Consumption Strikes Back?:
Measuring Long-Run Risk$^1$

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Abstract

We characterize and measure a long-run risk return tradeoff for the valuation of cash flows exposed to fluctuations in macroeconomic growth. This tradeoff features the risk prices of cash flows that are realized far into the future but are reflected in asset values. We apply this analysis to claims on aggregate cash flows, as well as to the cash flows from value and growth portfolios. Based on vector autoregressions, we characterize the dynamic response of cash flows to macroeconomic shocks and document that there are important differences in the long-run responses. We isolate those features of a recursive utility model and the consumption dynamics needed for the long run valuation differences among these portfolios to be sizable. Finally, we show how the resulting measurements vary when we alter the statistical specifications of cash flows and consumption growth.

Key words: Risk-return tradeoff, long run, asset pricing, macroeconomic risk
1 Introduction

In this paper we ask: how is risk exposure priced in the long run? Current period values of cash flows depend on their exposure to macroeconomic risks, risks that cannot be diversified. The risk exposures of cash flows are conveniently parameterized by the gap between two points in time: the date of valuation and the date of the payoff. We study how such cash flows are priced, including an investigation of the limiting behavior as the gap in time becomes large. While statistical decompositions of cash flows are necessary to the analysis, we supplement such decompositions with an economic model of valuation to fully consider the pricing of risk exposure in the long run.

Long-run contributions to valuation are of interest in their own right, but there is a second reason for featuring the long run in our analysis. Highly stylized economic models, like the ones we explore, are misspecified when examined with full statistical scrutiny. Behavioral biases or transactions costs, either economically grounded or metaphorical in nature, challenge the high frequency implications of pricing models. Similarly, while un-modeled features of investor preferences such as local durability or habit persistence alter short run value implications, these features may have transient consequences for valuation. While we could repair the valuation models by appending ad hoc transient features, instead we accept the misspecification and seek to decompose the implications.

Characterizing components of pricing that dominate over long horizons helps us understand better the implications of macroeconomic growth rate uncertainty for valuation. Applied time series analysts have studied extensively a macroeconomic counterpart to our analysis by characterizing how macroeconomic aggregates respond in the long run to underlying economic shocks. The unit root contributions measured by macroeconomists are a source of long-run risk that should be reflected in the valuation of cash flows. We measure this impact on financial securities.

Our study considers the prices of exposures to long run macroeconomic uncertainty, and the implications of these prices for the values of cash flows generated by portfolios studied previously in finance. These portfolios are constructed from stocks with different ratios of book value to market value of equity. It has been well documented that the one period average returns to portfolios of high book-to-market stocks (value portfolios) are substantially larger than those of portfolios of low book-to-market stocks (growth portfolios). We find that the cash flows of value portfolios exhibit positive comovement in the long run with macroeconomic shocks while the growth portfolios show little covariation with these shocks. Equilibrium pricing reflects this heterogeneity in risk exposure: risk averse investors must be compensated more to hold value portfolios. We quantify how this compensation depends on investor preferences and on the cash flow horizon.

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1 Analogous reasoning led Daniel and Marshall (1997) to use an alternative frequency decomposition of the consumption Euler equation.

2 For instance, Cochrane (1988) uses time series methods to measure the importance of permanent shocks to output, and Blanchard and Quah (1989) advocated uses restrictions on long run responses to identify economic shocks and measure their importance.

3 See, for example, Fama and French (1992)
The pricing question we study is distinct from the more common question in empirical finance: what is the short run tradeoff between risk and return measured directly from returns? Instead we decompose prices and returns by horizon. For instance, the one-period return to a portfolio is itself viewed as the return to a portfolio of claims to cash flows at different horizons. Moreover the price of a portfolio reflects the valuation of cash flows at different horizons. We use these representations to ask: when will the cash flows in the distant future be important determinants of the one-period equity returns and how will the long-run cash flows be reflected in portfolio values? From this perspective we find that there are important differences in the risks of value and growth portfolios that are most dramatic in the long run.

2 Overview

We illustrate our analysis in two figures. Figure 1 displays two of our cash flow series. Portfolio 1 is a growth portfolio and portfolio 5 is a value portfolio. The portfolios are re-balanced as in Fama and French (1992).4 The average returns to the portfolio of value stocks is much higher than the average return to the portfolio of growth stocks. For example, as reported in table 1 below, the expected quarterly returns to portfolios 1 and 5 are, respectively, 6.79% and 11.92% annually. In figure 1, both cash flows are depicted relative to aggregate consumption with the initial cash flows normalized to equal aggregate consumption. Notice that the cash flows of portfolio 1 grow much slower than those of portfolio 5. These differences in growth rates help govern the importance of each future cash flow in current values. Pricing also reflects risk and we find that the macroeconomic risk exposure of the cash flows from these portfolios are quite different.

To characterize the long-run components of pricing and required returns, we consider the contribution to portfolio value of future cash flows. This contribution slowly decays as the time to receipt of the cash flows increases. The limiting decay rate depends on both the expected growth of future cash flows and the risk-adjusted rate of return. Our analysis produces a vector of limiting risk prices associated with long-run cash flow exposure to aggregate risk. The prices that value this risk reflect investor preferences about uncertain prospects over long time horizons. These limit prices are robust to transient changes in valuation and in cash flow dynamics. Our study of limits is not meant to supplant the study of transition dynamics. Indeed the entire trajectory of risk adjustments are germane to the understanding of the valuation of risky cash flows that grow over time. We compute temporal decompositions that show how the risk adjustment depends on the discounting horizon of the cash flow. These trajectories, however, require more specificity in the modeling of macroeconomic risk and cash flow dynamics.

The top panel of figure 2 displays estimates of risk-adjusted returns for each portfolio based on statistical models of cash flows and a pricing model described below. Expected

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4Details of the construction of the portfolios and cash flows can be found in Hansen, Heaton, and Li (2005) and at http://www.bschool.nus.edu.sg/staff/biznl/bndata.html
Figure 1: The curves depict log of the ratio of portfolio cash flows relative to consumption. The plot given by --- is for cash flows from a portfolio of high book-to-market stocks (portfolio 5). The plot given by ------ is for flows from a portfolio of low book-to-market stocks (portfolio 1).
returns to holding each cash flow until maturity are given as a function of the horizon of future cash flows. The expected rates of return start at similar levels for both portfolios and then significantly separate as the horizon increases. In particular, the expected return to the value portfolio increases with horizon in contrast to the growth portfolio. This effect is due to important exposure to long-run macroeconomic risk in the value portfolio. Short-run risk exposures are not significantly different across the portfolios, however. This is reflected in similar expected short-horizon returns for each portfolio.

As an alternative depiction of the importance of long-run risk exposures we also consider a decomposition of returns into the one-period return to holding a claim on each of the future cash flows of a portfolio. The one-period expected return for the entire sequence of a portfolio’s cash flows is a weighted averaged of these holding-period returns. The bottom panel of figure 2 depicts this decomposition of expected returns for the growth and value portfolios as a function of horizon. The expected rate of return is much larger for the value portfolio once we look at the returns to holding portfolio cash flows beyond two years into the future. The limiting value of these figures are also good approximations to the entire figure after about five to size years (20 to 24 quarters).

Our value and return decompositions of course require a specific economic model and empirical inputs to characterize the growth and riskiness of cash flows. Following Epstein and Zin (1989b), Weil (1990), Tallarini (1998), Bansal and Yaron (2004) and many others, we use a recursive utility framework of Kreps and Porteus (1978). For these preferences, the intertemporal composition of risk matters to the decision maker. As emphasized by Epstein and Zin (1989b), these preferences also offer a convenient and appealing way to break the preference link between risk aversion and intertemporal substitution. Bansal and Yaron (2004) used such preferences to show how predictable components in consumption growth amplify the risk premia in security market prices. We study how long-run risk depends on intertemporal substitution, on risk aversion and on the predictable components to consumption growth.

In addition to an economic model, our decompositions require statistical inputs that quantify long-run stochastic growth in macroeconomic variables, particularly in consumption. The decompositions also require knowledge of the long-run link between stochastic cash flows and the macroeconomic risk variables. By their very nature, the components of financial risk due to macroeconomic shocks cannot be fully diversified and hence require nontrivial risk adjustments. The long-run nature of these risks makes measurement challenging, just as it does in the related macroeconomic literature. As many prior studies have done, we use log linear vector autoregressive (VAR) models of consumption and cash flows. Our focus on long-run risk deliberately stretches the VAR methods beyond their ability to capture transient dynamics. This leads us to explore the sensitivity of risk-measures to details in the specification of the time series evolution and the measurement accuracy of these components.

In section 3 we present our methodology for log-linear models and derive a long-run risk return tradeoff for cash flow risk. In section 4 we use the recursive utility model to show why the intertemporal composition of risk that is germane to an investor is reflected in both
Expected Returns for Portfolios 1 and 5

Figure 2: The plots given by solid line are for portfolio 5, a value portfolio and the plots given by dashed line are for portfolio 1, a growth portfolio. Top panel presents expected return to holding the individual cash flows at each horizon until maturity. Bottom panel presents expected one-quarter returns to holding cash flows from each horizon. Rates of return are given in annual percentage rates. Horizons are reported in quarterly units.
short run and long run risk-return tradeoffs. In section 5 we identify important aggregate shocks that affect consumption in the long run. Section 6 constructs the implied measures of the risk-return relation for portfolio cash flows. Section 7 explores statistical accuracy and section 8 provides a sensitivity analysis of alternative specifications of the long-run statistical relationship between consumption and portfolio cash flows. Section 9 concludes.

3 Long run risk in a log-linear economy

Characterization of long run implications through the analysis of steady states or their stochastic counterparts is a familiar tool in the study of dynamic economic models. We apply an analogous tool for the long-run valuation of cash flows. Cash flows grow randomly over time; but under general conditions, they have limiting growth rates. Moreover, the contributions to value of the cash flows have limiting decay rates as the date of cash flows moves forward in time. The decay rates reflect both risk-adjusted discounting and cash flow growth. By netting out the asymptotic growth rate, we obtain limiting risk adjusted discount rates of the cash flows. Repeating this exercise for alternative cash flows gives a long run risk return tradeoff. Our task in this section is to show formally how to characterize this tradeoff for a log-linear stochastic environment. Given the limiting nature of these calculations, our characterization applies more generally as we discuss in this and the next section.

3.1 Stochastic environment

The state of the economy is given by a vector $x_t$ which evolves according to a first-order vector autoregression:

$$x_{t+1} = Gx_t + Hw_{t+1}.$$  \hspace{1cm} (1)

The matrix $G$ has strictly stable eigenvalues (eigenvalues with absolute values that are strictly less than one). $\{w_{t+1} : t = 0, 1, \ldots\}$ is a vector of normal random variables that are independently and identically distributed over time with mean zero and covariance matrix $I$. The first-order specification is adopted for notational convenience. Higher-order systems can be represented as first order systems by augmenting the state vector.

One period asset payoffs are priced using a stochastic discount factor $S_{t+1,t}$. For instance, let $\psi(x_{t+1})$ be a claim to consumption at date $t+1$. This claim has a date $t$ price $E[\psi(x_{t+1})S_{t+1,t}|x_t]$. Multi-period claims are valued using multiples of the stochastic discount factor over the payoff horizon.

Initially we assume that the logarithm of the stochastic discount factor is linked to the state vector by:

$$s_{t+1,t} = \mu_s + U_s x_t + \xi_0 w_{t+1}$$  \hspace{1cm} (2)

where $s_{t+1,t} \equiv \log(S_{t+1,t})$. Later we explore more general specifications.
3.2 Risk return tradeoff

To develop a long-run tradeoff between risk and return, we first specify the long-run components of cash flows. Consider a hypothetical growth process modeled as the exponential of a random walk with drift:

\[ D_t^* = \exp \left( \zeta t + \sum_{j=1}^{t} \pi w_j \right). \] (3)

Using this growth process we introduce a transient or stationary component to produce the actual or observed cash flow:

\[ D_t = D_t^* \psi(x_t). \] (4)

Pricing of \( D_t \) requires valuation of both the transient and growth components. As we will see, the implications of the growth component for valuation and risk in the long run are invariant to the choice of the transitory component \( \psi \). The vector \( \pi \) measures long run exposure to risk. Changes in valuation due to changes in \( \pi \), give a characterization of long run risk.

Associated with the growth process (3) is the one-period valuation operator given by:

\[ \mathcal{P}\psi(x) = E \left[ \exp \left( s_{t+1,t} + \zeta + \pi w_{t+1} \right) \psi(x_{t+1}) | x_t = x \right]. \]

This operator assigns values to cash flows constructed with alternative functions \( \psi \) but with the same growth component. Formally, we view this operator as mapping functions of the Markov state into functions of the Markov state.\(^5\)

Prices of cash flows multiple periods in the future are inferred from this one-period pricing operator through iteration. For example, the time \( t \) value of date \( t+j \) cash flow (4) is given by:

\[ D_t^* \left[ \mathcal{P}^j \psi(x_t) \right] = D_t^* E \left( \exp \left[ \sum_{\tau=1}^{j} (s_{t+\tau,t+\tau-1} + \pi w_{t+\tau}) + j \zeta \right] \psi(x_{t+j}) | x_t = x \right) \]

where the notation \( \mathcal{P}^j \) denotes the application of the one-period valuation operator \( j \) times.

When the function \( \psi(x) \) is assumed to be an exponential function of the Markov state, the functions \( \{ \mathcal{P}^j \psi(x), j = 1, 2 \ldots \} \) are also exponential functions of the state. Iteration of two sets of equations yields the coefficients of these functions. To see this let \( \psi(x) = \exp(\omega x + \kappa) \) for some row vector \( \omega \) and some number \( \kappa \). Using the properties of the lognormal distribution:

\[ \exp(\omega^* x + \kappa^*) = \mathcal{P} \psi(x) = \mathcal{P}[\exp(\omega x + \kappa)] \]

where

\[ \omega^* = \omega G + U_s \] (5)

\(^5\)This operator is well defined for functions that are bounded functions of the Markov state, although it is well defined for other functions as well.
and

$$\kappa^* = \kappa + \mu_s + \zeta + \frac{|\omega H + \xi_0 + \pi|^2}{2}. \quad (6)$$

Iteration of (5) and (6) \(j\) times yields the coefficients for the function \(\mathcal{P}^j \psi(x)\).

Repeated iteration of (5) converges to a limit that is a fixed point of this equation:

$$\bar{\omega} = U_s (I - G)^{-1}.$$ 

The differences in the \(\kappa\)’s from (6) converge to:

$$-\nu \equiv \mu_s + \zeta + \frac{|\bar{\omega} H + \xi_0 + \pi|^2}{2}. \quad (7)$$

We include the minus sign in front of \(\nu\) because the right-hand side will be negative in our applications. In our present-value calculations the contribution to value from cash flows in the distant future become arbitrarily small.

While these iteration can be characterized simply for exponential functions of the Markov state, the same limits are obtained for a much richer class of functions. Moreover, the limits do not depend on the starting values for \(\omega\) and \(\kappa\), but \(\nu\) in particular depends on the exposure vector \(\pi\) to growth rate risk.

These limit points provide the following result:

**Result 3.1.** A solution to the equation:

$$\mathcal{P} \phi = \exp(-\nu) \phi$$

for a strictly positive function \(\phi\) is given by \(\phi(x) = \exp[U_s (I - G)^{-1}] x\) and \(-\nu\) by (7).

The equation in Result 3.1 is in form of an eigenvalue problem, and \(\phi\) is the unique (up to scale) solution that is strictly positive and satisfies a stability condition developed in the appendix for our application and more generally in Hansen and Scheinkman (2006). We use this characterization of the limit to investigate long-run risk. As \(j\) gets larger, \(\mathcal{P}^j (\psi)(x)\) approaches zero. The value of \(\nu\) gives the asymptotic decay rate of these values. This decay rate reflects two competing forces, the asymptotic rate of growth of the cash flow and the asymptotic, risk adjusted rate of discount.

To compute the limiting growth rate, we use the same approach as for the valuation operator, except now applied to the growth operator:

$$\mathcal{G} \psi(x) = E [\exp(\pi w_{t+1} + \zeta) \psi(x_{t+1}) | x_t = x].$$

Solve the equation:

$$\mathcal{G} \varphi(x) = \exp(\eta) \varphi(x)$$

where \(\varphi(x)\) is restricted to be positive. In this case \(\varphi\) can be normalized to unity and

$$\eta = \zeta + \frac{1}{2} \pi \cdot \pi. \quad (8)$$
Notice that $\eta$ in (8) could be computed directly as the growth rate in the process $\{D^*_t\}$ using the formula for the expectation of lognormal random variables.

The asymptotic rate of return is obtained by subtracting the growth rate $\eta$ from the decay rate $\nu$. As we show in the following theorem, this gives us a well defined price of long-run cash flow risk.

**Theorem 1.** Suppose that the state of the economy evolves according to (1) and the stochastic discount factor is given by (2), then the asymptotic rate of return is:

$$\eta + \nu = \zeta^* + \pi^* \cdot \pi$$

where

$$\pi^* \doteq -\xi_0 - U_s(I - G)^{-1}H$$
$$\zeta^* \doteq -\mu_s - \frac{\pi^* \cdot \pi^*}{2}.$$ 

The term $\pi^*$ is the price of exposure to long-run risk of cash flows as measured by $\pi$. By setting $\pi = 0$ we consider cash flows that do not grow over time and are stationary. An example is a discount bond, whose asymptotic pricing is studied by Alvarez and Jermann (2005). The asymptotic rate of return for such a cash flow with no long run risk exposure is: $\zeta^*$. Thus $\pi \cdot \pi^*$ is the contribution to the rate of return coming from the exposure of cash flows to long run risk. Since $\pi$ measures this exposure, $\pi^*$ is the corresponding price vector.

The logarithm of a stochastic discount factor over horizon $j$ is

$$\sum_{\tau=1}^{j} s_{t+\tau,t+\tau-1}.$$ 

As $j$ gets large, the long-run response to the shock vector $w_{t+1}$ converges to

$$(\xi_0 + U_s \sum_{j=0}^{\infty} G^j H)w_{t+1} = -\pi^* w_{t+1}.$$ 

Thus the long-run risk price vector has a simple characterization in this economy. It is the negative of the response coefficients for the long-run log stochastic discount factor to the underlying shocks.

One of our empirical ambitions is to measure $\pi^*$ and study its consequences. To do this we need a model for $s_{t+1,t}$. We turn to this task in the next section.

### 3.3 Limit implications

To understand the nature of our long run analysis, note that we can compute all of the iterates of the $P$. It is the limiting behavior that is of particular interest to us. In appendix
A we show how to construct a distorted probability measure with an expectation operator \( \hat{E} \) such that:

\[
\lim_{j \to \infty} \exp(\nu j) \mathcal{P}^j \psi(x) = \hat{E} \left( \frac{\psi}{\phi} \right) \phi(x).
\]  

(9)

This justifies using:

\[
\mathcal{P}^j \psi(x) \approx \exp(-\nu j) \hat{E} \left( \frac{\psi}{\phi} \right) \phi(x) = \hat{E} \left( \frac{\psi}{\phi} \right) \mathcal{P}^j \phi(x)
\]

for large \( j \). In this approximation the transient component of the cash flow determines only a scale factor: \( \hat{E} \left( \frac{\psi}{\phi} \right) \).

Thus for large \( j \) valuation of the transient components of the cash flows has an approximate one factor structure as a function of the Markov state \( x \).

Provided that this scale factor is positive, from a valuation perspective this justifies the use of a scaled version of \( \phi \) in place of \( \psi \) for large \( j \) because the implied values will be approximately the same. The scale factor divides out when computing the dividend price ratio.

### 3.4 Decomposition of returns by horizon

While we have characterized the limiting expected rate of return, it is of interest more generally to see how returns depends on the horizon of the payoffs. This will depend on the transient contribution, \( \psi \), in a more fundamental way and it is given by:

\[
\frac{1}{j} \left[ \log G^j \psi(x_t) - \log \mathcal{P}^j \psi(x_t) \right]
\]

which is the log of the ratio of the expected cash flow for horizon \( j \) to the correspond price scaled by the horizon. This is object reported in the top panel of our figure 2. The limit of this rate of return is what we computed as \( \nu + \eta \) in our long run analysis. Changing \( \psi \) alters how the rate return depends on horizon.

### 3.5 One-period returns

The one period return to a portfolio of cash flows is a weighted average of one-period returns to holding each cash flow. The holding period return to a security that pays off \( \psi(x_{t+j}) \) in period \( j \) is given by:

\[
R^i_{t+1,t} = \exp(\zeta + \pi w_{t+1}) \frac{\mathcal{P}_{j-1} \psi(x_{t+1})}{\mathcal{P}^j \psi(x_t)}.
\]

The logarithm of the expected gross returns for alternative \( j \) are what is reported in the bottom panel figure 2. As \( j \) gets large these returns are approximately equal to:

\[
R^d_{t+1,t} = \exp(\nu) \exp(\zeta + \pi w_{t+1}) \frac{\phi(x_{t+1})}{\phi(x_t)}
\]

(10)
which is the holding period return to a security that pays off the dominant eigenfunction $\phi$ over any horizon $j$. Thus for a given $\pi$ the holding period returns become approximately the same as the horizon increases. The weighting of these returns is dictated by the relative magnitudes of $P^j\psi$, which will eventually decay asymptotically at a rate $\nu$. Thus, $\nu$ gives us a measure of duration, the importance of holding period returns far into the future relative to holding period returns today. When $\nu$ is closer to zero, the holding period returns to cash flows far into the future are more important contributors to the portfolio decomposition of one period returns.$^6$

The logarithm of the return $R^d_{t+1}$ has two components: a cash flow component: $\zeta + \pi w_{t+1}$ determined by the reference growth process and a valuation component $\nu + \log \phi(x_{t+1}) - \log \phi(x_t)$ determined by the dominant eigenvalue and eigenfunction. While $\nu$ and the cash flow component change as we alter the cash flow risk exposure vector $\pi$, $\log \phi(x_{t+1}) - \log \phi(x_t)$ remains the same.

### 3.6 Other state variables with transient implications

As we have seen these decompositions are easy to compute when the transient component of the cash flow is $\exp(\omega x + \kappa)$. Our approximation results apply for a much larger family of cash flows. Consider a positive process $\{z_t\}$ that is jointly stationary and Markov with $\{x_t\}$. Unlike $\{x_t\}$, this process can have a nonlinear dynamic evolution. Our long run analysis continues to apply to cash flows of the following form:

$$D_t = D^*_t \psi(x_t) z_t.$$  \hfill (11)

An example of such cash flows is the share model of Santos and Veronesi (2006).

Notice that $\phi(x)/z$ solves the eigenvalue problem in Result 3.1 for the identical eigenvalue provided that we verify a side condition discussed in Hansen, Heaton, Roussanov, and Lee (2006). In solving this problem we consider functions of the composite Markov state $(x,z)$. The additional transient contribution to the cash flow does not alter the long-run risk return tradeoff of Theorem 1, and it does not alter limiting holding-period return in the one-period return decomposition (10).

Alternatively, we may also include transient contributions to the stochastic discount factor. Consider a positive process $\{z_t\}$ that is jointly stationary and Markov with $\{x_t\}$. Suppose that we replace $s_{t+1,t}$ by $s_{t+1,t} + \log z_{t+1} - \log z_t$.

Then $\phi(x)/z$ again solves the eigenvalue problem. Thus the risk-return characterization in Theorem 1 continues to apply. The limiting one-period holding period return (10) now includes a common valuation contribution induced by this transient departure in the stochastic discount factor. In the next section, we give examples from the asset pricing literature.

$Lettau and Wachter (2006)$ also consider the decomposition of returns into the holding period returns of the component cash flows. Their focus is different because they feature a single aggregate return with portfolio dynamics meant to capture differences in average returns across portfolios instead of differences in observed cash flow dynamics.
Our next task is specify an economic model to be used in valuation.

4 Stochastic discount factor

We focus on a recursive utility model that provides an important role for long-run consumption risk. The resulting specification of the stochastic discount factor provides some simplicity in characterizing long-run implications but is rich enough to imply differences in expected returns as they relate to long-run risk. The model is also used to evaluate the contribution to prices, due to cash flows at different horizons.

4.1 Preferences

We follow Kreps and Porteus (1978), Epstein and Zin (1989b) and Weil (1990) in choosing to examine recursive preferences. As we will see below, this specification of preferences provides a simple justification for examining the temporal composition of risk in consumption.

In our specification of these preferences, we use a CES recursion:

\[ V_t = \left[ (1 - \beta) (C_t)^{1-\rho} + \beta \mathcal{R}_t(V_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}}. \]  

(12)

The random variable \( V_{t+1} \) is the continuation value of a consumption plan from time \( t + 1 \) forward. The recursion incorporates the current period consumption \( C_t \) and makes a risk adjustment \( \mathcal{R}_t(V_{t+1}) \) to the date \( t + 1 \) continuation value. We use a CES specification for this risk adjustment as well:

\[ \mathcal{R}_t(V_{t+1}) = \left[ E (V_{t+1})^{1-\theta} | \mathcal{F}_t \right]^{\frac{1}{1-\theta}} \]

where \( \mathcal{F}_t \) is the current period information set. The outcome of the recursion is to assign a continuation value \( V_t \) at date \( t \).

The preferences provide a convenient separation between risk aversion and the elasticity of intertemporal substitution [see Epstein and Zin (1989b)]. For our purposes, this separation allows us to examine the effects of changing risk exposure with modest consequences for the risk-free rate. When there is perfect certainty, the value of \( 1/\rho \) determines the elasticity of intertemporal substitution (EIS). A measure of risk aversion depends on the details of the gamble being considered. As emphasized by Kreps and Porteus (1978), with preferences like these intertemporal compound consumption lotteries cannot necessarily be reduced by simply integrating out future information about the consumption process. Instead the timing of information has a direct impact on preferences and hence the intertemporal composition of risk matters. As we will see, this is reflected explicitly in the equilibrium asset prices we characterize. On the other hand, the aversion to simple wealth gambles is given by \( \theta \). Subject to these caveats, we will refer to \( \theta \) as a risk aversion coefficient as is common in the literature.
To analyze growth, we scale the continuation values in (12) by consumption:

\[
\frac{V_t}{C_t} = \left[ (1 - \beta) + \beta R_t \left( \frac{V_{t+1} C_{t+1}}{C_{t+1} C_t} \right)^{1-\rho} \right]^{1-\rho}.
\]

Since consumption and continuation values are positive, we find it convenient to work with logarithms instead. Let \( v_t \) denote the logarithm of the continuation value relative to the logarithm of consumption, and let \( c_t \) denote the logarithm of consumption. We rewrite recursion (12) as:

\[
v_t = \frac{1}{1-\rho} \log \left( (1 - \beta) + \beta \exp \left[ (1 - \rho) Q_t (v_{t+1} + c_{t+1} - c_t) \right] \right),
\]

where \( Q_t \) is:

\[
Q_t(v_{t+1}) = \frac{1}{1-\theta} \log E \left( \exp \left[ (1 - \theta) v_{t+1} \right] | \mathcal{F}_t \right).
\]

We will use this recursion to solve for \( v_t \) from an infinite horizon model.

### 4.2 Shadow Valuation

Consider the shadow valuation of a given consumption process. The utility recursion gives rise to a corresponding valuation recursion and implies stochastic discount factors used to represent this valuation. In light of the intertemporal budget constraint, the valuation of consumption in equilibrium coincides with wealth.

The first utility recursion (12) is homogeneous of degree one in consumption and the future continuation utility. Use Euler’s Theorem to write:

\[
V_t = (MC_t) C_t + E [(MV_{t+1}) V_{t+1} | \mathcal{F}_t]
\]

where:

\[
MC_t = (1 - \beta)(V_t)^\rho (C_t)^{-\rho},
MV_{t+1} = \beta (V_t)^\rho [R_t(V_{t+1})]^{\theta-\rho} (V_{t+1})^{-\theta}
\]

The right-hand side of (14) measures the shadow value of consumption today and the continuation value of utility tomorrow.

Let consumption be the numeraire, and suppose for the moment that we value claims to the future continuation value \( V_{t+1} \) as a substitute for future consumption processes. Divide both sides of (14) by \( MC_t \) and use marginal rates of substitution to compute shadow values. The shadow value of a claim to a continuation value is priced using \( \frac{MV_{t+1}}{MC_t} \) as a stochastic discount factor. Thus a claim to next period’s consumption is valued using

\[
S_{t+1,t} = \frac{MV_{t+1} MC_{t+1}}{MC_t} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{R_t(V_{t+1})} \right)^{\rho-\theta}
\]

(15)
as a stochastic discount factor. There are two (typically highly correlated) contributions to the stochastic discount factor in formula (15). One is the direct consumption growth contribution familiar from the Rubinstein (1976), Lucas (1978) and Breeden (1979) model of asset pricing. The other is the continuation value relative to its risk adjustment. The contribution is forward-looking and is present only when $\rho$ and $\theta$ differ.

Given the homogeneity in the recursion used to depict preferences, equilibrium wealth is given by $W_t = \frac{V_t}{MC_t}$. Substituting for the marginal utility of consumption, the wealth-consumption ratio is:

$$\frac{W_t}{C_t} = \frac{1}{1 - \beta} \left( \frac{V_t}{C_t} \right)^{1-\rho}.$$  

Taking logarithms, we find that

$$\log W_t - \log C_t = -\log(1 - \beta) + (1 - \rho)v_t$$  

(16)

When $\rho = 1$ we obtain the well known result that the wealth consumption ratio is constant.

A challenge in using this model empirically is to measure the continuation value, $V_{t+1}$, which is linked to future consumption via the recursion (12). When $\rho \neq 1$, one approach is to use the relationship between wealth and the continuation value, $W_t = V_t/MC_t$, to construct a representation of the stochastic discount factor based on consumption growth and the return to a claim on future wealth. In general this return is unobservable. An aggregate stock market return is sometimes used to proxy for this return as in Epstein and Zin (1989a); or other components can be included such as human capital with assigned market or shadow values (see Campbell (1994)).

In this investigation, like that of Restoy and Weil (1998) and Bansal and Yaron (2004), we base the analysis on a well specified stochastic process governing consumption and avoid the need to construct a proxy to the return on wealth. In contrast to Restoy and Weil (1998) and Bansal and Yaron (2004), we feature a role for continuation values that accommodates the case of logarithmic preferences ($\rho = 1$). We begin with this case as it is well understood that logarithmic preferences lead to substantial simplification in the calculation of equilibrium prices and returns [e.g. see Schroder and Skiadas (1999)].

Many other values for $\rho$ have been used in the literature. For example, Campbell (1996) argues for less intertemporal substitution and Bansal and Yaron (2004) argue for more. We explore deviations from logarithmic preferences by examining approximations to the continuation value and the stochastic discount factor around our initial assumption of $\rho = 1$.

### 4.3 The special case in which $\rho = 1$

The $\rho = 1$ limit in recursion (13) is:

$$v_t = \beta Q_t(v_{t+1} + c_{t+1} - c_t)$$

$$= \frac{\beta}{1 - \theta} \log E \left( \exp \left[ (1 - \theta)(v_{t+1} + c_{t+1} - c_t) \right] \mid F_t \right).$$  

(17)
The stochastic discount factor in this special case is:

\[ S_{t+1,t} \equiv \beta \left( \frac{C_t}{C_{t+1}} \right) \left[ \frac{(V_{t+1})^{1-\theta}}{\mathcal{R}_t(V_{t+1})^{1-\theta}} \right]. \]

Recursion (17) is used by Tallarini (1998) in his study of *risk sensitive* business cycles and asset prices.

Notice that the term of \( S_{t+1,t} \) associated with the risk-aversion parameter \( \theta \) satisfies:

\[ E \left[ \frac{(V_{t+1})^{1-\theta}}{\mathcal{R}_t(V_{t+1})^{1-\theta}} \bigg| \mathcal{F}_t \right] = 1. \]

This term can thus be thought of as distorting the probability distribution. The presence of this distortion is indicative of a rather different interpretation of the parameter \( \theta \). Anderson, Hansen, and Sargent (2003) argue that this parameter may reflect investors’ concerns about not knowing the precise riskiness that they confront in the marketplace instead of incremental risk aversion applied to continuation utilities. Under this view, the original probability model is viewed as a statistical approximation, but investors are concerned that this model may be misspecified. Although we continue to refer to \( \theta \) as a risk-aversion parameter, this alternative interpretation is germane to our analysis because we will explore sensitivity of our measurements to the choice of \( \theta \). Changing the interpretation of \( \theta \) alters what we might view as reasonable values of this parameter. Instead of focusing on the intertemporal composition of risk as in the Kreps and Porteus (1978) formulation, under this view we are led to consider potential misspecifications in probabilities that most challenge investors.

To make our formula for the marginal rate of substitution operational, we need to compute \( V_{t+1} \) using the equilibrium consumption process. Suppose that the first-difference of the logarithm of equilibrium consumption is given by:

\[ c_{t+1} - c_t = \mu_c + U_c x_t + \gamma_0 w_{t+1}. \quad (18) \]

This representation implies an impulse response function for consumption where the date \( t \) shock \( w_t \) adds \( \gamma_j w_t \) to consumption growth at date \( t + j \). The response vector is:

\[ \gamma_j = \begin{cases} 
\gamma_0 & \text{if } j = 0 \\
U_c G^{j-1} H & \text{if } j > 0 
\end{cases} \]

For this lognormal consumption growth process, the solution for the continuation value is:

\[ v_t = \mu_v + U_v x_t \]

where:

\[ U_v = \beta U_c (I - \beta G)^{-1}, \]

\[ \mu_v = \frac{\beta}{1 - \beta} \left[ \mu_c + \frac{(1 - \theta)}{2} \gamma(\beta) \cdot \gamma(\beta) \right], \]
and \( \gamma(\beta) \) is the discounted impulse response:

\[
\gamma(\beta) = \sum_{j=0}^{\infty} \beta^j \gamma_j = \gamma_0 + \beta U_c(I - G \beta)^{-1} H.
\]

The logarithm of the stochastic discount factor is:

\[
s_{t+1,t} = \mu_s + U_s x_t + \xi_0 w_{t+1}
\]

where:

\[
\begin{align*}
\mu_s &= \log \beta - \mu_c - \frac{(1 - \theta)^2 \gamma(\beta) \cdot \gamma(\beta)}{2}

U_s &= -U_c

\xi_0 &= -\gamma_0 + (1 - \theta) \gamma(\beta).
\end{align*}
\]

The stochastic discount factor includes both the familiar contribution from contemporaneous consumption plus a forward-looking term that discounts the impulse responses for consumption growth. For instance, the price of payoff \( \phi(w_{t+1}) \) is given by:

\[
E[\exp(s_{t+1,t})\phi(w_{t+1})|\mathcal{F}_t] = \frac{E[\exp(s_{t+1,t})\phi(w_{t+1})|\mathcal{F}_t]}{E[\exp(s_{t+1,t})|\mathcal{F}_t]}.
\]

The first term is a pure discount term and the second is the expectation of \( \phi(w_{t+1}) \) under the so-called risk neutral probability distribution. The negative of the \( w_{t+1} \) coefficient for the innovation to the logarithm \( s_{t+1,t} \) of the stochastic discount factor:

\[
\gamma_0 + (\theta - 1) \gamma(\beta).
\]

is the vector of risk prices for exposure to the shock vector \( w_{t+1} \). The term \( \gamma_0 \) is familiar from Hansen and Singleton (1983) and the term \( (\theta - 1) \gamma(\beta) \) is the adjustment for the intertemporal composition of consumption risk implied by the Kreps and Porteus (1978) specification of recursive utility. Large values of the risk aversion parameter \( \theta \) enhance the importance of this component. This latter effect is featured in the analysis of Bansal and Yaron (2004). Similarly, the term \( \theta \gamma(\beta) \) captures the “bad beta” of Campbell and Vuolteenaho (2004) except that they use measurements based on the market return instead of consumption.

Our interest is in the long-run consequences for cash flow risk. As we discussed in section 3, consider the valuation of alternative securities that are claims to the cash flows with permanent components \( \pi w_{t+1} \). The valuations of these components are dominated by a single factor. Applying Result 3.1, the dominant valuation factor is invariant to both the risk aversion parameter \( \theta \) and the cash-flow risk exposure parameter \( \pi \). It is the exponential of the discounted conditional expectation of consumption growth rates. From Theorem 1, the long-run cash-flow risk price is:

\[
\pi^* = \gamma_0 + U_c(I - G)^{-1} H + (\theta - 1) \gamma(\beta)
\]
\[ \gamma(1) + (\theta - 1)\gamma(\beta) \]  

where \( \gamma(1) \) is the cumulative growth rate response or equivalently the limiting consumption response in the infinite future. The comparison between one-period and long-run risk prices is informative. The long-run risk price uses the long-run consumption response vector \( \gamma(1) \) in place of \( \gamma_0 \), but the recursive utility contribution remains the same. As the subjective discount factor \( \beta \) tends to unity, \( \gamma(\beta) \) converges to \( \gamma(1) \), and hence the long-run risk price is approximately \( \theta \gamma(1) \).

Consider the limiting expected rate of return on a stationary cash flow \( \psi(x_t) \). In accordance with theorem 1 (with \( \pi = 0 \) and \( \zeta = 0 \)) it is given by:

\[ -\log \beta + \mu_c - \frac{\gamma(1) \cdot \gamma(1)}{2} - (\theta - 1)\gamma(\beta) \cdot \gamma(1) \]  

(20)

This is our cash flow counterpart to a riskless bond.

To implement these calculations we measure the risk exposure vector \( \pi \) and use assumed values for the parameters \( \beta \) and \( \theta \). Because aggregate consumption is not that variable, we are forced to use large values of \( \theta \) to generate important differences in risk prices. For example, figure 2 assumes that \( \theta = 20 \) and \( \beta = .97^{1/4} \). The limiting values in the top panel of this figure use the risk prices in (19) in conjunction with measurements of the risk exposure vector \( \pi \).

Under the alternative interpretation suggested by Anderson, Hansen, and Sargent (2003), \((\theta - 1)\gamma(\beta)\) is the contribution to prices induced because investors cannot identify potential model misspecification that is disguised by shocks that impinge on investment opportunities. An investor with this concern explores alternative shock distributions including ones with a distorted mean. He uses a penalized version of a max-min utility function. In considering how big the concern is about model misspecification, we might ask if it could be ruled easily with historical data. This lead us to ask how large is \(-\theta(1 - \beta)\gamma(\beta)\) in a statistical sense. To gauge this, when \( \theta = 10 \) and \(|(1 - \beta)\gamma(\beta)| = .01 \) a hypothetical decision maker asked to tell the two models apart would have about 24% chance of getting the correct answer given 250 observations. Doubling \( \theta \) changes this probability to about 6%.

In this sense \( \theta = 10 \) is in an interesting range of statistical ambiguity while \( \theta = 20 \) leads to an alternative model that considerably easier to discriminate based on historical data. Perhaps part of large choice of \( \theta \) can be ascribed to statistical ambiguity on the part of investors.

---

7 For the power utility specification \( \rho = \theta \), the long-run cash flow risk price vector \( \theta \gamma(1) \). Thus the expected excess rates of return are very close to those of the recursive utility model provided that the subjective discount factor \( \beta \) is close to unity. Just as with one period returns, there is an important difference in the long-run risk cash flow rate of return. By using recursive utility and keeping \( \rho \) close to one as we vary \( \theta \) the level of returns is controlled while risk premia vary.

8 These numbers are essentially the same if the prior probability across models is the same or if the min-max solution of equating the type I and type II errors is adopted.
4.4 Intertemporal substitution ($\rho \neq 1$)

Approximate characterization of equilibrium pricing for recursive utility have been produced by Campbell (1994) and Restoy and Weil (1998) based on a log-linear approximation of budget constraints. In what follows we use a distinct but related approach. We follow Kogan and Uppal (2001) by approximating around an explicit equilibrium computed when $\rho = 1$ and then varying the parameter $\rho$.

We start with a first-order expansion of the continuation value:

$$v_t \approx v^1_t + (\rho - 1) Dv^1_t$$

where $v^1_t$ is the continuation value for the case in which $\rho = 1$. While $v_t$ is linear in the state, the derivative, $Dv^1_t$, includes both a linear and quadratic function of the state. (See Hansen, Heaton, Lee, and Rousanov (2006) for the formulas and derivation.) The corresponding expansion for the logarithm of the stochastic discount factor is:

$$s_{t+1,t} \approx s^1_{t+1,t} + (\rho - 1) Ds^1_{t+1,t},$$

where:

$$Ds^1_{t+1,t} = \frac{1}{2} w_{t+1}' \Theta_0 w_{t+1} + w_{t+1}' \Theta_1 x_t + \vartheta_0 + \vartheta_1 x_t + \vartheta_2 w_{t+1}.$$  

Formulas for $\Theta_0$, $\Theta_1$, $\vartheta_0$, $\vartheta_1$ and $\vartheta_2$ are also given in Hansen, Heaton, Lee, and Rousanov (2006).

Finally, we use the stochastic discount factor expansion to determine how the decay rate, $\nu$, of the dominant eigenfunction changes with $\rho$. This calculation makes use of the formula:

$$\frac{d\nu}{d\rho} \bigg|_{\rho=1} = \hat{E} (Ds^1_{t+1,t})$$

where $\hat{E}$ is the distorted expectation operator used in formula (9) previously. The details of the justification and implementation of these formulas are given in appendix A. This derivative will depend on the assumed cash flow growth process through the distorted expectations. Since the asymptotic growth rate of cash flows does not depend on $\rho$ this same calculation can be used to study the sensitivity of the asymptotic rates of returns to changes in $\rho$.

4.5 Intertemporal complementarity

In subsection 3.2, we observed that if the (log) discount factor is modified by the first difference of a process $\{z_t\}$ that is jointly stationary with $\{x_t\}$ the implications for long-run risk are unaffected. This implies that many standard models have the same implications for the pricing of long-run risk.

As an example consider a model of intertemporal complementarity or habit persistence where the habit stock, $H_t$, evolves according to:

$$H_t = (1 - \lambda)C_t + \lambda H_{t-1}.$$
The parameter $\lambda$ governs the rate of depreciation in the habit stock and $H_t$ is a geometric average of current and past consumption. At time $t$ this habit stock is aggregated with current consumption to form:

$$A_t = \left[ \alpha C_t^{1-\theta} + (1 - \alpha) H_t^{1-\theta} \right] \frac{1}{1-\theta}.$$

The aggregate, $\{A_t\}$, is evaluated at time $t$ by the consumer according to utility function:

$$V_t = E \left[ (1 - \beta) \sum_{\tau=0}^{\infty} \beta^\tau (A_{t+\tau})^{1-\theta} \right] \frac{1}{1-\theta}.$$

For the purposes of valuation, we compute the static marginal utility of current period consumption as:

$$MC_t = \alpha (A_t/C_t)^{\theta} (A_t)^{-\theta}$$

and the static marginal utility of current period habits as:

$$MH_t = (1 - \alpha) (A_t/H_t)^{\theta} (A_t)^{-\theta}.$$

Since the consumer/investor accounts for the impact of the current consumption choice on future values of $H_t$, we are led to define an intermediate variable $z_t$:

$$z_t = MC_t + (1 - \lambda) \sum_{j=0}^{\infty} (\lambda \beta)^j E (MH_{t+j} \mid F_t) \left( C_t \right)^{\theta}.$$

Given our assumption for the consumption endowment and the homotheticity of preferences, $\{z_t\}$ is stationary. The resulting stochastic discount factor is:

$$S_{t+1,t} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\theta} \frac{z_{t+1}}{z_t}.$$

If the consumer views the process $\{H_t\}$ as exogenus we have a version of external habits similar to that of Abel (1990). In this case the stochastic discount factor is:

$$S_{t+1,t} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\theta} \frac{z_{t+1}}{z_t}.$$

where:

$$z_t = MC_t (C_t)^{\theta} = \alpha \left[ \alpha + (1 - \alpha) \left( \frac{H_t}{C_t} \right)^{1-\theta} \right] \frac{1}{1-\theta}.$$

We use a CES aggregator of consumption and the habit stock instead of a difference specification to guarantee that $A_t$ is positive even when consumption follows a log-normal evolution equation.
Under this alternative specification, the constructed process \( \{z_t\} \) is jointly stationary with \( \{x_t\} \). If this joint process satisfies the condition discussed in appendix A either the external or internal specification of habits the models, have the same implications for the pricing of long-run risk.\(^{10}\) Other models in the literature feature a \( \{z_t\} \) constructed as functions of consumption histories including the specification considered recently by Menzly, Santos, and Veronesi (2004).

5 Long-Run Consumption Risk

Our first measurement task is to estimate the consumption dynamics needed to characterize how risk exposure is priced. As in much of the empirical literature in macroeconomics, we use vector autoregressive (VAR) models to identify interesting aggregate shocks and estimate \( \gamma(z) \). In our initial model we let consumption be the first element of \( y_t \) and corporate earnings be the second element:

\[
y_t = \begin{bmatrix} c_t \\ e_t \end{bmatrix}.
\]

Our use of corporate earnings in the VAR is important for two reasons. First, it is used as a predictor of consumption and an additional source of aggregate risk.\(^{11}\) For example, changes in corporate earnings potentially signal changes in aggregate productivity which will have long-run consequences for consumption. Second, corporate earnings provide a broad-based measure of the ultimate source of the cash flows to capital. The riskiness of the equity claims on these cash flows provides a basis of comparison for the riskiness of the cash flows generated by the portfolios of stocks that we consider in section 6.

The process \( \{y_t\} \) is presumed to evolve as a VAR of order \( \ell \). In the results reported subsequently, \( \ell = 5 \). The least restrictive specification we consider is:

\[
A_0 y_t + A_1 y_{t-1} + A_2 y_{t-2} + \ldots + A_\ell y_{t-\ell} + B_0 = w_t,
\]

The vector \( B_0 \) is two-dimensional, and the square matrices \( A_j, j = 1, 2, \ldots, \ell \) are two by two. The shock vector \( w_t \) has mean zero and covariance matrix \( I \).

Form:

\[
A(z) \doteq A_0 + A_1 z + A_2 z^2 + \ldots + A_\ell z^\ell.
\]

We are interested in a specification in which \( A(z) \) is nonsingular for \( |z| < 1 \) implying that the process \( \{y_t\} \) is stationary. By inverting the matrix polynomial \( A(z) \) for the autoregressive

\(^{10}\) The model considered by Campbell and Cochrane (1999) does not satisfy this condition because of the fat tail in the distribution of \( \{z_t\} \). As a result their model has different implications for the evaluation of long-run risk. See Hansen (2006) for a formal comparison of the implications of Campbell and Cochrane (1999) and Menzly, Santos, and Veronesi (2004).

\(^{11}\) Whereas Bansal and Yaron (2004) also consider multivariate specification of consumption risk, they seek to infer this risk from a single aggregate time series on consumption or aggregate dividends. With flexible dynamics, such a model is not well identified from time series evidence. On the other hand, while our shock identification allows for flexible dynamics, it requires that we specify \textit{a priori} the important sources of macroeconomic risk.
representation, we obtain the power series expansion for the moving-average coefficients. The discounted consumption response is $u_cA(\beta)^{-1}$ where $u_c$ selects the first row, the row consisting the consumption responses. Multiplying by $(1 - \beta)$ gives the geometric average response:

$$\gamma(\beta) = (1 - \beta)u_cA(\beta)^{-1}$$

as required by our model.

Following Hansen, Heaton, and Li (2005), we restrict the matrix $A(1)$ to have rank one:

$$A(1) = \alpha [1 -1]$$

where the column vector $\alpha$ is freely estimated. This parameterization imposes two restrictions on the $A(1)$ matrix. It imposes a unit root in consumption and earnings, but restricts these series to grow together. In this system, both series respond in the same way to shocks in the long run, so they are cointegrated. Since the cointegration relation we consider is prespecified, the model can be estimated as a vector autoregression in the first-difference of the log consumption and the difference between the log earnings and log consumption. In section 8 we explore other growth restrictions specifications.

In our analysis, we will not be concerned with the usual shock identification familiar from the literature on structural VAR’s. This literature assigns structural labels to the underlying shocks and imposes a priori restrictions to make this assignment. Our primary interest is the intertemporal composition of consumption risk and not the precise labels attached to individual shocks. We construct two uncorrelated shocks as follows. One is temporary, formed as a linear combination of shocks that has no long run impact on consumption and corporate earnings. The second is permanent which effects consumption and earnings equally in the long run.\(^{12}\) Thus we impose $\gamma(1) = [0 1]$ as a convenient identifying restriction for the shocks.

For our measure of aggregate consumption we use aggregate consumption of nondurables and services taken from the National Income and Product Accounts. This measure is quarterly from 1947 Q1 to 2005 Q4, is in real terms and is seasonally adjusted. We measure corporate earnings from NIPA and convert this series to real terms using the implicit price deflator for nondurables and services. Using these series, we estimate the system with cointegration.

In figure 3 we report the response of consumption to permanent and temporary shocks. The immediate response of consumption to a permanent shock is approximately twice that of the response to a temporary shock. Permanent shocks are an important feature of aggregate consumption. The full impact of the permanent shock is slowly reflected in consumption and ultimately accumulates to a level that is more than twice the on-impact response.

With recursive utility, the geometrically weighted average response of consumption to the underlying shocks affects both short-run and long-run risk prices. For this reason, the predictable responses of consumption to shocks identified by the VAR with cointegration,

\(^{12}\)This construction is much in the same spirit as Blanchard and Quah (1989).
Figure 3: The two curves are impulse responses of consumption to shocks implied by bivariate VAR’s where consumption and earnings are assumed to be cointegrated. — depicts the impulse response to a permanent shock. -. depicts the impulse response to a temporary shock. Each shock is given a unit impulse. Responses are given at quarterly horizons.
affect risk prices at all horizons. In section 8 we demonstrate that cointegration plays an important role in identifying both the long-run impact of the permanent shock depicted in figure 3 and the temporal pattern of the responses to both shocks.

To examine the long-run risk components of aggregate consumption, consider pricing a claim to aggregate consumption. In this case $\pi$ is equal to the long-run exposure of consumption to the two shocks: $\gamma(1)$. With recursive preferences and $\rho = 1$, the asymptotic rate of return of Theorem 1 reduces to:

$$- \log \beta + \mu_c + \frac{\gamma(1) \cdot \gamma(1)}{2}$$

which does not depend on the risk-aversion parameter $\theta$.

The excess of the asymptotic return to the consumption claim over the riskless return is:

$$\gamma(1) \cdot [\gamma(1) + (\theta - 1)\gamma(\beta)] .$$

The expected excess return is essentially proportional to $\theta$ due to the dependence of the risk-free benchmark on $\theta$. Notice that when $\beta = 1$, the expected excess return reduces to $\theta \gamma(1) \cdot \gamma(1)$ and the proportionality is exact.

Even in the long-run, the consumption claim is not very risky. The VAR system implies that $\gamma(1) \cdot \gamma(1) = 0.0001$. Hence when $\beta = 1$, increases in $\theta$ has a very small impact on the excess return to the consumption claim. For example, even when $\theta = 20$ the expected excess return, in annual units, is .8% ($= 20 \times 0.0001 \times 4$). A similar conclusion holds when we use aggregate stock market dividends instead of consumption as a cash flow measure. We obtain a different conclusion when we consider cash flows from portfolios.

Many models in the literature impose cointegration between consumption and cash flows by assumption. For example, a discrete-time version of the share model of Santos and Veronesi (2006) can be depicted as in (11) with $D_t^* = C_t$. The long-run cash flow risk price is the same as that of consumption. While physical claims to resources may satisfy balanced growth restrictions, financial claims of the type we investigate need not as reflected in the trends displayed in figure 1.

6 Long-Run Cash Flow Risk

Theorem 1 characterizes the limiting factor risk prices $\pi^*$ for cash flow exposure to long-run risk. This relation in conjunction with our economic model 4 allow us to compute limiting valuation and risk adjustments. Our task in this section is to measure the long-run risk exposure of the cash flows from some portfolios familiar from financial economics and to consider the implied differences in values and the risk premia of returns.

Previously, Bansal, Dittmar, and Lundblad (2005) and Campbell and Vuolteenaho (2004) have related measures of long-run cash flow risk to one period returns using a log-linearization of the present value relation. Our aim is different, but complementary to their study. As we described in section 3, we study how long run cash flow risk exposure is priced in the context of economic models of valuation.
We consider cash flows that may not grow proportionately with consumption. This flexibility is consistent with the models of Campbell and Cochrane (1999), Bansal, Dittmar, and Lundblad (2005), Lettau, Ludvigson, and Wachter (2006), and others. It is germane to our empirical application because the sorting method we use in constructing portfolios can induce permanent differences in dividend growth. Consistent with our use of VAR methods, we consider a log-linear model of cash flow growth:

$$d_{t+1} - d_t = \mu_d + U_d x_t + \iota_0 w_{t+1}.$$ 

where $d_t$ is the logarithm of the cash flow. This growth rate process has a moving-average form:

$$d_{t+1} - d_t = \mu_d + \iota(L) w_{t+1}.$$ 

where:

$$\iota(z) = \sum_{j=0}^{\infty} \iota_j z^j$$

and:

$$\iota_j = \begin{cases} \iota_0 & \text{if } j = 0 \\ U_d G^j - 1 H & \text{if } j > 0 \end{cases}$$

6.1 Martingale extraction

In section 3, we considered benchmark growth processes that were geometric random walks with drifts. Empirically our cash flows are observed to have stationary components as well. This leads us to construct the random walk components to the cash flow process. Specifically, we represent the log dividend process as the sum of a constant, a martingale with stationary increments and the first difference of a stationary process. Write:

$$d_{t+1} - d_t = \mu_d + \iota(1) w_{t+1} - U_d^* x_{t+1} + U_d^* x_t$$

where:

$$\iota(1) = \iota_0 + U_d (I - G)^{-1} H$$

$$U_d^* = U_d (I - G)^{-1}$$

Thus $d_t$ has a growth rate $\mu_d$ and a martingale component with increment: $\iota(1) w_t$. To relate this to the development in section 3, $\iota(1) = \pi$, $\mu_d = \zeta$ and $\psi(x_t) = \exp(U_d^* x_t)$ in the cash flow representation (4). We will fit processes to cash flows to obtain estimates of $\iota(1)$ and $\mu_d$.

---

13 Martingale approximations are commonly used in establishing central limit approximations (e.g. see Gordin (1969) or Hall and Heyde (1980)), and are not limited to log-linear processes. For scalar linear time series, it coincides with the decomposition of Beveridge and Nelson (1981).
6.2 Empirical Specification of Dividend Dynamics

We identify dividend dynamics and, in particular, the martingale component $\iota(1)$ using VAR methods. Consider a VAR with three variables: consumption, corporate earnings and dividends (all in logarithms). Consumption and corporate earnings are modeled as before in a cointegrated system. In addition to consumption and earnings, we include in sequence the dividend series from each of the five book-to-market portfolios and from the market. Thus the same two shocks as were identified previously remain shocks in this system because consumption and corporate earnings remain an autonomous system. An additional shock is required to account for the remaining variation in dividends beyond what is explained by consumption and corporate earnings. As is evident from figures 1, these series have important stochastic low frequency movements relative to consumption. Cash flow models that feature substantial mean reversion or stochastically stable shares relative to aggregate consumption are poor descriptions of these data.

Formally, we append a dividend equation:

$$A_0^*y_t^* + A_1^*y_{t-1} + A_2^*y_{t-2} + \ldots + A_\ell^*y_{t-\ell} + B_0^* = w_t^*,$$

(22)

to equation system (21). In this equation the vector of inputs is

$$y_t^* = \begin{bmatrix} y_t \\ d_t \end{bmatrix} = \begin{bmatrix} c_t \\ e_t \\ d_t \end{bmatrix}$$

and the shock $w_t^*$ is scalar with mean zero and unit variance. This shock is uncorrelated with the shock $w_t$ that enters (21). The third entry of $A_0^*$ is normalized to be positive. We refer to (22) as the dividend equation, and the shock $w_t^*$ as the dividend shock. As in our previous estimation, we set $\ell = 5$. We presume that this additional shock has a permanent impact on dividends by imposing the linear restriction:

$$A^*(1) = [\alpha^* - \alpha^* 0].$$

In the next section we will explore sensitivity of our risk measures to alternative specifications of long-run stochastic growth in the cash flows.

A stationary counterpart to this log level specification can be written in terms of the variables $(c_t - c_{t-1})$, $(e_t - c_t)$, $(d_t - d_{t-1})$. We estimate the VAR using these transformed variables with four lags of the growth rate variables and five lags of the logarithmic differences between consumption and earnings.

6.3 Book to Market Portfolios

We use five portfolios constructed based on a measure of book equity to market equity, and characterize the time series properties of the dividend series as it covaries with consumption and earnings. We follow Fama and French (1993) and construct portfolios of returns by sorting stocks according to their book-to-market values. We use a coarser sort into 5 portfolios.
Properties of Portfolios Sorted by Book-to-Market

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>One-period Exp. Return (%)</td>
<td>6.79</td>
<td>7.08</td>
<td>9.54</td>
<td>9.94</td>
<td>11.92</td>
<td>7.55</td>
</tr>
<tr>
<td>Long-Run Return (%)</td>
<td>8.56</td>
<td>8.16</td>
<td>10.72</td>
<td>10.84</td>
<td>13.01</td>
<td>8.77</td>
</tr>
<tr>
<td>Avg. B/M</td>
<td>0.32</td>
<td>0.61</td>
<td>0.83</td>
<td>1.10</td>
<td>1.80</td>
<td>0.65</td>
</tr>
<tr>
<td>Avg. P/D</td>
<td>51.38</td>
<td>34.13</td>
<td>29.02</td>
<td>26.44</td>
<td>27.68</td>
<td>32.39</td>
</tr>
</tbody>
</table>

Table 1: Data are quarterly from 1947 Q1 to 2005 Q4 for returns and annual from 1947 to 2005 for B/M ratios. Returns are converted to real units using the implicit price deflator for nondurable and services consumption. Average returns are converted to annual units using the natural logarithm of quarterly gross returns multiplied by 4. “One-period Exp. Return,” we report the predicted quarterly gross returns to holding each portfolio in annual units. The expected returns are constructed using a separate VAR for each portfolio with inputs: \((c_t - c_{t-1}, e_t - c_t, r_t)\) where \(r_t\) is the logarithm of the gross return of the portfolio. We imposed the restriction that consumption and earnings are not Granger caused by the returns. One-period expected gross returns are calculated conditional on being at the mean of the state variable implied by the VAR. “Long-Run Return” reports the limiting value of the logarithm of the expected long-horizon return from the VAR divided by the horizon. “Avg. B/M" for each portfolio is the average portfolio book-to-market over the period computed from COMPUSTAT. “Avg. P/D” gives the average price-dividend for each portfolio where dividends are in annual units.

to make our analysis tractable. In addition we use the value-weighted CRSP return for our “market” return.

Summary statistics for these portfolios are reported in table 1. The portfolios are ordered by average book to market values where portfolio 1 has the lowest book-to-market value and portfolio 5 has the highest. Both one-period and long-run average returns generally follow this sort. For example, portfolio 1 has much lower average returns than portfolio 5. It is well documented that the differences in these average returns are not explained by exposure to contemporaneous covariance with consumption.

In this section we are particularly interested in the behavior of the cash flows from the constructed portfolios. The constructed cash flow processes accommodate changes in the classification of the primitive assets and depend on the relative prices of the new and old asset in the book-to-market portfolios. The monthly cash flow growth factors for each portfolio are constructed from the gross returns to holding each portfolio with and without dividends. The difference between the gross return with dividends and the one without dividends times
the current price-dividend ratio gives the cash flow growth factor. Accumulating these factors over time gives the ratio of the current period cash flow to the date zero cash flow. We normalize the date zero cash flow to be unity. The measure of quarterly cash flows in quarter $t$ that we use in our empirical work is the geometric average of the cash flows in quarter $t - 3, t - 2, t - 1$ and $t$. This last procedure removes the pronounced seasonality in dividend payments. Details of this construction are given in Hansen, Heaton, and Li (2005), which follows the work of Bansal, Dittmar, and Lundblad (2005). The geometric averaging induces a transient distortion to our cash flows, but will not distort the long run stochastic behavior.

We estimate $\iota(1)$ from the VAR inclusive of portfolio dividends which gives us a measure of $\pi$. We the limiting rates of returns using the methods described in section 3. Table 2 gives long-run average rates of return for the five book-to-market portfolios. We explore formally sensitivity to the risk aversion parameter $\theta$ and report derivatives with respect to the intertemporal elasticity parameter $\rho$.

We first consider the implied logarithm of the expected return decomposed and scaled by horizon. These are reported in figure 4 and discussed previously. The figures are computed assuming that the Markov state is set to its unconditional mean. This figure reproduces the decompositions depicted in the upper panel of 2, but here we include sensitivity to changes in the parameter $\rho$. Using the horizon counterpart to the $\rho$ derivatives discussed previously, we compute approximations for $\rho = 1.5$ and $\rho = .5$. Changing $\rho$ leads to a roughly parallel shift in the curves, with a larger value of $\rho$ increasing the overall returns.

In table 2 we report the limiting cash flow discount rates or long-run expected returns. These adjust the asymptotic decay in valuation for dividend growth. Recall from table 1 that one-period and long-horizon reinvestment expected returns are similar for each portfolio. The asymptotic cash flow discount rates only achieve comparable dispersion for large values of $\theta$, say $\theta = 20$. While the discount rates in table 2 are lower, common changes in these rates can be achieved by simply altering the subjective discount factor $\beta$.

Portfolio 1 has low long-run cash flow covariation with consumption relative to portfolio five. This results in larger risk adjustments for the high book-to-market portfolios. Complementary to many other asset pricing studies, differences in the average rates of return on long-run cash flows are small except for large values of the risk aversion parameter $\theta$, say $\theta = 20$. In contrast to aggregate securities, the implied heterogeneity in the the limiting expected returns are now substantial when $\theta$ is large. For the reasons we gave earlier, changing $\theta$ alters the expected excess returns almost proportionately.

The derivatives with respect to $\rho$ are similar across securities so that modest movements in $\rho$ have very little impact on the excess long-run returns, but rather substantial impact on the returns. Notice that for large values of $\theta$, increasing $\rho$ above one reduces some of the expected excess long-run rates of return. Recall that under recursive utility, the temporal pattern of risk matters. Shifts in intertemporal preferences change the way these patterns are evaluated and therefore affect attitudes toward risk and risk premia.

Up until now, we have focused on the return implications of the cash flows. In the remainder of this section we consider implications for portfolio values. We consider two
Figure 4: Expected returns to holding cash flows from portfolios 1 and 5 at different horizons. -- assumes $\rho = 1/1.5$, _ _ assumes $\rho = 1$, · · · assumes $\rho = 1.5$. Expected returns are in annualized percentages.
## Limiting Cash Flow Discount Rates

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Rate of Return</th>
<th>Excess Return</th>
<th>Return Derivative</th>
<th>Excess Return Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>θ = 1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6.51</td>
<td>-0.02</td>
<td>3.50</td>
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<td>2</td>
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<td>3.50</td>
<td>0.00</td>
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<tr>
<td>3</td>
<td>6.67</td>
<td>0.14</td>
<td>3.52</td>
<td>0.01</td>
</tr>
<tr>
<td>4</td>
<td>6.70</td>
<td>0.17</td>
<td>3.52</td>
<td>0.01</td>
</tr>
<tr>
<td>5</td>
<td>6.75</td>
<td>0.22</td>
<td>3.52</td>
<td>0.02</td>
</tr>
<tr>
<td>market</td>
<td>6.60</td>
<td>0.06</td>
<td>3.51</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>θ = 5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6.27</td>
<td>-0.10</td>
<td>3.45</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>6.42</td>
<td>0.05</td>
<td>3.44</td>
<td>-0.00</td>
</tr>
<tr>
<td>3</td>
<td>7.03</td>
<td>0.66</td>
<td>3.35</td>
<td>-0.09</td>
</tr>
<tr>
<td>4</td>
<td>7.16</td>
<td>0.79</td>
<td>3.36</td>
<td>-0.08</td>
</tr>
<tr>
<td>5</td>
<td>7.42</td>
<td>1.05</td>
<td>3.33</td>
<td>-0.11</td>
</tr>
<tr>
<td>market</td>
<td>6.68</td>
<td>0.30</td>
<td>3.41</td>
<td>-0.03</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>θ = 20</td>
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<td>1</td>
<td>5.39</td>
<td>-0.39</td>
<td>3.24</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>5.98</td>
<td>0.21</td>
<td>3.18</td>
<td>-0.01</td>
</tr>
<tr>
<td>3</td>
<td>8.37</td>
<td>2.59</td>
<td>2.75</td>
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<tr>
<td>4</td>
<td>8.89</td>
<td>3.12</td>
<td>2.81</td>
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<tr>
<td>5</td>
<td>9.92</td>
<td>4.15</td>
<td>2.67</td>
<td>-0.52</td>
</tr>
<tr>
<td>market</td>
<td>6.98</td>
<td>1.20</td>
<td>3.02</td>
<td>-0.17</td>
</tr>
</tbody>
</table>

Table 2: Limiting expected returns to holding the cash flows of portfolio 1 through 5. Excess returns are measured relative to the return on a long horizon discount bond. The derivative entries in columns four and five are computed with respect to ρ and evaluated at ρ = 1. Returns are reported in annualized percentages.
Limiting Decay Rates and Price-Dividend Ratios

<table>
<thead>
<tr>
<th>Preference Parameters</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limiting Decay Rates (annual %)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta = 5, \rho = 1$</td>
<td>4.16</td>
<td>4.48</td>
<td>2.71</td>
<td>3.14</td>
<td>0.40</td>
</tr>
<tr>
<td>$\theta = 20, \rho = 1$</td>
<td>3.27</td>
<td>4.04</td>
<td>4.04</td>
<td>4.87</td>
<td>2.90</td>
</tr>
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<td>2.68</td>
<td>3.74</td>
<td>4.93</td>
<td>6.02</td>
<td>4.57</td>
</tr>
<tr>
<td>$\theta = 30, \rho = 1/1.5$</td>
<td>1.66</td>
<td>2.74</td>
<td>4.13</td>
<td>5.19</td>
<td>3.83</td>
</tr>
</tbody>
</table>

Price-Dividend Ratios

<table>
<thead>
<tr>
<th>Preference Parameters</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = 5, \rho = 1$</td>
<td>23.6</td>
<td>22.1</td>
<td>36.9</td>
<td>32.0</td>
<td>251.4</td>
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<tr>
<td>$\theta = 20, \rho = 1$</td>
<td>28.5</td>
<td>23.8</td>
<td>26.6</td>
<td>21.9</td>
<td>35.9</td>
</tr>
<tr>
<td>$\theta = 30, \rho = 1$</td>
<td>33.6</td>
<td>25.1</td>
<td>22.8</td>
<td>18.4</td>
<td>23.4</td>
</tr>
<tr>
<td>$\theta = 30, \rho = 1/1.5$</td>
<td>54.4</td>
<td>34.4</td>
<td>27.7</td>
<td>21.7</td>
<td>28.4</td>
</tr>
</tbody>
</table>

Table 3: The limited decay rates are multiplied by 400 to produce annual percentages. The reported price-dividend ratios use the mean value of the state vector $x_t = 0$. These calculations assumed asubjective factor of $\beta = 0.97^{1/4}$.

calculations for alternative parameter configurations. First we report the limiting expected decay rate in value, $\nu$, for each of the five portfolios. Under a continuous-time approximation, $1/\nu$ is the corresponding limiting price dividend ratio.\(^{14}\) Moreover, in the one-period return decomposition reported in bottom panel of figure 2, $\frac{\exp(-\nu)}{1-\exp(-\nu)}$ gives an approximation to the weighting of the holding period returns in the total return. For smaller values of $\nu$, the one-period holding period return on a cash flow that pays off far into the future is a more important contributor to the overall one-period return. We also report the implied price-dividend ratios when $x_t = 0$.\(^{15}\)

In table 3 we report the value of decay rates and price-dividend ratios for several different parameter configurations. We consider three alternative values of $\theta$. Interestingly, the value decay rate portfolio 1 is much higher than that of portfolio 5 when $\theta = 5$. Bigger values of $\theta$ are required to ensure that price-dividend ratios are larger for low book-to-market portfolios than for high ones. Recall that low book-to-market portfolios generate less cash flow growth than high book-to-market portfolios. The differential risk adjustments have to

\(^{14}\)The discrete time counterpart is $\frac{\exp(-\nu)}{1-\exp(-\nu)}$.

\(^{15}\)These are not the same as the average price-dividend ratios even though they are evaluated at the average value of $x_t$. This is because the price-dividend ratios are nonlinear functions of $x_t$. 

30
more than compensate for the growth rate disparity. For example, the second and third rows of each panel consider larger values of $\theta$. Even $\theta = 20$ is not large enough for limiting price-dividend ratio of the low book-to-market portfolios to exceed those of the high book-to-market portfolios. By further increasing $\theta = 30$ (fourth row of each panel), the value decay for portfolio one is noticeably less than that of portfolio five. Recall that lower decay rates in value imply that the contributions of the holding period returns for cash flows far into the future are more pronounced.

Next, we consider the impact of changing the intertemporal substitution parameter $\rho$. As we saw in table 2 a decrease in $\rho$ lowers all expected rates of returns. As a consequence decreasing $\rho$ to $1/1.5$ (fourth row of each panel in table 3) reduces all of the decay rates. Correspondingly, price-dividend ratios are increased. The increase is particularly pronounced for the low book-to-market portfolios.

The decay rates and price-dividend ratios are depicted for a single value of the subjective discount factor $\beta$. Changing $\beta$ gives an alternative mechanism for altering price-dividend ratios. Decreasing $\beta$ results in a roughly parallel shift in the decay rates for all of the portfolios and corresponding increases the price-dividend ratios. Unfortunately, this change with low risk aversion will not induce relatively low decay rates for the low-book to market portfolios. Moreover, this decrease increases the one-period risk free rate, which is arguably high for our baseline value of $\beta = 0.97^{1/4}$.

All of this discussion abstracts from errors in estimating the cash flow growth rates and risk exposure. We address estimation accuracy and sensitivity to specification in the next two sections.

7 Measurement accuracy

Thus far we have not discussed the statistical accuracy of our measurements. Given the long-run nature of our calculations, statistical accuracy is an important issue. In this section we report some aspects.

7.1 Consumption responses

Using the cointegration specification, we explore the statistical accuracy of the estimated responses. Following suggestions of Sims and Zha (1999) and Zha (1999), we impose Box-Tiao priors on the coefficients of each equation in the VAR and simulate histograms for the parameter estimates. This provides an approximation for Bayesian posteriors with a relatively diffuse (and improper) prior distribution. These “priors” are chosen for convenience, but they give us a simple way to depict the sampling uncertainty associated with the estimates. We use these priors in computing posterior distributions for the immediate response of consumption to a temporary shock and for the long-run response of the permanent shock

\footnote{For instance, the quarterly risk-free rate is 6.6% in annual units with $\theta = 1$.}
to consumption. This long-run response is identical to $|\gamma(1)|$. Figure 5 gives the posterior histogram for both responses.

As might be expected, the short-run response estimate is much more accurate than the long-run response. Notice that the horizontal scales of the histograms differ by a factor of ten. In particular, while the long-run response is centered at a higher value, it also has a substantial right tail. Consistent with the estimated impulse response functions, the median long-run response is about double that of the short-term response. In addition nontrivial probabilities are given to substantially larger responses.\(^{17}\) Thus, from the standpoint of sampling accuracy, the long-run response could be even more than double that of the immediate consumption response.

### 7.2 Long run risk prices

We consider sampling uncertainty in some of the inputs used to measure long run risk. Recall that these inputs are based in part on low frequency extrapolation of VAR systems fit to match transition dynamics. As in the related macroeconomics literature, we expect a substantial degree of sampling uncertainty. We now quantify how substantial this is for our application.

When $\rho = 1$, the expected excess returns are approximately equal to:

$$\theta \gamma(1) \cdot \pi.$$ 

We now investigate the statistical accuracy of $\gamma(1) \cdot \pi$ for the five portfolios, and for the difference between portfolios 1 and 5. The vector $\pi$ is measured using $\iota(1)$. In table 4 we report the approximate posterior distribution for $\gamma(1) \cdot \pi$ computed using the approach of Sims and Zha (1999) and Zha (1999). As before we use Box-Tiao priors. We scale the values of $\gamma(1) \cdot \pi$ by 400 just as we did when reporting predicted annualized average returns in percentages. While there is a considerable amount of statistical uncertainty in these risk measures, there are important differences in the risk relative risk exposures of portfolios 1 and 5. We conclude that, from a long-run perspective, there is significant evidence that the portfolio of value stocks is riskier than the portfolio of growth stocks.

\(^{17}\)The accuracy comparison could be anticipated in part from the literature on estimating linear time series models using a finite autoregressive approximation to an infinite order model (see Berk (1974)). The on impact response is estimated at the parametric rate, but the long-run response is estimated at a considerably slower rate that depends on how the approximating lag length increases with sample size. Our histograms do not confront the specification uncertainty associated with approximating an infinite order autoregression, however.
Approximate posterior distributions for immediate and long-run responses

Figure 5: Top figure gives the posterior histogram for the immediate response of consumption to the temporary shock. Bottom figures gives the long-run response of consumption to the permanent shock. The histograms have sixty bins with an average bin height of unity. They were constructed using Box-Tiao priors for each equation. The vertical axis is constructed so that on average the histogram height is unity.
### Table 4: Accuracy of estimates of $\gamma(1) \cdot \pi$ scaled by 400

Quantiles were computed by simulating 100,000 times using Box-Tiao priors. The quantiles were computed using only simulation draws for which the absolute values of the eigenvalues were all less than .999. The fraction of accepted draws ranged from .986 to .987. The quantiles were computed using VARs that included consumption, corporate earnings and a single dividend series with one exception. To compute quantiles for the 5 – 1 row, dividends for both portfolios were included in the VAR.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Quantile</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>.05</td>
</tr>
<tr>
<td>1</td>
<td>-.63</td>
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<tr>
<td>2</td>
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<tr>
<td>5-1</td>
<td>.02</td>
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</tbody>
</table>
8 Specification sensitivity

So far our measurements and inferences are conditioned on particular models of stochastic growth. In this section we explore the impact of changing the growth configurations. We consider first the consumption dynamics, and then we consider the cash flow dynamics.

8.1 Consumption Dynamics

Our results so far feature a model in which consumption and earnings have a common long run response to shocks. To achieve this we were required to impose two restrictions on the matrix \( A(1) \). We compare the discounted consumption response \( |\gamma(\beta)| \) from this model with two alternatives. The first imposes no restrictions on \( A(1) \) and is implemented by running a VAR in log levels. The second one restricts \( A(1) \) to be identically zero. This one is implemented by running the VAR in log differences. In figure 6 we depict \( |\gamma(\beta)| \) as a function of \( \beta \) for the baseline model and for two alternative specifications. The log-level VAR is estimated to be stable, and as a consequence the implied \( |\gamma(1)| = 0 \). This convergence is reflected in the figure, but only for values of \( \beta \) very close to unity. For more moderate levels of \( \beta \), the log level specification reduces the measure of \( |\gamma(\beta)| \) by a third. The first difference specification gives results that are intermediate relative to the baseline specification and the log-level specification. In summary, our restriction that consumption and earnings respond to permanent shocks in the same way ensures that a larger value of \( |\gamma(\beta)| \).

8.2 Dividend Dynamics

Our calculations so far have been based on one model of cash flow growth. We now explore some alternative specifications used in other research and check for sensitivity. All of these specifications allow for the dividends from the portfolios to have growth components that are different from consumption to accommodate the growth heterogeneity that is evident in figure 1.

In our baseline model, we identified dividend dynamics and, in particular, the martingale component \( \iota(1) \) using VAR methods. We used a VAR with three variables: consumption, corporate earnings and dividends (all in logarithms). Consumption and earnings were restricted to the same long-run response to permanent shocks. In addition, dividends had their own stochastic growth component.

We now consider two alternative specifications of dividend growth. Both are restrictions on the equation:

\[
A_0^t y_t^* + A_1^t y_{t-1} + A_2^t y_{t-2} + \ldots + A_\ell^t y_{t-\ell} + B_0^* + B_1^* t = w_t^*,
\]

where the shock \( w_t^* \) is scalar with mean zero and unit variance and uncorrelated with the shock vector \( w_t \) that enters (21). The third entry of \( A_0^* \) is normalized to be positive. As in our previous estimation, we set \( \ell = 5 \).
Figure 6: Norm of $\gamma(\beta)$ for different values of $\beta$ and three different VAR systems. The solid line — is for the cointegrated system, the dashed-dot line $\ldots$ is for the system without cointegration, the dotted line $\cdots$ is for the specification with first differences used for all variables.
The first alternative specification restricts that the trend coefficient $B_1^*$ equal zero, and is the model used by Hansen, Heaton, and Li (2005). Given our interest in measuring long-run risk, we measure the permanent response of dividends to the permanent shock. While both consumption and corporate earnings continue to be restricted to respond to permanent shocks in the same manner, the dividend response is left unconstrained. There is no separate growth component for dividends in this specification. The second alternative specification includes a time trend by freely estimating $B_1^*$. A model like this, but without corporate earnings, was used by Bansal, Dittmar, and Lundblad (2005). We refer to this as the time trend specification. In this model the time trend introduces a second source of dividend growth.

The role of specification uncertainty is illustrated in the impulse responses depicted in figure 7. This figure features the responses of portfolios 1 and 5 to a permanent shock. For each portfolio, the measured responses obtained for each of the three growth configurations are quite close up to about 12 quarters (3 years) and then they diverge. Both portfolios initially respond positively to the shock with peak responses occurring in about seven quarters periods. The response of portfolio 5 is much larger in this initial phase than that of portfolio 1. The two alternative models for portfolio 5 give essentially the same impulse responses. The time trend is essentially zero for portfolio 5. The limiting response of the alternative models are much lower than that of the baseline specification.

For portfolio 1 there are important differences in the limiting responses of all three models. While the limiting response of the baseline model is negative, when a time trend is introduced in place of a stochastic growth component, the limit becomes substantially more negative. The time trend specification implies that portfolio 1 provides a large degree of consumption insurance in the long run in contrast to the small covariation measured when the additional growth factor is stochastic, as in our baseline dividend growth model. When consumption/earnings is the sole source of growth, the limiting response is positive but small. While the limiting responses are sensitive to the growth specification, the differences in the long-run responses between portfolios 1 and 5 are approximately the same for the time trend model and for our baseline dividend growth model.\(^\text{18}\)

While the use of time trends as alternative sources of cash flow growth alters our results, it requires that we take these trends literally in quantifying long run risk. Is it realistic to think of these secular movements, that are independent of consumption growth, as deterministic trends when studying the economic components of long-run risk? We suspect not. While there may be important persistent components to the cash flows for portfolio 1, it seems unlikely that these components are literally deterministic time trends known \emph{a priori} to investors. The time trend for this portfolio is in part offset by the negative growth induced by cointegration. We suspect that the substantially negative limiting response for portfolio

\(^{18}\)Bansal, Dittmar, and Lundblad (2005) use their estimates with a time trend model as inputs into a cross sectional return regression. While estimation accuracy and specification sensitivity may challenge these regressions, the consistency of the ranking across methods is arguably good news, as emphasized to us by Ravi Bansal. As is clear from our previous analysis, we are using the economic model in a more formal way than the running of cross-sectional regressions.
Figure 7: This figure depicts the impulse responses to a permanent shock to consumption of the cash flows to portfolios 1 and 5. The \ldots{} curve is generated from the level specification for dividends; the \ldots{} is generated from the level specification with time trends included; and the \ldots{} curve is generated from the first difference specification.
1 is unlikely to be the true limiting measures of how dividends respond to consumption and earnings shocks.\textsuperscript{19}

In summary, while there is intriguing heterogeneity in the long run cash flow responses and implied returns, the implied risk measures are sensitive to the growth specification. Given the observed cash flow growth, it is important to allow for low frequency departures from a balanced growth restriction. The simple cointegration model introduces only one free growth parameter for each portfolio, but results in a modest amount of cash flow heterogeneity. The time trend growth models impose additional sources of growth. The added flexibility of the time trend specification may presume too much investor confidence in a deterministic growth component, however. The dividend growth specification that we used in our previous calculations, while \textit{ad hoc}, presumes this additional growth component is stochastic and is a more appealing specification to us.\textsuperscript{20}

\textsuperscript{19}Sims (1991) and Sims (1996) warn against the use of time trends using conditional likelihood methods because the resulting estimates might over fit the the initial time series, ascribing it to a transient component far from the trend line.

\textsuperscript{20}In the specifications we have considered, we have ignored any information for forecasting future consumption that might be contained in asset prices. Our model of asset pricing implies a strict relationship between cash flow dynamics and prices so that price information should be redundant. Prices, however, may reveal additional components to the information set of investor and hence a long-run consumption risk that cannot be identified from cash flows. When we consider an alternative specification of the VAR where we include consumption, corporate earnings, dividends as well as prices, we obtain comparable heterogeneity.
9 Conclusion

Growth-rate variation in consumption and cash flows have important consequences for asset valuation. The methods on display in this paper formalize the long-run contribution to value of the stochastic components of discount factors and cash flows and quantify the importance of macroeconomic risk. We used these methods to isolate features of the economic environment that have important consequences for long-run valuation and heterogeneity across cash flows. We made operational a well defined notion of long-run cash flow risk and a well defined limiting contribution to the one-period returns coming from cash flows in the distant future.

In our empirical application we showed that the stochastic growth of growth portfolios has negligible covariation with consumption in the long run while the cash flow growth of value portfolios has positive covariation. For these differences to be important quantitatively for our long run risk-return calculations, investors must be either highly risk averse or highly uncertain about the probability models that they confront.\footnote{This latter conclusion can be made precise by using detection probabilities in the manner suggested by Anderson, Hansen, and Sargent (2003).}

In this paper we used an ad hoc VAR model to identify shocks. In contrast to VAR methods, an explicit valuation model is a necessary ingredient for our analysis; and thus we analyzed the valuation implications through the lens of a commonly used consumption-based model. There are important reasons for extending the scope of our analysis in future work, and the methods we described here are amenable to such extensions. One next step is to add more structure to the macroeconomic model, structure that will sharpen our interpretation of the sources of long-run macroeconomic risk.

While the recursive utility model used in this paper has a simple and usable characterization of how temporal dependence in consumption growth alters risk premia in the long run, other asset models have interesting transient implications for the intertemporal composition of risk, including models that feature habit persistence (\textit{e.g.} Constantinides (1990), Heaton (1995), and Sundaresan (1989)) and models of staggered decision-making (\textit{e.g.} see Lynch (1996) and Gabaix and Laibson (2002).)

The model we explore here focuses exclusively on time variation in conditional means. Temporal dependence in volatility can be an additional source of long-run risk. Time variation in risk premia can be induced by conditional volatility in stochastic discount factors.\footnote{It can also be induced by time variation in risk exposure.} While the direct evidence from consumption data for time varying volatility in post war data is modest, the implied evidence from asset pricing for conditional volatility in stochastic discount factors is intriguing. For instance, Campbell and Cochrane (1999) and others argue that risk prices vary over the business cycle in ways that are quantitatively important. The models of Campbell and Cochrane (1999) and Lettau and Wachter (2006) are alternative ways to alter the long-run risk characterization through volatility channels.

While the methods we have proposed aid in our understanding of asset-pricing models, they also expose measurement challenges in quantifying the long-run risk-return tradeoff.
Important inputs into our calculations are the long-run riskiness of cash flows and consumption. As we have shown, these objects are hard to measure in practice. Statistical methods typically rely on extrapolating the time series model to infer how cash flows respond in the long-run to shocks. This extrapolation depends on details of the growth configuration of the model, and in many cases these details are hard to defend on purely statistical grounds. Also there is pervasive statistical evidence for growth rate changes or breaks in trend lines, but this statistical evidence is difficult to use directly in models of decision-making under uncertainty without some rather specific ancillary assumptions about investor beliefs. Many of the statistical challenges that plague econometricians presumably also plague market participants. Naive application of rational expectations equilibrium concepts may endow investors in these models with too much knowledge about future growth prospects. Learning and model uncertainty are likely to be particularly germane to understanding long run risk.
A Eigenfunction results

In what follows we use the notation:

\[ M_{t+1,t} = \exp(s_{t+1,t} + \pi w_{t+1}) \]

A.1 Eigenfunctions and stability

We follow Hansen and Scheinkman (2006) by formalizing the approximation problem as a change in measure. Our analysis is in discrete time in contrast to their continuous-time analysis. Moreover, we develop some explicit formulas that exploit our functional forms.

Write the eigenfunction problem as:

\[ E[M_{t+1,t} \phi(x_{t+1}) | x_t] = \exp(-\nu) \phi(x_t) \]

where \( M_{t+1,t} = \exp(s_{t+1,t} + \pi w_{t+1}) \). Then

\[ \hat{M}_{t+1,t} = \exp(\nu t) M_{t+1,t} \left[ \frac{\phi(x_{t+1})}{\phi(x_t)} \right] \]

satisfies:

\[ E(\hat{M}_{t+1,t} | x_t) = 1. \]

As a consequence \( \hat{M}_{t+1,t} \) induces a distorted conditional expectation operator. Recall our solution \( \phi(x) = \exp(-\bar{\omega} x) \) to this problem. Then by the usual complete the square argument, \( \hat{M}_{t+1,t} \) changes the distribution of \( w_{t+1} \) from being a multivariate standard normal to being a multivariate normal with mean:

\[ \hat{\mu}_w = H' \bar{\omega}' + \pi' + \xi_0' \quad (23) \]

and covariance matrix \( I \). This adds a constant term to the growth rate of consumption. Let the implied distorted expectation operator \( \hat{E} \).

We use this distorted shock distribution in our computations. For instance,

\[ E[M_{t+1,t} \psi(x_{t+1}) | x_t] = \exp(\rho) \phi(x_t) \hat{E} \left[ \frac{\psi(x_{t+1})}{\phi(x_{t+1})} | x_t \right]. \]

Iterating, we obtain:

\[ E[M_{t+j,t} \psi(x_{t+j}) | x_t] = \exp(\rho j) \phi(x_t) \hat{E} \left[ \frac{\psi(x_{t+j})}{\phi(x_{t+j})} | x_t \right]. \]

The limit that interests us is:

\[ \lim_{j \to \infty} \hat{E} \left[ \frac{\psi(x_{t+j})}{\phi(x_{t+j})} | x_t \right] = \hat{E} \left[ \frac{\psi(x_t)}{\phi(x_t)} \right]. \]
provided that \( \{x_t\} \) has a well defined stationary distribution under the \( \hat{E} \) probability distribution and the conditional expectation operator converges the corresponding unconditional expectation operator.

Let \( q \) and \( \hat{q} \) denote the stationary densities of \( \{x_t\} \) under \( E \) and the \( \hat{E} \) measures. Define \( \varphi = \hat{q}/(q\hat{\phi}) \) implying that
\[
\hat{E} \left[ \frac{\psi(x_t)}{\phi(x_t)} \right] = E[\varphi(x_t)\psi(x_t)] .
\]
The density \( q \) is normal with mean zero and covariance matrix:
\[
\Sigma = \sum_{j=0}^{\infty} (G^j)HH'(G^j)',
\]
which can be computed easily using a doubling algorithm. The density \( \hat{q} \) is normal with mean
\[
\hat{\mu}_x \doteq (I - G)^{-1}H(-H'\omega' + \pi' + \xi_0'),
\]
and the same covariance matrix as \( q \).

Consider now a joint Markov process \( \{(x_t, z_t) : t \geq 0\} \), and the equation:
\[
E \left[ M_{t+1,t} \left( \frac{z_{t+1}}{z_t} \right) \left( \frac{\phi(x_{t+1})}{z_{t+1}} \right) |x_t \right] = \exp(\rho) \left[ \frac{\phi(x_t)}{z_t} \right] .
\]
While this amounts to a rewriting of the initial eigenvalue equation, it has a different interpretation. The process \( \{z_t\} \) is a transient contribution to the stochastic discount factor, and the eigenfunction equation is now expressed in terms of the composite state vector \( (x, z) \) with the same eigenvalue and an eigenfunction \( \phi(x)/z \). The limit of interest is now:
\[
\lim_{j \to \infty} \hat{E} \left[ \frac{\psi(x_{t+j})z_{t+j}}{\phi(x_{t+j})} \right] |x_t \right] = \hat{E} \left[ \frac{\psi(x_t)z_t}{\phi(x_t)} \right] .
\]
To study this limit we require that the process \( \{(x_t, z_t)\} \) be stationary under the distorted probability distribution and that \( \psi(x_t)z_t \) have a finite expectation under this distribution.

In the special case in which \( G = 0 \), and \( \phi = 1 \) it suffices to study \( \psi(x_t)z_t \).

### A.2 Eigenvalue derivative

We compute this derivative using the approach developed in Hansen (2006). Suppose that \( \hat{M}_{t+1,t} \) depends implicitly on a parameter \( \rho \). Since each member of the parameterized family has conditional expectation equal to unity,
\[
E \left( \frac{\partial \log \hat{M}_{t+1,t}}{\partial \rho} |x_t \right) = E \left( \frac{\partial \hat{M}_{t+1,t}}{\partial \rho} |x_t \right) = 0.
\]
Note that
\[
\hat{E} \left( \frac{\partial \log M_{t+1,t}}{\partial \rho} | x_t \right) = \hat{E} \left( \frac{\partial \log M_{t+1,t}}{\partial \rho} | x_t \right) - \frac{\partial \nu}{\partial \rho} + \hat{E} \left( \frac{\partial \log \phi(x_{t+1})}{\partial \rho} | x_t \right) - \frac{\partial \log \phi(x_t)}{\partial \rho}.
\]
Since the left-hand side is zero, applying the Law of Iterated Expectation under the \( \hat{\cdot} \) probability measure:
\[
0 = \hat{E} \left( \frac{\partial \log M_{t+1,t}}{\partial \rho} \right) - \frac{\partial \nu}{\partial \rho} + \hat{E} \left( \frac{\partial \log \phi(x_{t+1})}{\partial \rho} \right) - \hat{E} \left( \frac{\partial \log \phi(x_t)}{\partial \rho} \right).
\]
Since \( \{x_t\} \) is stationary under the \( \hat{\cdot} \) probability measure,
\[
\frac{\partial \nu}{\partial \rho} = \hat{E} \left( \frac{\partial \log M_{t+1,t}}{\partial \rho} \right).
\]
To apply this formula, write
\[
\log M_{t+1,t} = s_{t+1,t} + \pi w_{t+1}
\]
Differentiating with respect to \( \rho \):
\[
Ds_{t+1,t}^1 = \frac{1}{2} w_{t+1}' \Theta_0 w_{t+1} + w_{t+1}' \Theta_1 x_t + \vartheta_0 + \vartheta_1 x_t + \vartheta_2 w_{t+1}.
\]
Recall that under the distorted distribution \( w_{t+1} \) has a constant mean \( \hat{\mu}_w \) conditioned on \( x_t \) given by (23) and \( x_t \) has a mean \( \hat{\mu}_x \) given by (24). Taking expectations under the distorted distribution:
\[
\hat{E} \left( Ds_{t+1,t}^1 \right) = \frac{1}{2} (\hat{\mu}_w)' \Theta_0 \hat{\mu}_w + \frac{1}{2} trace(\Theta_0) + (\hat{\mu}_w)' \Theta_1 \hat{\mu}_x + \vartheta_0 + \vartheta_1 \hat{\mu}_x + \vartheta_2 \hat{\mu}_w.
\]
References


