**Factor Pricing Models - CAPM, ICAPM, APT**

1. **Introduction/Overview**

\[
E(R^e_t) = \beta_{t,e} \gamma_t + \epsilon_t
\]

\[
R^e_i = \alpha_i + \beta_{i,R} R^e + \epsilon_i
\]

**WHAT CAN WE USE FOR \( f_t \)?**

- **Rules of the Game**: Avoid EMH or HN?
- **Explain** \( E(R^e) \)?

**So Far**

\[
f_t = \Delta C_t
\]

**Goal Other \( f_t \)**

- **CAPM**

\[
f_t = \text{EXCESS RETURN ON MARKET PORTFOLIO } R^{e,M}_t
\]

\[
E(R^e_t) = \beta_{t,M} \gamma_t
\]

**Note**

\[
\beta_{t,M} = 1, \quad E(R^{e,M}) = 1 \gamma_t \Rightarrow E(R^e_t) = \beta_{t,M} E(R^{e,M})
\]

- **ICAPM**

\[
E(R^e_t) = \beta_{t,M} \gamma_t + \epsilon_t
\]

**When the Factor is a Traded Excess Return, the Mean of the Factor Should Equal the Factor Risk Premium**

\[
\lambda_t = E(\epsilon_t)
\]

**APT**

- **Time Series Intercept** is the cross-sectional error

\[
E(\epsilon_t) = \beta_{t,M} E(R^{e,M})
\]

- **CAPM**: Intercept should be zero

**Not If the Factor is NOT Traded, e.g., \( f_{\alpha} \)**
ICAPM: \[ f_t = \text{'INNOVATIONS TO STATE VARIABLES FOR INVESTMENT OPPORTUNITIES / OUTSIDE INCOME'} \]

FAHGA + FRENCH TABLE 2

\[ R^2_t = a_t + b_t (CMR_t) + \varepsilon_t \text{, } \text{[HML + SMB + C]SMB + C} \]

\[ E(R^2_t) = \mu_t + \beta_t \text{, } \text{[HML + SMB + C]} \]

E(HML) = E(SMB) = \text{EXHML}

HML: VALUE - GROWTH
SMB: SMALL - BIG

WHY?

TODAY: DERIVE.

A) EQUILIBRIUM

\[ \Delta \zeta_t - \lambda_t \rightarrow \mu_t \]

\[ E(R^2_t) = \beta_t f_t \]

B) APT

WARNING: NOT CLEAN!

ICAPM \rightarrow FCAPM \rightarrow APT \rightarrow CS C HISTORY. VS \rightarrow, WITH LESS CLEAR "AVOID C DATA"
3. \textit{Law of Motion of Price of Consumption.}

\[ U = E \int_{0}^{\infty} e^{-\delta t} \frac{\ln C_t}{1 - \gamma} dt \]

\[ n_t = e^{-\delta t} C_0 \]

\[ \frac{d n_t}{n_t} = -r_f dt - \frac{\lambda}{C_t} \frac{d C_t}{C_t} + \frac{1}{2} \sigma^2 (\sigma) \frac{d \sigma^2}{C_t} \]

\[ E_t (d R^w_t) = r_f^w dt + \sigma E_t (d R^s_t) \]

\[ P_t^w = k C_t \rightarrow \frac{d P_t^w}{P_t^w} = \frac{d C_t}{C_t} + \frac{d \sigma^2}{\sigma^2} + \sigma^2 \frac{d \sigma^2}{\sigma^2} \]

\[ \Rightarrow E_t (d R^w_t) = \sigma E_t (d R^s_t) \]

\[ \text{Consumption Claim: } P_t^w = E_t \int_{0}^{\infty} e^{-\delta s} \left( \frac{C_{s+1}}{C_s} \right)^{\frac{1}{\gamma}} C_{s+1} \, ds \]

A) \[ 11a, \quad \frac{d x_t}{x_t} = \mu dt + \sigma d z_t \rightarrow f (C_{t+1}, C_t) \text{ same} \]

\[ \frac{p_t}{c_t} = E_t \int_{0}^{\infty} e^{-\delta s} \left( \frac{C_{s+1}}{C_s} \right)^{\frac{1}{\gamma}} ds = \lambda \]

B) \[ k = \text{assumed even if not IID} \]

\[ \frac{P_t^w}{c_t} = E_t \int_{0}^{\infty} e^{-\delta s} \left( \frac{C_{s+1}}{C_s} \right)^{\frac{1}{\gamma}} ds \]

\[ = \int_{0}^{\infty} e^{-\delta s} ds = \frac{1}{\delta} \]

\[ \text{Cashflow: } \]

\[ \text{Discount Rate} \]

\[ \text{Assumptions: } \]

- What's \( PV? \) MARKET RETURN if NO JOB, INCOME, REAL ESTATE...
- "\( PV \)^{\text{proxy}}"
- NO JOB - LOSS \( \Delta \) and \( R^w \)
- \( X = 1 \) or IID, LINK \( \Delta C_t, M_t \rightarrow R^w \text{ not news,} \)
- LINEARITY \(-\) CONTINUOUS TIME. DISCRETE \( M_t = \frac{R^w}{R^{w+1}} - a - b M_{t-1} \)
4. ACPM / "STATE VARIABLES"
- NOT IID, \( x = 1 \)
- \( \beta_{t,n} = a + b \left( \frac{\beta_{t+1}}{\beta_{t}} \right) + \epsilon_t \)
- \( \Delta \beta_{t,n} = M \left( \beta_{t,n} \right) \Delta t + \left( C \beta_{t,n} \right) \Delta z \)
- "STATE VARIABLE FOR INVESTMENT OPPORTUNITIES"
- \( \text{MONEY} \rightarrow \text{GOOD NEWS} \rightarrow \text{INVEST} \rightarrow \text{VARIOUS} \)
- \( V(W_t, X_t) = \max \{ E \left[ \sum_{s=1}^{\infty} \frac{\delta^s V_{W_{t+s}}}{\delta^s W_t} \right] | W_t, X_t \} \)
- "ENVIRONMENT THEOREM" \( \frac{\partial V(W_t, X_t)}{\partial W_t} = \frac{\partial V_{W_{t+1}}}{\partial W_t} \)
- \( \text{DISCOVERED} = \$1 \text{ CONSUMED} \)
- USE TO SUBSTITUTE \( x = -W_t, \ X_t \)
- (CPM) (NEW)

\[ e^{\delta t} V_{W_t}(x, y, z) = e^{\delta t} V_{W_{t+1}}(z, y, x) =: \varphi \]

\[ \frac{\Delta \beta_{t,n}}{\Delta t} = -\delta \frac{\Delta W_t}{\Delta W_t} \frac{\Delta V_{W_t}}{\Delta W_t} - \frac{\Delta V_{W_t}}{\Delta W_t} \Delta z_t \]

\[ E_t(\sigma^2_t) = E_t \left( \frac{\sigma^2_{t+1}}{\sigma^2_t} \right) \]

\[ \text{RISK MEASUREMENT} \frac{\sigma^2_{t+1}}{\sigma^2_t} \]

\[ E_t \left( \frac{\sigma^2_{t+1}}{\sigma^2_t} \right) = \frac{1}{\sqrt{\sigma^2_{t+1}}} \left( \frac{V_{W_{t+1}}}{V_{W_t}} \right) E_t \left( \frac{\sigma^2_{t+1}}{\sigma^2_t} \right) - \frac{\sigma^2_{t+1}}{\sigma^2_t} E_t \left( \frac{\sigma^2_{t+1}}{\sigma^2_t} \right) \]

\[ \text{PROPERTY OF WHOLE ENVIRONMENT, ALREADY} \]

\[ E_t(R_{t+1}^* | \omega_t) = \beta_t \left( \frac{V_{W_{t+1}}}{V_{W_t}} \right) \cos \left( \beta_t R_{t+1}^* R_{t+1} \right) + \left( \frac{V_{W_{t+1}}}{V_{W_t}} \right) \cos \left( \beta_t R_{t+1}^* R_{t+1} \right) \]

\[ \beta_t \left( \frac{V_{W_{t+1}}}{V_{W_t}} \right) + \beta_t \left( \frac{V_{W_{t+1}}}{V_{W_t}} \right) \]

A TWO FACTOR MODEL, LIKE FF3M!
S. MULTIPLIER MODELS / OUTSIDE INCOME

A. OUTSIDE INCOME

\[ C_t = W_t + Y_t + u_t \]
\[ W_t = R^e_t (w_t - C_t) \]
\[ M_t = \beta \frac{C_t - R^e_t (w_t - C_t) - Y_t}{C_t - C_t} \]
\[ M_t = \frac{\beta C_t - R^e_t w_t - Y_t}{C_t - C_t} \]
\[ M_t = a - b_t R^e_t + d_t + Y_t \]

B. STATE VARIABLES THAT FORECAST OUTSIDE INCOME

\[ d_t = M_t (x_t, y_t) dt + C_t (x_t, y_t) d\zeta_t \]

... just like IS-AMP.

6. MULTIPLIER MODELS / PORTFOLIO INTUITION

A. "HEDGING DEMAND" PORTFOLIO LOGIC

- OUTSIDE INCOME:
  \[ A, B, E, g, \beta, \ldots, E(R^e_t) = \beta_t \]
  they are

  - In a recession, you lose job. AP, BI
    \[ R^e_t - \beta_t + \beta_t w_t^2 + \varepsilon_t^2, E^A_t, E^B_t \]

  - Buy A? \[ P^A_t \rightarrow E(R^e_t) \]
    looks like 2:
    \[ P^A_t \rightarrow E(R^e_t) \]

  - \[ E(R^e_t) = \beta_{kt} w_t + \beta_{kt} \varepsilon_t \]

  - STATE VARIABLES:
    - News of future job loss just as bad
    - News of poor investment opportunities

  - MUST BE ALLOCATE HEDGE DEMAND \( \rightarrow \) MARKET PRICES.
7. INTEREST RATES
Example: Long Term Bonds
- T-Bill 17%
- Stocks 7%
- Bonds 2%

E(r) vs. Actual Returns

Bond Price ↑ → Bad News to E(r)

→ Good News to E(r_{n+1})

8. MULTIFACTOR MODELS
- Intuition, Halo, Mimicking Portfolios

M_{n+1} = \beta^T r_{n+1} + \epsilon_n

\beta_{n+1}' = \beta_{n}' R_{n+1} + \epsilon_{n+1}

→ "Tricky", C_{n}, P_{n}, News, \eta_{n}

\eta_{n} = GDP, Investment, VR, Interest Rates... All \eta_{n}

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Mimicking Portfolios

\eta_{n} = \alpha + \beta \eta_{n-1}

x = \text{Prox}(\eta_{n}, x)

\eta_{n} = \beta' R_{n} + \epsilon_{n}

\eta_{n} = \beta' R_{n} + \epsilon_{n}

... Zoo! "Fishing Expedition"
COMMENTS

- All replace $\Delta C_{t,n}$ with determinants.
  
  Special cases of $\Delta C$, not alternatives to basic idea.

- Assumptions: Very special.

- Practice: Inspiration, not checking/testing.
  
  - CAPM: $\lambda = \delta(\Delta C)$? $\Delta C_{t,n} = R_{t+n}^{w} - 16.7.6^{2}$
  
  - ICAPM: Do x forecast? Solve $\frac{V_{wx}}{V_{w}}$? No.

  - Macroe, mimicking -- "Factors might be."

- Point: Practice, work. But... x? "Rules of game"?
  
  Art. Read pp?

- How many assumptions do you use? Example:
  
  $E(r_{x}^{i})$, anomaly or fund. $E(r_{x}^{i}) = \beta_{i,m}E(r_{m}^{m})$. $r_{x}^{i} = \beta_{i,m}r_{m}^{m} + \varepsilon_{i}^{i}$

  ✓ "Can get with index." "Not a new anomaly." "Can hedge w. index".
  
  "Is as rational as the market." X Deep "explain". "CAPM is wrong"?
16. APT

Goal

\[ R_t^i = \beta_1 f_t^1 + \beta_2 f_t^2 + \varepsilon_t \]

\( R_t^2 \)

\[ \Rightarrow E(R_t^i) = \beta_1 E(f_t^1) + \beta_2 E(f_t^2) \]

\( \varepsilon_t \)

\( \varepsilon_t \)

APT Different Logic: "Small" \( \varepsilon \) \( \Rightarrow \) "Small" \( \varepsilon_t \), No \( \nu(\varepsilon) \)

\( R_t^p = R_t^i - \beta_2 f_t^2 - \beta_3 f_t^3 = \lambda_t + \varepsilon_t \) \( \Rightarrow \) Portable \( \lambda_t \)

A Portfolio. "Optimal" Hedge

Minimum Variance

\[ \rightarrow E(R_t^{op}) = \lambda_t \]

\[ \sigma^2(R_t^{op}) \]

\[ \sigma^2(\varepsilon_t) \]

Assumption \( |\text{SR}| < \text{MAX} = A \)

"Small" \( \varepsilon_t \) \( \Rightarrow \) "Small" \( \lambda_t \)

IF \( |\text{SR}| \leq A \) THEN AS \( \sigma^2(\varepsilon) \rightarrow 0 \) \( |\lambda_t| \rightarrow 0 \)

\( \sigma^2(\varepsilon) \leq \delta \Rightarrow |\lambda_t| \leq \delta \)

- Alphas Should Be Small When \( R^2 \) Are Large
APT VS EQUILIBRIUM MODELS (APM)

- **Absolute vs Relative Pricing.**
- **Does $R^2$ Matter?**
  - CAPM: NO
  - APT: YES!
- Downgraded factor structure for factor premium
  - $R_{it} = \alpha_i + \beta_i \cdot R^f + \varepsilon_{it}$
  - $\text{Cov}(R_{it}^i) = \beta_i \cdot \sigma_i^2 + \varepsilon$

APT YES, CAPM NO

- APT is often used as APT!

- $E(R^i) = \beta_i \cdot E(R^f)$

- $P(\bar{X}) = P(\bar{S}) \quad (P_L = 0?)$
  - $x = m$? Extend

- **Pure Arbitrage Failed.**
- APT only for large premiums, small $\varepsilon$. 

- $11m^2 \gg SR big$