CONTINGENT CLAIMS, STATE PRICES, RISK-NEUTRAL PROBABILITIES

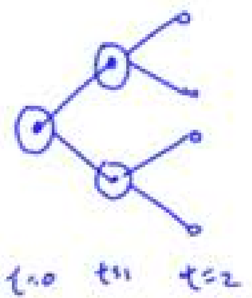
1. STATES + COMPLETE MARKETS

- DEEPER M; MARKET STRUCTURE; "COMPLETE" vs "INCOMPLETE"; USEFUL VIEWS

- TERMS
  - STATES OF NATURE $s = 1, 2 \cdots S$. RAIN / SHINE . VALUE OF S+P500.
  - CONTINGENT CLAIM PAYS $X(s) = 1$ in S ONLY. PRICE $PC(s)$ $PC(2)$ — $X(s)$: 0, 1, 0, 0

    TYPICALLY $PC(s) < 1$.
  - COMPLETE MARKET ALL C. CLAIMS AVAILABLE (OR SYNTHESIZED)
  - INCOMPLETE MARKET NOT " . EXAMPLE 1 — u 1.20, d 0.9 "STOCK"
  - SPANNING CAN CREATE C. CLAIMS. EXAMPLE 1 — R^f 1, R^f 1 "BOND"   S+B = CLAIMS

- SPANNING BY DYNAMIC TRADING



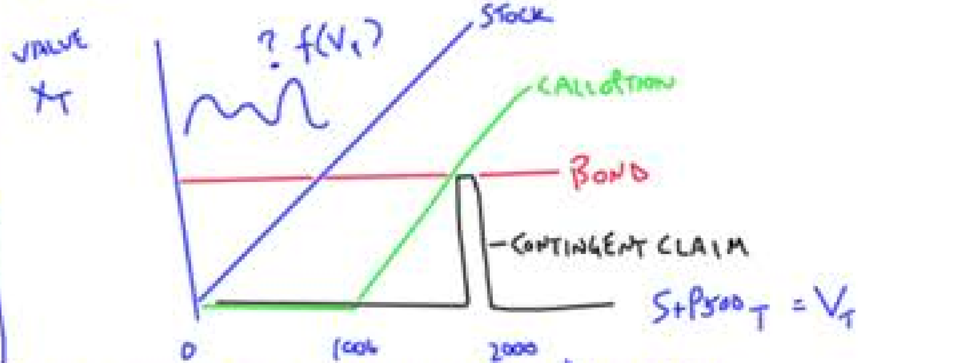
2 SECURITIES [S, B]

AND TRADING AT 0, 1

=> 4 STATES SPANNED

- "COMPLETE" CLAIMS ON, SAY, S+P, NOT RAIN

- OPTION PRICING $S = $ S+P500 AT T



→ ALL CALL+PUT = C.CLAIMS = "ANY" FUNCTION

→ DYNAMIC TRADING STOCK, BOND = C.CLAIMS = "ANY" FUNCTION

2. RISK NEUTRAL PROBABILITY + DISCOUNT FACTOR IN COMPLETE MARKETS

DISCOUNT FACTOR ↔ CCLAIMS

CONTINGENT CLAIMS → $P(x) = \sum_s \underline{pc}(s)\, X(s)$

$$P(x) = \sum_s \pi(s) \underbrace{\left[\frac{pc(s)}{\pi(s)}\right]}_{m(s)} X(s) = E(mx)$$

- INGREDIENT : LOOP
- "A DISCOUNT FACTOR EXISTS" - CAN REPRESENT $\{P, x\}$ BY $P = E(mx)$
- COMPLETE, LOOP ⇒ m EXISTS. $m = pc/\pi$
- $P = E(mx)$ INNOCUOUS. $m = f(\text{DATA})$!
- CONTINUOUS STATES. $\int_s f(s)\, m(s)\, x(s)\, ds$

  " m IS A STATE PRICE DENSITY "

RISK-NEUTRAL PROBABILITIES

$$P(x) = \sum_s pc(s)\, X(s) = \sum_s \pi(s)\, m(s)\, X(s)$$

DEFINE $\pi^*(s) = \dfrac{pc(s)}{\sum_s pc(s)} = \dfrac{m(s)\pi(s)}{\sum_s m(s)\pi(s)} = \dfrac{m(s)}{E(m)}\pi(s) = R^f m(s)\pi(s)$

$0 \le \pi^*(s) \le 1$, PROB's

$$\Rightarrow P(x) = \frac{1}{R^f}\sum_s \pi^*(s)\, x(s) = \frac{E^*(x)}{R^f}$$

- π^* ARE "RISK NEUTRAL PROBABILITIES" $\pi^*(s) = R^f m(s)\pi(s)$ IS "CHANGE OF MEASURE"
- $\pi^*(s) = R^f \beta \dfrac{u'(c_t)}{u'(c)}\pi(s)$ OVERWEIGHT UNPLEASANT STATES
- USE RISK NEUTRAL FORMULAS ARE SIMPLE. USE WHEN RISK PREMIUMS DON'T MATTER—ARBITRAGE
- SUMMARY $\boxed{P = E(mx) = \sum_s \pi(s)m(s)x(s) = \sum_s pc(s)x(s) = \frac{1}{R^f}\sum_s \pi^*(s)\, x(s)}$

  "STOCHASTIC DISCOUNT FACTOR" = "STATE-PRICE DENSITY" = TRANSFORM TO RISK-NEUTRAL PROBABILITIES" ; $\boxed{m \leftrightarrow pc \leftrightarrow \pi^*}$

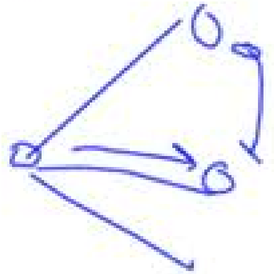
3. INVESTORS IN A COMPLETE MARKET.

$$\text{MAX } U(C_t) + \beta E_t U(C_{t+1}) \quad \text{S.T.} \quad C_t + E_t(M_{t+1} C_{t+1}) = W$$

$$U(C_t) + \beta \sum_s \pi(s)\, U[C_{t+1}(s)] \quad \text{S.t.} \quad C_t + \sum_s \pi(s) M(s)\, C_{t+1}(s) = W$$

$$\sum_s P(s)\, C_{t+1}(s) = W \leftarrow \text{BUY CLAIMS FOR FUTURE CONSUMPTION}$$

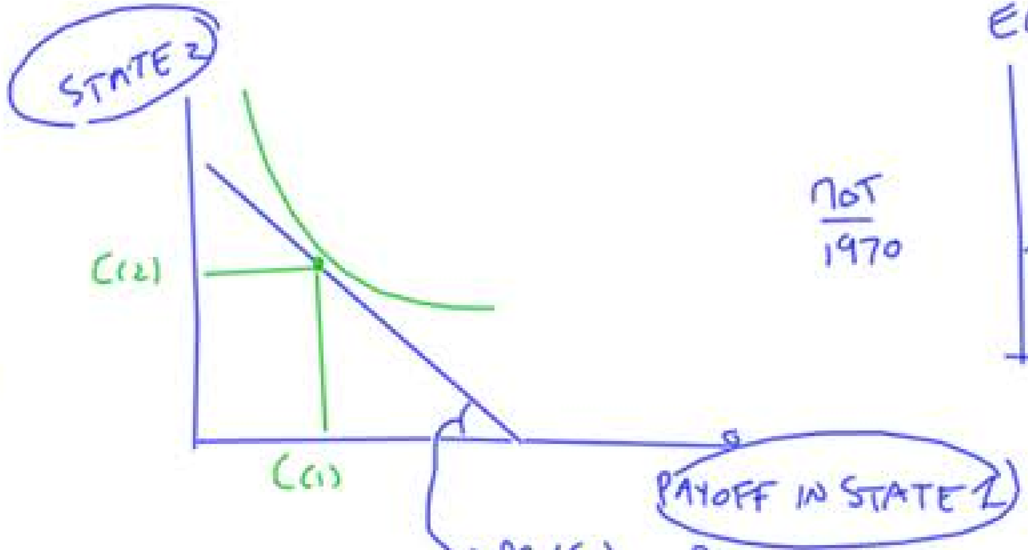
$\partial / \partial C_t \,;\, \partial / \partial C_{t+1}(s)$



$$M_{t+1}(s) = \beta \frac{U'[C_{t+1}(s)]}{U'[C_t(s)]}$$

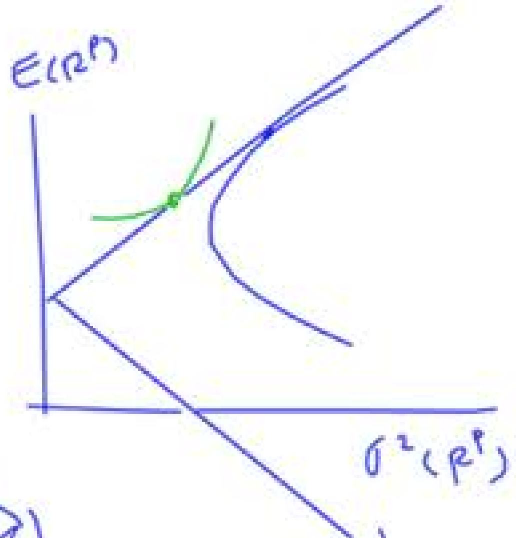
$$M_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)} \quad \text{IN EACH STATE}$$

$$\frac{M(S_1)}{M(S_2)} = \frac{U'(C(S_1))}{U'(C(S_2))} = MRS(1,2)$$



$$\frac{M(S_1)}{M(S_2)} = \frac{P(1)/\pi(1)}{P(2)/\pi(2)}$$

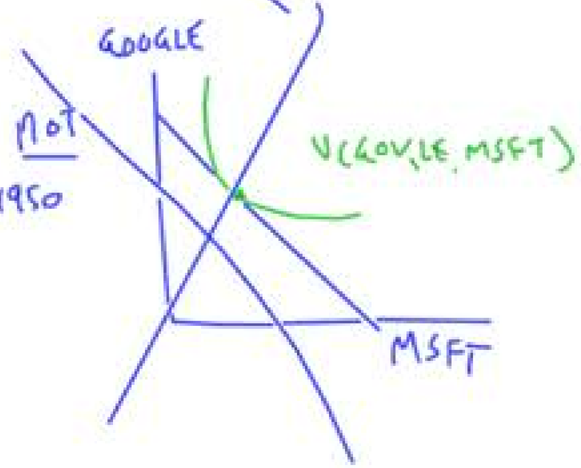
NOT 1970



- $P = E(MX)$ = BUNDLING

  $M = \beta \frac{U'(C_{t+1})}{U'(C_t)}$ = INVESTOR.

NOT 1950



- $U'(c) > 0 \rightarrow M_{t+1} > 0$ IN EVERY STATE OF NATURE

  $(\pi^* = M(s)\,\pi(s) \cdot R^f > 0)$

4. RISK SHARING (← INVESTORS, COMPLETE MARKET)

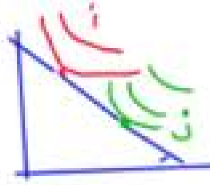
- THERE IS ONLY ONE P^c, M, SAME FOR EVERYONE
- i - PEOPLE

$$\beta^i \frac{u_i'(c^i_{t+1})}{u_i'(c^i_t)} = \frac{P^c}{\pi} = M_{t+1} = \beta^j \frac{u_j'(c^j_{t+1})}{u_j'(c^j_t)}$$

STATE BY STATE

MARGINAL UTILITY GROWTH IS THE SAME FOR ALL INVESTORS

STATE 2



STATE 1

- IF β, u(·) ARE THE SAME, POWER

$$\beta\left(\frac{c^i_{t+1}}{c^i_t}\right)^{-\gamma} = \beta\left(\frac{c^j_{t+1}}{c^j_t}\right)^{-\gamma} = \boxed{\frac{c^i_{t+1}}{c^i_t} = \frac{c^j_{t+1}}{c^j_t}}$$

- ASSET MARKETS SERVE TO SHARE RISKS

log c

COMPLETE ASSET MARKETS ALLOW PERFECT RISK SHARING.

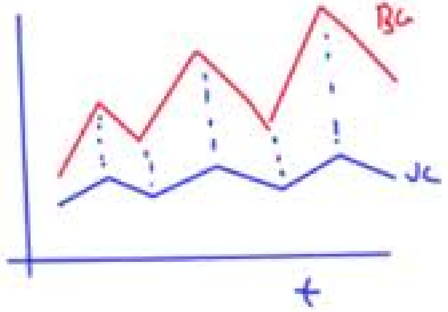
- NOT LEVELS
- INCOMES VARY!

- IF $\gamma_i \neq \gamma_j$

$$\left(\frac{c^i_{t+1}}{c^i_t}\right)^{-\gamma_i} = \left(\frac{c^j_{t+1}}{c^j_t}\right)^{-\gamma_j}$$

$$\gamma_i \log\left(\frac{c^i_{t+1}}{c^i_t}\right) = \gamma_j \log\left(\frac{c^j_{t+1}}{c^j_t}\right)$$

low γ_i → MORE VOLATILE ΔC
→ HIGHER ΔC GROWTH!



- ASSET MARKETS ALLOCATE RISK (+ REWARD) TO THOSE WILLING TO BEAR IT

- WHY AGGREGATE SHOCKS (AGG. CONSUMPTION, R^{MARKET}, ETC) APPEAR
- "PARETO OPTIMAL" ALLOCATION.
- "AGGREGATION" AS IF

$$u\left(\sum c^i_t\right) = u(c^{AGG}_t)$$

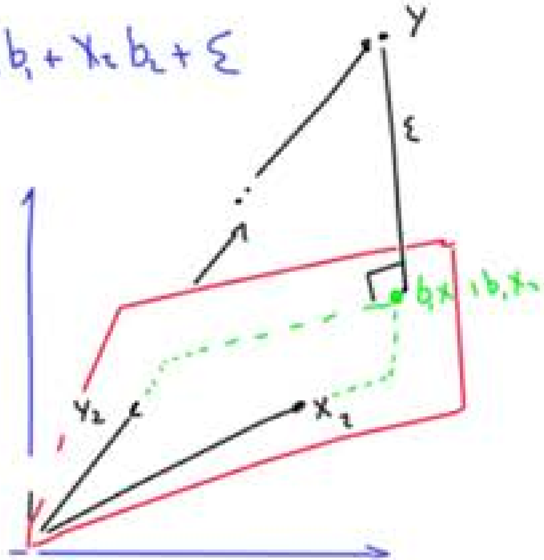
5. STATE SPACE GEOMETRY

- $S = (1, 2 \dots S')$
- $m = [\, m(1)\ \ m(2) \dots m(S') \,] \in \mathbb{R}^S \quad (L^2)$
  $x = [\, x(1)\ \ x(2) \dots x(S') \,] \in \mathbb{R}^S \quad (L^2)$
- ANALOGY : REGRESSION $\quad y = xb + \epsilon = x_1 b_1 + x_2 b_2 + \epsilon$

  xb = "PROJECTION" OF y ON x

  MIN $E(\epsilon^2)$ = "SIZE" OF RESIDUAL

  $E(x\epsilon) = 0$ ERROR IS "ORTHOGONAL" TO RHV

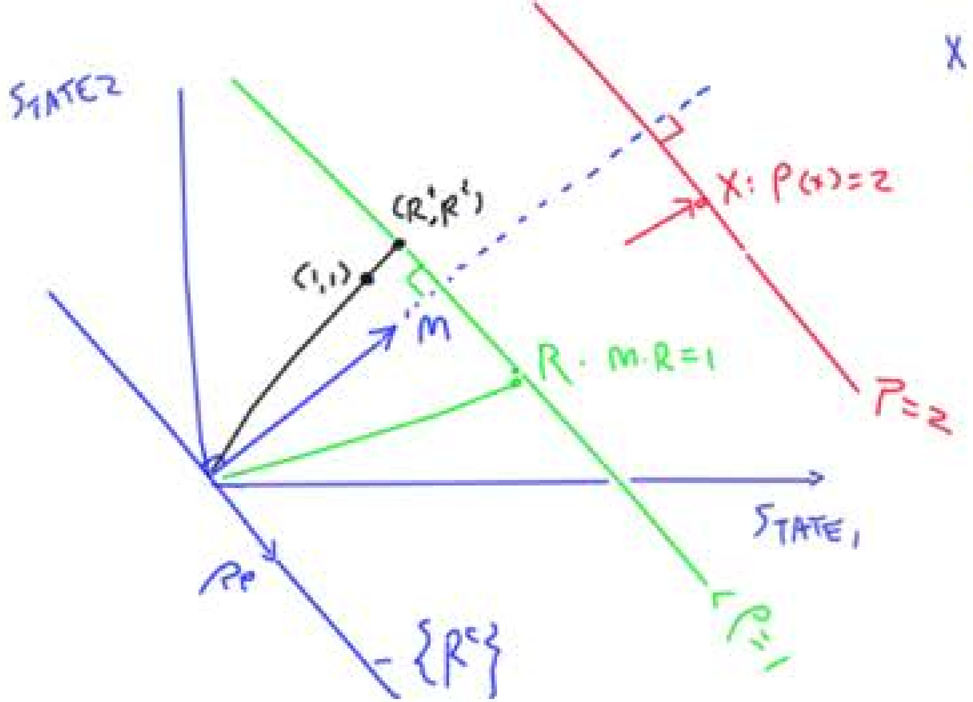


- VS

  $m(s) > 0$ IN ALL STATES $\to \in \mathbb{R}^{S+}$

  $E(mx) = m \cdot x = \langle m \mid x \rangle$ = inner product

  $0 = E(mR^e) \to m \cdot R^e$ ARE ORTHOGONAL



$x \to p(x)$ IS A LINEAR FUNCTION FROM $\mathbb{R}^S (L^2) \Rightarrow \mathbb{R}$

6. COMPLETE MARKETS SUMMARY

1. $\{X, P\}$ COMPLETE, LOOP $\rightarrow \exists\ p_c : p(x) = \sum_s p_c(s)\, x(s)$; $\exists\ m : p(x) = \sum_s \pi(s)\, m(s)\, x(s)$; $\exists\ \pi^*(s) : p(x) = \sum_s \frac{\pi^*(s)\, x(s)}{R^f}$ P(x) IS A LINEAR FUNCTION

2. $p_c \longleftrightarrow m \longleftrightarrow \pi^*$ USE WHAT WORKS

3. $\pi^*(s) = R^f\, m(s)\, \pi(s)$ RISK AVERSION = PROBABILITY DISTORTION

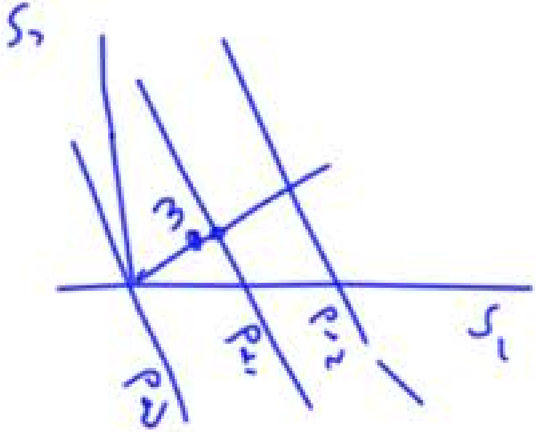
   $m(s) = \frac{\pi^*(s)}{\pi(s)} \cdot \frac{1}{R^f}$ DISCOUNT FACTOR = CHANGE OF MEASURE

4. $p = E(mx)$ ASSETS ARE BUNDLES OF CLAIMS. $m = f$ (DATA) MATTERS

5. $u'(c) > 0 \Rightarrow m(s) > 0$ $m_{t+1} > 0$ IN ALL STATES

6. GEOMETRY

   INNER PRODUCT



DISCOUNT FACTOR IN INCOMPLETE MARKETS

1. QUESTION

Q? You see $\{P, x\}$ IN COMPLETE MARKETS. $\exists\, m : P(x) = E(mx)$?

CONSTRUCT m?

THEOREMS

1. LAW OF ONE PRICE $\rightarrow \exists$ unique $x^* \in \underline{X}$ S.T. $P(x) = E(x^* x) \;\forall\, x \in \underline{X}$
2. $\exists$ A STRICTLY POSITIVE $m > 0 \iff$ LOOP AND NO ARBITRAGE

2. LOOP $\rightarrow x^*$

PAYOFF SPACE $\underline{X}$ INCOMPLETE $\underline{X} \subset \mathbb{R}^S$ EXAMPLES

NOT RETURNS. $P = 2$, $2 \cdot R$ too

A1 PORTFOLIOS $x_1, x_2 \in \underline{X} \rightarrow a x_1 + b x_2 \in \underline{X}$ $\rightarrow$ NO SHORT CONSTRAINTS

B/D/ASK

T. COSTS

CAN GENERALIZE

A2 LOOP, LINEAR $P(a x_1 + b x_2) = aP(x_1) + bP(x_2)$

MILD "RATIONALITY"

$\rightarrow 0 \in \underline{X}$

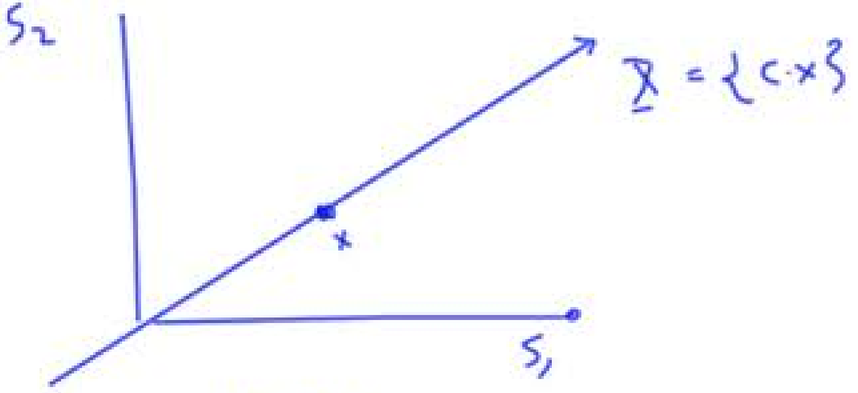


FIG. 4.1

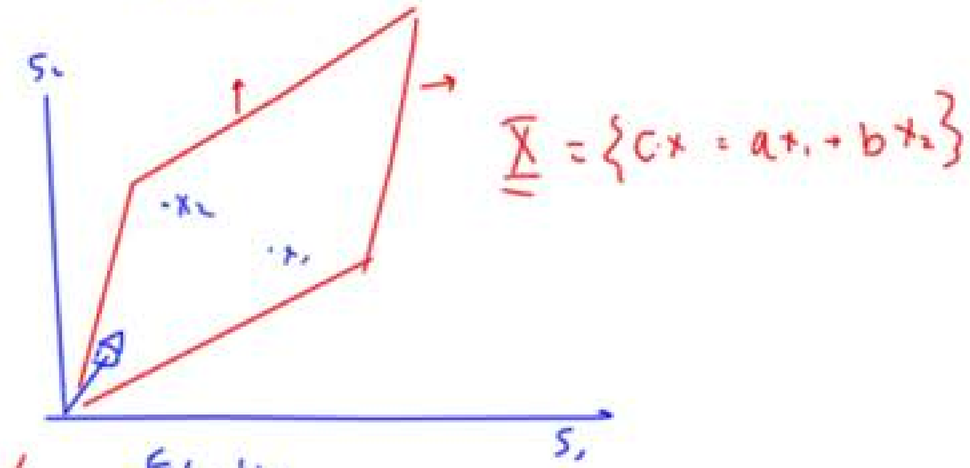
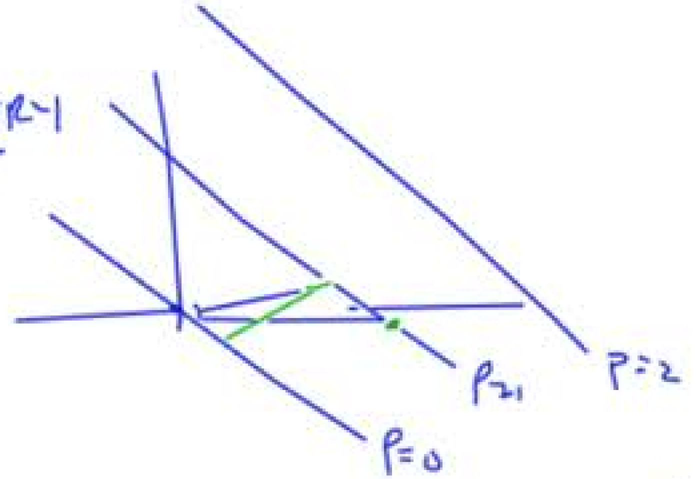


FIG. 4.2

THEOREM A1, A2 $\rightarrow \exists$ A UNIQUE $x^* \in \underline{X}$ s.t. $p(x) = E(x^* x) \ \forall x \in \underline{X}$

GEOMETRY



ALGEBRA WHEN S FINITE

EACH ARV $\in R^S$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in R^N \qquad \underline{X} = \{c'x\} \ c \in R^N$$

$$x^* = \underbrace{p' E(xx')^{-1}}_{c^{*\prime}} x = c^{*\prime} x$$

PROOF $E(x^* x') = E(p' E(xx')^{-1} x x') = p'$

EXAMPLE $x^* = 1' E(RR')^{-1} R$

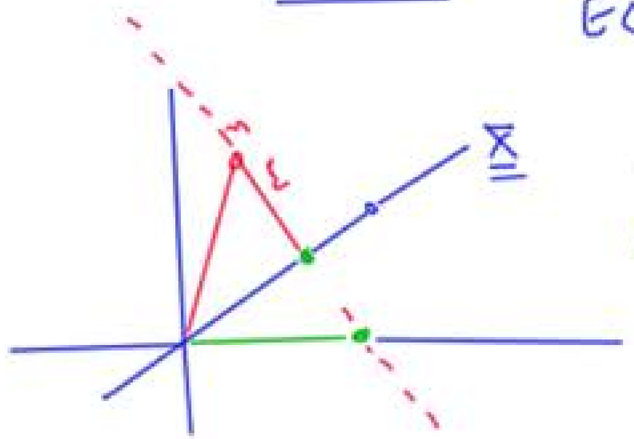
WHEN S INFINITE "RIESZ REP. THEOREM"

3. WHAT IT DOES + DOES NOT SAY

- UNIQUE $x^* \in \underline{X}$

$p = E(mx) = E[(\overbrace{m+\varepsilon}^{m'}) x]$

$E(x\varepsilon) = 0$



$\langle m | x \rangle = \langle x^* | x \rangle = p(x)$

$E(mx) = E(x^* x) = p(x)$

- ALL m CAN BE FORMED THIS WAY $\{m : p(x) = E(mx)\}$

$= \{m = x^* + \varepsilon \ \ E(x\varepsilon) = 0\}$

(many $m \leftrightarrow$ NOT COMPLETE $\underline{X}$)

- $x^* = \mathrm{proj}(m | \underline{X})$ FOR ANY m! $\left(m = \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)$

4. POSITIVE M / ARBITRAGE

THEOREM $\exists$ A $m > 0$ (ALL STATES) $\Leftrightarrow$ NO ARBITRAGE OPPORTUNITIES + LOOP

DEF $\underline{X}$, $P(x)$ LEAVE NO ARBITRAGE IF $x \geq 0$, $x > 0$ WITH $\pi > 0$, $\rightarrow P(x) > 0$

(LOTTERY TICKET)

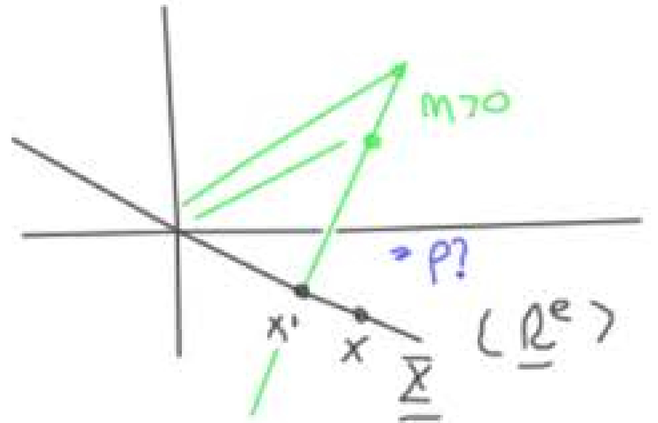
"ARBITRAGE".

IDEA $\leftarrow$ $m(s) > 0$ $x(s) \geq 0$ $P = \sum \pi(s) m(s) x(s) > 0$

COMPLETE UNIQUE x^* $x^* < 0$ FOR SOME s? $\sum_{s:\, x^* < 0} x^*(s) \cdot 1 = P < 0$

$\rightarrow$ Book.

5. WHAT IT DOES + DOES NOT SAY



- $m > 0$ IS NOT NEC. UNIQUE, OR IN $\underline{X}$, OR $= x^*$
- USE "NO ARBITRAGE EXTENSION" $P(x)$?
  NOT UNIQUE, BUT CAN'T GAME IT!

r

6. FORMULAS FOR x°

$\exists$ UNIQUE $x^\circ \in \underline{X}$ s.t. $P(x) = E(x^\circ x)\ \forall x \in \underline{X}$

$$x^\circ = P' E(xx')^{-1} x$$

$$x' = 1' E(RR')^{-1} R$$

5. FORMULAS FOR x^*

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad P = \begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix}$$

$$R = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix} \quad 1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

- $x^* = p' E(xx')^{-1} x$

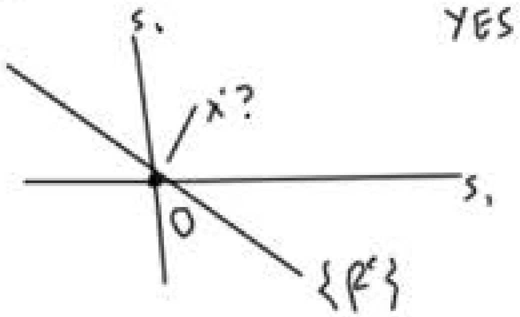
  $x^* = 1' E(RR') R$

- $x^* = \frac{1}{R^f} - \frac{1}{R^f}[E(R) - R^f]' \Sigma^{-1} [R - E(R)]$

$$1 = E(x^* R') = \frac{E(R')}{R^f} - \frac{1}{R^f}[E(R) - R^f]' \Sigma^{-1} E\underbrace{\left[[R - E(R)] R'\right]}_{\Sigma} = 1$$

- R^e? $x^* = 0' E(R^e R^{e\prime})^{-1} R^e$?

  YES! BUT $E(x^*) = 0$ 'R^f' $\to \infty$!



BETTER MAKEUP R^f

$$x^* = \frac{1}{R^f} - \frac{1}{R^f} E(R^e)' \Sigma^{-1} [R^e - E(R^e)]$$

$\to E(x^*) = 1/R^f \quad \to E(x^* R^e) = 0$

- CONTINUOUS TIME

$$dR_t = \mu\, dt + \sigma\, dZ_t$$

$$\begin{bmatrix} dR^1 \\ \vdots \\ dR^N \end{bmatrix} = \begin{bmatrix} \mu^1 \\ \vdots \\ \mu^N \end{bmatrix} dt + \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} dz \end{bmatrix}$$

$$E(dR_t\, dR_t') = \sigma\sigma' dt = \Sigma\, dt$$

this should be a dZ_t, not dt

$$\frac{d\Lambda}{\Lambda} = -r^f dt - [\mu - r^f]' \Sigma^{-1} \sigma\, dt$$

$\to \in \Sigma \quad \to E_t\left(\frac{d\Lambda}{\Lambda}\right) = -r^f dt \quad \to E_t(dR_t) = r^f dt - E_t\left(\frac{d\Lambda}{\Lambda} dR_t\right)$

$\frac{d\Lambda}{\Lambda}$ IS POSITIVE!