough revenue to offset the decline in the present value of the surplus. With \( P_{t-1} = P \),

\[ \frac{P_t - P y}{P_t v} = 0.1 \frac{R}{R-1}s \]

\[ 1 - \frac{P}{P_t} = 0.1 \frac{R}{R-1}sv \]

\[ P_t = \frac{P}{1 - 0.1 \frac{R}{R-1}sv} \]

If the Fed raises all the necessary revenue at time 10,

\[ \frac{1}{R^{10}} \left( \frac{P_{t+10} - P y}{P_{t+10} v} \right) = 0.1 \frac{R}{R-1}s \]

\[ P_{10} = \frac{P}{1 - R^{10}0.1 \frac{R}{R-1}sv} \]
We need to plot histories, and debt in particular. Real debt evolves as
\[
b_t = R\left(b_{t-1} - \frac{M_t - M_{t-1}}{p_t} - s_t\right) \\
= R\left(b_{t-1} - \frac{P_t - P_{t-1}y}{P_t} - s_t\right)
\]

The top graphs in my figure below show the results. I scaled each series by its steady state value so they would fit on the same plot. You can see that if the Fed chooses “inflation now” the price level goes up to about 1.4. Debt (V in the graph) declines. The one shot of seigniorage has to buy back just enough of the outstanding debt so that the new lower surplus is just equal to the interest cost of carrying the new, lower stock of debt forever. If the Fed chooses “inflation later” it has to raise the price level to 2. During the 10 year delay, the stock of debt got bigger, as there were fewer surpluses. Now, a bigger amount of seigniorage is needed to reduce the larger stock of debt to the new steady state.

b) With nominal debt, the government valuation equation is
\[
\frac{B_{t-1}(t)}{P_t} = \sum_{j=0}^{\infty} \frac{1}{R^j} \left(s_{t+j} + \frac{P_{t+j} - P_{t+j-1}y}{P_{t+j}}\right)
\]

Again, \(Mv = py\) means that the Fed controls the price level. However, there’s a new channel. If it changes \(P_t\) it can also change the left hand side of the equation directly. Again looking at the difference relative to the steady state,
\[
\frac{B_{t-1}}{P_t} - \frac{B_{t-1}}{P} = -0.1 \frac{R}{R - 1} s + \sum_{j=0}^{\infty} \frac{1}{R^j} \left(\frac{P_{t+j} - P_{t+j-1}y}{P_{t+j}}\right)
\]

If the Fed does it all in period 1,
\[
\frac{B_{t-1}}{P_t} - \frac{B_{t-1}}{P} = -0.1 \frac{R}{R - 1} s + P_t - P \frac{y}{v}
\]
\[
\left(\frac{y + B}{v} + \frac{P}{P_t}\right) = -0.1 \frac{R}{R - 1} s + \frac{y}{v} + \frac{B}{P}
\]
\[
P_t = P_t - p_t = \frac{y}{v} + \frac{B}{P}
\]
\[
P_t = P_t - p_t = \frac{1}{0.1 \frac{R}{R - 1} s}
\]
\[
P_t = P_t - p_t = \frac{1}{0.1 \frac{R}{R - 1} y}
\]
\[
P_t = P_t - p_t = \frac{1}{0.1 \frac{M}{M + B} \frac{R}{R - 1} y}
\]
Comparing this to the previous answer,
\[ P_t = \frac{P}{1 - 0.1 \frac{R^s}{R^y}} \]
you see that a large real stock nominal debt makes the term in the denominator smaller, and thus mitigates the required inflation. If there is no nominal debt outstanding, the equations are the same. The Fed can get away with a lot less inflation now, since inflating not only raises seigniorage, but “defaults” on the outstanding nominal debt. As you can see in the picture, we now get much smaller inflation – a price level jump to 1.09 instead of about 1.4! The debt decline is the same.

How is this possible? In the SW regime, the initial debt was paid back and then reduced. In this regime, the initial debt was defaulted on! Either way gets you less debt. (Note I plotted the real value of the debt here.)

I plotted the stock of debt as before, except we have to account for the “default” in period 1. Thus, in period 1 reset the debt to its new value and then track it forward as before. Debt evolves as

\[
\begin{align*}
B_{t-1} + M_{t-1} &= p_t s_t + M_t + Q_t B_t \\
B_{t-1} + M_{t-1} &= p_t s_t + M_t + \frac{1}{R P_{t+1}} B_t \\
\frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} &= s_t + \frac{M_t}{P_t} + \frac{1}{R P_{t+1}} B_t \\
\frac{B_t}{P_{t+1}} &= R \left( \frac{B_{t-1}}{P_t} - \frac{M_t - M_{t-1}}{P_t} - s_t \right)
\end{align*}
\]

If the Fed chooses inflation in period 10 however, there is no difference. We get the same result as before. What’s going on? The Fed can unexpectedly inflate, and unexpectedly “default” on outstanding nominal debt in period \( t \), when the shock happens. If it does not do so, then people holding bonds at period \( t \) get paid, and they are gone. The Fed cannot plan to “unexpectedly” default in period 10 after the shock is known. Put another way, once the shock is known, the government cannot sell bonds at full value. People in period 9 know that the government will inflate in period 10, so the government does not raise revenue in period 9 that it can then “default away.” The government can always “unexpectedly default” – this is the core of the time-consistency problem – but it can’t plan to unexpectedly default in period 10 in response to a period 1 shock. (Long term debt changes this picture. If there is any debt outstanding in period 10 that was issued before the shock, the plan to inflate in period 10 will give a shock devaluation to these bonds in period 1, and that will help reduce the seigniorage the government must raise.)

In short, the existence of outstanding debt gives another channel with which the government can respond to the surplus decline. However, it has to act quickly; once that debt is rolled over, the opportunity is gone. The tradeoff for “inflation later” vs. “inflation now” is much worse with nominal bonds outstanding.

An application worth thinking of is exchange rates. Governments seem to devalue all of a sudden, rather than reacting to bad shocks by a schedule of gradual future devaluations.
(which means inflation). By doing so, they effectively default on short-term domestically-denominated debt held by foreigners, which they could not do with a slow and widely expected devaluation.

The picture: Notice the difference in vertical scale between the graphs!

![Graphs showing different inflation scenarios for SW and nominal debt regimes.](image)

2. a) Real value of debt

\[ b_t = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} = \frac{s_t}{1 - \rho \beta} \]  

(3)

b) The answer is, you can’t reconcile these. Try it. Find \( b_{t+1} \) from the identity, and see if you can get it to equal \( s_{t+1}/(1 - \rho \beta) \).

\[ b_{t+1} = R(b_t - s_t) = R \left( \frac{s_t}{1 - \rho \beta} - R s_t \right) \]

\[ = R \left( \frac{1}{1 - \rho \beta} - 1 \right) s_t = R \left( \frac{\rho \beta}{1 - \rho \beta} \right) s_t \]

Now you see the problem. This is in the time \( t \) information set. There is just no way this can equal a function of \( s_{t+1} \) which is in the time \( t+1 \) information set. The answer is, with real debt or no price level shocks, the real value of debt is predetermined. Then the surplus process cannot follow an AR(1). Shocks to surpluses at \( t \), which imply increases in the real value of debt from \( t \) to \( t+1 \), must come with a negative shock to future surpluses to pay off that increased debt. Thus, for real debt, we must have

\[ (E_t - E_{t-1}) \sum_{j=0}^{\infty} \beta^j s_{t+j} = 0. \]

This follows directly from (3) by taking innovations and recognizing that \( b_t \) is in the time \( t-1 \) information set.
c)

\[ s_{t+1}^c = \rho s_t^c + \varepsilon_{t+1}^c \]
\[ s_{t+1}^l = \phi s_t^l + \varepsilon_{t+1}^l \]
\[ s_t = s_t^c + s_t^l \]

The condition we need for this to work is that real debt is predetermined – \( E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} \) must be in the time \( t-1 \) information set.

\[
\begin{align*}
E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} &= \frac{s_t^l}{1 - \phi \beta} + \frac{s_t^c}{1 - \rho \beta} \\
(E_t - E_{t-1}) \sum_{j=0}^{\infty} \beta^j s_{t+j} &= \frac{\varepsilon_t^l}{1 - \phi \beta} + \frac{\varepsilon_t^c}{1 - \rho \beta} = 0 \\
\varepsilon_t^l &= -\frac{1 - \phi \beta}{1 - \rho \beta} \varepsilon_t^c
\end{align*}
\]

For debt to be predetermined, we need to have the permanent component offset the transitory. As explained, we can think of this as a choice by the government, that satisfies the Ricardian proposition that debt be paid off. If they’re going to run a deficit today, they need to promise higher future taxes to eventually pay off that debt!

To check,

\[
\begin{align*}
b_t &= E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} = \frac{s_t^l}{1 - \phi \beta} + \frac{s_t^c}{1 - \rho \beta} \\
b_{t+1} &= R(b_t - s_t) = \frac{1}{\beta} \left( \frac{s_t^l}{1 - \phi \beta} + \frac{s_t^c}{1 - \rho \beta} - s_t - s_t \right) \\
&= \frac{1}{\beta} \left( \frac{\phi \beta s_t^l}{1 - \phi \beta} + \frac{\rho \beta s_t^c}{1 - \rho \beta} \right) \\
&= \phi s_t^l + \frac{\rho s_t^c}{1 - \rho \beta} \\
&= \frac{s_{t+1}^l - \varepsilon_{t+1}^l}{1 - \phi \beta} + \frac{s_{t+1}^c - \varepsilon_{t+1}^c}{1 - \rho \beta} \\
&= \frac{s_{t+1}^l}{1 - \phi \beta} + \frac{s_{t+1}^c}{1 - \rho \beta}
\end{align*}
\]

Here’s my simulation. You see how periods of surplus pay down the debt and vice versa
d) 

\[ s_{t+1}^c = \rho s_t^c + \varepsilon_{t+1}^c \]

\[ s_{t+1}^d = s_t^d + \varepsilon_{t+1}^d \]

\[ s_t = \frac{1}{1 - \phi \mu_L} \varepsilon_t^d + \frac{1}{1 - \rho L} \varepsilon_t^c \]

\[ s_t = \left( \frac{1 - \mu_L}{1 - \phi \mu_L} \right) \left( \frac{1 - \phi^L}{1 - \rho \mu_L} \right) \varepsilon_t \]

\[ s_t = \left( \frac{1 - \phi L}{1 - \rho L} \right) \left( \frac{1 - \beta^L}{1 - \phi L} \right) \varepsilon_t^c \]

\[ s_t = \left( \frac{1 - \phi L}{1 - \rho L} \right) \left( \frac{1 - \beta^L}{1 - \phi L} \right) \varepsilon_t^c \]

\[ (1 - \rho L) (1 - \phi L) s_t = \beta \left( \frac{\phi - \rho}{1 - \rho \beta} \right) (1 - \beta^{-1} L) \varepsilon_t^c \]

This is not the Wold representation. The Wold needs to have an invertible MA root, so it’s

\[ (1 - \rho L) (1 - \phi L) s_t = (1 - \beta L) v_t \]

The plot:
You can see that the “structural” response a big deficit today is paid for by a long string of surpluses. In the univariate response, it’s like the AR(1). The deficit is apparently never paid off. Thus the project of fitting a univariate time series model to $s_t$ and then taking present values to match to the value of the debt is fundamentally flawed. It can’t work.

e) Response of PV  The sum of the MA coefficients is

$$s_t = A(L)\varepsilon^e_t$$

$$\sum \beta^j A_j = A(\beta)$$

For the structural representation

$$A(\beta) = \left( \frac{\phi - \rho}{1 - \rho \beta} \right) \left( \frac{\beta - L}{(1 - \rho L) (1 - \phi L)} \right) |_{L = \beta}$$

$$= \left( \frac{\phi - \rho}{1 - \rho \beta} \right) \left( \frac{\beta - \beta}{(1 - \rho \beta) (1 - \phi \beta)} \right) = 0$$

Thus there is no response whatsoever in the impact period. Well, we constructed it that way! This representation retains the property

$$(E_t - E_{t-1}) \sum \beta^j s_{t+j} = 0$$

For the Wold representation

$$s_t = \frac{(1 - \beta L)}{(1 - \rho L) (1 - \phi L)} v_t$$

we have

$$A(\beta) = \frac{(1 - \beta^2)}{(1 - \rho \beta) (1 - \phi \beta)} > 0$$

As you can see the univariate representation misses the key restriction! It behaves like the AR(1). This representation loses the property

$$(E_t - E_{t-1}) \sum \beta^j s_{t+j} = 0$$
f) This is pretty devastating. No matter how well you estimate the univariate surplus project, the whole idea of fitting a univariate surplus model and then calculating sums $E_t \sum \beta^j s_{t+j}$ falls apart. If you compare to real data you’re going to get the wrong $b$. If you compare to nominal data, you’re going to predict price-level shocks that aren’t there.

The answer is the real value of debt will help to predict the surplus, to us econometricians who can’t see real shocks. Even though the surplus is completely autonomous here, and not “causally responding” to the value of the debt, $\{b_t, s_t\}$ do, it turns out, reveal the structural shocks, in a way that $\{s_t\}$ alone does not. But I’ll leave “how to do it right” for another day.

None of this is a technical problem with nominal debt. Of course, to make sense we don’t want to have a negative price level, so let’s add a constant,

$$s_{t+1} = \bar{s} + \rho(s_t - \bar{s}) + \varepsilon_{t+1}$$

$$\frac{B_{t-1}(t)}{P_t} = E_t \sum \beta^j s_{t+j} = \frac{\bar{s}}{1 - \beta} + \frac{s_t - \bar{s}}{1 - \rho \beta}$$

The fact that debt is predetermined means we must have price level shocks,

$$B_{t-1}(t) (E_t - E_{t-1}) \frac{1}{P_t} = E_t - E_{t-1} \sum \beta^j s_{t+j} = \frac{\varepsilon_t}{1 - \rho \beta}$$

The previous solution is the case $P = \text{constant}$.

That’s ok, but the price level shocks are “too big.” We still have a prediction that the real value of the debt, measured including the price level shocks, is real value $= B_{t-1}(t)/P_t$, and that should move in perfect tandem with $s_t$. The real value of US nominal debt does nothing of the sort. We’re much closer to the real debt case with little inflation. And for good reasons! Once we add any price stickiness, the government is going to try hard to avoid price level shocks. In this case, the government will do that, even with nominal debt, by trying to offset cyclical $s$ shocks with long-run promises. When that fails, as we’ll see next week, it can use long term debt to smooth.

In sum, while it’s technically possible to use an AR(1), if you want to match the data you need to start with something much more like the two-part model we thought about for real debt (no price level shocks).

Whew! As you see, specifying a sensible surplus process and a procedure for thinking about the data is much more subtle than you might have thought!

Extra I didn’t ask for this, but it’s interesting to compute the present value of this univariate representation and check we get the right answer, and show what answer you would get from the Wold representation. If you don’t know these tricks, they’re useful.

$$s_t = \beta \left( \frac{\phi - \rho}{1 - \rho \beta} \right) \frac{(1 - \beta^{-1} L)}{(1 - \rho L)(1 - \phi L)} \varepsilon_t$$

$$E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} = E_t \left( \frac{1}{1 - \beta L^{-1} s_t} \right)$$
Digression: given $s_t = A(L)\varepsilon_t$ how do you calculate $E_t \left( \frac{1}{1 - \beta L^{-1}} s_t \right)$? There is a very cool Hansen-Sargent formula that answers this question

$$E_t \left( \frac{1}{1 - \beta L^{-1}} s_t \right) = \frac{A(L) - \beta L^{-1}A(\beta)}{1 - \beta L^{-1}} \varepsilon_t$$

$E_t$ of a moving average just zeros out the future errors. The $-\beta L^{-1}A(\beta)$ term neatly gets rid of those. You can see this just by expanding

$$\frac{A(L)}{1 - \beta L^{-1}} = a_0 + a_1 L + a_2 L^2 + ...$$

$$+ \beta a_0 L^{-1} + \beta a_1 + \beta a_2 L + \beta a_3 L^2 + ...$$

$$+ \beta^2 a_0 L^{-2} + \beta^2 a_1 L^{-1} + \beta^2 a_2 + \beta^2 a_3 L + \beta^2 a_4 L^2 + ...$$

If you collect the future terms vertically, you’ll see they are

$$\beta^3 A(\beta) L^{-3} + \beta^2 A(\beta) L^{-2} + \beta A(\beta) L^{-1} = \frac{\beta L^{-1}}{1 - \beta L^{-1}} A(\beta)$$

Subtracting those off, you just have the terms with current and past errors, which is what we want. Ok, back to work,

$$E_t \left( \frac{1}{1 - \beta L^{-1}} s_t \right) = \beta \left( \frac{\varphi - \rho}{1 - \rho \beta} \right) \frac{1}{1 - \beta L^{-1}} \left[ \frac{(1 - \beta^{-1} L)}{(1 - \rho L)(1 - \phi L)} - \beta L^{-1} \frac{(1 - \beta^{-1} \beta)}{(1 - \rho \beta)(1 - \phi \beta)} \right] \varepsilon_t$$

Notice, the $A(\beta)$ term was already zero – we already saw this. Thus, there were no future terms to begin with

$$E_t \left( \frac{1}{1 - \beta L^{-1}} s_t \right) = \beta \left( \frac{\varphi - \rho}{1 - \rho \beta} \right) \frac{L}{1 - \beta} \left[ \frac{(\beta - L)}{(1 - \rho L)(1 - \phi L)} \right] \varepsilon_t$$

The $L$ in the numerator shows that debt does not respond to $\varepsilon_t$, as we knew it should not do. In sum, we get

$$b_t = - \left( \frac{\varphi - \rho}{1 - \rho \beta} \right) \frac{L}{(1 - \rho L)(1 - \phi L)} \varepsilon_t$$

With

$$s_t = \beta \left( \frac{\varphi - \rho}{1 - \rho \beta} \right) \frac{(1 - \beta^{-1} L)}{(1 - \rho L)(1 - \phi L)} \varepsilon_t$$

you can see transparently that

$$b_{t+1} - Rb_t = (1 - \frac{1}{\beta} L)b_{t+1} = -(1 - \beta^{-1} L) \left( \frac{\varphi - \rho}{1 - \rho \beta} \right) \frac{1}{(1 - \rho L)(1 - \phi L)} \varepsilon_t = s_t$$

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This does not work for the Wold representation

\[ s_t = \frac{(1 - \beta L)}{(1 - \rho L)(1 - \phi L)} v_t \]

\[
E_t \left( \frac{1}{1 - \beta L^{-1} s_t} \right) = \frac{1}{1 - \beta L^{-1}} \left[ \frac{(1 - \beta L)}{(1 - \rho L)(1 - \phi L)} - \frac{\beta L^{-1} (1 - \beta^2)}{(1 - \rho \beta)(1 - \phi \beta)} \right] v_t
\]

\[
= \frac{1}{1 - \beta L^{-1}} \left[ \frac{(1 - \beta L)(1 - \rho \beta)(1 - \phi \beta) - \beta L^{-1} (1 - \beta^2)(1 - \rho L)(1 - \phi L)}{(1 - \rho L)(1 - \phi L)(1 - \rho \beta)(1 - \phi \beta)} \right] v_t
\]

\[
= \frac{1}{1 - \beta L^{-1}} \left[ \frac{(1 - \beta L^{-1})(1 - \beta^2) - \beta (1 - \beta (\phi + \rho) + \phi \rho) L}{(1 - \rho L)(1 - \phi L)(1 - \rho \beta)(1 - \phi \beta)} \right] v_t
\]

\[
= \frac{(1 - \beta^2)}{(1 - \rho \beta)(1 - \phi \beta)} \left[ 1 - \frac{\beta(1 - \beta (\phi + \rho) + \phi \rho) L}{1 - \beta^2} \right] v_t
\]

I don’t get much out of this except to note that the present value does depend on \( v_t \) and hence is not predetermined based on the information set of past \( s \) only.

3. a) Before, we have \( c/P = s \), or \( P = c/s \). This remains true afterward for all but \( t = 0 \) and \( t = T \). At time \( T \), we have \( (c + B)/P_T = s \) or

\[
\frac{c + B}{P_T} = s
\]

At time 0 we have

\[
c - BQ_0^{(T)} = P_0 s
\]

\[
c = P_0 s + B E_0 \beta^T \frac{P_0}{P_T}
\]

\[
\frac{c}{P_0} = s + B E_0 \beta^T \left( \frac{s}{c + B} \right)
\]

\[
\frac{c}{P_0} = s + \frac{B}{c + B} \beta^T s
\]

Here is the price path – the government shifted the price level from period 2 to period 7.
b) At any time we have

\[ Q_t^{(j)} = E_t \beta^j \frac{P_t}{P_{t+j}} \]

\[ Y_t^{(j)} = \left[ Q_t^{(j)} \right]^{-\frac{1}{\beta}} \]

\[ y_t^{(j)} = Y_t^{(j)} - 1; \]

For times that don’t intersect 0, T this is just \( \beta^j \) with yield \( R \).

\[ Y_t^{(j)} = \left[ Q_t^{(j)} \right]^{-\frac{1}{\beta}} = 1/\beta = R \]

the others are pretty obvious, I just did it numerically.

In the period of the shock - 1 here – the current price level is low, so you see the generally high nominal interest rates reflecting inflation when we return to the usual price level. The blip in 1-6 yield curve reflects the fact that current price levels are 1, but each expects the big one period price level jump. Then in the jump period, we expect deflation when the jump is over.

c) You know the answer – at any date, total real value of debt is unaffected,

\[ \sum \beta^j s_{t+j} = \frac{1}{1-\beta} s = \frac{R}{R-1} s = \frac{1+r}{r} s \]

The value of the perpetuity is just \( \left( \sum_{j=0}^{\infty} Q_t^{(j)} c \right)/P_t \). The value of the debt due at \( T \) is just \( Q_t^{(T-t)} B/P_t \). Note we don’t include it during the period of the shock. This is good practice
to get the timing straight.