PORTFOLIO THEORY: CLASSIC

- WHAT TO BUY? CLASSIC: RENS (HN)R MODEL
  \[ \text{Expected Utility} \]
  \[ E(R), \mu_{w}, \sigma^2 \text{ is } \int \mu_{w} w \mu_{w}^2. \]
  \[ \text{Best?} \]

- LOGNORMAL ID POWER UTILITY
  \[ \max_{w_0, w_t} \int u(w) dw \text{ s.t. } \]
  \[ dw_t = w_t \left[ r dt + \sigma dw_t \right] - \gamma dw_t \]
  \[ \gamma = \text{risk aversion} \]

- SOLVE BY DYNAMIC PROGRAMMING:
  \[ V(w_t) = \max_{w_{t+1}} E_t \left[ E^{s+1} V(w_{t+1}) - \gamma dw_t \right] \]
  \[ s_{t+1} = \max_{w_{t+1}} u(w_{t+1}) \Delta + \left[ E^{s+1} V(w_{t+1}) \right] \]

- INVESTOR HOLDS A CONSTANTLY REBALANCED MEAN-VARIANCE EFFICIENT PORTFOLIO
  \[ \text{(Reminder: Min Var}(w_r^2) \text{ St at } \text{We}(t)) = \epsilon \]
  \[ w_{SW} = \frac{w}{2} \rightarrow w = \frac{1}{2} \epsilon (M + r) \]

- FACT: IF \( u(c) = \frac{c^{1-\gamma}}{1-\gamma} \) THEN \( V(w) = \frac{1}{1-\gamma} \)

- FRACTION \( w_t \) IS CONSTANT \( (w_{t+1}, \text{now}) \) \( \rightarrow c_t = h_{t+1} \)

- HOW?
  \[ \frac{c_t}{p_t} = \frac{1}{1-\gamma} \text{ \( \rightarrow P V - V W \left[ E^{s+1} (M + r) - \epsilon \right] \frac{1}{2} \epsilon \] \text{ \( w_t \) t. \( 0 \) \( \text{CONSUME (CONSTANT FRACTION) OF WEALTH!} \)}}
Portfolio Theory 2

Mean-Variance

Relative to Market.

If all like this with different $\gamma$

$$w_i = \frac{x_i^\gamma}{\sum x_i^\gamma}$$

$$r_i^w = r_i^e + \gamma m^e$$

$w_i, r_i^w, m^w$

If...

CARM $E(r_i^e) = r_i^e + \beta_m E(m^e)$

Alphas.

Write statistical model C whether or not $\beta_m = \text{CARM}$

$$r_i^e = \frac{1}{n} \sum_{i=1}^{n} w_i r_i^m + \sum_{i=1}^{n} w_i (r_i^m - \bar{r})$$

$$E(r_i^e) = \frac{1}{n} \sum_{i=1}^{n} w_i E(r_i^m) + \sum_{i=1}^{n} w_i (E(r_i^m - \bar{r}))$$

$$w_i = \frac{1}{n} \sum_{i=1}^{n} w_i E(r_i^m)$$

$$E(r_i^e) = \frac{1}{n} \sum_{i=1}^{n} w_i E(r_i^m)$$

$$\sum_{i=1}^{n} w_i = 1$$

- Two-Fund Theorem

- $\Sigma^2 (A-Y)$ same for all - no HMT, tailored portfolios.

- Only $E, \sigma$ matter. Not names, styles.

- Think of portfolio not assets in isolation.

- Still true. Widely ignored.

Leverage.

Leverage is very unstable.

$$\frac{1}{C_r} k W_r \rightarrow \frac{\sigma (\frac{1}{C_r})}{C_r} = \frac{\sigma (W_r)}{W_r} = 0.20$$

- No job? (CARM False)

- $E/P$ varies $(\sigma^2)$
Portfolios 3: Héron:

\[
\max \int_0^\infty U(C_t) \, dt \text{ s.t. } W_0,
\]

\[
dW_t = W_t \left[ r_t \, dt + \omega_t \, (dR_t - r_t \, dt) \right] + (Y_t - \kappa_t) \, dt
\]

\[
dW_t = W_t \left[ r_t \, dt + \omega_t \, (dR_t - r_t \, dt) \right] + (Y_t - \kappa_t) \, dt
\]

Time-Varying E \& \Sigma:

\[
\frac{dR_t}{M(Y_t)} = \frac{\Sigma(t)}{\Sigma(t)} \, dt
\]

State Variables:

\[
\begin{align*}
\frac{dX_t}{M(X_t)} &= X_t \, dt + \sigma_t(X_t) \, dW_t \\
\frac{dV_t}{v_t} &= v_t \, dW_t
\end{align*}
\]

\[
V(W_0, X_0) = \max \int_0^\infty e^{-r_t} U(C_t) \, ds \text{ s.t. } W_0, X_0
\]

\[
0 = \max U(C_t) \cdots + V_x \, dx + V_w \, dw + \frac{1}{2} \partial^2 V \frac{\partial^2 C}{\partial x^2} \omega^2 \, dr \, dx
\]

\[
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\]
Portfolios 4: Merton Examples

- Thing: \( w, \frac{1}{\delta} \frac{E(R_{t+n})}{\delta^2} = \frac{1}{\delta} \alpha + \beta (DIP) + \varepsilon_{t+n} \)

\( R_{t+n}^u = \alpha + \beta (DIP) + \varepsilon_{t+n} \)

- Cost: \( \alpha + \beta (DIP) \approx 5\% \). \( E(R) \approx 7\% \).

Huge! Trustworthy?

(Chart):

- Hedging.

Huge! Period M, E is 100%, wrong.

Portfolios 5: Comments

- Big Picture

\( E[U(C_{1+n})] \)

\( C_{1+n} = w_{1+n} = [R_f + w_{1+n} R_{1+n}] \)

\( \frac{\partial}{\partial w} E[U(C_{1+n}) R_{1+n}] = 0 \)

- Before: Fix \( E[R] \) \( \rightarrow E(R) \) "What \( E[R] \) must be to induce investor to just hold market"

- Now: Fix \( E(R), \delta \) \( \rightarrow C_{1+n} \)!
- Conundrum
- Average investor holds market
- Average $\xi$ is zero - $\xi$ is zero sum game, relative to market
- Cannot all rebalance
- $\Rightarrow$ different? "Smarter"? $\xi = \eta$?
  $\Rightarrow$ Merton - lots of risks! More important than $\xi$:
  Express as differences.
- $\xi^{\prime}$ is horribly unstable
  $\Rightarrow$ differences, economic function.

Paradox. Merton hedge not used.

$- N = \frac{V_{W_0}}{V_{W}}$ Too nervous? Hard only numerical

- Bond example. Look at it wrong?

Discount factors

$E \left( R_{t} \right) \xi_{t} \rightarrow \frac{M_{t+1}}{M_{t}}$

May $E \left( C_{t} \right) = S + E \left( M_{t+1} C_{t+1} \right) = W_{0}$

$UC_{t} = \gamma M_{t+1}$

$C_{t} = U^{\prime} \left( \gamma M_{t+1} \right) \quad C_{t} \sim \left( \gamma M_{t+1} \right)^{1/2}$

Options? Dynamic portfolio?

Separate final payoff from financial engineering