Assignment 2 Answers

1. The model is

\[ i_t = E_t \pi_{t+1} \]
\[ i_t = \phi_\pi \pi_t + x_t \]
\[ x_t = \rho x_{t-1} + \varepsilon_t \]

With \( \phi_\pi > 1 \) we find the forward-looking solution

\[ \phi_\pi \pi_t + x_t = E_{t} \pi_{t+1} \]
\[ \pi_t = -E_t \sum_{j=0}^{\infty} \phi_\pi^{-j} x_{t+j} \]
\[ = -\sum_{j=0}^{\infty} \phi_\pi^{-j} \rho^j x_t \]
\[ = -\frac{\phi_\pi^{-1}}{1 - \rho \phi_\pi^{-1}} x_t \]
\[ = -\frac{1}{\phi_\pi - \rho} x_t \]

and interest rates follow.

\[ i_t = -\frac{\rho}{\phi_\pi - \rho} x_t \]

In the “fiscal solution” we have

\[ \phi_\pi \pi_t + x_t = E_t \pi_{t+1} \]
\[ (E_t - E_{t-1}) \pi_t = \delta_t = 0 \]

(A full model solution would include fiscal shocks as well as monetary policy shocks. Here I am just deriving the response to the monetary policy shock with all fiscal shocks turned off.) Thus, the response function is given by

\[ \pi_{t+1} = \phi_\pi \pi_t + x_t \]
\[ \pi_t = \sum_{j=0}^{\infty} \phi_\pi^j x_{t-j-1} \]

\[ i_t = \phi_\pi \pi_t + x_t \]

(There are several ways to write the same thing.)

Note the fiscal equation removes the indeterminacy of equilibria. However, if \( \phi_\pi > 1 \) we’d have an explosive solution, so the Fed should follow \( \phi_\pi < 1 \). I think that’s good – our Fed makes lots of noises about how they like to stabilize inflation, not threaten hyperinflation if we don’t behave.

Now, let’s plot the response functions. The NK solutions are in blue. Inflation goes down, which initially sounds appealing as a response to a monetary tightening. Note inflation...
goes down right away, in period 1, the same time as the shock. Inflation jumps to a new equilibrium path in order to offset the explosion which would follow if it did not jump.

The really weird thing is that interest rates jump down as well! Wait, this was a positive interest rate shock, what happened? Yes, the shock is positive, but it induced such a big endogenous and contemporaneous decline in inflation that $i_t = \phi \pi_t + x_t$ falls like a stone even though $x_t$ rose. You might ask, how would anyone tell that this was a tightening and not a dramatic loosening, and you would have a good point. ($x_t$ is not observable from $\{i_t, \pi_t\}$ data in this equilibrium).

Now look at the fiscal path. Here interest rates do rise. In fact, (with no fiscal shock), ex-post inflation cannot change, so inflation does not move in the period of the shock. There’s no jumping to another equilibrium path here. This actually sounds pretty reasonable given the data, in which inflation seems pretty sluggish, rather than seeing price level jumps on the same day as FOMC announcements. Then interest rates follow obvious dynamics generated from the shock and inflation.

Now, the fiscal solution gives inflation in response to monetary tightening. Isn’t this bad –aren’t we supposed to see lower inflation in response to monetary tightening, as the NK solution showed (granted that measuring “tightening” might be hard given data from that model)? No, of course not! This is a purely frictionless model! Real rates are constant, and there is no mechanism for real rates to lower “demand!” In a totally frictionless model, all the Fed can do when it raises the nominal rate is to raise expected inflation. So of course raising the nominal rate raises inflation! They mystery solution here is the NK one. Here we have a completely frictionless model, with fixed real rate, output, and super-neutrality, yet somehow raising the nominal rate lowers inflation, instantly? What magic is that? The answer is, the threat of hyperinflation (which is what the rate rise does) induces a jump to
a different equilibrium. The response function is not playing out the dynamics of the model equilibrium, it’s playing out how the model jumps from one equilibrium to another.

Obviously, there’s a lot more to do here, namely to contrast fiscal and NK solutions’ impulse response functions to a wide variety of shocks, and in model structures that are capable of delivering anything like the data. I’ll leave that as this week’s thesis topic suggestion.

2. Now, let’s solve the New-Keynesian model

\[
\begin{align*}
    y_t &= E_t y_{t+1} - \sigma (i_t - E_t \pi_{t+1}) + x_{dt} \\
    \pi_t &= \beta E_t \pi_{t+1} + \gamma y_t + x_{\pi t} \\
    i_t &= \phi_\pi \pi_t + x_{it}
\end{align*}
\]

(I left out \( \bar{y} \), but you can include that in \( x_{\pi t} \))

\[
\begin{bmatrix}
    y_{t+1} \\
    \pi_{t+1} \\
    x_{dt+1} \\
    x_{\pi t+1} \\
    x_{it+1}
\end{bmatrix} = \begin{bmatrix}
    \frac{1}{\beta} (\beta + \sigma) & -\frac{1}{\beta} (1 - \beta \phi_\pi) & -1 & \frac{\sigma}{\beta} & \sigma \\
    -\frac{\gamma}{\beta} & 0 & -\frac{1}{\beta} & 0 & 0 \\
    0 & 0 & \rho_d & 0 & 0 \\
    0 & 0 & 0 & \rho_\pi & 0 \\
    0 & 0 & 0 & 0 & \rho_i
\end{bmatrix}
\begin{bmatrix}
    y_t \\
    \pi_t \\
    x_{dt} \\
    x_{\pi t} \\
    x_{it}
\end{bmatrix} + \begin{bmatrix}
    \delta_{yt+1} \\
    \delta_{\pi t+1} \\
    \varepsilon_{dt+1} \\
    \varepsilon_{\pi t+1} \\
    \varepsilon_{it+1}
\end{bmatrix}
\]

I’ll proceed analytically, using Scientific Workplace’s eigenvector calculations. You’re welcome to go numerical at any point. The eigenvalues are

\[
\lambda = \frac{1}{2\beta} \left( 1 + \beta + \sigma \gamma \pm \sqrt{(1 + \beta + \sigma \gamma)^2 - 4\beta (1 + \sigma \gamma \phi_\pi)} \right), \rho_d, \rho_\pi, \rho_i
\]

Using SW’s eigenvector calculation, the eigenvectors of the unstable eigenvalues are

\[
\begin{bmatrix}
    1 - \beta - \sigma \gamma + \sqrt{(1 + \beta + \sigma \gamma)^2 - 4\beta (1 + \sigma \gamma \phi_\pi)} \\
    2\gamma \\
    0 \\
    0 \\
    0
\end{bmatrix} \leftrightarrow \lambda_-
\]

\[
\begin{bmatrix}
    1 - \beta - \sigma \gamma - \sqrt{(1 + \beta + \sigma \gamma)^2 - 4\beta (1 + \sigma \gamma \phi_\pi)} \\
    2\gamma \\
    0 \\
    0 \\
    0
\end{bmatrix} \leftrightarrow \lambda_+
\]

The eigenvectors of the stable eigenvalues are

\[
\begin{bmatrix}
    1 - \rho_d \beta \\
    (1 - \rho_d) (1 - \beta \rho_d) + \sigma \gamma (\phi_\pi - \rho_d) \\
    0 \\
    0
\end{bmatrix} \leftrightarrow \rho_d
\]
The model dynamics are then

\[
\begin{bmatrix}
\sigma (\rho_\pi - \phi_\pi) \\
1 - \rho_\pi \\
0 \\
(1 - \rho_\pi) (1 - \beta \rho_\pi) + \sigma \gamma (\phi_\pi - \rho_\pi) \\
0 \\
-\sigma (1 - \rho_i \beta) \\
-\sigma \gamma \\
0 \\
0 \\
(1 - \rho_i) (1 - \rho_i \beta) + \sigma \gamma (\phi_\pi - \rho_i)
\end{bmatrix}
\leftrightarrow \rho_\pi
\]

\[
\begin{bmatrix}
-\sigma (1 - \rho_i \beta) \\
-\sigma \gamma \\
0 \\
0 \\
(1 - \rho_i) (1 - \rho_i \beta) + \sigma \gamma (\phi_\pi - \rho_i)
\end{bmatrix}
\leftrightarrow \rho_i
\]

where the \( z \) and the \( x \) are related by

\[
x_{dt} = \left[ (1 - \rho_d) (1 - \rho_d \beta) + \sigma \gamma (\phi_\pi - \rho_d) \right] z_{dt}
\]

\[
x_{xt} = \left[ (1 - \rho_\pi) (1 - \rho_\pi \beta) + \sigma \gamma (\phi_\pi - \rho_\pi) \right] z_{xt}
\]

\[
x_{it} = \left[ (1 - \rho_i) (1 - \rho_i \beta) + \sigma \gamma (\phi_\pi - \rho_i) \right] z_{it}
\]

Similarly the shocks \( v \) are related to fundamental shocks \( x \) by

\[
\varepsilon_{dt} = \left[ (1 - \rho_d) (1 - \rho_d \beta) + \sigma \gamma (\phi_\pi - \rho_d) \right] v_{dt}
\]

\[
\varepsilon_{xt} = \left[ (1 - \rho_\pi) (1 - \rho_\pi \beta) + \sigma \gamma (\phi_\pi - \rho_\pi) \right] v_{xt}
\]

\[
\varepsilon_{it} = \left[ (1 - \rho_i) (1 - \rho_i \beta) + \sigma \gamma (\phi_\pi - \rho_i) \right] v_{it}
\]

It’s interesting to carry along the \( i \) response. From

\[ i_t = \phi_\pi \pi_t + x_{it}, \]

we can simply append the \( i \) to the response variables as

\[
\begin{bmatrix}
1 - \rho_d \beta \\
\gamma \\
\gamma \phi_\pi \\
1 - \rho_\pi \\
1 - \rho_i \beta
\end{bmatrix}
\begin{bmatrix}
\sigma (\rho_\pi - \phi_\pi) \\
\sigma (\rho_\pi - \phi_\pi, 0) \\
-\sigma (1 - \rho_i \beta) \\
-\sigma \gamma \\
-\sigma \gamma \rho_i + (1 - \rho_i) (1 - \rho_i \beta)
\end{bmatrix}
\leftrightarrow \begin{bmatrix}
z_{dt} \\
z_{xt} \\
z_{it}
\end{bmatrix}
\]

For the money impulse response function, we only need the last column. In the end, we just simulate

\[
\begin{bmatrix}
y_t \\
\pi_t \\
i_t
\end{bmatrix}
= \begin{bmatrix}
-\sigma (1 - \rho_i \beta) \\
-\sigma \gamma \\
-\sigma \gamma \rho_i + (1 - \rho_i) (1 - \rho_i \beta)
\end{bmatrix}
\begin{bmatrix}
z_{it}
\end{bmatrix}
\]

\[
z_{it} = \rho z_{it-1} + v_{it}
\]

\[
v_{it} = \frac{\varepsilon_{it}}{(1 - \rho_i) (1 - \rho_i \beta) + \sigma \gamma (\phi_\pi - \rho_i)}
\]
As I read it, this is working very much like the frictionless model. The interest rate and inflation lines are just about the same. The real rate rises slightly, and output falls a bit as a result. Output falls because inflation is lower than future inflation (Phillips curve), and this is why the real rate is higher.

The fiscal solution for your interest. Here again I just picked the equilibrium in which $E_t - E_{t-1} \pi_t = 0$, but changed $\phi_x$ to $<1$.

This change produces radically different responses. Of course inflation cannot now “jump” down during the period of the shock. That’s the whole point. The tightening now produces an actual rise in nominal rates. This is a rise in real rates as well, with expected inflation declining. Output has two influences: the higher real rates want less output today than tomorrow, but higher inflation today than in the future wants a boom today relative to tomorrow. The Phillips curve wins (and you can see how some lags in inflation might help here) I’d judge the inflation response as sensible, the output response as not – we’d expect a general decline in output to match what people think these responses should look like.

The big point though is that the responses are very different. Assuming that fiscal policy validates monetary policy makes a big difference to this model’s predictions for policy actions.

Alas, data can’t tell the responses apart very easily, since we don’t easily observe fiscal shocks directly. We can’t tell how much $E_t \sum \beta^j S_{t+j}$ changed.
Response to monetary tightening without fiscal change — 3 equation model