DIFFUSION MODELS

$\varepsilon_t \sim iid\ N(0,1)$

$X_{t+1} = \rho X_t + \sigma \varepsilon_{t+1}$ $\quad E_t(X_{t+i}) = \rho^i X_t$

$X_t = \sum_{j=0}^{\infty} \rho^j \sigma \varepsilon_{t-j}$ $\rightarrow$ $\sigma_t^2(X_{t+i}) = (1+\rho^2+\dots+\rho^{2(i-1)})\sigma^2$

RANDOM WALK → BROWNIAN

$Z_t - Z_0 = \sum_{j=1}^{t} \varepsilon_{t-j}$

$E(Z_t - Z_0) = 0$

$\sigma^2(Z_t - Z_0) = t$ $\quad \sum VAR(\varepsilon_{t-j})$

DEFINE $Z_{t+\Delta} - Z_t \sim N(0, \Delta)$ FOR ANY Δ

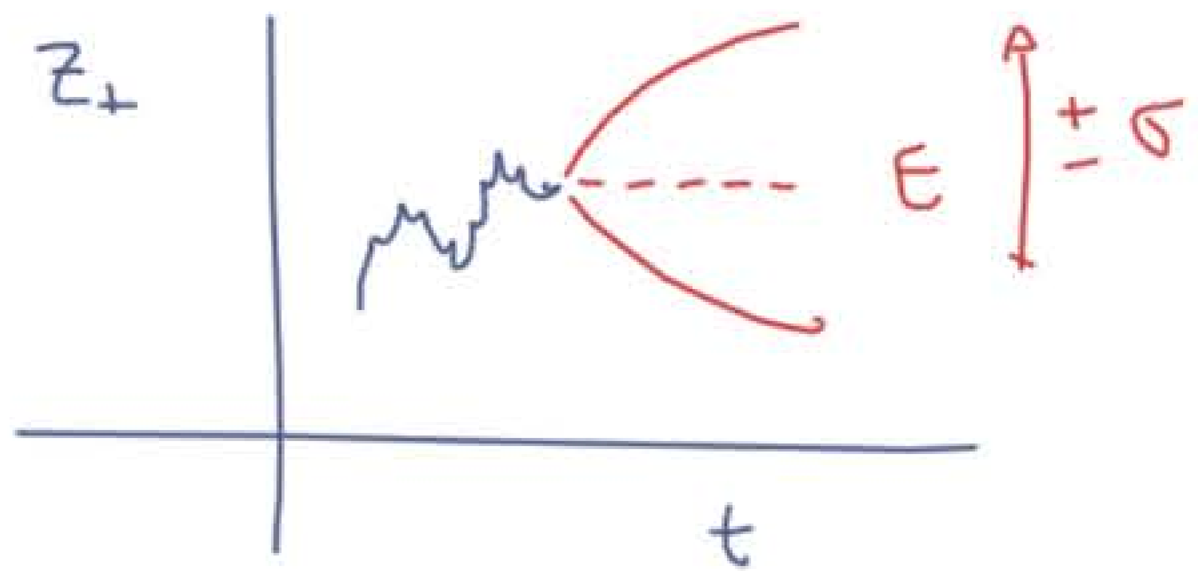
dZ_t

$dZ_t =$ "$\lim_{\Delta \searrow 0}$" $(Z_{t+\Delta} - Z_t) \iff \varepsilon_{t+1} = Z_{t+1} - Z_t$

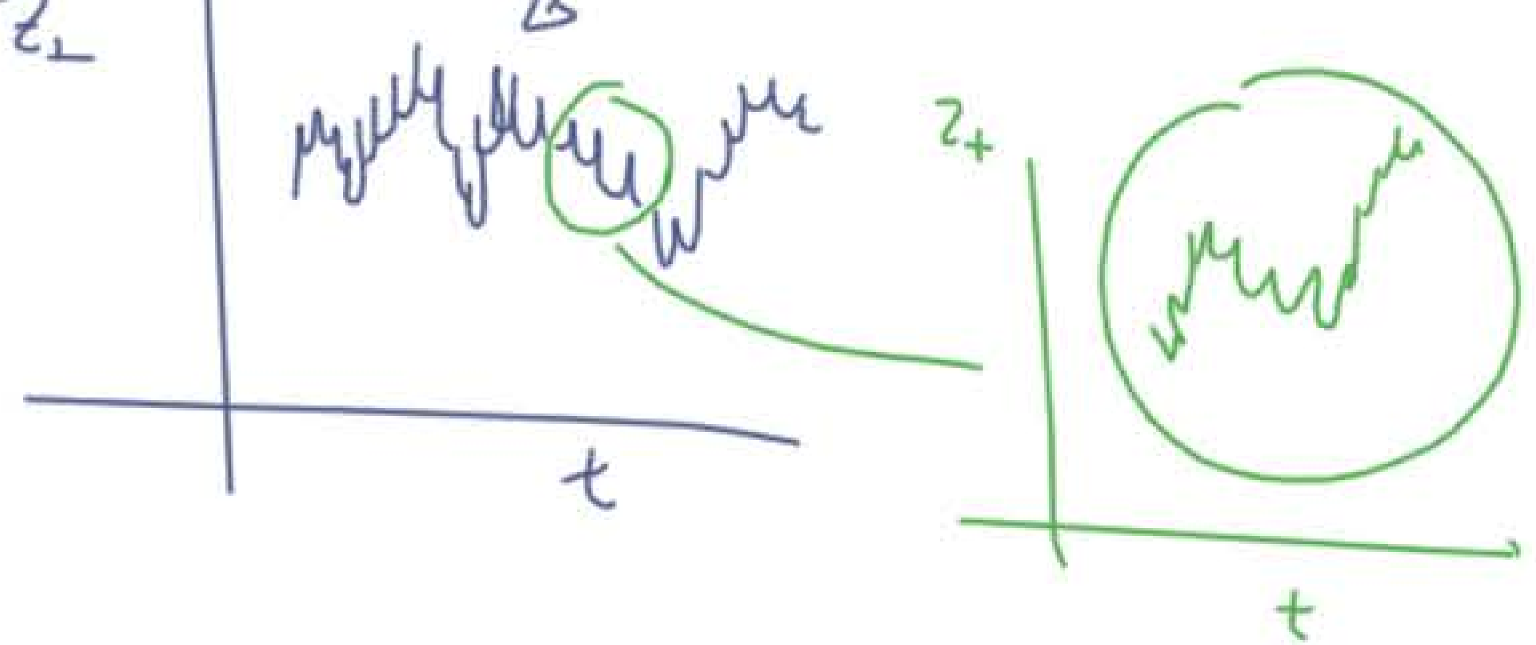
a) $dz \sim \sqrt{dt}$! $\quad \sigma^2(Z_{t+\Delta} - Z_t) = \Delta \quad \sigma(Z_{t+\Delta} - Z_t) = \sqrt{\Delta}$

FORWARD DIFFERENCE

$Z_{t+\Delta} - Z_t$



NOT DIFFERENTIABLE!



b) $E_t(dZ_t) = 0 \longleftrightarrow E_t(\varepsilon_{t+1}) = 0$

$VAR_t(dZ_t) = E_t(dZ_t^2) = dZ_t^2 = dt \Leftarrow dz \sim \sqrt{dt}$!

DIFFUSIONS

(1) $X_{t+1} - X_t = \mu + \sigma \varepsilon_{t+1}$

$dX_t = \mu\, dt + \sigma\, dz_t$

$E_t(dX_t) = \mu\, dt$ $\qquad dz^2 = dt$

$dX_t^2 = \mu^2 \cancel{dt^2} + \sigma^2 dt + 2\mu\sigma \cancel{dt\, dz_t}$ $\qquad dt^{3/2}$

$\sigma_t^2(dX_t) = E_t(dX_t^2) = \sigma^2 dt$

(2) $\frac{dP_t}{P_t} = \underbrace{\mu\, dt}_{0.05} + \underbrace{\sigma\, dz_t}_{0.10}$

RETURN!

$dP_t = \mu P_t\, dt + \sigma P_t\, dz_t$

(3) $X_{t+1} = \rho X_t + \sigma \varepsilon_{t+1}$

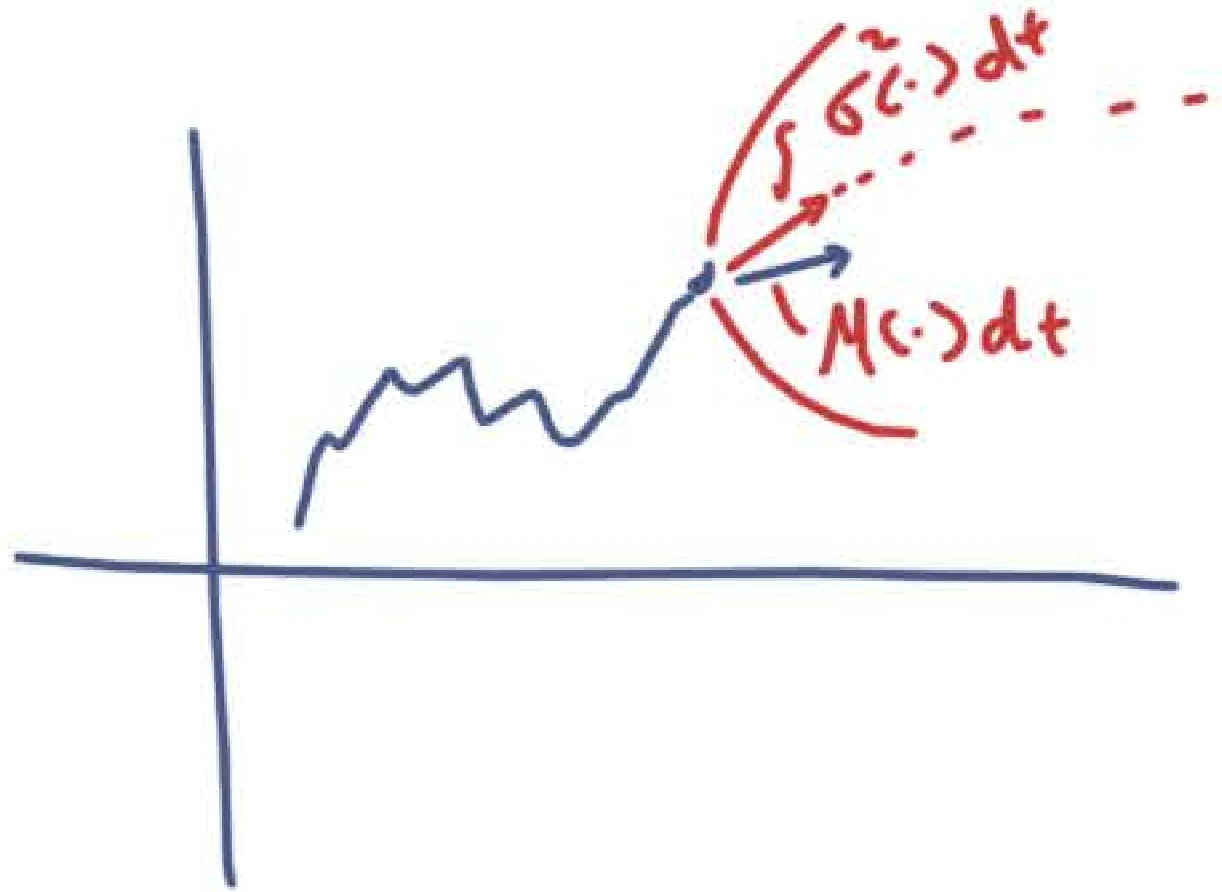
$X_{t+1} - X_t = -(1-\rho) X_t + \sigma \varepsilon_{t+1}$

$dX_t = -\phi X_t\, dt + \sigma\, dz_t$

$E_t dX_t = -\phi X_t\, dt$ $\qquad \sigma_t^2(dX_t) = \sigma^2 dt$

(4) $dX_t = \overbrace{\mu(X_t, t, \dots)\, dt}^{\text{drift}} + \overbrace{\sigma(X_t, t, \dots)\, dz_t}^{\text{diffusion}}$

$\sigma\sqrt{X_t}\, dz_t$



REVIEW

- DEFINE z_t, dz_t "=" $z_{t+\Delta} - z_t$
- $dz_t \approx \sqrt{dt}$; $dz_t^2 = dt$
- BUILD X_t FROM z_t
- FIND $E_t(dz_t)$ $\sigma_t^2(dz_t)$

II. ITO'S LEMMA

- STOCHASTIC CALCULUS IS JUST LIKE CALCULUS. BUT... DO 2nd ORDER EXPANSIONS, KEEP dz, $dz^2 = dt$.

$$\mu(x_t, t, \ldots)$$

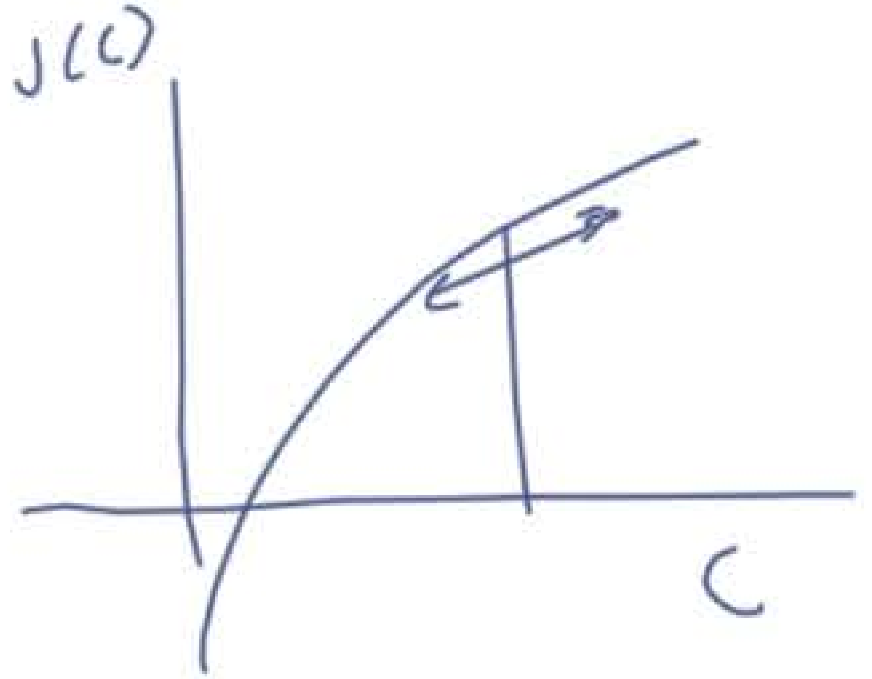
$$dx_t = \mu_t\, dt + \sigma_t\, dz_t$$

$$y_t = f(x_t, t)$$

[illegible]

$$dx_t = \mu\, dt + \sigma\, dz_t \qquad y_t = f(x_t) \qquad dy_t\ ?$$

$$dy_t = \frac{\partial f}{\partial x}\underbrace{dx_t}_{\mu dt + \sigma dz_t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\underbrace{dx_t^2}_{\sigma^2 dz^2 = \sigma^2 dt} + \cdots \cancel{\frac{\partial^3 f}{\partial x^3} dx_t^3}$$



$$dy_t = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2\right] dt + \frac{\partial f}{\partial x}\sigma_t\, dz_t$$

① $\dfrac{dP_t}{P_t} = \mu\, dt + \sigma\, dz_t$

$y_t = \log P_t$? $\quad dy_t = \dfrac{1}{P_t} dP_t - \dfrac{1}{2}\dfrac{1}{P_t^2} dP_t^2$

$= \mu\, dt + \sigma\, dz_t - \frac{1}{2}\sigma^2 dt$

$\underline{dy_t} = \underbrace{\left[\mu - \frac{1}{2}\sigma^2\right]}_{\text{LOWER DRIFT!}} dt + \sigma\, dz_t$

② $d(x_t y_t)$?

$$= \frac{\partial(xy)}{\partial x} dx_t + \frac{\partial(xy)}{\partial y} dy_t + \frac{1}{2} \times 2 \times \frac{\partial^2(xy)}{\partial x_t \partial y_t} dx_t dy_t \cancel{+ \frac{1}{2}\frac{\partial^2(xy)}{\partial x^2} dx^2}$$

$$d(x_t y_t) = y_t\, dx_t + x_t\, dy_t + \underline{\underline{dx_t\, dy_t}}\,!$$

• $\underline{\text{SUMMARY}}$ $\quad dz,\ dz^2 = dt,\ \cancel{dz\, dt},\ \cancel{dt^2}\ f' + f''$

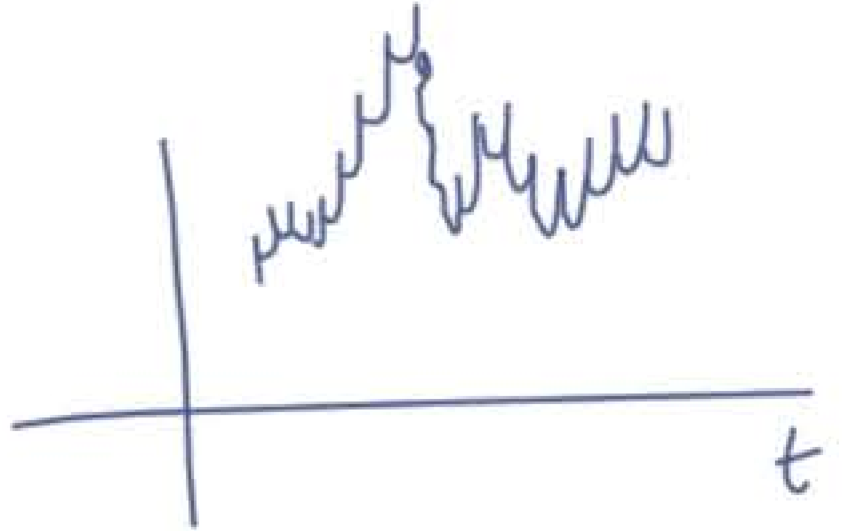
III Solving Stochastic Differential Equations

$$x_t = \rho x_{t-1} + \varepsilon_t \rightarrow x_t = \rho^t x_0 + \sum_{j=0}^{t} \rho^j \varepsilon_{t-j}$$

Stochastic Integral

$$Z_T - Z_0 = \int_{t=0}^{T} dZ_t \longleftrightarrow Z_T - Z_0 = \sum_{t=1}^{T} \varepsilon_t$$

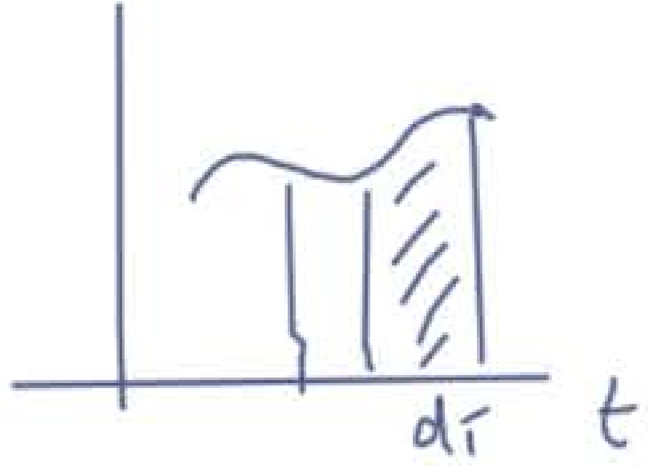
YES $\int_0^t dZ_t$



$$\int_{t=0}^{T} dZ_t = (Z_\Delta - Z_0) + (Z_{2\Delta} - Z_\Delta) + \cdots + (Z_T - Z_{T-\Delta})$$

"ADD UP CHANGES"

NO $\int_0^t \frac{dZ_t}{dt} \cdot dt$



this should be a sqrt (T), not T.

$$\int_{t=0}^{T} dZ_t = Z_T - Z_0 = N(0,T) = T\varepsilon \qquad \varepsilon \sim N(0,1)$$

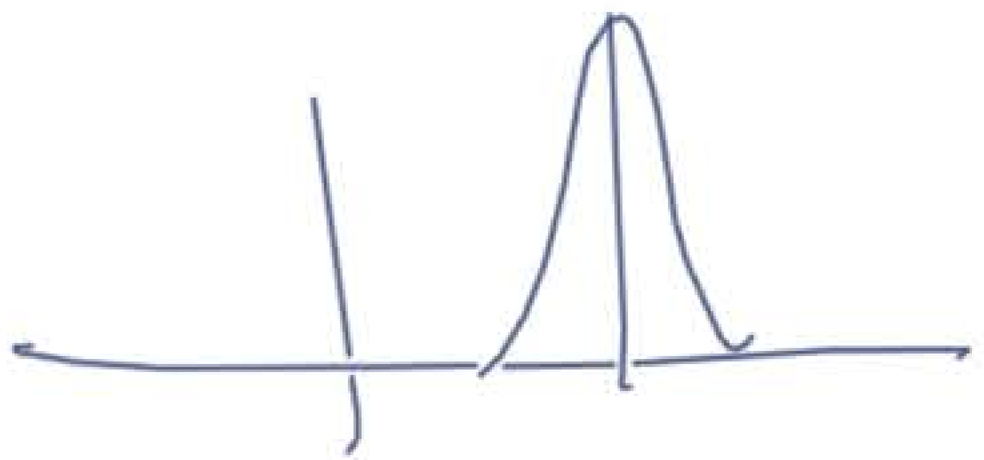
DIFFUSIONS

$$dX_t = \mu\, dt + \sigma\, dZ_t$$

① BM

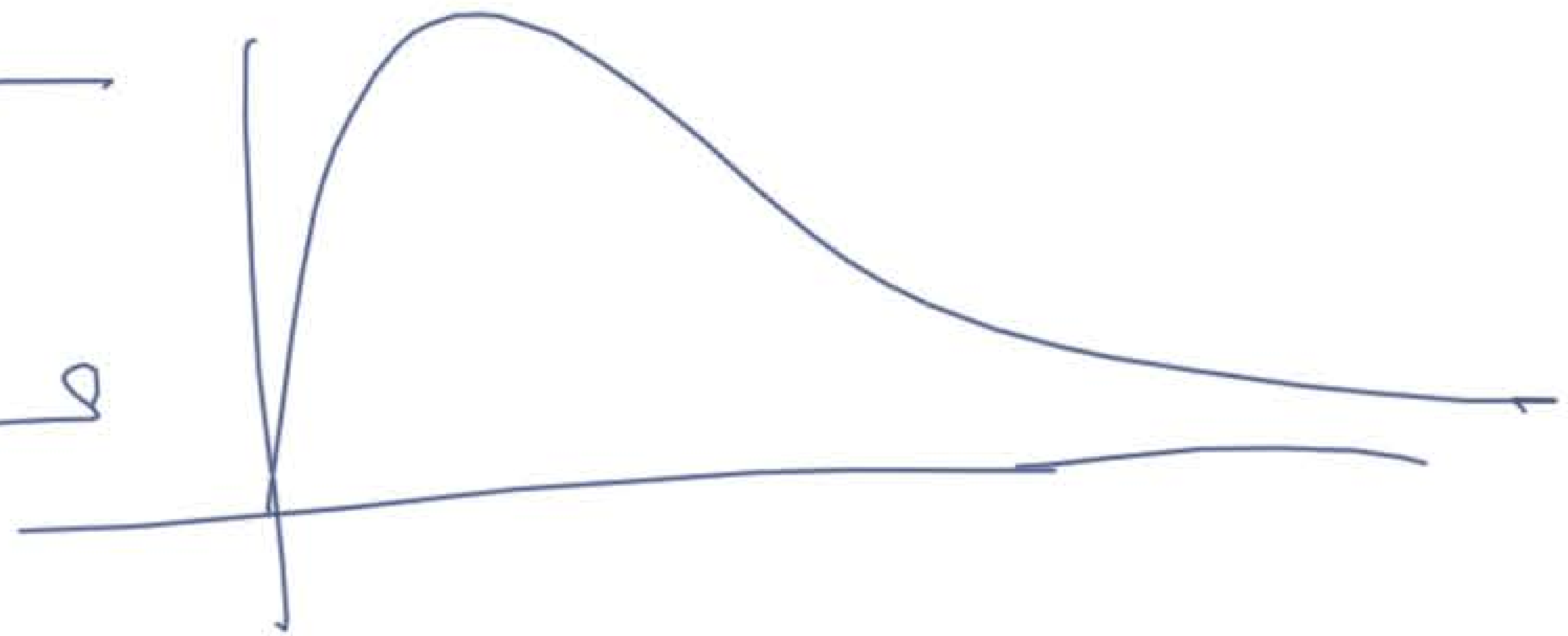
$$\int_{t=0}^{T} dX_t = \mu \int_{t=0}^{T} dt + \sigma \int_{t=0}^{T} dZ_t$$

$$X_T - X_0 = \mu T + \sigma \int_{t=0}^{T} dZ_t \to E_0(X_T - X_0) = \mu T \qquad \sigma_0^2(X_T - X_0) = \sigma^2 T$$



② GBM

$$\frac{dP_t}{P_t} = \mu\, dt + \sigma\, dZ_t \to$$



$$\int_{t=0}^{T} d\log P_t = (\mu - \tfrac{1}{2}\sigma^2)\int_{t=0}^{T} dt + \sigma \int_{t=0}^{T} dZ_t$$

$$\log P_T - \log P_0 = (\mu - \tfrac{1}{2}\sigma^2)T + \sigma \int_{t=0}^{T} dZ_t$$

$$R_{0\to T} = \frac{P_T}{P_0} = e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma \int_{t=0}^{T} dZ_t}$$

$$\to E_0\left(\frac{P_T}{P_0}\right) = e^{(\mu - \frac{1}{2}\sigma^2)T + \frac{1}{2}\sigma^2 T} = e^{\mu T} \qquad \left[E(e^{x}) = e^{E(x) + \frac{1}{2}\sigma^2(x)}\right]$$

(3) AR(1) $dX_t = -\phi X_t dt + \sigma dZ_t$

$$X_T = e^{-\phi T} X_0 + \sigma \int_{t=0}^{T} e^{-\phi(T-t)} dZ_t \Rightarrow E_0(X_T) = e^{-\phi T} X_0$$

$$\sigma_0^2(X_T) = \sigma^2 \int_{t=0}^{T} e^{-2\phi(T-t)} \quad dZ_t^2 = dt$$

$$X_T = \rho^t X_0 + \sigma \sum_{j=0}^{T} \rho^j \varepsilon_{t-j}$$

$$= \frac{1-e^{-2\phi T}}{2\phi} \sigma^2$$

(4) GENERAL $dX_t = \mu(X_t, \cdot)dt + \sigma(X_t, \cdot)dZ_t$ NOT SO EASY!

<u>SUMMARY</u>

- $\int_{t=0}^{T} dX_t = X_T - X_0$ "ADDING CHANGES" STOCHASTIC INTEGRAL
- $\int_{t=0}^{T} dZ_t \sim N(0,T)$
- SOLUTION $\rightarrow$ MOMENTS $E(X_T)$ $\sigma^2(X_T)$