Problem set 3 Answers

1. a) We start with a linearized model.

\[ m_t - p_t = -\alpha i_t \]
\[ i_t = r + E_{t+1} \pi_{t+1} \]
\[ m_t - p_t = -\alpha (r + p_{t+1} - p_t) \]

\[ m_t = -\alpha r - \alpha p_{t+1} + (1 + \alpha) p_t \]
\[ p_{t+1} = \frac{1 + \alpha}{\alpha} p_t - \frac{1}{\alpha} m_t - r \]

The conventional wisdom: solve the unstable root forward:

\[ \frac{1 + \alpha}{\alpha} p_t = p_{t+1} + \frac{1}{\alpha} m_t + r \]
\[ p_t = \frac{\alpha}{1 + \alpha} p_{t+1} + \frac{1}{1 + \alpha} m_t + \frac{\alpha}{1 + \alpha} r \]

\[ p_t = \sum_{j=0}^{\infty} \left( \frac{\alpha}{1 + \alpha} \right)^j \frac{1}{1 + \alpha} m_{t+j} + \sum_{j=0}^{\infty} \left( \frac{\alpha}{1 + \alpha} \right)^{j+1} r \]
\[ p_t = \sum_{j=0}^{\infty} \left( \frac{\alpha}{1 + \alpha} \right)^j \frac{1}{1 + \alpha} m_{t+j} + \alpha r \]

This seems pretty cool. The price level today reflects expected future money growth, not just today’s money growth.

The intuition – if you know there will be inflation in the future (between t+1 and t+2), then you know there will be less money demand in the future (t+1). Since money supply is given, that means the price level will be higher in the future (t+1). For a given price level today (t), that means inflation from t to t+1 will be higher. But higher inflation from t to t+1 depresses money demand today, thus raising the price level today. By this mechanism expected future monetization leads to higher prices today.

Notice the future is discounted at the interest-elasticity of money demand, not at the discount rate. Typical numbers for this elasticity are quite small, 0.1 or so. So this geometric sum may not look that far in the future.

Sargent’s “unpleasant monetarist arithmetic” warned that the deficits of the 1980s would have to be monetized; when they were going to be monetized that would cause inflation, and that inflation would spill back to inflation in the early 1980s.

This didn’t happen. I’ll buy beer for the student who has the courage to ask Sargent why "unpleasant monetarist arithmetic" predictions did not come true.

b) Now let \( m_t = m \)

\[ p_{t+1} = \frac{1 + \alpha}{\alpha} p_t - \frac{1}{\alpha} m - r \]
The steady state, or “unique locally bounded solution” in this case

\[
p = \frac{1 + \alpha}{\alpha}p - \frac{1}{\alpha}m - r
\]

\[
\alpha p = (1 + \alpha)p - m - \alpha r
\]

\[
p = m - \alpha r
\]

The other solutions are

\[
p_{t+1} = \frac{1 + \alpha}{\alpha}p_t - \frac{1}{\alpha}m - r
\]

Obviously we're going to have a whole family and they will be explosive. I find it nice to characterize differences between the generic solution and the steady state or forward looking solution.

\[
p_{t+1} - p = \left(\frac{1}{\alpha} + 1\right) p_t - p - \frac{p}{\alpha}
\]

\[
(p_{t+1} - p) = \left(\frac{1 + \alpha}{\alpha}\right) (p_t - p)
\]

Thus there is a whole family of explosive solutions, indexed by \(p_0\), with

\[
(p_{t+1} - p) = \left(\frac{1 + \alpha}{\alpha}\right)^j (p_0 - p)
\]

Important: The right way to think of this is not that a little mistake today causes inflation to rise in the future. Rather, expected inflation in the future causes you to economize on money demand today, since you will lose money holding it over night. People trying to hold less money than there is today makes the price level today go up.

This is all in logs so we do not have any infinities.

2)

a)

\[
\max \sum \beta^t U(c_t, M_t/P_t)
\]

s.t. \(M_{t-1} + B_{t-1} + P_t y_t = M_t + Q_t B_t + P_t c_t + P_t s_t\)

The first order conditions, where \(\lambda\) is the Lagrange multiplier on the time \(t\) budget constraint

\[
\frac{\partial}{\partial c_t} : U_c(t) = \lambda_t P_t
\]

\[
\frac{\partial}{\partial M_t} : U_M(t)/P_t = -\lambda_t + \lambda_{t+1}
\]

\[
\frac{\partial}{\partial B_t} : \lambda_{t+1} = Q_t \lambda_t
\]

Substituting out the \(\lambda\), we get the usual intertemporal first order condition for bond pricing

\[
U_c(t) = \frac{(1 + i_t)}{(1 + \pi_{t+1})} \beta U_c(t + 1)
\]
we also get a first order condition for money vs. consumption at time t

\[
\frac{U_M(t)}{U_c(t)} = \left( \frac{1}{1 + i_t} - 1 \right) = \frac{i_t}{1 + i_t}
\]

Algebra:

\[
\begin{align*}
\lambda_{t+1} &= Q_t \lambda_t \\
\beta U_c(t + 1) / P_{t+1} &= Q_t U_c(t) / P_t \\
U_c(t) &= \frac{P_t}{Q_t} \beta U_c(t + 1) / P_{t+1} \\
U_c(t) &= (1 + i_t) \frac{P_t}{P_{t+1}} \beta U_c(t + 1) \\
U_c(t) &= \frac{(1 + i_t)}{(1 + \pi_{t+1})} \beta U_c(t + 1)
\end{align*}
\]

\[
\begin{align*}
U_M(t) / P_t &= \lambda_t - \lambda_{t+1} \\
U_M(t) / P_t &= \lambda_t - Q_t \lambda_t \\
U_M(t) / P_t &= (1 - Q_t) \lambda_t = (1 - Q_t) \frac{U_c(t)}{P_t} \\
\frac{U_M(t)}{U_c(t)} &= \left( 1 - \frac{1}{1 + i_t} \right) = \frac{i_t}{1 + i_t}
\end{align*}
\]

Using the power utility function

\[
U = c_t^{-\gamma} + \left( \frac{M_t}{P_t} \right)^{1-\gamma}
\]

the second first-order condition is

\[
\frac{(M_t)^{-\gamma}}{c_t^{-\gamma}} = \frac{i_t}{1 + i_t}
\]

\[
M_t / P_tc_t = \left( \frac{i_t}{1 + i_t} \right)^{-\frac{1}{\gamma}}
\]

You can read this as a “money demand function.” As interest rates rise, real money demand declines. \(-1/\gamma\) controls the interest elasticity of money demand.

b) Equilibrium

\[
C_t = Y
\]


gives us

\[
\begin{align*}
U_c(t) &= \frac{(1 + i_t)}{(1 + \pi_{t+1})} \beta U_c(t + 1) \\
1 &= \frac{(1 + i_t)}{(1 + \pi_{t+1})} \frac{1}{1 + r} \\
(1 + i_t) &= (1 + \pi_{t+1})(1 + r)
\end{align*}
\]
Look for the steady state with $M_t = M, P_{t+1} = P_t = P$ so $\pi = 0$.

$$
\frac{M}{Py} = \left( \frac{r}{1+r} \right)^{-\frac{1}{\gamma}} = (r\beta)^{-\frac{1}{\gamma}}
$$

c) Look for steady states with $M_{t+1}/M_t = 1 + \mu$. For it to be a steady state, $i$ will be constant so $M/Py$ is constant, so we have $\pi = \mu$. Then $1 + i = (1 + \mu)(1 + r)$

$$
\frac{M}{Py} = \left( \frac{i}{1+i} \right)^{-\frac{1}{\gamma}} = \left( \frac{(1 + \mu)(1 + r) - 1}{(1 + \mu)(1 + r)} \right)^{-\frac{1}{\gamma}}
$$

Here’s the plot.

i) As you can see, real money demand is lower for higher inflation = mean money growth = nominal interest rate. Money demand goes to infinity when $i = 0$; $(1 + \mu)(1 + r) = 1$ i.e. $\mu + r + \mu r = 0$ or $\mu \approx -r$. In this case the interest cost of holding money is zero, so you satiate in money balances.

Extremely large money balances now are not that big a surprise at zero rates, nor is the nonlinearity of this relationship. See the second graph
There is a secular downward shift in money demand, as people use more electronic transactions. But on top of that, note that everytime the interest rate goes down, people start to hold more money. There is a bit of hysterisis; once rates rise again it takes them a while to start economizing on money but sooner or later they do.

The huge and cry about “banks are holding too much money and not lending it out,” or “companies are sitting on piles of cash and not investing” ignores the standard interest-elasticity of money demand.

ii) As $\mu = \pi \to \infty, \frac{M}{P_{t+1}} \to 1$. This makes no sense at all. What’s going on? Look at the basic first order condition, written this way

$$\frac{M_t}{P_{t+1}} = \left(1 - \frac{1}{1+i}\right)^{\frac{1}{i}}$$

If you buy some money $M_t$ it is useful today, but it is also useful tomorrow. However, it’s worth less tomorrow by inflation. Now, as inflation goes to infinity in this model, the value of money tomorrow goes to zero, so the second term goes to zero. But the value of money today remains. You’re willing to hold money for one day, even if it becomes completely useless tomorrow. That’s because we don’t have any intra-day inflation. In reality, inflation “during the day” would make money even less valuable. So this is a poor model for studying very high inflation.

Another way to see the issue, as suggested by ii, is to consider the first order condition if you know money is useless tomorrow:

$$\frac{U_M(t)}{P_t} = \lambda_t - \lambda_{t+1}$$

$$\frac{U_M(t)}{P_t} = \frac{U_c(t)}{P_t} - \frac{U_c(t+1)}{P_{t+1}}$$

If $P_{t+1} = \infty$, you still hold money for one day’s purchases.

The answer to this problem is to solve the corresponding continuous time model,

$$\int e^{-rt}U(c_t, M_t/P_t)dt$$

The corresponding first order condition is

$$\left(\frac{M_t}{P_{t+1}}\right)^{-\gamma} = i_t = r + \frac{1}{P_t} \frac{dP_t}{dt}$$

$$\frac{M_t}{P_{t+1}} = i_t^{\frac{1}{\gamma}} = (r + \mu)^{\frac{1}{\gamma}} = (r + \pi)^{\frac{1}{\gamma}}$$

Now, as inflation rises to infinity, money demand really goes to zero as it should. This is all much prettier. I didn’t assign the continuous time model simply because it’s a little more challenging mathematically, and it’s good to solve the discrete model first. But the right way to do this is in continuous time!
So, don’t use the discrete time model to study high inflations! I include the continuous-time results in red in the graphs so you can see the difference. Doing the continuous time version of this problem is a really good idea.

d) Let’s investigate the difference equation for prices

\[
\left( \frac{M_t}{P_{ty}} \right)^{-\gamma} = \left( 1 - \frac{1}{1 + h_t} \right) = \left( 1 - \beta \frac{P_t}{P_{t+1}} \right)
\]

\[
\frac{1}{\beta} \left( \frac{M_t}{P_{ty}} \right)^{-\gamma} = \left( \frac{1}{\beta} \frac{P_t}{P_{t+1}} - \frac{1}{P_{t+1}} \right)
\]

\[
\frac{1}{\beta} \left[ 1 - \left( \frac{M_t}{P_{ty}} \right)^{-\gamma} \right] = \frac{1}{P_{t+1}}
\]

Rather than look at the equation itself, let’s look at the deviations from the steady state.

\[
\frac{M}{P_y} = \left( \frac{r}{1 + r} \right)^{-\frac{1}{\gamma}} = (r\beta)^{-\frac{1}{\gamma}}
\]

Yes the steady state satisfies the difference equation

\[
\frac{1}{\beta} \left[ 1 - \left( \frac{M}{P_y} \right)^{-\gamma} \right] = \frac{1}{P}
\]

\[
\frac{1}{\beta} \left[ 1 - \left( (r\beta)^{-\frac{1}{\gamma}} \right)^{-\gamma} \right] = 1
\]

\[
\frac{1}{\beta} \left[ 1 - (r\beta) \right] = 1
\]

\[
\frac{1}{\beta} \left[ 1 - \left( \frac{r}{1 + r} \right) \right] = 1.
\]
e) OK, dividing the difference equation by the steady state, we obtain

\[
\frac{1}{\beta} P_t \left( 1 - \left( \frac{M}{Y} \right)^{-\gamma} \frac{1}{P_t^{-\gamma}} \right) = \frac{1}{P_{t+1}}
\]
\[
\frac{1}{\beta} P \left( 1 - \left( \frac{M}{Y} \right)^{-\gamma} \frac{1}{P^{-\gamma}} \right) = \frac{1}{P}
\]
\[
P \left( 1 - \left( \frac{M}{Y} \right)^{-\gamma} \frac{1}{P_t^{-\gamma}} \right) = \frac{P}{P_t}
\]
\[
\frac{P \left( \frac{PY}{M} \right)^{-\gamma} - \left( \frac{P}{P_t} \right)^{-\gamma}}{P_t \left( \frac{PY}{M} \right)^{-\gamma} - 1} = \frac{P}{P_{t+1}}
\]
\[
P \left( \frac{r\beta}{1 - r\beta} \right)^{-1} - \left( \frac{P}{P_t} \right)^{-\gamma} = \frac{P}{P_{t+1}}
\]
\[
\frac{P \left( 1 - \frac{r\beta}{P_t} \right)^{-\gamma}}{P_t} \left( 1 - \frac{r\beta}{1 - r\beta} \right) = \frac{P}{P_{t+1}}
\]

The next figure plots this transition equation.

As you can see, it cuts from the bottom — unstable, as in the linear case.

The top end just goes off to eternal deflation \((1/P \text{ rises})\). Since it cuts from below, the bottom end shows that \(1/P \text{ goes to zero, or } P \text{ goes to infinity, in finite time.} \) Except for the integer problem, there’s nothing wrong with this. The utility function allows people to hold money for one period even if they know it will be completely useless the next period.

The next graph shows price paths.
As you see, hyperinflation occurs in finite time in this model. There are many price paths, all of them perfectly valid equilibria.

3) a) Now, the government valuation equation in this economy is

\[
\frac{B_{t-1}}{P_t} \cdot M_{t-1} = B_t \cdot B_t + M_t + P_t s_t
\]

\[
\frac{B_{t-1}}{P_t} = \frac{1}{1 + r} \left( P_{t+1} \right) + \frac{1}{1 + r} \left( \frac{M_t - M_{t-1}}{P_t} \right) + s_t
\]

\[
\frac{B_{t-1}}{P_t} = \sum_{j=0}^{\infty} \left( \frac{1}{1 + r} \right)^j \left( \frac{M_{t+j} - M_{t+j-1}}{P_{t+j}} \right) + s_{t+j}
\]

For \( M_t = M \),

\[
\frac{B_{t-1}}{P_t} = \sum_{j=0}^{\infty} \left( \frac{1}{1 + r} \right)^j s_{t+j}
\]

b) For fixed \( s \), this picks the initial \( P_t \) and solves the indeterminacy issue. What if this isn’t the steady state, you ask? Well, then we see inflation or deflation. Why don’t we see
inflation or deflation, you ask? A: Because no government is stupid enough to sit with a constant money supply and debt, and watch inflation or deflation break out.

c) If there is not debt, this equation does not work to select the equilibrium! In that case the flow equation is

\[ M_{t-1} = M_t + P_t s_t \]

i.e. the government can only run a surplus/deficit by printing or retiring money. \( M_t = M \) implies \( s_t = 0 \) forever, the government just leaves a stock of money outstanding.

To determine the price level, we must use the existence of government bonds, and the fact that inflation devalues outstanding government bonds.

I conclude from this exercise that inflation and the price level are just as indeterminate under a constant money rule \( MV(i) = PY \) as it is under interest rate targets, except for the case that \( V \) is a constant and there is no interest elasticity at all.

This is important in my overall thinking about the issue. We’ve shown that Taylor rules don’t determine the price level. Now, apparently \( MV = PY \) doesn’t do so either, unless \( V \) is constant. As I read it, not only is the fiscal theory of the price level a nice possibility, it is the only logically coherent theory that can determine the price level in an economy a bit like ours.