Most Important Rule

\[ E(R) = \alpha + \beta (S_H, L_H) \]

FAMA: FRENCH model says small/value stocks get higher returns!

Options

1. Intro and Payoffs

- Object: Option price \( P \rightarrow \) stock/bond; "relative pricing" not "explanation"
- Method: \( P = E(M) \) learn about \( M \) from \( S, B \)
- Was "arbitrage" now less so

Payoffs \( x_t \)

Value at \( t \)

\[ C_t = \max(S_t - x, 0) \]

\[ P_t = \max(x - S_t, 0) \]

\[ R_t \]

Write Call

Write Put

Buy disaster insurance

\[ \text{Stock/put/pit} \]

Profit

\[ \text{Write Call, Put} \]

\[ \text{’autres? Bank expensive?} \]
OPTIONS 2: ARBITRAGE BOUNDS

PRICE ≤ S, B? AVOID v(c.)

OBJECT: PRICE, "6"

WHY:
A) IDEAL TRADING VEHICLE
- \( S=100, C=10, \delta=\frac{3}{2} \rightarrow \beta=S \)
- CAN'T DEFAULT!
B) HEDGING, BETAVOLATILITY, ETC

"SYNTHETIC STOCK"

\[ X := C - P_T = S_T - X \quad \text{(T)} \]

\[ P = E(M^v): C - P = S - X / \text{RE} \quad \text{or} \]

PUT-CALL PARITY (M: LOOP)
OMITTED: BLACK-SCHOLES

**Continuous Trading**

\[
\frac{dS}{S} = r dt + \sigma dW
\]

Find \( x' \) that prices \( S, B \) and all payoffs formed by dynamic trading.

\[ C = \text{E} (s' \cdot x') \]

\[
x' = \frac{dS}{\frac{S}{\eta}} = r dt - \frac{\mu + r}{\eta} dt
\]

(check: \( \text{E}(\frac{dS}{\eta}) = -r dt \))

\[
\text{E}(\frac{dS}{S}) - r dt = \text{E}(\frac{dS}{\eta} \cdot \frac{S}{\eta})
\]

\[ C_0 = \frac{S_0}{\eta} \int \frac{S_T(x) \cdot S_T(y) \cdot e^{-\frac{1}{2} \sigma^2 t} \cdot x \cdot y \cdot e^{rT}}{\eta} \cdot \xi dt \]

\[ \phi (M - \xi) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2} (e - \eta)^2} dx \]

\[ S_T(x) \to S_0 = \frac{x \cdot e^{-\frac{1}{2} \sigma^2 T} \cdot \xi \cdot \text{E} \cdot \text{N}(\phi)}{\eta} \]

\[ \text{N}(\phi) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2} (\eta - x)^2} dx \]
\[ C_0 = \Phi \left( \frac{\ln(S_0/k) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) \cdot e^{-rT} \Phi \left( \frac{\ln(S_0/k) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) \]

- **Note:** M disappeared! "Arbitrage" M ins!
- \( \sigma \) here! More \( \sigma \rightarrow \sigma^2 \)
- "Implied vol" \( \rightarrow \sigma \) to justify \( C \), standardizes across \( X, T \)
- \( \rightarrow \) Derivatives, "Greeks" (Like Yield)

Options IV: Other Methods

- Differential Equation. Solve Price Not Discount Factor

Guess \( C(S,T) \)

Ito \( \frac{dc}{c} = C\,dr + C\,dS + \frac{1}{2}C_{SS}dS^2 \)

\[ dc = \left[ \frac{C_u + C_S S + \frac{1}{2}C_{SS} S^2}{2} \right] dt + C_S S dS \]

**Price by A**

\[ E \left( \frac{dc}{c} \right) - rdt = E \left( \frac{dc}{c} \right) \lambda \]

\[ x = \frac{C + C_S S + \frac{1}{2}C_{SS} S^2}{r - C} \]

\[ rC - s + C_S + C_x + \frac{1}{2}C_{SS} S^2 = 0 \]

"Black-Scholes Differential Eqn"

Any asset whose price \( (S_T) \) solves

Boundary \( C(S_T, T) = \frac{1}{\lambda} = \max(S_T - x, 0) \)

Solve Back

\( C(S, t-\Delta t) = C(S_t, S) - \frac{\partial C(S_t, S)}{\partial t} \)

\( = C(S_t, S) + \left[ rS \frac{\partial C(S_t, S)}{\partial S} + \frac{1}{2}S^2 \frac{\partial^2 C(S_t, S)}{\partial S^2} - rC(S_t, S) \right] \Delta t \)

- Guess + Check
- Transform to integral → Arbitrage
- Numerical.
Options 5

Spanning, State Prices + Current Models

Even if B-S does not hold SPANNING "ANY" Full set of C, P options
dynamic trading S, B

Finding λ (even when B-S fails)

\[ x = \varphi'(E(x|x) x) \]

Fact: You can find λ from 2nd derivative of C wrt x

Payoff \( E^x \)

Price:

\[ V = \left[ \begin{array}{c} C(x-E) \\ -2C(x) + C(x+E) \end{array} \right] \rightarrow E^x \frac{d^2c(x)}{dx^2} \]

Buy \( \frac{1}{E} \rightarrow \frac{dx}{dx} = C \) claim price

\[ P = \int_{S_T}^{x} \left( \frac{\partial^2 c(x,y)}{\partial x^2} \right) x(S_T) dS_T \]

\[ P(x(S_T)) ? \]
OPTIONS: DATA, SMILE, MODELS

- OTM Puts "CHEAP"?
- OTM Puts "EXPENSIVE"!
  "WRITING DISASTER INSURANCE"
- Fix BLACK-SCHOLES?
  - Risk Aversion?
  - Probability too!

\[
\frac{d S_t}{S_t} = M_s \, dt + \sqrt{V_t} \, dZ_t
\]
- Correlation

\[
dV_t = -\phi(V_t - V_0) + \sigma V_t \, dW_t
\]

\[
d\eta_t = \gamma dt - \frac{\eta_t - \eta_0}{\tau} dZ_t - \gamma \omega dW_t
\]

**MARKET PRICE OF VOLATILITY RISK**

**STOCHASTIC VOLATILITY**

- Fat Tails
- Fat Left Tail

\[
\text{You can do this!}
\]

\[
\frac{\partial C}{\partial s} = \frac{1}{S_t} \left( \frac{\partial C}{\partial \sigma} \sigma \sqrt{V_t} \right)
\]

\[
\frac{\partial C}{\partial \sigma^2} = \frac{1}{S_t^2} \left( \frac{\partial C}{\partial \sigma} \right)^2
\]

\[
\text{\textsc{moneyness of normals}} = \text{\textsc{fat tails}}
\]

\[
\text{IF} \quad \text{WHEN} \quad \Rightarrow \text{\textsc{fat left tail}}
\]

\[
\text{\textsc{moneyness of normals}} = \text{\textsc{fat tails}}
\]

\[
\text{\textsc{fat left tail}}
\]

\[
\text{\textsc{fat left tail}}
\]
BJ Jumps
\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \epsilon_t
\]
\[ \text{Market price of jump risk} \]

- Pricing by mostly arbitrage,
- A disciplined way to price "nearby" securities
- Detect "mispricing"
- Construct hedge
- Isolate premiums

1 Bonds
- US: Default free, cashflow known
  \[ P = E(\rho, t) \]
- Data plot
  - Up and down with inflation
  - With recessions!
  - Spread high in recession, low in booms
  - "Level", "slope" capture most movements
  - "Yield curve" / "forward curve"
  - Can slope up/down
  - Why? (Economics) - model just like Black-Scholes?
  - Price all maturities
  - Price/hedge TS options