In this problem set we’ll investigate the parallel determinacy issues with fixed money supply and an interest-elastic money demand.

1. Start with a linear approximation. This makes the algebra easier but you don’t see the deeper economics. Assume the money demand curve is

\[ m_t - p_t = -\alpha i_t \]

where \( i \) is the nominal interest rate, \( m \) and \( p \) are log money demand and the log price level; and assume

\[ i_t = r + E_t \pi_{t+1}. \]

Typical numbers for \( \alpha \) are around 0.1 to 0.2 — a 5-10 percentage point rise in interest rates (big) corresponds to changing log \( M \) by 1, i.e. 100%. A graph on the solutions will show a slope of this order in the data.

a) Find a difference equation for \( p_t \) and \( m_t \). Show that this is an “unstable” equation, and solve forward for \( p_t \) as a function of future \( m_t \). Explain intuitively why future money growth causes a higher price level today.

b) Suppose \( m_t = m \), a constant. Find the steady state (or forward-looking solution as in a) for \( p \). Express possible other solutions as differences \( p_t - p \). Are such differences always zero? Are they “stable” — multiple local equilibria? Are they “unstable” — the steady state is a unique local equilibrium, and the others explode? Can you see any economic argument to rule out multiple equilibria at this stage?

2) Let’s do it right, paralleling the treatment of interest rate rules in class. Model money in the utility function

\[
\max \sum_{t=0}^{\infty} \beta^t u(c_t + v(M_t/P_t)) \\
\text{s.t. } M_{t-1} + B_{t-1} = M_t + Q_t B_t + P_t y_t - P_t c_t - P_t s_t
\]

\( Q_t \) is the one-period nominal bond price \( Q_t \equiv 1/(1+i_t) \). The equilibrium is a constant endowment, \( c_t = y \). Also specialize to \( u(c_t + v(M_t/P_t)) = c_t^{\gamma} + (M_t/P_t)^{1-\gamma} \) when necessary. I used \( r = 0.05, \beta = 1/(1+r) \) and \( \gamma = 5 \) to make plots.

(It may help in places to allow real bonds as well, i.e.

\[
M_{t-1} + B_{t-1} + b_{t-1} = M_t + Q_t B_t + q_t b_t + P_t y_t - P_t c_t - P_t s_t
\]

I want to allow complete markets. Since \( c_t = y \) the allocation will be the same with any market structure, so the “budget constraint” only ends up defining asset prices. If you want to define a real interest rate carefully, adding a real bond to the budget constraint will help.)
In this problem I have used the timing convention that $M_{t-1}$ and $B_{t-1}$ are money and debt bought at the end of period $t-1$ held overnight to period $t$.

a) Find the equilibrium conditions for this problem. One will be the familiar intertemporal condition for consumption. The other based on $U_M/U_c$ will look like an interest-elastic money demand curve. There will also be a valuation equation or equilibrium present value “budget constant”

b) Find the equilibrium ($c = y$) steady state(s) for prices $P_t = P$ when money is constant $M_t = M$ and the government follows a “passive” fiscal policy, setting $s$ as needed. Find the steady state $P$, and relate $r, i,$ and $\pi$ in equilibrium.

c) Look also for steady states with $M_{t+1}/M_t = 1 + \mu$. Plot real money demand $M/Py$ as a function of $i$.

i) Explain the limit as $i \to 0$. This plot may help you to make sense of all the complaining that “banks are sitting on cash and not lending it out,” or “companies are sitting on cash and not investing.”

ii) Find and explain the limit of money demand as $i \to \infty$. Hint: In this discrete-time model, would someone hold any money this period $t$ even if $P_{t+1} = \infty$?

d) Now, revert to $M_t = M$ and a passive fiscal policy. Find the nonlinear difference equation for prices. It will be easier to find the difference equation for the inverse price level $1/P_{t+1} = f(1/P_t)$. Verify that the steady state you found above satisfies this difference equation.

e) Now, find the difference equation for deviations from the steady state, $P/P_{t+1} = f(P/P_t)$. Plot the phase diagram, $P/P_{t+1}$ on the y axis and $P/P_t$ on the x axis. Plot also $P_{t+1}/P$ as a function of $P/P_t$ so you understand the mapping from inverse-price dynamics to price dynamics. Make plots of $P/P_t$ and $P_t/P$ over time as well, for a variety of initial $P_0$.

i) Is the steady state a “stable” or “unstable” solution? Are there “multiple locally bounded equilibria” or a “unique locally bounded equilibrium” of this model?

ii) Are the equilibrium paths other than the steady state valid equilibria of this model, or does this model also produce global indeterminacy?

NOTE: you will find that this model produces inflation to $P_{t+1} = \infty$ in finite time. If you think very hard about the period just before $P_{t+1} = \infty$ it’s a mess which confused Obstfeld and Rogoff and everyone else for 25 years, so you are not required to get it straight on a problem set. Yes, solutions in which $P_{t+1} = \infty$ but $P_t < \infty$ are valid, and the point of the problem is not for you to worry about these. In Part c ii you figured out that such solutions are ok. The point of the problem is to look at the multiple solutions and convince yourself they are ok.

3) Now, let’s examine the fiscal side of this economy.

a) Write the equilibrium government debt valuation equation. You’re looking for $B_{t-1}/P_t$ on the left hand side and a present value on the right hand side, for an arbitrary $M_t$ process. Show how seigniorage $(M_{t+j+1} - M_{t+j})/P_{t+j}$ acts in the present value equation.
b) Specializing to $M_t = M$, and reverting to fixed $s$ rather than passive $s$, show how this equation can now select a unique equilibrium. Puzzle: What if that equilibrium is not the steady state – what if there is a little too much or too little $B_{t-1}$ outstanding?

c) Consider the case $B_{t-1} = 0$, and $B_t = 0$ for all $t$. Write and interpret the flow budget constraint. How does the government pay for surpluses/deficits? Exhibit the “passive” policy that supports any equilibrium in the $M_t = M$, multiple equilibrium cases. That’s useful, as now you check that there really is such a passive $s$ that works. (We skipped that step before). What happens to the idea of using “active” fiscal policy to wipe out the alternative equilibria in this case? Explain the role of nominal government debt in selecting the price level path.