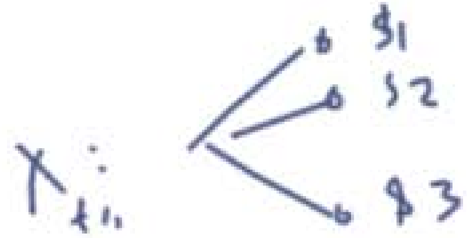
1. OVERVIEW



$u(c_t) + \beta E_t u(c_{t+1})$

$$P = E(mx)$$

$$P_t = \underbrace{E_t(m_{t+1} x_{t+1})}_{\text{GENERAL}} = E_t\Big[\underbrace{\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}}_{\text{UTILITY}}\Big] = E_t\Big[\underbrace{\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} x_{t+1}}_{\text{POWER UTILITY EXAMPLE}}\Big]$$

WILLINGNESS TO PAY

→ RETURNS, EXCESS RETURNS, PRESENT VALUES, CONTINUOUS TIME

→ CLASSIC RESULTS / LANGUAGE OF FINANCE

- INTEREST RATES
- RISK PREMIUMS – COVARIANCE MATTERS
- $E(R)$ AND β
- MEAN-VARIANCE FRONTIER
- RANDOM WALKS
- GENERAL EQUILIBRIUM + CAUSALITY

2. ALTERNATIVE REPRESENTATIONS $P=E(mx) \Rightarrow$

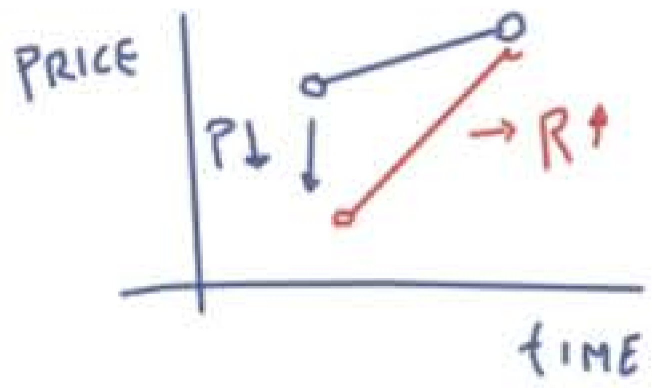
$\Rightarrow$ RETURNS, PRESENT VALUES, CONTINUOUS TIME

(MEET THE PLAYERS)

RETURN

$$P=1 \rightarrow \boxed{1=E_t(m_{t+1}R_{t+1})}$$

LOW PRICE = HIGH R



$$R_{t+1} = \frac{P_{t+1}+D_{t+1}}{P_t} = 1.10 \text{ NOT } 0.10 \text{ OR } 10\%.$$

RISKFREE RATE $\quad 1 = E(mR^f) = E(m)R^f \rightarrow \boxed{R^f = 1/E(m)}$

$$p^{(1)} = E(m \times 1) = E(m) \quad ; \quad R^f = 1/p^{(1)}$$

EXCESS RETURN $\quad R^e = R - R^f$ or $R^i - R^j$. $\quad P=0$!

BET. LEVERAGE. LONG-SHORT. FOCUS ON RISK NOT TIME

$$\boxed{0=E(mR^e)}$$

PRESENT VALUES $\quad \{x_t\}$ = STREAM OF DIVIDENDS $[d_t!]$

$$P_t = E_t\sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} x_{t+j} = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} x_{t+j} = \boxed{E_t\sum_{j=1}^{\infty} m_{t,t+j}\, x_{t+j} = P_t}$$

$$P_t = E_t\int_{s=0}^{\infty} e^{-\delta s}\frac{u'(c_{t+s})}{u'(c_t)}\, x_{t+s}\,ds = E_t\int_{s=0}^{\infty} e^{-\delta s}\left(\frac{c_{t+s}}{c_t}\right)^{-\gamma}x_{t+s}\,ds = \boxed{E_t\int_{s=0}^{\infty} \frac{\Lambda_{t+s}}{\Lambda_t}\,x_{t+s}\,ds = P_t}$$

$$m_{t,t+j} = m_{t+1}m_{t+2}\cdots m_{t+j} = \frac{\Lambda_{t+j}}{\Lambda_t} \quad ; \quad \beta = e^{-\delta}$$

CONTINUOUS TIME LIKES LEVELS (Λ_t, Z_t)

(PRESENT VALUES)

SOURCE $\quad \max E \sum_{t=0}^{\infty} \beta^t u(c_t)$ or $E \int_{t=0}^{\infty} e^{-\delta t} u(c_t)\, dt$

then $C_t - \xi P_t \;,\; C_{t+s} + \xi X_{t+s}$

SPLIT

RETURNS ⟷ PRESENT VALUES ; DISCRETE TIME

⟺

$$P_t = E_t \sum_{j=1}^{\infty} M_{t,t+j} X_{t+j} = E_t\left[M_{t+1}\left(P_{t+1} + X_{t+1}\right)\right]$$

$$\Leftrightarrow \quad 1 = E_t\left(M_{t+1} \frac{P_{t+1} + X_{t+1}}{P_t}\right) = E_t\left(M_{t+1} R_{t+1}\right)$$

RETURN IN CONTINUOUS TIME

$$dR_t = \frac{P_{t+\Delta} - P_t + X_t \Delta}{P_t} \Rightarrow \boxed{dR_t = \frac{dP_t}{P_t} + \frac{X_t}{P_t} dt = \frac{dV_t}{V_t}}$$

TYPICALLY $dR_t = \mu\, dt + \sigma\, dz_t$

A NET RETURN $\quad$ CUMULATIVE VALUE PROCESS

0.10

R ⟷ PV ; CONTINUOUS TIME

$$P_t = E_t \int_{s=0}^{\infty} \frac{\Lambda_{t+s}}{\Lambda_t} X_{t+s}\, ds = E_t \int_{s=0}^{\Delta} \frac{\Lambda_{t+s}}{\Lambda_t} X_{t+s}\, ds + \overbrace{\frac{\Lambda_{t+\Delta}}{\Lambda_t}}^{\Delta} \underbrace{\int_{s=\Delta}^{\infty} \frac{\Lambda_{t+s}}{\Lambda_{t+\Delta}} X_{t+s}\, ds}_{P_{t+\Delta}}$$

(...)

$$\boxed{0 = E_t\left(\frac{d\left[\Lambda_t V_t\right]}{\Lambda_t V_t}\right) = \frac{X_t}{P_t} dt + E_t\left[\frac{d(\Lambda_t P_t)}{\Lambda_t P_t}\right] ;\; 0 = E_t\left[d(\Lambda_t V_t)\right]}$$

$1 = E(MR)$

RISKFREE RATE, CONTINUOUS TIME

$$\boxed{dR_t = r_t^f\, dt}$$

(no dz_t = riskfree . r_t^f can change)

A) $\dfrac{dB_t}{B_t} = r_t^f\, dt \quad [V]$

B) $P = 1 \;;\; X_t\, dt = r_t^f\, dt$.

CLASSIC ISSUES IN FINANCE

3. RISK FREE RATE AND MACROECONOMICS

$$R^f_t = 1/E_t(M_{t+1}) = 1/E\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]$$

APPROXIMATE? CONTINUOUS TIME!

$$\boxed{0 = E_t\left(\frac{d(\Lambda_t V_t)}{\Lambda_t V_t}\right) = E_t\left(\frac{d\Lambda_t}{\Lambda_t} + \frac{dV_t}{V_t} + \frac{d\Lambda_t}{\Lambda_t}\frac{dV_t}{V_t}\right)}$$

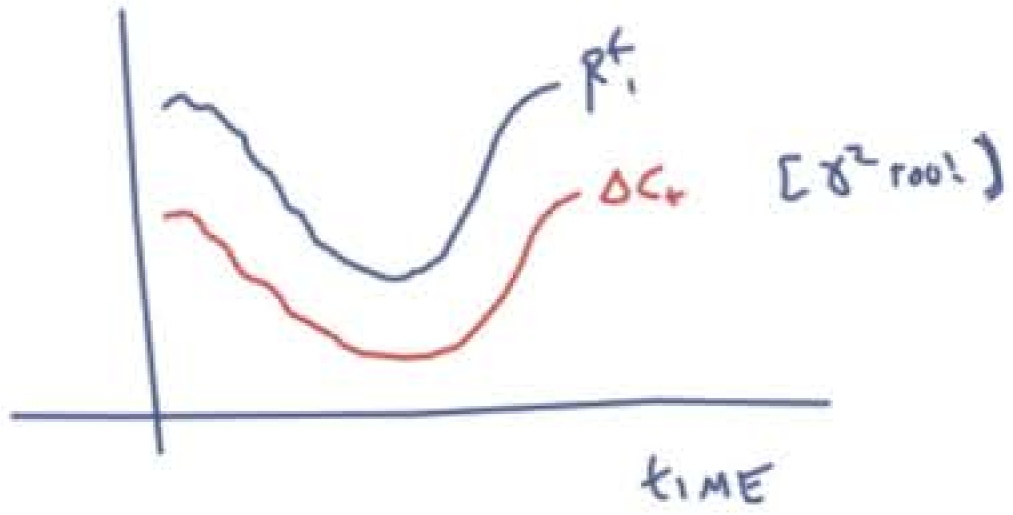
$$\frac{dB_t}{B_t} = r^f_t\, dt \rightarrow \boxed{r^f_t = -E_t\left(\frac{d\Lambda_t}{\Lambda_t}\right)} \quad \text{-- not } 1/\ !$$

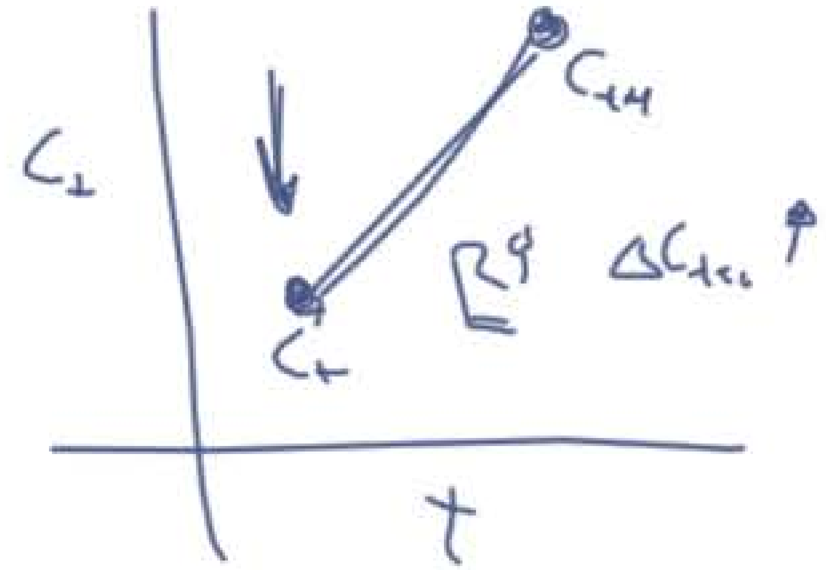
$$\Lambda_t = e^{-\delta t} C_t^{-\gamma} \rightarrow \boxed{\frac{d\Lambda_t}{\Lambda_t} = -\delta\, dt - \gamma\frac{dC_t}{C_t} + \frac{1}{2}\gamma(\gamma+1)\frac{dC_t^2}{C_t^2}}$$

I.E. IF $\frac{dC_t}{C_t} = \mu_c\, dt + \sigma_c\, dz_t \rightarrow \frac{d\Lambda_t}{\Lambda_t} = -\delta\, dt - \left[\gamma\mu_c - \frac{1}{2}\gamma(\gamma+1)\sigma_c^2\right]dt - \gamma\sigma_c\, dz_t$

$$\rightarrow r^f_t = \delta + \gamma E_t\left[\frac{dC_t}{C_t}\right] - \frac{1}{2}\gamma(\gamma+1)\sigma_t^2\left[\frac{dC_t}{C_t}\right]$$

$$R^f_t \approx 1 + \underbrace{\delta}_{\text{IMPATIENCE}} + \underbrace{\gamma E_t(\Delta C_{t+1})}_{\text{INTERTEMPORAL SUBSTITUTION}} - \underbrace{\frac{1}{2}\gamma(\gamma+1)\sigma_t^2(\Delta C_{t+1})}_{\text{PRECAUTIONARY SAVING}}$$





4. RISK PREMIUMS

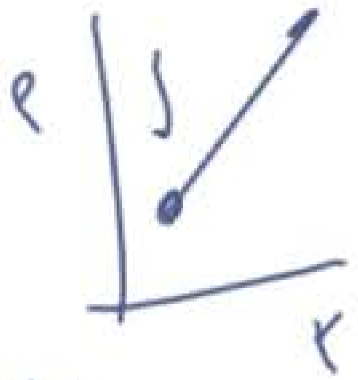
PRICE DISCOUNT

$$P = E(mx) = E(m)E(x) + \text{cov}(m,x)$$

$$P = \underbrace{\frac{E(x)}{R^f}}_{\text{TIME}} + \underbrace{\text{cov}(m,x)}_{\text{RISK}}$$

RETURN

$$R = R^f + (R - R^f) = R^f + R^e$$



$$0 = E(mR^e) = \frac{E(R^e)}{R^f} + \text{cov}(m, R^e)$$

$$\boxed{\begin{aligned} E(R^e) &= -R^f \cdot \text{cov}(m, R^e) \\ E(R) &= R^f - R^f \cdot \text{cov}(m, R) \end{aligned}}$$

$$\text{cov}\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t+1}\right)?$$

→ CONTINUOUS!

- COVARIANCE DRIVES RISK PREMIUM NOT VARIANCE $\sigma^2(R)$
- LOW P ↔ HIGH R
- IS HIGH E(R) GOOD OR BAD?
- $R = R^f + R^e$
  8% 2% 6%

CONTINUOUS

$$0 = E_t\left[\frac{d\Lambda}{\Lambda} + \frac{dv}{v} + \frac{d\Lambda\, dv}{\Lambda v}\right] = -r^f dt + E_t(dR) + \text{cov}\left(\frac{d\Lambda}{\Lambda}, dR\right)$$

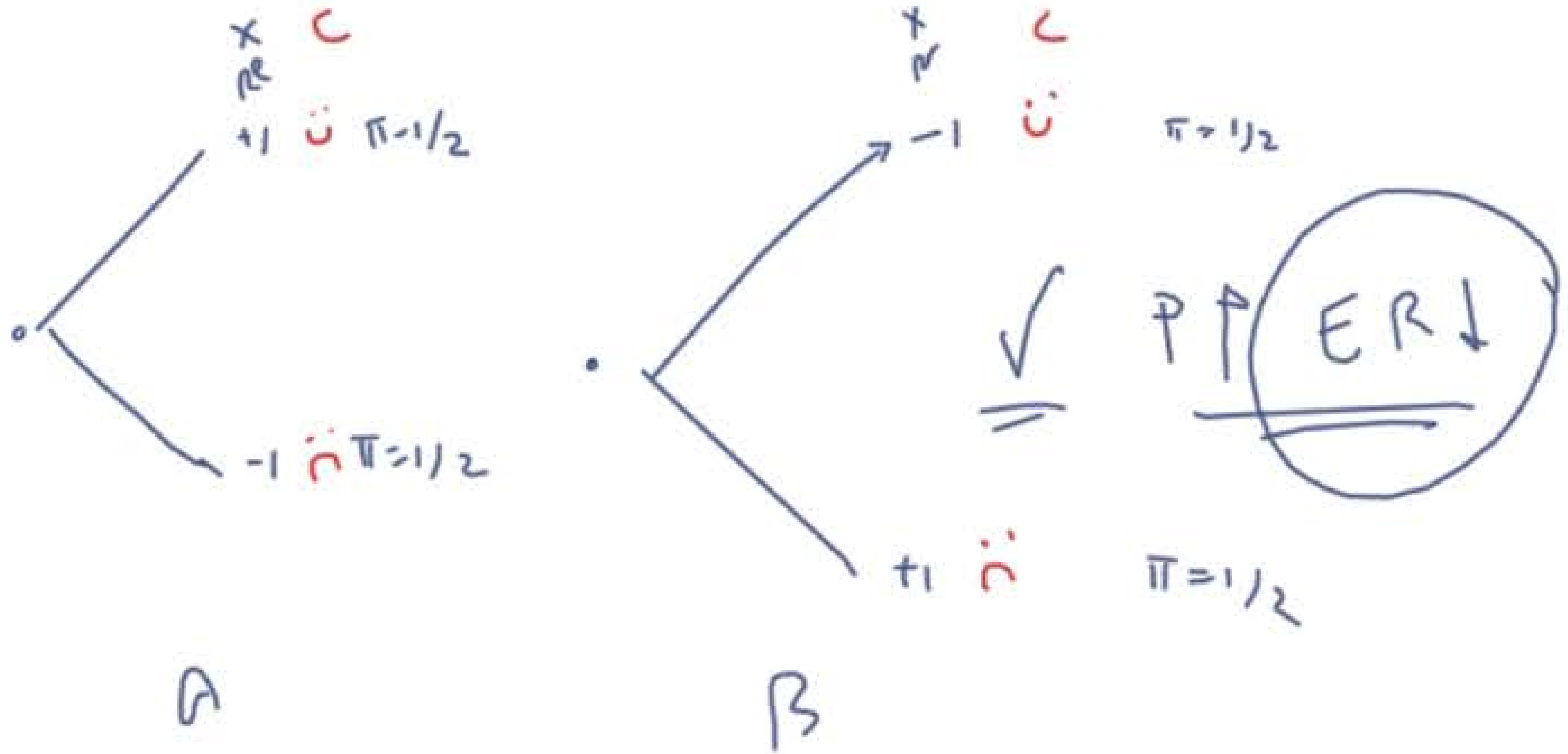
$$\boxed{E_t(dR_t) = r^f_t\, dt - \text{cov}_t\left(\frac{d\Lambda_t}{\Lambda_t}, dR_t\right)}$$

5. CONSUMPTION + RISK PREMIUMS

$$E_t(dR_t) = r_t^f + \gamma \, cov_t\left(\frac{dc_t}{c_t}, dR_t\right)$$

$$\boxed{E_t(R_{t+1}) \simeq R_t^f + \gamma \, cov_t(\Delta c_{t+1}, R_{t+1})}$$

- INVESTORS CARE ABOUT c, NOT R^i, NOT R^P, NOT $\sigma(IBM, MSFT)$
- COV: HOW MUCH A LITTLE MORE R^i AFFECTS $\sigma^2(c)$
- INSURANCE



6 RISK PREMIUMS AND BETAS

$$E(R^{ei}) = -\,cov(m, R^{ei}) = \frac{cov(m, R^{ei})}{var(m)}\,(-\,var(m)) = \beta_{i,m} \cdot \lambda_m$$

— MARKET PRICE OF RISK

$$E(R^{ei}) = \gamma\, cov(\Delta c_{t+1}, R^{ei}_{t+1}) = \frac{cov(\Delta c_{t+1}, R^{ei}_{t+1})}{var(\Delta c_{t+1})} \cdot [\gamma\, var(\Delta c_{t+1})] = \beta_{i,\Delta c} \cdot \lambda_{\Delta c}$$

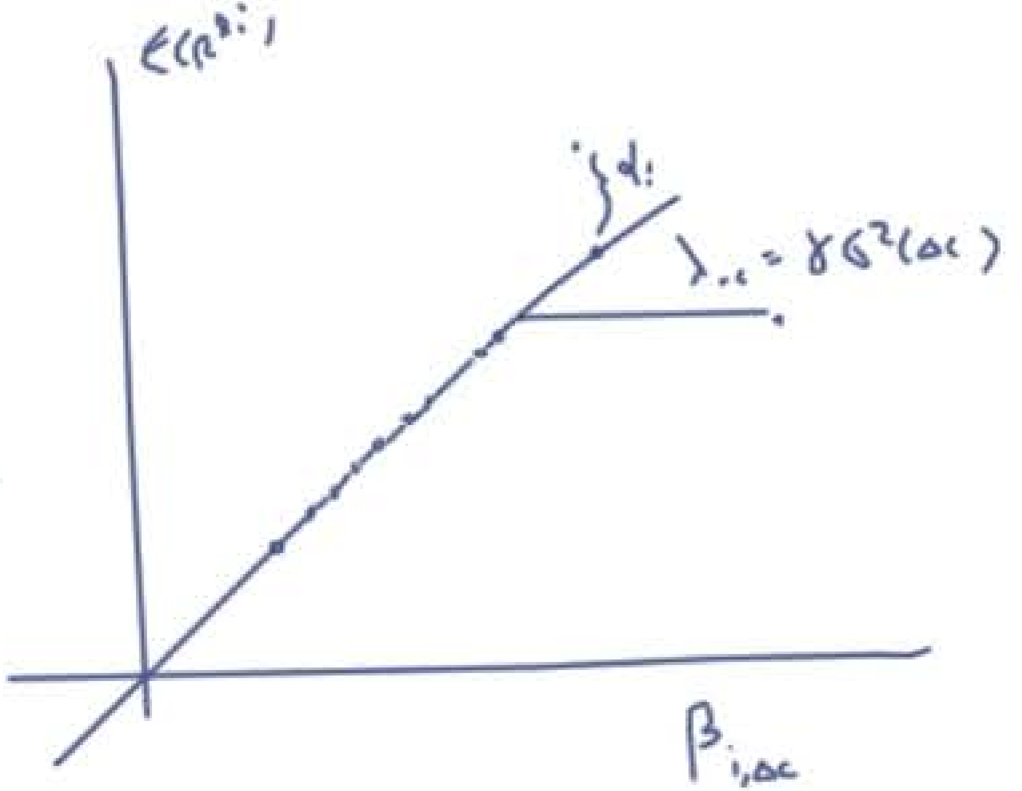
1.2 ① $R^{ei}_{t+1} = a_i + \beta_{i,\Delta c}\, \Delta c_{t+1} + \varepsilon^i_{t+1} \quad t = 1, 2 \ldots T \quad \forall i$

② $E(R^{ei}) = \beta_{i,\Delta c} \cdot \lambda_{\Delta c} \quad (\forall i)$

$\Rightarrow$ ⊙ $y_i = x_i\, b$

6 λ IS SAME FOR ALL i. ← bigger for $\gamma, \sigma^2_{\Delta c}$

- ABOUT WHY $E(R^{ei}) > E(R^{ei})$ [VALUE]
- NOT WHY R_{t+1} VARIES OVERTIME, PREDICTING $E_t R_M$ OVERTIME



- $\sigma^2(R^{ei})$ DOES NOT MATTER
- "ONLY SYSTEMATIC RISK IS PRICED"

$$R^{ei}_{t+1} = \beta_{im} M_{t+1} + \varepsilon^i_{t+1}$$

$$\sigma^2(R^{ei}_{t+1}) = \underbrace{\beta^2_{iM}\, \sigma^2(M_{t+1})}_{\text{"SYSTEMATIC"}} + \underbrace{\sigma^2(\varepsilon^i_{t+1})}_{\text{"IDIOSYNCRATIC" "DIVERSIFIABLE"}}$$

- "ASSET PRICING MODEL"

$$M \leftarrow u'(c) \sim \Delta c_{t+1}$$

TO … HML, SMB ETC $M = f(\text{DATA})$

7. MEAN VARIANCE FRONTIER AND ROLL THEOREM

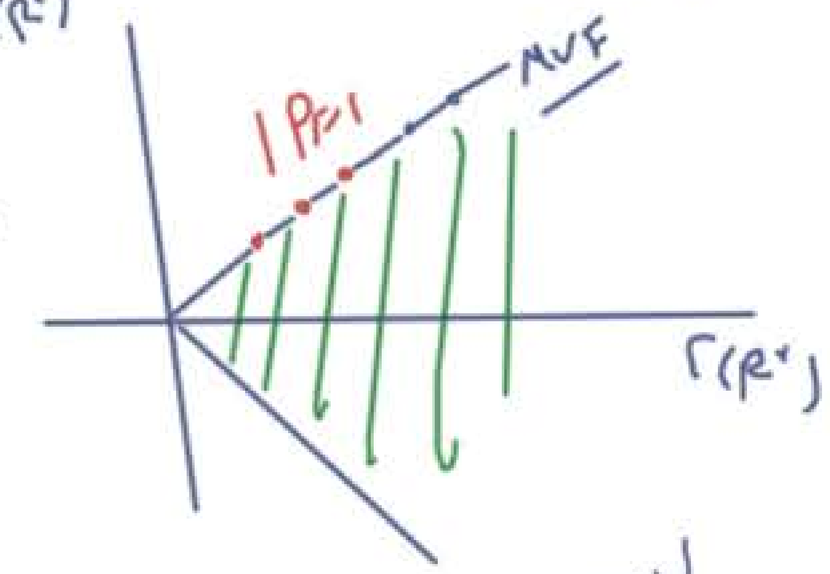
$$\boxed{0 = E(mR^e)} = E(m)E(R^e) + cov(m,R^e) = E(m)E(R^e) + \rho\,\sigma(m)\sigma(R^e)$$

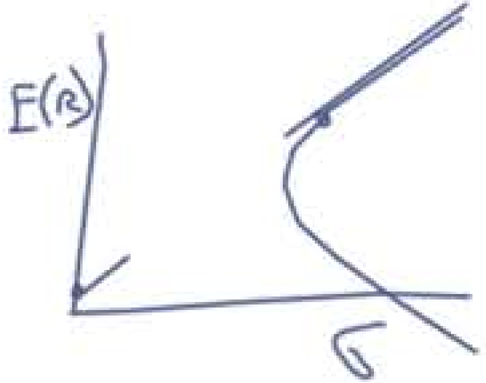
$$\frac{E(R^e)}{\sigma(R^e)} = -\frac{\sigma(m)\rho(m,R^e)}{E(m)}$$

SHARPE RATIO - INVARIANT TO LEVERAGE $\Rightarrow \frac{E(2R^e)}{\sigma(2R^e)} = \frac{E(R^e)}{\sigma(R^e)}$

$$\|\rho\| < 1 \Rightarrow \frac{\|E(R^e)\|}{\sigma(R^e)} < \frac{\sigma(m)}{E(m)} = \gamma\,\sigma(\Delta c_{t+1})$$

$\Rightarrow$





① FRONTIER!

② SLOPE IS HIGHER FOR MORE RISK ($\sigma(\Delta c)$) or RISK AVERSION (γ)

③ ALL FRONTIER RETURNS HAVE $\rho = 1$ W. m + W. EACHOTHER

④ "TWO FUND THEOREM" R^{mv} ON MVF → ALL MVF $R^e = w R^{emv}$
$= w\,[R^{mv} - R^f]$

⑤ "ROLL THEOREM"

$$E(R^e) = \beta_{R^e, R^{mv}} \cdot \lambda_{R^{mv}} \leftrightarrow R^{mv} \text{ IS ON MVF}$$

(PROOF $\rho = 1 \leftrightarrow R^{emv}$ ON MVF $\leftrightarrow m = a + bR^{emv}$)

$\Rightarrow$ ANY ASSET PRICING MODEL = SOME R IS ON MVF

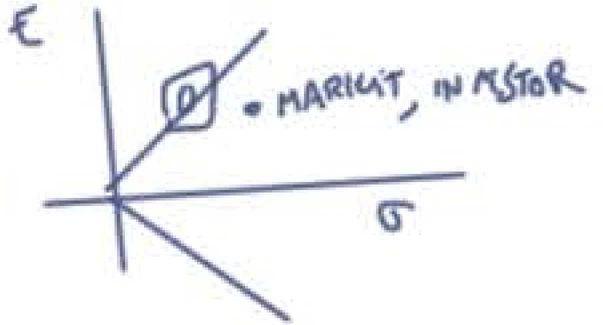
EXAMPLE
CAPM R^{emkt} ON MVF

$\Leftrightarrow E(R^{ei}) = \beta_{i,mkt}\,\underbrace{E(R^{emkt})}_{\lambda_{mkt}}$

HISTORY

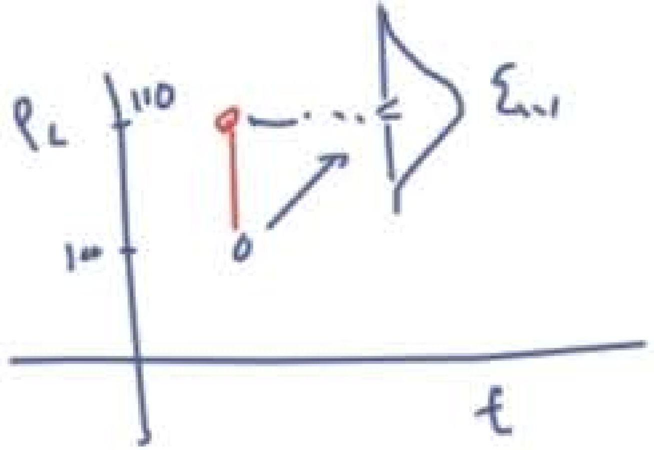
⑥ NOT $R \sim N$; APPLIES TO ALL ASSETS ; ~~NOT~~ ANY U

NOT MARKET IS ON MVF
INVESTOR WANTS MVF



8. Random walks + time varying risk premiums

- Behavior over time



① $P_t = 100 \quad E_t(P_{t+1}) = 110$?

$\rightarrow P_t = E_t(P_{t+1})$

$P_{t+1} = P_t + \varepsilon_{t+1}$ "random walk"

③ Right $\quad P_t = E_t(M_{t+1}(P_{t+1} + D_{t+1}))$

$M_{t+1}(P_{t+1} + D_{t+1}) = P_t + \varepsilon_{t+1}$

$\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}[P_{t+1} + D_{t+1}] = P_t + \varepsilon_{t+1}$

$\approx P_{t+1} = P_t + \varepsilon_{t+1}$ daily

④ $E_t(dR_t) = E_t\left(\frac{dV_t}{V_t}\right) = \mu dt$

$d\log V_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz_t$

$\log V_{t+\Delta} = \log V_t + \underbrace{(\mu - \frac{1}{2}\sigma^2)\Delta}_{\text{small (a) high frequency}} + \varepsilon_{t+1}\sqrt{\Delta}$

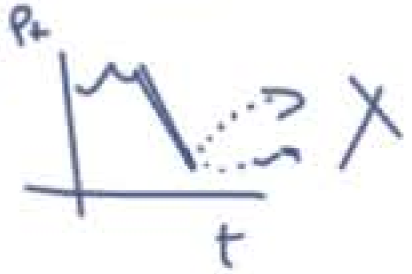
$\hookrightarrow$ constant ER

$\Rightarrow$ Nothing else matters

$P_{t+1} = a + 1 \cdot P_t + b x_t + \varepsilon_{t+1}$

$b = 0$

- "Efficiency" price contains all information $\Leftarrow$ competition
- Techincal trading, price pressure, bounceback, cong run



Graph

FACT $b \neq 0$!

$$R^e_{t+1} = a + \underline{b} \cdot (D/P_t) + \varepsilon_{t+1}$$

$$\Downarrow$$

THEORY $\underline{E_t(R^e_{t+1})} = \underline{cov_t(M_{t+1}, R^e_{t+1})} = \rho_t \cdot \gamma_t \cdot \sigma_t(\Delta c_{t+1})$

$$= \rho_t \cdot \underline{\sigma_t(M_{t+1})}$$

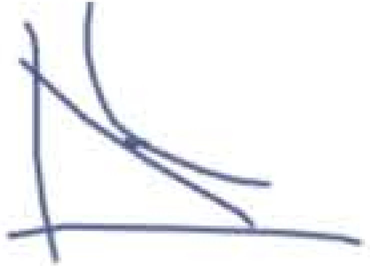
- $\sigma_t(M_{t+1})$ VARIES OVER TIME!

- "INEFFICIENCY"? γ_t, σ_t CAN VARY OVER TIME. NO STORIES, HOW?

- "FORECAST RETURNS" = DOES $E_t(R^e_{t+1})$ VARY?

9. GENERAL EQUILIBRIUM + CAUSALITY

PUZZLE:



FINANCE $R^f = \dfrac{1}{E\left(\beta \frac{u'(c_{t+1})}{u'(c_t)}\right)}$ $\{c_t\} \to R^f$

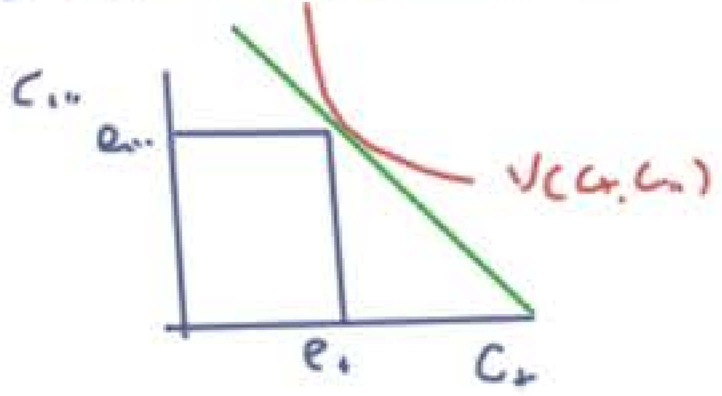
MACRO
MICRO $u'(c_t) = R^f E(\beta u'(c_{t+1}))$ $R^f \to \{c_t\}$

$R^f, E_t c_{t+1} \to c_t$

$E(R^e) = -\gamma \, cov(\Delta c_{t+1}, R^e_{t+1})$ $\Delta c_t, cov, R_{t+1} \to ?$

GENERAL EQUILIBRIUM = SUPPLY!

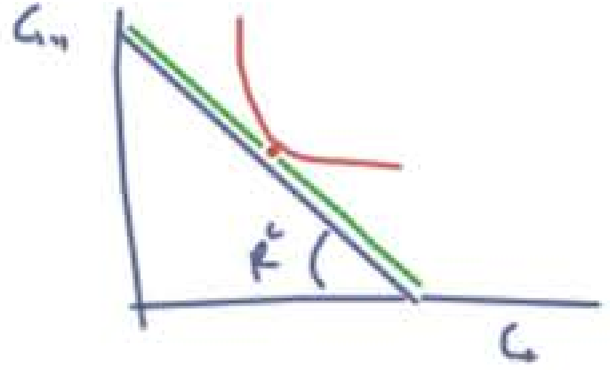
CASE 1 ENDOWMENT ECONOMY (LUCAS)



INDIVIDUAL $R^f \to c$

ECONOMY $e \to c \to R^f$

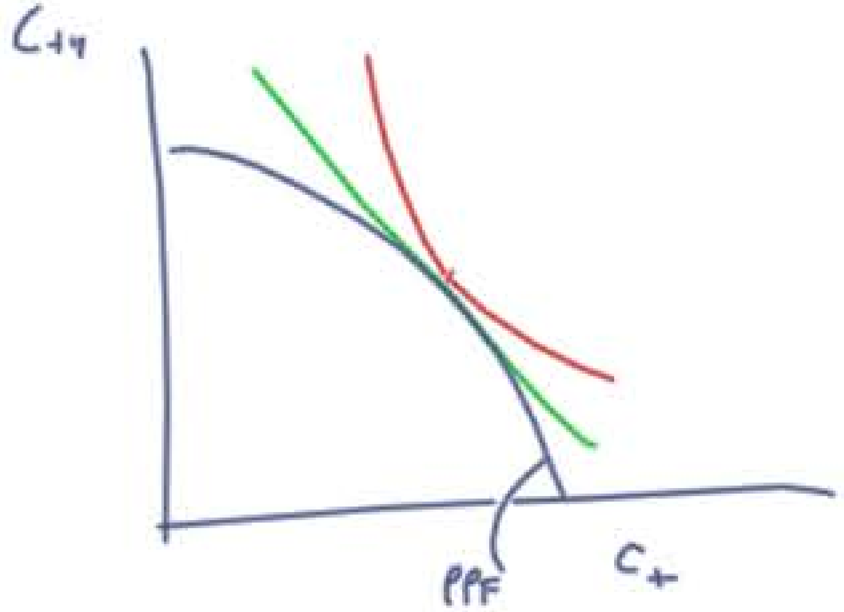
CASE 2 TRADITIONAL FINANCE



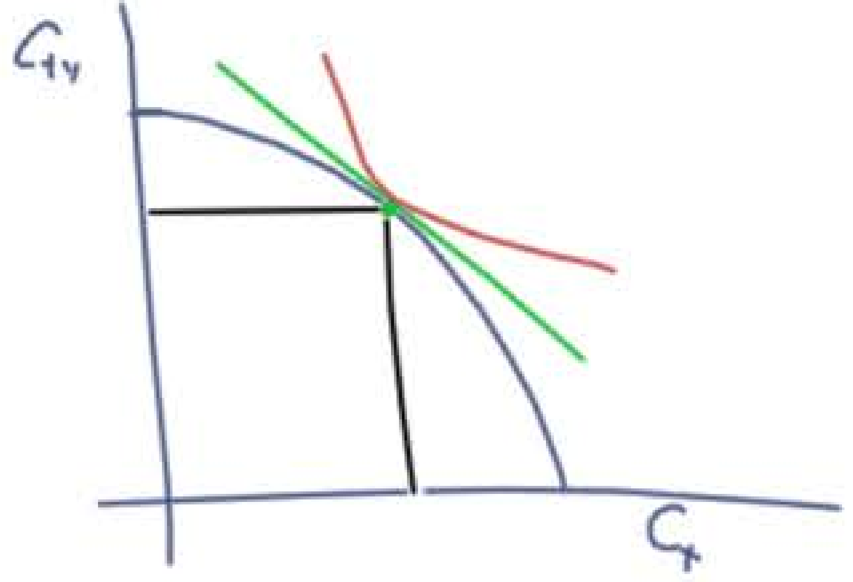
INDIVIDUAL + ECONOMY

$R \to C$. DETERMINE COMPOSITION OF MARKET PORTFOLIO

CASE 3 REALITY



LUCAS THEOREM



- IF YOU MODEL $\{C_t\ C_{t+1}\}$ CORRECTLY, NO ERROR

- IF YOU MODEL R_{t+1} CORRECTLY NO ERROR

EXOGENOUS BETAS

R. FUTURE ....

FUTURE ...

$$E(R^i) = \cdot \operatorname{cov}\left(\beta \frac{u'(C_{t+1})}{u'(C_t)} \cdot \frac{P_{t+1} + D_{t+1}}{P_t}\right)$$

BOTTOM LINE

- $P = E(mx)$ IS A CONDITION THAT MUST HOLD, PART OF GE
- $P = E(mx)$ IS USEFUL IN MANY APPLICATIONS, CAN STOP
- BEWARE "EXPLAIN", "CAUSE", "EXOGENOUS"
- GENERAL EQ. MODELS THAT SPECIFY DEMAND (UTILITY) AND SUPPLY (TECHNOLOGY, INVESTMENT), AND TRULY EXOGENOUS SHOCKS (?)

10 SUMMARY + PREVIEW

$P = E(mx) \longrightarrow$

RETURNS

$E(R)$ + β ; COVARIANCE, SYSTEMATIC RISK

MEAN-VARIANCE FRONTIER, ROLL THEOREM + PRICING

RANDOM WALKS

GENERAL EQUILIBRIUM

TODO

ALL MORE CAREFULLY

ASSET PRICING MODELS $m = f(\text{DATA})$

· CAPM, ETC

· OTHER PRICES $\longrightarrow$ OPTIONS, BONDS

"ARBITRAGE"

FACTS

PORTFOLIO THEORY