

18 Ch12 Classic linear regressions

18.1 Summary

1. General steps

- Model: $E(R^{ei}) = (\alpha_i +) \beta'_i \lambda$
- Estimate parameters. $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$.
- Standard errors of parameter estimates. $\sigma(\hat{\alpha}), \sigma(\hat{\beta}), \sigma(\hat{\lambda})$
- Test the model. $\hat{\alpha}' V^{-1} \hat{\alpha}$ “too big?”
- Evaluate, diagnose the model. Is the SML “too flat?” Do betas drive out characteristics? Does σ^2, β^2 matter? Is $\hat{\lambda}$ reasonable?
- Test one model vs. another. Can we drop a factor, e.g. sml?

2. Time series

$$R_t^{ei} = \alpha_i + \beta_i f_t + \varepsilon_{it}$$

- Procedure: Just run this, $t = 1, 2, \dots, T \forall i$.
- Estimates and standard errors:
 - $\hat{\alpha}, \hat{\beta}, \sigma(\hat{\alpha}, \hat{\beta})$ from OLS (OLS/GMM) formulas.
 - $\hat{\lambda} = E_T(f); \sigma(\hat{\lambda}) = \sigma(f)/\sqrt{T}$ (iid) or GMM
- Test: Econometric issue is the joint distribution of intercepts $\hat{\alpha}_i$ from N regressions $cov(\hat{\alpha}, \hat{\alpha}')$. Then the statistic is

$$\hat{\alpha}' cov(\hat{\alpha}, \hat{\alpha}')^{-1} \hat{\alpha}$$

iid

$$\hat{\alpha}' cov(\hat{\alpha})^{-1} \hat{\alpha} = T \left[1 + \bar{f}' \Sigma_f^{-1} \bar{f} \right]^{-1} \hat{\alpha}' \Sigma^{-1} \hat{\alpha} \sim \chi_N^2$$

iid normal “GRS Test.”

$$\frac{T - N - K}{N} \left[1 + \bar{f}' \Sigma_f^{-1} \bar{f} \right]^{-1} \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha} \sim F_{N, T-N-K}$$

- GMM. Doing N regressions at once is easy – stack them up, let the S matrix handle the correlation!

$$g_T(b) = \begin{bmatrix} E_T(\varepsilon) \\ E_T(f_t \varepsilon_t) \end{bmatrix} = \begin{bmatrix} E_T(R_t^e - \alpha - \beta f_t) \\ E_T((R_t^e - \alpha - \beta f_t) f_t) \end{bmatrix}$$

$$d = \frac{\partial g_T}{\partial [\alpha \ \beta]'} = \begin{bmatrix} 1 & E(f_t) \\ E(f_t) & E(f_t^2) \end{bmatrix} \otimes I_N$$

$$S = \sum_{j=-\infty}^{\infty} \begin{bmatrix} E(\varepsilon_t \varepsilon'_{t-j}) & E(\varepsilon_t \varepsilon'_{t-j} f_{t-j}) \\ E(f_t \varepsilon_t \varepsilon'_{t-j}) & E(f_t f_{t-j} \varepsilon_t \varepsilon'_{t-j}) \end{bmatrix}$$

$$var \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \frac{1}{T} d^{-1} S d^{-1'}$$

$$\hat{\alpha}' var(\hat{\alpha})^{-1} \hat{\alpha} \sim \chi_N^2$$

(e) Reduces to standard formulas for ε iid and independent of f . But why do that?

3. Cross section

(a) Method: First run time series to get β , then run

$$E(R^{ei}) = \gamma + \beta_i \lambda + \alpha_i$$

$i = 1, 2, \dots, N$ in the cross section.

(b) Econometric issues:

- i. β are generated regressors.
- ii. α are correlated across assets

(c) Classic iid formulas with no intercept $\gamma = 0$

$$\begin{aligned}\sigma^2(\hat{\lambda}_{OLS}) &= \frac{1}{T} \left[(\beta' \beta)^{-1} \beta' \Sigma \beta (\beta' \beta)^{-1} \left(1 + \lambda' \Sigma_f^{-1} \lambda \right) + \Sigma_f \right] \\ \sigma^2(\hat{\lambda}_{GLS}) &= \frac{1}{T} \left[(\beta' \Sigma^{-1} \beta)^{-1} \left(1 + \lambda' \Sigma_f^{-1} \lambda \right) + \Sigma_f \right]\end{aligned}$$

$$\text{cov}(\hat{\alpha}_{OLS}) = \frac{1}{T} (I - \beta(\beta' \beta)^{-1} \beta') \Sigma (I - \beta(\beta' \beta)^{-1} \beta') \left(1 + \lambda' \Sigma_f^{-1} \lambda \right)$$

$$\text{cov}(\hat{\alpha}_{GLS}) = \frac{1}{T} (\Sigma - \beta(\beta' \Sigma^{-1} \beta)^{-1} \beta') \left(1 + \lambda' \Sigma_f^{-1} \lambda \right)$$

$$\hat{\alpha}' \text{cov}(\hat{\alpha}, \hat{\alpha}')^{-1} \hat{\alpha} \sim \chi_{N-K-1}^2.$$

$$\hat{\alpha}'_{GLS} \text{cov}(\hat{\alpha}_{GLS})^{-1} \hat{\alpha}_{GLS} = T \left(1 + \lambda' \Sigma_f^{-1} \lambda \right) \hat{\alpha}_{GLS} \Sigma^{-1} \hat{\alpha}_{GLS} \sim \chi_{N-K}^2$$

(d) Classic iid formulas with an intercept, OLS

$$X = \begin{bmatrix} 1_N & \beta \end{bmatrix}$$

$$\sigma^2 \left(\begin{bmatrix} \hat{\gamma} \\ \hat{\lambda} \end{bmatrix} \right) = \frac{1}{T} \left[(X' X)^{-1} X' \Sigma X (X' X)^{-1} \left(1 + \lambda' \Sigma_f^{-1} \lambda \right) + \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_f \end{bmatrix} \right]$$

$$\text{cov}(\hat{\alpha}) = \frac{1}{T} (I - X(X' X)^{-1} X') \Sigma (I - X(X' X)^{-1} X') \left(1 + \lambda' \Sigma_f^{-1} \lambda \right)$$

(e) GMM approach

$$ag_T = \begin{bmatrix} I_N & & \\ & I_N & \\ & & \beta' (\Sigma^{-1}) \end{bmatrix} \begin{bmatrix} E_T(R^e - a - \beta f) \\ E_T[(R^e - a - \beta f) f] \\ E_T[(R^e - \beta \lambda)] = E_T(\alpha) \end{bmatrix} = 0$$

note $\hat{\alpha}$ = last N rows of $g_T(a, \beta, \lambda)$.

$$\hat{\alpha}' \text{cov}(\hat{\alpha} \hat{\alpha}')^{-1} \hat{\alpha}$$

- (f) Fact: if the factor is a return, and including the factor as a test asset, GLS cross-section = time-series. If returns are iid normal, also = ML.
- (g) Fact: "Efficient GMM" is not GLS when returns are not iid.

4. Test comments

- (a) Run $\hat{\alpha}' cov(\hat{\alpha})^{-1} \hat{\alpha}$ tests for cross sections! (but don't necessarily pay too much attention)
- (b) Don't test a model by $\lambda = 0$. Don't test for one model vs. another by $\lambda_2 = 0$. $E(f_2) = \alpha_2 + \beta_{21} E(f_1)$ is the right test

5. **Fama MacBeth** (For asset pricing)

- (a) Run TS to get betas.

$$R_t^{ei} = \alpha_i + \beta_i' f_t + \varepsilon_t^i \quad t = 1, 2, \dots, T \text{ for each } i.$$

- (b) Run a cross sectional regression *at each time period*,

$$R_t^{ei} = \beta_i' \lambda_t + \alpha_{it} \quad i = 1, 2, \dots, N \text{ for each } t.$$

$$R_t^{ei} = \gamma_t + \beta_i' \lambda_t + \alpha_{it} \quad i = 1, 2, \dots, N \text{ for each } t.$$

- (c) Then, *Estimates* of λ, α are the *averages across time*

$$\hat{\lambda} = \frac{1}{T} \sum_{t=1}^T \hat{\lambda}_t; \quad \hat{\alpha}_i = \frac{1}{T} \sum_{t=1}^T \hat{\alpha}_{it}$$

- (d) *Standard errors* use $\sigma^2(\bar{x}) = \sigma^2(x)/T$

$$\sigma^2(\hat{\lambda}) = \frac{1}{T} var(\hat{\lambda}_t) = \frac{1}{T^2} \sum_{t=1}^T (\hat{\lambda}_t - \hat{\lambda})^2$$

$$cov(\hat{\alpha}) = \frac{1}{T} cov(\hat{\alpha}_t) = \frac{1}{T^2} \sum_{t=1}^T (\hat{\alpha}_{it} - \hat{\alpha}_i) (\hat{\alpha}_{jt} - \hat{\alpha}_j)$$

This one main point. These standard errors are easy to calculate

- (e) *Test*

$$\hat{\alpha}' cov(\hat{\alpha}, \hat{\alpha}')^{-1} \hat{\alpha} \sim \chi_{N-1}^2.$$

- (f) Fact: FMB corrects for α correlated across time. (It's a panel with clustering). FMB does not correct for generated betas. ("Shanken correction")
- (g) Fact: FMB assumes errors are uncorrelated over time. It's easy enough to add correlation over time,

$$\sigma^2(\hat{\lambda}) = \frac{1}{T} \sum_j E_T \left[(\hat{\lambda}_t - \hat{\lambda}) (\hat{\lambda}_{t-j} - \hat{\lambda}) \right]$$

- (h) Fact: When β_{it} are constant over time, FMB = cross section=pooled estimate.