Long-Run Investment Horizons

and Implications for

Mixed-Asset Portfolio Allocations

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December 30, 2011

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The author thanks Tomohiro Ando, Bassam Barazi, John Cochrane, George Constantinides, Elroy Dimson, David Geltner, Narasimhan Jegadeesh, Johannes Ledolter, Greg MacKinnon, Luboš Pástor, Maarten van der Spek and Ruey Tsay for helpful comments and Savina Rizova and Zahra Siddique for excellent research assistance. However, all errors and omissions are the author’s responsibility.
Abstract

The reported returns on private-market assets (such as commercial real estate and private equity) often display higher levels of autocorrelation than their public-market counterparts. When different asset classes display varying degrees of autocorrelation, the investment horizon may substantially alter optimized mixed-asset portfolio allocations. Consequently, the one-year returns typically used in mixed-asset portfolio optimization procedures often generate excessive allocations to private-market asset classes. By examining the three components in the standard portfolio optimization technique as the investment horizon lengthens, the allocations to commercial real estate – as one example of private-market assets – are much reduced and more consistent with allocation levels generally found in large institutional pension plans.

Key words: Long-horizon variance and correlation, autocorrelation, portfolio optimization.
I. Introduction

If the reported returns on all asset classes displayed the same serial correlation, then short- and long-term risk measures would display the same relative rankings amongst asset classes. Alternatively stated, long-term measures of variance and correlation amongst asset classes (as compared to short-term measures) would be unitless.

However, it is well known that the broad asset classes considered by most institutional investors vary considerably with regard to the levels of their autocorrelation. While public-market assets most often display very little serial correlation, private-market assets (e.g., private (commercial) real estate, leveraged buyouts, venture capital, etc.) display significant levels of serial correlation. These private-market investments, often broadly referred to as alternative investments, now comprise about 20% of the typical U.S. (defined-benefit) pension plan. The impact of varying levels of autocorrelation in the return series impacts the risk characteristics of returns over long investment horizons. Consequently, the issue of how to consider asset classes with varying levels of autocorrelation matters to those who seek to optimize mixed-asset portfolios.

The standard practice of using one-year returns (and one-year investment horizons) has many fine qualities (e.g., consistency with financial-reporting cycles). Indeed, the investment landscape when viewed through the prism of the public markets (and assumed frictionless trading) seems quite rational in light of yearly portfolio rebalancing. However, there is nothing sacrosanct about one-year horizons. Moreover, the standard practice of one-year returns obscures the changing risk/return (and, therefore, portfolio-optimizing) characteristics associated with increasing
investment horizons – again, once one considers asset classes with varying levels of serial correlation.

Much of this paper will focus on private-market commercial real estate returns as a particular example of the issues associated private-market assets more generally. This is not to suggest that the high levels of autocorrelation often found in other private-market asset classes (e.g., private equity) are unimportant; they are indeed quite important. Instead, it reflects the longer (and generally more reliable) time series of returns available with private real estate, which is important in the empirical analyses that follow.

As a result, annual returns – as typically used in a mixed-asset portfolio optimization – often generate excessive allocations to private real estate. Excessive in the sense that: 1) the autocorrelation of the private real estate return series understates: a) the long-run volatility of such returns and b) its correlation with most other asset classes, and 2) the suggested allocations to real estate are substantially higher than the allocations generally found in large institutional pension plans (even among those with significant allocations to private real estate). As an incomplete approach to addressing the varying levels of serial correlation, many researchers and practitioners advocate either \textit{ad hoc} rules designed to constrain the maximum allocation to private-market (commercial) real estate and/or the use of some de-smoothing procedure designed to inject additional volatility into the observed (appraisal-based) real estate return series (thereby reducing the optimization’s allocation to private real estate). This paper takes a different approach by examining the three components in the standard portfolio optimization technique – average return, variance and correlation – as the investment horizon lengthens. Examining the volatility and correlation of long-run returns (when
the return series displays significant autocorrelation) helps to illustrate the potential changes in optimized mixed-asset portfolio settings.

It should be reinforced that the aim of this paper is primarily to explore misapplied risk measures (i.e., volatility and correlation) and why institutional investors hold far less commercial real estate than is suggested by a mere mixed-asset portfolio optimization using historical, annual returns. In so doing, this paper provides an alternative perspective to the many popular techniques for estimating “true” private (commercial) real estate volatility. This is not to suggest that these alternative techniques do not have important applications for other objectives (e.g., better assessing market inflection points); rather, these techniques say little about the treatment of other private-market alternative investments (e.g., private equity) and say nothing about the treatment of Treasury bills – which also displays significant autocorrelation – in the context of portfolio optimization. Therefore, a more robust treatment of the problem is to examine long-horizon characteristics.

The balance of the paper is organized as follows: Section II examines the optimal mixed-asset portfolio allocations using historical one-year returns as motivation for examining longer investment horizons. Section III examines the three main components to portfolio optimization in light of longer investment horizons. Sections IV and V examine the optimal mixed-asset portfolio allocations using longer investment horizons. Section VI concludes.
II. Portfolio Allocations using One-Year Returns

Partly because returns to private (commercial) real estate\(^1\) display low levels of volatility and significant autocorrelation, there has been an ongoing debate about real estate’s optimal role in a mixed-asset portfolio. This debate extends back at least as far as Firstenberg, Ross and Zisler (1988), who were among the first to acknowledge the problems relating to potentially understated volatility of the serially correlated (observed) private real estate returns. When returns follow a “random walk,” this implies that (in short time intervals) future prices cannot be determined by the path of past prices\(^2\) or, more broadly, that expected appreciation returns are constant combined with a zero-mean error term. While the serial correlation parameter is approximately zero for most public-market financial assets, this is not true for many private-market financial assets. This disparity in the levels of serial correlation unfortunately causes, as noted earlier, the \textit{ex post} optimal portfolio allocations to vary with the length of the investment horizon.

II.A. The Data & \textit{Ex Post} Portfolio Optimization

In order to approximate the portfolio optimization process of institutional investors, it is helpful to have an understanding of the types of assets in which they invest. While the majority of pension fund assets is allocated to stocks and bonds, alternative investments represent an increasing minority. For a representative sample of large institutional investors, consider Table 1:

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\(^1\) For purposes of this paper, the commercial real estate is proxied by the index provided by the National Council of Real Estate Investment Fiduciaries (NCREIF), which publishes performance indices of institutionally owned, stabilized, private unlevered U.S.-based real estate. Over the five years ended in 2009, the NCREIF index averaged approximately 5,650 properties and more than $250 billion of gross asset value.

\(^2\) A more complete discussion of the random walk is outside the scope of this article. However, the interested reader is suggested to refer to chapter 2 of Campbell, Lo and MacKinley (1997) as one of many possible sources.
Table 1
Defined-Benefit Pension-Plan Sponsors’ Portfolio Allocations

Represents the percentage allocations of defined-benefit pension plan assets, for the years ended December 31st, based upon a sample of large, defined-benefit pension plans that reported data for all years. Sources: Pension Real Estate Association and Standard & Poor’s *Money Market Directories.*

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<td>3.3</td>
<td>3.2</td>
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<td>3.0</td>
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<td>2.8</td>
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<tr>
<td>Equities (unspeccified)</td>
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<td>15.8</td>
<td>14.7</td>
<td>15.8</td>
<td>16.5</td>
<td>16.5</td>
<td>20.8</td>
<td>23.4</td>
<td>26.1</td>
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<td>19.9</td>
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<tr>
<td>Other U.S. Equities</td>
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<td>13.3</td>
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<td>14.5</td>
<td>13.3</td>
<td>14.0</td>
<td>12.1</td>
<td>11.2</td>
<td>10.0</td>
<td>9.0</td>
<td>10.3</td>
<td>8.1</td>
<td>7.9</td>
<td>6.4</td>
</tr>
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<td>U.S. Indexed Equities</td>
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<td>11.9</td>
<td>12.9</td>
<td>14.1</td>
<td>15.5</td>
<td>14.2</td>
<td>11.2</td>
<td>6.8</td>
<td>6.7</td>
<td>4.5</td>
<td>2.5</td>
<td>2.8</td>
<td>2.9</td>
<td>3.5</td>
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<tr>
<td>International Equities</td>
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<td>9.0</td>
<td>9.9</td>
<td>9.9</td>
<td>10.4</td>
<td>11.8</td>
<td>10.0</td>
<td>10.1</td>
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<td>12.4</td>
<td>15.2</td>
<td>13.9</td>
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<td>Company Stock</td>
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<td>4.5</td>
<td>4.8</td>
<td>4.4</td>
<td>3.6</td>
<td>3.6</td>
<td>3.5</td>
<td>2.8</td>
<td>2.6</td>
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<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>0.3</td>
<td>0.8</td>
<td>1.3</td>
<td>1.6</td>
<td>1.4</td>
<td>1.9</td>
<td>2.1</td>
<td>2.6</td>
<td>1.4</td>
<td>4.3</td>
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<tr>
<td>Bonds</td>
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</tr>
<tr>
<td>Bonds (other)</td>
<td>20.0</td>
<td>18.7</td>
<td>17.9</td>
<td>17.7</td>
<td>17.5</td>
<td>17.9</td>
<td>19.0</td>
<td>20.5</td>
<td>18.9</td>
<td>19.2</td>
<td>17.6</td>
<td>19.4</td>
<td>19.1</td>
<td>20.3</td>
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<tr>
<td>U.S. Government Bonds</td>
<td>6.7</td>
<td>6.2</td>
<td>5.9</td>
<td>5.0</td>
<td>4.3</td>
<td>3.7</td>
<td>4.5</td>
<td>4.8</td>
<td>4.0</td>
<td>2.7</td>
<td>3.1</td>
<td>3.4</td>
<td>3.5</td>
<td>5.0</td>
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<tr>
<td>Corporate Bonds</td>
<td>3.8</td>
<td>3.4</td>
<td>3.4</td>
<td>3.4</td>
<td>3.1</td>
<td>2.5</td>
<td>2.3</td>
<td>2.0</td>
<td>1.3</td>
<td>0.5</td>
<td>0.6</td>
<td>0.8</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>International Bonds</td>
<td>2.3</td>
<td>1.8</td>
<td>1.9</td>
<td>1.5</td>
<td>1.4</td>
<td>1.3</td>
<td>1.1</td>
<td>0.7</td>
<td>2.2</td>
<td>1.8</td>
<td>1.9</td>
<td>0.9</td>
<td>1.0</td>
<td>2.5</td>
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<tr>
<td>Real Estate Equity</td>
<td>2.9</td>
<td>2.9</td>
<td>2.7</td>
<td>2.6</td>
<td>2.6</td>
<td>2.8</td>
<td>3.1</td>
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<td>3.2</td>
<td>3.6</td>
<td>4.3</td>
<td>4.4</td>
<td>4.6</td>
</tr>
<tr>
<td>Mutual Funds</td>
<td>1.2</td>
<td>1.5</td>
<td>1.7</td>
<td>1.9</td>
<td>1.9</td>
<td>2.2</td>
<td>2.3</td>
<td>2.2</td>
<td>2.1</td>
<td>2.8</td>
<td>3.5</td>
<td>4.1</td>
<td>4.2</td>
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<tr>
<td>Other Assets</td>
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<td>6.2</td>
<td>5.4</td>
<td>5.2</td>
<td>5.0</td>
<td>4.7</td>
<td>5.4</td>
<td>6.8</td>
<td>6.3</td>
<td>7.6</td>
<td>8.0</td>
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<td>Total Assets</td>
<td>100.0</td>
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<td>100.0</td>
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</table>

So, at the risk of oversimplifying the investment set available to institutional investment managers, this paper will examine the optimal ex post portfolio allocations by utilizing the historical annual returns from the nine asset classes found in Table 2:
Table 2
Summary Statistics for the Annual Returns of Selected Asset Classes

Summary statistics of annual returns for the years ended 1978-2009. The selected asset classes are: the Standard & Poor's 500 large-capitalization stocks; U.S. small stocks as proxied by the total return achieved by the Dimensional Fund Advisors (DFA) Small Company 9/10 (for ninth and tenth deciles) Fund; international equity returns are proxied by the MSCI (Morgan Stanley Capital International) EAFE ("Europe, Australia and Far East") Index; U.S. LT (long-term) Government Bonds are constructed with data from The Wall Street Journal; U.S. LT Corporate Bonds are represented by the Citigroup Long-Term High-Grade Corporate Bond Index (nearly all Aaa- and Aa-rated bonds with at least 10 years to maturity); Domestic High-Yield Corporate Bonds are represented by the Lehman Brothers High Yield Index (which covers the universe of fixed rate, noninvestment grade debt); U.S. 30-Day Treasury Bills returns represent data from The Wall Street Journal; public real estate returns are proxied by annual returns reported by the National Association of Real Estate Investment Trusts ("NAREIT") (for those REITs with 75% or greater of their gross invested book assets invested directly or indirectly in the equity ownership of real estate); and private real estate returns are proxied by the National Council of Real Estate Investment Fiduciaries ("NCREIF") Property Index (which is an unlevered index of properties acquired on behalf of tax-exempt institutions.) [U.S. Inflation, which of course is not an asset class (but included as a matter of perspective), is represented by the Consumer Price Index for All Urban Consumers.] Source: Morningstar

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Geometric Mean</th>
<th>Geometric Median</th>
<th>Arithmetic Mean</th>
<th>Arithmetic Median</th>
<th>Standard Deviation</th>
<th>Serial Correlation</th>
<th>Sharpe Ratio</th>
<th>Highest Return</th>
<th>Lowest Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>11.31%</td>
<td>12.78%</td>
<td>16.20%</td>
<td>17.37%</td>
<td>-0.01%</td>
<td>0.346</td>
<td>37.58%</td>
<td>-37.00%</td>
<td></td>
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<tr>
<td>U.S. Small Stocks</td>
<td>13.52%</td>
<td>15.56%</td>
<td>19.68%</td>
<td>21.14%</td>
<td>-17.12%</td>
<td>0.350</td>
<td>60.70%</td>
<td>-36.72%</td>
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</tr>
<tr>
<td>MSCI EAFE</td>
<td>8.02%</td>
<td>10.44%</td>
<td>9.80%</td>
<td>14.14%</td>
<td>-0.67%</td>
<td>0.153</td>
<td>66.80%</td>
<td>-45.09%</td>
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<tr>
<td>U.S. LT Government Bonds</td>
<td>8.96%</td>
<td>9.69%</td>
<td>8.28%</td>
<td>12.91%</td>
<td>-31.96%</td>
<td>0.520</td>
<td>40.36%</td>
<td>-14.90%</td>
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<tr>
<td>U.S. LT Corporate Bonds</td>
<td>8.87%</td>
<td>9.37%</td>
<td>8.75%</td>
<td>10.83%</td>
<td>-7.61%</td>
<td>0.508</td>
<td>42.56%</td>
<td>-7.45%</td>
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<tr>
<td>Domestic High-Yield Bonds</td>
<td>9.64%</td>
<td>10.73%</td>
<td>10.06%</td>
<td>16.10%</td>
<td>-20.26%</td>
<td>0.323</td>
<td>58.21%</td>
<td>-26.16%</td>
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<tr>
<td>U.S. 30-Day Treasury Bills</td>
<td>5.69%</td>
<td>5.74%</td>
<td>5.36%</td>
<td>3.31%</td>
<td>84.13%</td>
<td>0.115</td>
<td>14.71%</td>
<td>0.10%</td>
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<tr>
<td>NAREIT-Equity</td>
<td>12.41%</td>
<td>13.99%</td>
<td>17.18%</td>
<td>18.01%</td>
<td>2.58%</td>
<td>0.375</td>
<td>37.13%</td>
<td>-37.73%</td>
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<tr>
<td>NCREIF Property</td>
<td>8.77%</td>
<td>9.10%</td>
<td>9.97%</td>
<td>8.31%</td>
<td>73.17%</td>
<td>0.534</td>
<td>20.46%</td>
<td>-16.86%</td>
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<tr>
<td>U.S. Inflation</td>
<td>3.97%</td>
<td>4.01%</td>
<td>3.29%</td>
<td>2.98%</td>
<td>76.96%</td>
<td>13.31%</td>
<td>0.09%</td>
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While there are certainly other asset classes to consider, the asset classes identified above meet the dual criteria of: 1) represents a significant allocation by typical pension fund managers and 2) has a sufficiently long time series of returns such that it permits the long-horizon calculations subsequently performed. As such, these asset classes represent a fairly typical mix of stocks (domestic and foreign), bonds, cash and real estate as found in institutional portfolios as proxied by
large defined-benefit pension plans.\textsuperscript{3} The correlation matrix for these assets (using one-year returns) is shown in Table 3:

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<tbody>
<tr>
<td>S&amp;P 500</td>
<td>1.00</td>
<td>0.69</td>
<td>0.63</td>
<td>0.05</td>
<td>0.21</td>
<td>0.72</td>
<td>0.25</td>
<td>0.51</td>
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<td>0.14</td>
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<tr>
<td>U.S. Small Stocks</td>
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<td>0.45</td>
<td>-0.16</td>
<td>0.03</td>
<td>0.83</td>
<td>0.18</td>
<td>0.74</td>
<td>0.18</td>
<td>0.28</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSCI EAFE</td>
<td>0.63</td>
<td>0.45</td>
<td>1.00</td>
<td>-0.07</td>
<td>0.03</td>
<td>0.61</td>
<td>0.07</td>
<td>0.38</td>
<td>0.35</td>
<td>0.06</td>
<td></td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>U.S. LT Government Bonds</td>
<td>0.05</td>
<td>-0.16</td>
<td>-0.07</td>
<td>1.00</td>
<td>0.93</td>
<td>0.10</td>
<td>-0.01</td>
<td>-0.05</td>
<td>-0.32</td>
<td>-0.38</td>
<td></td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>U.S. LT Corporate Bonds</td>
<td>0.21</td>
<td>0.03</td>
<td>0.03</td>
<td>0.93</td>
<td>1.00</td>
<td>0.38</td>
<td>0.05</td>
<td>0.14</td>
<td>-0.28</td>
<td>-0.37</td>
<td></td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>Domestic High-Yield Corporate Bonds</td>
<td>0.72</td>
<td>0.83</td>
<td>0.61</td>
<td>0.10</td>
<td>0.38</td>
<td>1.00</td>
<td>0.13</td>
<td>0.74</td>
<td>-0.03</td>
<td>0.03</td>
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<td></td>
</tr>
<tr>
<td>U.S. 30-Day Treasury Bills</td>
<td>0.25</td>
<td>0.18</td>
<td>0.07</td>
<td>-0.01</td>
<td>0.05</td>
<td>0.13</td>
<td>1.00</td>
<td>0.15</td>
<td>0.38</td>
<td>0.69</td>
<td></td>
<td>0.19</td>
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</tr>
<tr>
<td>NAREIT-Equity</td>
<td>0.51</td>
<td>0.74</td>
<td>0.38</td>
<td>-0.05</td>
<td>0.14</td>
<td>0.74</td>
<td>0.15</td>
<td>1.00</td>
<td>0.25</td>
<td>0.19</td>
<td></td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>NCREIF Property</td>
<td>0.27</td>
<td>0.18</td>
<td>0.35</td>
<td>-0.32</td>
<td>-0.28</td>
<td>-0.03</td>
<td>0.38</td>
<td>0.25</td>
<td>1.00</td>
<td>0.47</td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>

Average Coefficient *: 0.42, 0.37, 0.31, 0.06, 0.19, 0.43, 0.15, 0.36, 0.10

* excluding inflation and itself.

These three necessary inputs\textsuperscript{4} – average return, risk and correlation – produce the \textit{ex post} efficient frontier, based on the technique popularized\textsuperscript{5} by Markowitz (1952), shown in Figure 1.

\textsuperscript{3} Inflation is clearly not an asset class in which pension funds invest; instead its inclusion in Tables 2 and 3 is meant merely to provide the reader with some sense of the \textit{ex post} real (or inflation-adjusted) rates of return.

\textsuperscript{4} While some argue for considering higher moments of the return distribution – for example, see: Harvey, et al. (2010) – this paper considers just the standard inputs identified above.

\textsuperscript{5} The optimization process itself was conducted using Morningstar© software. The only constraint placed on this \textit{ex post} optimization process was to confine each asset’s weight ($w$) to [0, 1].
Figure 1. Efficient Frontier based on Selected Asset Classes – Using Annual Returns. Plots the (arithmetic) average return and standard deviation and represents the (constrained) optimized ex-post efficient frontier for the selected asset classes shown in Table 2 for the period 1978-2009 (where the constraint confines each asset's weight \( w \) to \([0, 1]\)) – using the arithmetic mean and standard deviation shown in Table 2 and the correlation matrix shown in Table 3.

![Efficient Frontier Chart]

Figure 2 displays the ex post allocations to the various asset classes as one moves from the efficient lowest-risk/lowest-return portfolio to the efficient highest-risk/highest-return portfolio. At the low end of the efficient risk/return spectrum, Treasury bills and private-market real estate dominate the portfolio allocations. Towards the middle of the efficient risk/return spectrum, Treasury bonds,
small stocks and real estate (first in its private-market form and later in its public-market form\(^6\)) dominate the portfolio allocations. At the high end of the efficient risk/return spectrum, small stocks dominate the portfolio allocations. Interestingly, real estate – in either its private- or public-market form – occupies an average portfolio allocation in excess of 25% (as shown by the white dashed line in Figure 2); real estate appears as a substantial portfolio weighting in all but the most extreme edges of the efficient frontier.

Why are the (private) real estate allocations so high? The answers lie in Tables 2 and 3, where real estate offers the highest Sharpe ratio (Table 2) and the lowest average correlation\(^7\) (Table 3) with any of the other asset classes. But as to be subsequently addressed, the volatility measure of the appraisal-based private real estate series is thought to understate the true (one-year) volatility of the asset class. Moreover and as noted earlier, these substantial portfolio allocations to (public and private) real estate are inconsistent with the levels witnessed in institutional portfolios (Table 1).

---

\(^6\) As proxied by the all-equity component of the NAREIT (National Association of Real Estate Investment Trusts) Index. For the five years ended in 2009, the NAREIT Equity Index averaged 130 publicly traded firms, with an aggregate annual market capitalization of approximately $287 billion.

\(^7\) The optimization process does not use the average correlation coefficient of one asset class to all other asset classes to construct the efficient frontier. Nevertheless, the average correlation coefficient provides some crude proxy for the diversification-enhancing capabilities of a given asset class.
Figure 2. Components of the Efficient Frontier – Using Annual Returns. Represents the optimal (ex-post) portfolio allocations among the selected asset classes, as one moves from the efficient low-risk/low-return portfolios (shown in Figure 1) to the efficient high-risk/high-return portfolios for the selected asset classes shown in Table 2 for the period 1978-2009. The white dashed line represents the average (efficient) portfolio allocation to both public and private real estate (as proxied by the NAREIT and NCREIF indices).

II.B. Appraisal-Based Real Estate Return Series

That private real estate occupies such a significant portion of the efficient frontier (Figure 2) often produces concerns about the understated volatility associated with a primarily appraisal-based
return series,\(^8\) due to appraisal "smoothing" and lags. In response to these concerns, a plethora of "de-smoothing" procedures (and criticisms) have been proposed. While a full-scale discussion of these various procedures is beyond the scope of this article, a general overview is a helpful counterpoint to the assertion made in this paper that long investment horizons are a preferable prism through which portfolio allocations can be viewed. Therefore, Appendix #1 provides a sketch of some important aspects of the de-smoothing literature which generally asserts that, due to the infrequent trading of private real estate, the lack of publicly available information and the imprecision with which the "true" prices are observed, appraisers anchor on past appraised values and only partially update their valuation estimates. Consequently, the de-smoothing approaches attempt to recover true market prices from the observed appraisal-based return series and, in so doing, typically inject additional volatility in the return series while reducing (if not eliminating) autocorrelation (without biasing the first moment of the return distribution).

II.C. Private Real Estate’s Nexus to REITs and Private Equity

Interestingly, private real estate\(^9\) has connections to both the public real estate market (i.e., publicly traded real estate investment trusts (REITs)) and to the larger world of private equity (e.g., hedge funds, leveraged buyouts, venture capital, etc.). Let’s take a moment to examine these connections:

\(^8\) Because these commercial assets are privately held (primarily by institutional investors), they trade infrequently. So, in order to construct a quarterly return series, NCREIF requires that the market values of the properties are periodically estimated by appraisals.

\(^9\) Sometimes also referred to as direct (v. indirect) or unsecuritized (v. securitized) real estate.
II.C.1. Public- v. Private-Market Real Estate

While publicly traded REITs are an imperfect substitute for private real estate (as the former often includes a service and/or investment component\(^\text{10}\) bundled in the security acquired by shareholders), these imperfections seem fairly small when viewed from a long-run perspective. Alternatively stated, the underlying assets of most (equity) REITs and private real estate funds are predominately buildings (and the leases of tenants occupying those buildings). As such, we would expect that – once we control for the differences in the composition of the indices used to proxy public- and private-market real estate returns – the long-run performance of these two investment avenues are quite similar. In support of this assertion, Pagliari, \(et \ al.\) (2005) found – using a paired-comparison \(t\) test – that the returns from the NCREIF and de-levered NAREIT indices were not notably different, while Oikarinen, \(et \ al.\) (2009) found essentially the same result using a co-integration test. From a somewhat different perspective, Horrigan, \(et \ al.\) (2009) argue that REITs can be used to replicate private-market real estate portfolios. These three papers suggest, from varying perspectives, that public- and private-market real estate investments are substitutes for one another\(^\text{11}\) – again, provided that care is taken to control for the substantive differences in the reported indices.

The biggest of these differences may be leverage: the NAREIT returns are reported with leverage (which averages roughly 40-50\%) whereas the NCREIF returns are reported on an

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\(^{10}\) For example, REITs may engage in: third-party management services, limited real estate services to tenants, the fund-management business (\(i.e.,\) act as the sponsoring entity of a syndicated venture), real estate development, \(etc.\)

\(^{11}\) There are dissenting points of view, including: Riddioough, \(et \ al.\) (2005), who attribute the possible causes of their observed gap in public- \(v.\) private-market real estate performance to “…liquidity and geography as missing factor adjustments, an unrepresentative sample period or \(efficiencies \ to \ securlizing \ commercial \ real \ estate\)” (emphasis added).
unlevered basis. Then, as would be expected, this gearing simply increases the average return and the volatility of return for the levered index. And, consequently, both public- and private-market real estate indices (NAREIT and NCREIF, respectively) are considered as part of the investor’s overall real estate allocation – as shown in Figure 2.

II.C.2. Private-Market Real Estate v. Private Equity

Because of their non-traded and illiquid nature, there are also many similarities between direct property investments and the larger investment arena generally described as private equity. Like private real estate, hedge-fund returns (and similarly intended investment strategies) often display highly autocorrelated returns and dampened volatilities – see, for example, a series of similar-themed papers: Lo (2001), Lo (2002) and Getmansky, Lo and Makarov (2004) which argue that asset classes (e.g., hedge-fund returns) which do not display identically and independently distributed (iid) returns must be treated in a more sophisticated fashion when producing evaluative statistics (e.g., Sharpe ratios) and/or understanding long-run performance – as compared to those asset classes which do display iid returns.

Due to illiquidity and large transaction costs, “marked” asset values for non-traded securities are slow to recognize true market values. With regard to hedge-fund returns (or, more accurately, the mark-to-market process with regard to asset values), Getmansky, Lo and Makarov (2004) offer several possible explanations: managers simply extrapolate past markets values for thinly traded securities for which a bona fide quote is unavailable, managers obtain value estimates from broker/dealers who simply extrapolate past market values, and/or managers look to smooth performance (and thereby reduce volatility). [Note that these reasons are quite similar in spirit with much of the private-market real estate literature (see Appendix #1) which suggests that real estate
Consequently, it is important to understand the implications of the return-generating process for private equity and its potential applicability to direct real estate investments. For example, Getmansky, Lo and Makarov (2004) argue that the highly auto-correlated returns observed with respect to hedge funds are due to illiquidity. They posit four possible additional causes of this serial correlation: a) market inefficiencies, b) time-varying expected returns, c) time-varying leverage and d) incentive fees tied to cumulative returns.

The authors dismiss the first of these arguments on the grounds that associating significant serial correlation with market inefficiencies naively extrapolates the lessons found in traded, liquid securities to the characteristics of non-traded, illiquid securities. The very nature of highly competitive markets such as private equity – with well-trained and compensated professionals directing substantial funds – suggests that any apparent arbitrage opportunities (due to the persistence in returns, as suggested by their substantive serial correlation) would be immediately exploited. Instead, it must be that high transaction costs and the illiquidity of the securities prevent these arbitrage opportunities from being profitably exploited. Indeed, the very nature of private-

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12 These possible causes are quite apart from the non-synchronous trading problem often associated micro-market structure and thinly traded securities. The non-synchronous trading literature tends to measure minutes, hours or days – not the months or quarters often found with private equity and/or private real estate.

13 Consider the admonition of Samuelson (1965) that “… properly anticipated prices fluctuate randomly” as a seminal description of market efficiency in the context of exchange-traded, liquid securities with few transaction costs.
market assets operates counter to the tenets of much of mainstream finance (which utilizes a framework of zero transaction costs, perfectly divisible securities, etc. to derive many of its most widely accepted theorems).\textsuperscript{14}

The authors also dismiss the second of these arguments. They assert that variations in expected returns over time cannot realistically generate the autocorrelation (30-50\%) observed in monthly hedge fund returns. The authors support their argument with an econometric model of the return-generating process and then calibrate the parameters of their model with plausible values to test whether or not time-varying return requirements give rise to the levels of serial correlation observed in hedge-fund returns.

While their last two arguments (time-varying leverage and incentive fees) are inapplicable to private real estate returns as reported in the NCREIF index,\textsuperscript{15} the authors also create econometric models to examine whether either of these two arguments might lead to the levels of observed serial correlation. Here too they find both possibilities unable to plausibly produce the levels of serial correlation observed in monthly hedge-fund returns.

So, like much of the private-market real estate literature, Getmansky, Lo and Makarov (2004) assert that “true” returns, as determined by market equilibria, are unobservable and that observed

\textsuperscript{14} As the authors point out, substantive serial correlation cannot per se be an indication of market inefficiency. If so, how would one explain the high serial correlation observed in the return series of U.S. Treasury bills? Certainly, the U.S. bond market is one of the world’s most efficient financial markets.

\textsuperscript{15} The NCREIF indices are reported without leverage and/or investment management fees. Nevertheless, these indices report returns that display high serial correlation.
returns represent weighted averages of current and previous true returns\textsuperscript{16} – such that the observed returns are a “smoothed” byproduct of true returns: while the expected value of the return is unbiased, the volatility and correlation measures are biased downward.

**II.D. Private Equity → Brief & Unreliable Return Series**

A return series for private equity is noticeably missing from Tables 2 and 3. The reason is twofold: any decent return series for private equity is relatively brief and the reliability of the data is questionable.

As to the first reason, consider one of the better known and more respected indices: Cambridge Associates’ quarterly return series of U.S. venture capital. The first full year of the return series begins in 1982. Instead, Tables 2 and 3 start in 1978 – the inception date for the NCREIF Property Index (with all other shown asset classes starting earlier). There are two cautionary aspects of any brief return series worth noting:

First, it is often the case that a relatively short time series does not adequately display the “true” risk/return characteristics of an index. As an example, see Figure 3 and consider the path of average return and volatility as the history of the NCREIF Property Index (or NPI) series grows:

\textsuperscript{16} There are differences, including: the model of Getmansky, Lo and Makarov (2004) provides for a finite weighted average (whereas Geltner (1993) assumes an infinite weighted average) and for different weighting schemes: straight-line, sum-of-the-years digits and geometric (whereas Geltner (1993) assumes only a geometric (or exponential) weighted average).
Figure 3. Historical Path of Private Real Estate Returns. The upper line, beginning with the point labeled 1981, represents the NPI's average nominal return and volatility after three years have elapsed (i.e., from the NPI's inception in 1978), whereas the nominal return labeled 2009 represents the NPI's average return and volatility after thirty-two years have elapsed – the interim points represent the cumulative average return and volatility as the Index aged. The lower line represents the same evolution in risk and return, but shown for real (i.e., inflation-adjusted) returns.

As the NPI history grew, volatility increased while its average return declined.\(^\text{17}\) There are two explanations for this behavior: a) As the sample size of observed returns increases with time, the larger sample – absent any “regime” changes – is more representative of the “true” population.

\(^\text{17}\) On a nominal basis, the NPI’s average return declined with the passage of time. However, much of this decline is attributable to the decline in inflation over the same period – from the double-digit inflation levels of the late 1970s/early 1980s to the more modest levels thereafter. Consequently, the NPI’s average real (or inflation-adjusted) return showed more stability.
characteristics. While the early estimates of the sample average return are unbiased, those same early estimates of the population volatility (i.e., the sample standard deviation) tend to be biased downward. b) The nature of the (purported) attractive risk/return characteristics is the very thing that attracts institutional capital to seek a verifiable index of returns and to invest additional capital. Since it is often the case that investor enthusiasm for the “next new thing” swings too far, it is not surprising that the volatility increases over time. In this regard, the NPI seems to reflect such swings in enthusiasm. Moreover, any asset class that has yet to experience a full market cycle ought to be justifiably viewed with suspicion. However, the real estate data series is more than 32 years old and has experienced at least one full market cycle.

Second, it was decided – for the reasons indicated above – best not to truncate time periods of the other asset classes to that of the private-equity returns series in order to perform the mixed-asset portfolio optimization which follows (see sections IV and V) such that all asset classes having the same time period – despite Stambaugh (1997), for example, proposing an approach to address investment histories with differing lengths.

That being said, it is worthwhile to report (Table 4) the summary statistics for the reported private-equity return series over the (somewhat) abbreviated time period of 1982-2009:
Table 4

Summary Statistics for the Annual Returns of Selected Asset Classes

Summary statistics of annual returns for the years ended 1982-2009 for the selected asset classes shown in Table 2, plus the inclusion of private-equity venture capital returns provided by Cambridge Associates. Source: Morningstar.

<table>
<thead>
<tr>
<th>Geometric Mean</th>
<th>Arithmetic Mean</th>
<th>Median</th>
<th>Standard Deviation</th>
<th>Serial Correlation</th>
<th>Sharpe Ratio</th>
<th>Highest Return</th>
<th>Lowest Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>11.17%</td>
<td>12.73%</td>
<td>16.20%</td>
<td>17.82%</td>
<td>3.50%</td>
<td>0.448</td>
<td>37.58%</td>
</tr>
<tr>
<td>U.S. Small Stocks</td>
<td>11.39%</td>
<td>13.47%</td>
<td>18.01%</td>
<td>21.34%</td>
<td>-28.73%</td>
<td>0.393</td>
<td>60.70%</td>
</tr>
<tr>
<td>MSCI EAFE</td>
<td>7.68%</td>
<td>10.33%</td>
<td>9.81%</td>
<td>23.73%</td>
<td>16.47%</td>
<td>0.224</td>
<td>66.80%</td>
</tr>
<tr>
<td>U.S. LT Government Bonds</td>
<td>10.49%</td>
<td>11.24%</td>
<td>9.78%</td>
<td>13.08%</td>
<td>-48.34%</td>
<td>0.508</td>
<td>40.36%</td>
</tr>
<tr>
<td>U.S. LT Corporate Bonds</td>
<td>10.53%</td>
<td>11.00%</td>
<td>10.02%</td>
<td>10.59%</td>
<td>-18.19%</td>
<td>0.622</td>
<td>42.56%</td>
</tr>
<tr>
<td>Domestic High-Yield Corporate Bonds</td>
<td>10.53%</td>
<td>11.73%</td>
<td>11.24%</td>
<td>16.92%</td>
<td>-25.25%</td>
<td>0.393</td>
<td>58.21%</td>
</tr>
<tr>
<td>U.S. 30-Day Treasury Bills</td>
<td>4.98%</td>
<td>5.01%</td>
<td>5.04%</td>
<td>2.66%</td>
<td>82.93%</td>
<td>10.54%</td>
<td>0.10%</td>
</tr>
<tr>
<td>Private Equity</td>
<td>13.01%</td>
<td>19.45%</td>
<td>10.83%</td>
<td>54.07%</td>
<td>12.44%</td>
<td>0.267</td>
<td>278.38%</td>
</tr>
<tr>
<td>NAREIT-Equity</td>
<td>11.55%</td>
<td>13.26%</td>
<td>17.19%</td>
<td>18.63%</td>
<td>2.70%</td>
<td>0.442</td>
<td>37.13%</td>
</tr>
<tr>
<td>NCREIF Property</td>
<td>7.53%</td>
<td>7.85%</td>
<td>9.21%</td>
<td>8.12%</td>
<td>70.10%</td>
<td>0.372</td>
<td>20.06%</td>
</tr>
</tbody>
</table>

| U.S. Inflation | 3.02% | 3.02% | 2.98% | 1.23% | 18.28% | 6.11% | 0.09% |

Finally and as earlier noted, the statistics for private equity should be viewed cautiously.

Sound data is notoriously difficult to come by. Moreover, there are a good number of investment styles (e.g., venture capital, leveraged buyouts, mezzanine financing, domestic vs. foreign, etc.) within the broad rubric of private equity,\(^{18}\) substantial persistence in returns by sponsor (for example, see Kaplan and Schoar (2005)) and there are well known problems with the data, including inconsistencies and selection bias – the latter can substantially overstate the reported risk/return characteristics (for example, see Asness, et al. (2001) and Cochrane (2005)). Indeed, the maximum reported annual return of the venture capital time series was over 278%; a figure that deserves further exploration/explanation.

III. Long-Run Investment Horizons

For the reasons described above, private-market investments – real estate or otherwise – typically display significant autocorrelation in the observed return series; as a result, the observed

periodic volatility understates the “true” volatility. As an alternative to the de-smoothing models, investors can optimize their mixed-asset portfolios using long-run investment horizons of the asset classes.

Using long-run investment horizons seems compelling from the following perspectives:

1. Long horizons may be preferable when investors decide to allocate to private-market investments, as the illiquidity and transaction costs of such investments often render frequent portfolio rebalancing impractical.

2. Despite their elegance, the de-smoothing models often rely upon unknowable parameters and/or difficult-to-ascertain observations. Using long horizons allows the data to “speak for themselves” (e.g., without having to make assumptions appraisers’ behavior).

3. The longer holding period also captures the serial correlation observed in Treasury bills (e.g., see Table 2), which the various de-smoothing models are not designed to address.

4. The use of long-run investment horizons provides investors with another prism by which they can make judgments about their ex ante portfolio allocations. For all the mathematical precision that surrounds ex post portfolio optimization, the practice of ex ante portfolio management is decidedly imprecise.
While the standard practice of using one-year returns has many fine qualities (e.g., consistent with financial-reporting cycles), there is nothing sacrosanct about one-year horizons. In fact, the standard practice of one-year returns tends to obscure the changing risk/return (and, therefore, portfolio-optimizing) characteristics associated with increasing investment horizons when one considers asset classes with varying levels of serial correlation. Said another way [Cochrane (2011), p.1083]: “To some extent, ‘short-run volatility’ does not matter to long-run investors.”

The proposal to use long-run investment horizons in the context of portfolio allocations is not new. However, older papers generally assume zero autocorrelation among returns – for example, see Levy (1972), Levhari and Levy (1975) and Lee, Wu and Wei (1990) – and/or tend to examine the long-run horizon in the context of the capital asset pricing model. While Geltner, Rodriguez and O’Connor (1995) specifically examine autocorrelated returns, they use regression-based estimates of annual (public- and private-market) real estate returns to form long-horizon, mixed-asset portfolios. There is also a vast literature of those who tend to examine (and/or look to exploit) the long-run predictability of asset-class returns given various state variables. Consider as recent examples: Barberis (2000), Campbell and Viceira (2005) and Hoevenaars, et al. (2007). Among those who recently examine long-run volatility measures directly in the context of private-market real estate, at least three papers are notable: MacKinnon and Zaman (2009), Cheng, et al. (2010) and Rehring (2011); these papers all utilize the framework of Campbell and Viceira (2005), with Rehring (2011) including transaction costs and marketing-period risk. There are also papers which examine

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19 Though, of course, the one-year cycle also aligns itself with the cosmos – as it takes our planet approximately one year to orbit the sun.
long-run risks in light of consumption persistence (e.g., Bansal and Yaron (2004) and Hansen, Heaton and Li (2008)) as it relates to the equity premium.

In the alternative, I suggest that a simple, tractable model of auto-correlated returns – without resorting to state variables – provides mixed-asset investors with the opportunity to better understand the effects of long-run horizons. To do so, we need to identify long-run estimates of the three inputs used in the standard portfolio-optimization process: average return, volatility of return and covariance.

III.A. A Simple Model of Auto-Correlated Returns

Since non-traded and illiquid investment vehicles – such as (private) real estate and private equity – display significant autocorrelation in their observed returns (as do Treasury bills), let’s build a simple model of returns which permits autocorrelation in the return series; moreover, let’s do so for two investments or asset classes – such that we can also model the covariance between them. Let’s define our investments as \( x \) and \( y \) and model the nature of their return-generating processes as:

\[
\begin{align*}
    r_{x,t} &= \alpha_x + \varphi_x r_{x,t-1} + \varepsilon_{x,t} \quad \text{where: } |\varphi_x| < 1 \text{ and } \varepsilon_{x,t} \sim \mathcal{N}(0, \sigma_{\varepsilon_x}^2) \\
    r_{y,t} &= \alpha_y + \varphi_y r_{y,t-1} + \varepsilon_{y,t} \quad \text{where: } |\varphi_y| < 1 \text{ and } \varepsilon_{y,t} \sim \mathcal{N}(0, \sigma_{\varepsilon_y}^2)
\end{align*}
\]

Equation (1) permits a return-generating process in which there is a constant return \( \alpha \), autocorrelation (when \( \varphi \neq 0 \)) with the prior-period return and an independent and identically (normally) distributed, mean-zero error term. It is a first-order autoregressive [AR(1)] model, which also implies certain other statistical characteristics – see Appendix #2.
A special case of Equation (1) is the absence of autocorrelation (i.e., $\phi = 0$). In such cases, the return series conforms to the independent and identically distributed (iid) returns associated with the “random walk” – as illustrated by the S&P 500’s near-zero serial correlation shown in Table 2. In the presence of autocorrelation (i.e., $\phi \neq 0$), the return series is dependent upon prior returns (i.e., it displays inertia) – as illustrated by NCREIF’s substantial serial correlation shown in Table 2.

### III.B. Long-Horizon Returns: Geometric v. Arithmetic Average

While the arithmetic average ($\bar{r}$) of the periodic (or discrete) return is an unbiased sample of the one-period return, the arithmetic average overstates the compounded return (or the geometric average) over multi-period horizons. The geometric average ($\bar{r}^G$) is never greater than the arithmetic average; moreover, the geometric average can be approximated as:

$$\bar{r}^G \approx \bar{r} - \frac{\sigma^2}{2} \quad (2)$$

As an example, see Bodie, et al. (1992) for this well-known result. This approximation holds for any long horizon; for example, see Sharpe (1999), who assumes iid returns. In Monte Carlo simulations, I find that this approximation holds equally well when the AR(1) returns are dependent (i.e., $\phi \neq 0$)

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20 At its essence, the arithmetic average ignores the interaction terms associated with long-horizon returns and, therefore, overstates, the geometric average when the variance is non-zero – and, this overstatement increases with the variance of those periodic returns. To be exact about Equation (2):

$$\bar{r}^G = \sqrt{T \prod_{t=1}^{T} (1 + r_t)} - 1 = \exp \left[ \frac{\sum_{t=1}^{T} \ln (1 + r_t)}{T} \right] - 1; \quad \bar{r} = \frac{\sum_{t=1}^{T} r_t}{T} \quad \text{and} \quad \sigma^2 = \frac{\sum_{t=1}^{T} (r_t - \bar{r})^2}{T}.$$
and, therefore, violate the iid assumption. Messmore (1995) refers to the difference between the geometric and arithmetic averages as the “variance drain” while Arnott (2005), in a slightly different context, refers to this difference as the “cost of risk.”

Clearly this cost of risk increases with the variance of returns. Consequently, those asset classes which display the highest volatility are those that most overstate the contribution to long-horizon returns, when the arithmetic average is used to optimize long-run portfolio allocations.

Finally, given the construction of long-horizon returns (see §III), the arithmetic average converges to the geometric average as the holding period lengthens.

### III.C. Long-Horizon Volatility

Let’s model long-horizon (from the initial period, $t = 1$, through the final period, $T$) variance, assuming our simple AR(1) model of returns is a reasonable representation of the long-horizon return-generating process.

#### III.C.1. A First Pass at Long-Horizon Volatility

In turn, we are asking: What is the volatility ($\sigma_T$) of $\prod_{t=1}^{T}(1+r_t) = \sum_{t=1}^{T} \ln(1+r_t)$, when $r_t = \alpha + \varphi r_{t-1} + \varepsilon_t$? For purposes of this exercise, we will ignore the interaction terms (or, alternatively, assume that returns are expressed in logarithmic form). In the simplest of cases, consider the variance of a two-period return: $\sigma_{t=2}^2 = \sigma_{t+1}^2 = \sigma_t^2 + \sigma_{t+1}^2 + 2\sigma_{t+1}$. Of course, we are interested in longer horizons; so, for expositional purposes, let’s generalize to any $T$-period horizon:
\( \sigma_t^2 = w' \Sigma w \), where \( T \geq 2 \), \( w \) is a vector of ones, and \( \Sigma \) is the auto-variance matrix. Most importantly to our perspective on long-horizon volatility, this auto-variance matrix\(^{21}\) has the following elements:

\[
\Sigma = \begin{bmatrix}
\sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & \sigma_{1,4} & \cdots & \sigma_{1,T} \\
\sigma_{2,1} & \sigma_{2,2} & \sigma_{2,3} & \sigma_{2,4} & \cdots & \sigma_{2,T} \\
\sigma_{3,1} & \sigma_{3,2} & \sigma_{3,3} & \sigma_{3,4} & \cdots & \sigma_{3,T} \\
\sigma_{4,1} & \sigma_{4,2} & \sigma_{4,3} & \sigma_{4,4} & \cdots & \sigma_{4,T} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{T,1} & \sigma_{T,2} & \sigma_{T,3} & \sigma_{T,4} & \cdots & \sigma_{T,T}
\end{bmatrix}
\]  

(3)

Because of the AR(1) assumption above (for either \( x \) or \( y \)), \( \sigma_{t,t+i} = \sigma_{t} \sigma_{t+i} \phi_{t,t+i} = \sigma^2 \varphi^i \), equation (3) can be rewritten as:

\[
\Sigma = \sigma^2 \begin{bmatrix}
1 & \varphi & \varphi^2 & \varphi^3 & \cdots & \varphi^{T-1} \\
\varphi & 1 & \varphi & \varphi^2 & \cdots & \varphi^{T-2} \\
\varphi^2 & \varphi & 1 & \varphi & \cdots & \varphi^{T-3} \\
\varphi^3 & \varphi^2 & \varphi & 1 & \cdots & \varphi^{T-4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\varphi^{T-1} & \varphi^{T-2} & \varphi^{T-3} & \varphi^{T-4} & \cdots & 1
\end{bmatrix}
\]  

(4)

Inserting Equation (4) into long-horizon variance \( \left( \sigma_T^2 = w' \Sigma w \right) \) produces (as shown more explicitly in Appendix #3) the following result:

\[ \sigma_t^2 = \sigma^2 \rightarrow \sigma_t^2 = \sigma^2 \]

21 Based upon our AR(1) model, we assume that the variance \( \sigma_t^2 = \sigma^2 \ \forall t \in \{1, 2, \ldots, T\} \) and serial correlation \( \phi_{t,t+1} = \sigma_{t} \sigma_{t+k} \phi_{t+k+1} = \sigma^2 \ \forall k \in \{1, 2, \ldots, T - 1\} \) are constant.
\[ \sigma^2_T = T \sigma^2 + 2(T - 1) \phi \sigma^2 + 2(T - 2) \phi^2 \sigma^2 + 2(T - 3) \phi^3 \sigma^2 + \ldots + 2 \phi^{T-1} \sigma^2 \]

\[ = T \sigma^2 + 2 \sigma^2 \sum_{i=1}^{T-1} \phi^i (T - i) \]  

\[ = T \sigma^2 \left[ 1 + 2 \left[ \frac{\phi}{1-\phi} - \frac{\phi(1-\phi^{-1})}{T(1-\phi)} \right] \right] \]

The special case \( \left( \sigma^2_T = T \sigma^2 \Leftrightarrow \sigma_T = \sqrt{T} \sigma \right) \) of Equation (5) is well known:

In the absence of auto-correlated returns (i.e., \( \phi = 0 \)), the long-horizon variance equals the periodic variance \( (\sigma^2) \) multiplied by the number of periods \( (T) \). Or, equivalently stated, the long-horizon standard deviation equals the periodic standard deviation \( (\sigma) \) multiplied by the square root of the number of periods \( (\sqrt{T}) \) – the so-called “square root” rule.

However, Equation (5) also highlights the portfolio-optimization problem when asset-class returns have varying levels of autocorrelation \( (0 < |\phi| < 1) \); in such cases, the investment horizon is no longer unitless (i.e., long-horizon variance is no longer simply periodic variance multiplied by a scalar \( (T) \)). Accordingly, different asset classes (see Table 2) will produce long-horizon volatilities that grow at varying rates. Private real estate (along with private equity and Treasury bills) displays this

\[ \text{22} \] Simply put, all of the off-diagonal elements of \( \Sigma \) vanish when \( \phi = 0 \).

\[ \text{23} \] Wang, et al. (2011) also critique the use of the square-root-of-time rule in the context of daily returns used to estimate value at risk (VaR) over long horizons and, accordingly, propose a modified scaling method – based upon the serial correlation in returns – in order to estimate multi-period VaR.
complication (i.e., auto-correlated returns) in a material way and is the motivation for examining the long-horizon characteristics of asset-class returns.

Because portfolio optimizers tend to think of asset-class volatility in terms of standard deviations – rather than variance – Figure 4 provides a stylized view of Equation (5) but recalibrated in terms of standard deviations\textsuperscript{24} (for illustrative purposes, assuming all asset classes have the same one-period standard deviation of 10%):

\[ \sigma_T = \sqrt{T \sigma^2} \left[ 1 + 2 \frac{\varphi}{1-\varphi} \frac{\varphi(1 - \varphi^{T+1})}{T(1-\varphi)^2} \right]. \]

\textsuperscript{24} The standard deviation form of Equation (5) is: \[ \sigma_T = \sqrt{T \sigma^2} \left[ 1 + 2 \frac{\varphi}{1-\varphi} \frac{\varphi(1 - \varphi^{T+1})}{T(1-\varphi)^2} \right]. \]
Figure 4. Illustration of Long-Horizon Standard Deviation. Illustrates how the long-horizon standard deviation varies as a function of time \( (T) \) and autocorrelation \( (\varphi) \). The dashed curve represents the special case of \textit{iid} returns (for which \( \varphi = 0 \)). Whereas other curves presume higher \( (\varphi > 0) \) or lower \( (\varphi < 0) \) autocorrelation in the return series. For the sake of illustration, all curves begin by assuming that the one-period volatility equals 10%.

As Figure 4 illustrates, those asset classes which display significantly autocorrelated return series (e.g., those curves which lie above the dashed curve, for which \( \varphi = 0 \)) also display markedly higher long-horizon volatility.

Indeed, the very point about considering long-horizon volatility is to examine a simple approximation of the range of likely ending portfolio values. In that regard, picking any arbitrary values for \( \bar{r}, T, \varphi \) and \( \sigma \) as well as an arbitrary percentile ranking \( (Z) \) generates a long-horizon average returns \( (\bar{r}_T) \) along with confidence bands, as illustrated in Figure 5:
Figure 5. Illustration of Long-Horizon Confidence bands. Given initial values of return ($\bar{r}$), volatility ($\sigma$), autocorrelation ($\phi$) as well as a percentile ranking ($Z$), Figure 5 illustrates the potential variation in ending portfolio values – for given time horizons ($T$).

III.C.2. Annualized Long-Horizon Volatility

When thinking about long-horizon volatility, it is helpful to rescale the long-horizon volatility ($\sigma_f^2$) by the horizon length ($T$), in order to produce an estimate of annualized (or, if you prefer, periodic) long-term volatility:  

Equation (6) can also be viewed as a variation of earlier work performed by Shiller (1981) and Kleidon (1986) with respect to variance ratios $\left[ VR = \frac{\sigma_f^2}{T\sigma^2} \right]$ and variance-bounds tests, respectively.
\[
\frac{\sigma_T^2}{T} = \sigma^2 + 2\sigma^2 \left[ \frac{\varphi}{1-\varphi} - \frac{\varphi(1-\varphi^{T+1})}{T(1-\varphi)^2} \right]
\] (6)

As the horizon approaches infinity, annualized (or scaled) long-horizon variance further simplifies (for \(\varphi < 1\)) to:

\[
\lim_{T \to \infty} \left( \frac{\sigma_T^2}{T} \right) = \sigma^2 \left( \frac{1+\varphi}{1-\varphi} \right)
\] (7)

As before, it is also helpful to think in terms of standard deviations; in this case, it is the scaled long-horizon standard deviation which is of interest:

\[
\frac{\sigma_T}{\sqrt{T}} = \frac{\sigma}{\sqrt{T}} \sqrt{1+2 \left[ \frac{\varphi}{1-\varphi} - \frac{\varphi(1-\varphi^{T+1})}{T(1-\varphi)^2} \right]}
\] (8)

The special case \(\left( \frac{\sigma_T}{T} = \frac{\sigma}{\sqrt{T}} \right)\) of Equation (8) is also well known:

In the absence of auto-correlated returns (i.e., \(\varphi = 0\)), the scaled long-horizon standard deviation equals the periodic standard deviation \((\sigma)\) divided by the square root of the number of periods \((\sqrt{T})\) – another variation of the so-called “square root” rule.

---

26 If working with time periods shorter than one year, then the “annualized” standard deviation is also as shown in Equation (8) – except that \(T\) now represents a fraction of a year. See Giliberto (2003) for further discussion with regard to annualizing quarterly returns.
Viewed from another perspective, Equation (8) provides a “decay” function for scaled long-horizon standard deviations. The relative decay rate (i.e., holding $\sigma_s$ constant) is a function of each asset class’ autocorrelation ($\varphi$, where $n = 1, 2, \ldots, N$ asset classes) as illustrated in Figure 6:

**Figure 6. Illustration of Annualized Long-Horizon Standard Deviation.** Figure 6 replicates Figure 4, except the former scales long-horizon volatility by the time horizon ($T$). Consequently, Figure 6 illustrates how the scaled or annualized long-horizon standard deviation varies as a function of time ($T$) and autocorrelation ($\varphi$). Again, the dashed curve represents the special case of iid returns (for which $\varphi = 0$). Whereas other curves presume higher ($\varphi > 0$) or lower ($\varphi < 0$) autocorrelation in the return series. For the sake of illustration, all curves begin by assuming that the one-period volatility equals 10%.

To many, this notion about annualized long-term volatility is more easily understood in the context of annualized long-run average returns. For any investment horizon ($T$) and given our
earlier assumptions, $\tilde{r} = T \tilde{r}$ (again, either ignoring interaction terms or using the logarithmic form of returns). For purposes of the illustration in Figure 7, the average return (like the periodic standard deviation) is also assumed to be 10%; additionally, two possible autocorrelation factors are assumed: zero and 50% – as another (and perhaps more familiar) way to contrast the slower decay associated with the larger autocorrelation factor.\(^{27}\)

\[^{27}\text{Yet another perspective is to consider the ratio of the long-horizon coefficient of variation }\left(\frac{CV_T = \frac{\sigma_T}{T \tilde{r}}}{CV_i = \frac{\sigma_i}{\tilde{r}}}\right)\text{ to the periodic (or, for our purposes, annual) coefficient of variation }\left(\frac{CV_i = \frac{\sigma}{\tilde{r}}}{CV_i = \frac{\sigma}{\tilde{r}}}\right).\text{ This ratio}\]

\[
\left(\frac{CV_L^i}{CV_i} = \left(1 + \frac{1 - \phi (1-\phi^T)}{T (1-\phi)}\right)\right)\text{ equals } \frac{1}{\sqrt{T}}, \text{ when } \phi = 0, \text{ and approaches zero as } T \text{ increases; however, this ratio also approaches zero as } T \text{ increases for any } \phi \neq 0, \text{ but does so more slowly.}\]
Figure 7. Illustration of Annualized Long-Horizon Standard Deviation. Figure 7 replicates Figure 5, except: 1) The former scales long-horizon volatility by the time horizon \((T)\), where Figure 7 arbitrarily assumes initial values of return \((\bar{r} = 10\%)\), volatility \((\sigma = 10\%)\), autocorrelation \((\phi)\) as well as a percentile ranking \((Z = 90^{\text{th}}\, \text{percentile} \, \text{and} \, 10^{\text{th}}\, \text{percentile})\), Figure 7 illustrates the potential variation in scaled or annualized ending portfolio values – for given time horizons \((T)\). 2) The former illustrates the variation in (scaled) ending portfolio values by the use of two assumed levels of autocorrelation \((\phi = 0 \text{ (the } iid \text{ assumption)} \, \text{and} \, \phi = .5 \text{ (the momentum/inertia assumption))}.

An empirical corollary to Figure 7 is shown in Figure 8, which reproduces an exhibit from Dimson, et al. (2002, 2011) in which they calculate the volatility of holding-period returns on U.S. equities for the period 1900-2010, using holding periods ranging from 10 to 111 years (with the shaded areas representing differing percentiles):
Figure 8. Annualized Real Returns on U.S. Equities. Provides inflation-adjusted annual returns on U.S. equities, for holding periods \((T)\) of 10-111 years, for the period 1900-2010. The dark-shaded regions represent the 90th and 10th percentiles – as is the case in Figure 7 – and the thin blue line represents the median return – as is also the case in Figure 7. Source: *Credit Suisse Global Investment Returns Yearbook 2011*. Permission generously granted by Drs. Dimson, Marsh and Staunton.

From this vantage point, we can see empirically the narrowing of likely (annualized) returns as the holding periods lengthens for U.S. common stocks. However, the parallels between Figures 7 and 8 should not be taken too far. For example, there is only one 111-year horizon return; therefore, the annualized standard deviation of Figure 8 collapses to zero by necessity. Moreover, the 111-year holding period represents, again by necessity, overlapping periods.
III.C.3. A Cautionary Note about the Model’s Simplifications

While Figure 8 provides a pattern of annualized long-horizon standard deviations that is similar in characteristic to Figure 7, it also clear that Figure 8 highlights that our simplifying assumptions do not completely hold in practice. Therefore, a cautionary note is appropriate: That is, our simple model of (returns and) risk contravenes the work of Pástor and Stambaugh (2011), who – in addition to \textit{iid} uncertainty – identify four other potential sources of uncertainty in long-run predictive variance: (i) mean reversion, (ii) uncertainty about future expected returns, (iii) uncertainty about current expected return and (iv) estimation risk. Engle (2009) similarly raises the issue that, over the long run, there is the risk that risks will change.

That said, it is unclear as to how these sources of uncertainty might differentially effect various asset classes, though – to be fair – estimation risk might seem even more significant for private-market investment vehicles. In other words, these notions of long-term volatility are a simplification. However, as Box (1979) noted: “All models are false, but some are useful.” The hope here is that examining various aspects of long-horizon volatility provides additional insights about the art of portfolio management when asset classes display varying levels of auto-correlation in their return series.

III.C.4. Two Other Perspectives on Annualized Long-Horizon Volatility

\footnote{28 “Whereas the mean-reversion component is strongly negative, the other components are all positive, and their combined effect outweighs that of mean reversion” (p.2).}

\footnote{29 Much earlier, the work of Mandelbrot (1963) and Fama (1965) questioned the very notion of \textit{iid} normal returns.}

\footnote{30 Jacquier, et al. (2003) point out similar problems with estimating long-run returns, using the geometric \textit{v.} arithmetic average, when these averages are subject to sampling error. They also point out problems when the \textit{ex ante} investment horizon exceeds the \textit{ex post} observed time series of return.}
Two other perspectives on long-horizon volatility are useful. First, the ratio of annualized (or scaled) long-horizon standard deviation to annual (or periodic) standard deviation provides another perspective on the relationship between the volatility of long-run returns (as $T \to \infty$) and of annual returns as influenced by the degree of serial correlation:

$$\lim_{T \to \infty} \left( \frac{\sigma_T}{\sqrt{T}} \right) = \sqrt{1 + 2 \left( \frac{\varphi}{1 - \varphi} \right)}.$$  

**Figure 9. Ratio of Scaled Long-Horizon Volatility to Annual Volatility.** Illustrates the ratio of annualized long-horizon standard deviation to the annual standard deviation for various levels of serial correlation ($\varphi$) as the horizon lengthens to infinity ($T \to \infty$).

Figure 9 indicates the powerful effect that high serial correlation has on prolonging the dampening of scaled volatility otherwise introduced by long holding periods. More specifically, when the level of
autocorrelation equals zero, the scaled long-term volatility equals the periodic volatility; that is, their ratio equals one. However, the ratio \( \lim_{t \to \infty} \left( \frac{\sigma_t}{\sqrt{T}} \right) = \sqrt{1 + 2 \left( \frac{\varphi}{1 - \varphi} \right)} \) doubles at \( \varphi = .6 \) (and halves at \( \varphi = -.6 \)); it triples at \( \varphi = .8 \) – increasing rapidly thereafter as \( \varphi \) increases\(^{31} \) (however, \( \varphi > .85 \) is a rare occurrence among most asset classes – see Table 2).

Second, let’s consider a stylized version of a comparison between the volatilities of common stocks and private real estate. For convenience, let’s assume that the standard deviation of the annualized returns for common stocks is 16%, with no serial correlation. On the other hand, let’s assume that the standard deviation of the annualized returns for private real estate is 6%, with serial correlation of .75. (These are stylized approximations of the figures shown in Table 2 for the S&P 500 and the NCREIF Property Index, respectively.) The choice of asset classes is designed to highlight the contrast between one asset with high annual volatility but no serial correlation in its return series with another asset class with near-opposite characteristics (i.e., low annual volatility but high serial correlation). As a result, the decay function for the annualized long-horizon volatility of common stocks \( \frac{\sigma_{CSIT}}{T} \) is much steeper than that of real estate \( \frac{\sigma_{REIT}}{T} \) – due to the differences in

\[ \lim_{t \to \infty} \left( \frac{\sigma_t}{\sqrt{T}} \right) = \sqrt{1 + 2 \left( \frac{\varphi}{1 - \varphi} \right)} \]

\(^{31} \) We begin with modifying Equation (6): \[ \frac{\sigma_T^2}{\sigma^2} = \frac{\sigma_T}{\sqrt{T}} = \sqrt{1 + 2 \left( \frac{\varphi}{1 - \varphi} - \frac{\varphi(1 - \varphi^2)}{T(1 - \varphi^2)} \right)} \]. This ratio differs had we begun with modifying Equation (8): \[ \frac{\sigma_T}{\sigma} = \sqrt{1 + 2 \left( \frac{\varphi}{1 - \varphi} - \frac{\varphi(1 - \varphi^2)}{T(1 - \varphi^2)} \right)} \]. The use of the former avoids the ratio trivially approaching zero for all \( T \) of substantial length – irrespective of \( \varphi \).
the autocorrelation of their return series. And, even though the annual volatility of common stocks is more than twice that of real estate, the highly auto-correlated real estate return series after 25 years displays long-horizon volatility very similar to that of common stocks\textsuperscript{32} – see Figure 10:

\textsuperscript{32} MacKinnon and Zaman (2009) present a similar analysis, but using (equity) REITs rather than common stocks. The results are quite similar and, consequently, represent another approach by which the similarity of long-run returns from public and private real estate can be judged.
**Figure 10. Annualized Long-Horizon Volatility: Common Stocks v. Real Estate.** This stylized illustration of the annualized volatility of common stocks and (private) real estate assumes an annual volatility for common stocks ($\sigma_{CS}$) of 16% and for real estate ($\sigma_{RE}$) of 6% and that the common stocks returns are iid ($\varphi_{CS} = 0$) while the real estate returns are autocorrelated ($\varphi_{RE} = .75$). The stylized comparison displays the annualized volatility for $1 \leq T \leq 25$ for common stocks $\left(\frac{\sigma_{CS}}{T}\right)$ and for real estate $\left(\frac{\sigma_{RE}}{T}\right)$.

**III.C.4 The Long-Horizon Controversy**

Figures 6, 7 and 10 might mistakenly give the impression that long-run volatility vanishes. This is most definitely not the case (e.g., see Figure 4). The dispersion in outcomes widens over all extended time periods (except in extreme cases of negative autocorrelation). However, annualizing this long-horizon volatility may give the appearance that long-run volatility vanishes – sometimes also referred to “time diversification.”
There have been several notable critics of time diversification, including Samuelson (1994) who demonstrates that the investment horizon is irrelevant to investor’s preferred allocation between the risky and risk-free assets when the investor’s utility function maximizes the logarithm of wealth. Kritzman (1994) and Thorley (1995) contradict Samuelson’s conclusion by asserting other forms of investors’ utility functions.

My aim is not to weigh in on this time-diversification controversy other than to make two points: First, it seems unclear as to whether or not investors know the form of their utility function (or, more importantly, act as if they do). Additionally, institutional investors with (presumably) perpetual lives may have utility functions different from individual investors. If the form of investors’ utility functions cannot be identified, then it is difficult to resolve the time-diversification controversy. Second, it is clear that many public (defined-benefit) pension plans face significant shortfalls (e.g., see Novy-Marx and Rauh (2011)). In some instances, rebalancing a portfolio towards higher-return/higher-risk strategies may be the most politically palatable approach – at least in the short run. There is already some evidence of such reaction, with the largest public pension plans doubling their target allocations to alternative investments in the past five years – such that the median allocation now stands at 20% – see Cunningham (2010) and as supported by Table 1. The move towards alternative investments often involves a move into privately traded assets – such as direct real estate, venture capital, leveraged buyouts, etc. These securities often reflect very different serial correlation characteristics from their publicly traded counterparts and, consequently, their

33 In agreement with Samuelson, Bodie (2011) takes another tact; he estimates the cost of the put option necessary to insure against shortfall risk as the investment horizon lengthens.
long-term portfolio-enhancing characteristics may be very different from their short-term characteristics; understanding these differences is the central argument of this paper.

III.D. Long-Horizon Correlation

For the third leg of our portfolio-optimization process, let’s examine the long-horizon correlation between any two asset classes, where they each may have differing levels of autocorrelation in their own return series.

III.D.1. Scaled Long-Horizon Correlations

The general formula for the periodic correlation of returns between two asset classes (say, \(x\) and \(y\)) is given by \(\rho_{x,y} = \frac{\sigma_{x,y}}{\sigma_x \sigma_y}\); by extension, \(\rho_{x,y}^{T} = \frac{\sigma_{x,y}^{T}}{\sigma_x^{T} \sigma_y^{T}}\) represents the long-horizon correlation. Because the values in the denominator were identified in Equation (8), let’s next consider long-horizon co-variance \((\sigma_{x,y}^{T})\). Similar to variance, let’s generalize to any \(T\)-period horizon: \(\sigma_{x,y}^{T} = w_x^T \Sigma_{x,y}^{T} w_y\), where \(w_x\) and \(w_y\) is each a vector of ones and \(\Sigma_{x,y}^{T}\) is the auto-covariance matrix of \(x\) and \(y\) of length \(T\), which can be written as:

\[
\Sigma_{x,y}^{T} = \begin{bmatrix}
\sigma_{x_1,y_1} & \sigma_{x_1,y_2} & \cdots & \sigma_{x_1,y_T} \\
\sigma_{x_2,y_1} & \sigma_{x_2,y_2} & \cdots & \sigma_{x_2,y_T} \\
\sigma_{x_3,y_1} & \sigma_{x_3,y_2} & \cdots & \sigma_{x_3,y_T} \\
\sigma_{x_T,y_1} & \sigma_{x_T,y_2} & \cdots & \sigma_{x_T,y_T}
\end{bmatrix}
\]
As shown more explicitly in Appendix #4, the infinite-horizon correlation is produced by inserting Equation (9) into long-horizon (i.e., \( T \geq 2 \)) co-variance and evaluating as \( T \) approaches infinity:

\[
\lim_{T \to \infty} \left( \rho_{x,y,T} \right) = \rho_{x,y} \frac{1 - \phi_x \phi_y}{\sqrt{1 - \phi_x^2} \sqrt{1 - \phi_y^2}} \tag{10}
\]

As shown in (equation (2.4) of) Appendix #2, \( \rho_{x,y} = \frac{\sqrt{1 - \phi_x^2} \sqrt{1 - \phi_y^2}}{1 - \phi_x \phi_y} \); consequently, equation (10) can be further simplified to \( \lim_{T \to \infty} \left( \rho_{x,y,T} \right) = \rho_{x,y} \). However, the latter is unobservable (though estimable).

**III.D.2. Long-Horizon Correlations Relative to Periodic Correlations**

The usefulness of Equation (10) is enhanced if we use it to examine the ratio of the scaled, infinite-horizon correlation to that of the single-period correlation – as given by:

\[
\lim_{T \to \infty} \left( \frac{\rho_{x,y,T}}{\rho_{x,y}} \right) = \frac{1 - \phi_x \phi_y}{\sqrt{1 - \phi_x^2} \sqrt{1 - \phi_y^2}} \tag{11}
\]

This ratio (or multiple) reaches its minimum when \( \phi_x = \phi_y \), regardless of their level. In such cases, the multiple equals unity and, equivalently, then \( \rho_{x,y,T} = \rho_{x,y} \). As another perspective, consider when either \( x \) or \( y \) (but not both) exhibits a random walk (e.g., \( \phi_y = 0 \)), then Equation (11) becomes:

\[
\frac{\rho_{x,y,T}}{\rho_{x,y}} = \frac{1}{\sqrt{1 - \phi_x^2}} \tag{12}
\]

Clearly, Equation (12) approaches its maximum as \( |\phi_x| \to 1 \). As such, the maximum difference between long-horizon correlation and periodic correlation – as embodied by Equations (11) and (12) – occurs when the difference between \( \phi_x \) and \( \phi_y \) is greatest. This relationship is illustrated in Figure 11:
Figure 11. Long-Horizon Correlation Multiple. The ratio between the long-horizon correlation \( \left( \rho_{x,y|T} \right) \) of return series \( x \) and \( y \) and the annual (or single-period) correlation \( \left( \rho_{x,y} \right) \) is illustrated for varying degrees of autocorrelation in the return series of \( x \) (\( \varphi_x \)) and of autocorrelation in the return series of \( y \) (\( \varphi_y \)) as the horizon length approaches infinity \( (T \to \infty) \).

The surface of Figure 11 can be crudely thought of as an inverted saddle. At its base (or valley), the ratio (or multiple) of scaled, long-horizon correlation to that of the single-period correlation is approximately equal to unity when the serial correlation of \( x \)'s return series is approximately equal to the serial correlation of \( y \)'s return series (and, as noted above, is exactly equal to unity when \( \varphi_x = \varphi_y \)) – regardless of their level. However, the ratio (or multiple) jumps dramatically as you move towards the edges of the saddle (i.e., instances in which the difference between \( \varphi_x \) and \( \varphi_y \) is greatest). Given our assumptions, the maximum difference between \( \varphi_x \) and \( \varphi_y \)
approaches 2.0; however, as an empirical matter, the maximum difference is closer to 1.0. In such cases, the multiple ranges from approximately 1.5 to 2.5. Consequently, those asset classes which display significantly different autocorrelation characteristics have a long-term correlation coefficient which is roughly twice as large as their single-period correlation and, as a result, their beneficial diversification characteristics may be markedly diminished.

IV. Ex Post Portfolio Optimizations using Multi-Year Returns

So, let’s revisit our early portfolio-optimization analysis and assume that the investment horizon is four years. As with one-year horizons, there is nothing sacrosanct about four-year horizons. However, they do fit nicely the available data (32 years of data produces eight non-overlapping 4-year holding periods) and many of our earlier formulae involve $\sqrt{T}$ (and, as such, the square root of four is a convenient number which aids our intuition involving certain results). Finally, four years may more accurately represent the approximate holding period of real assets held by institutional investors.

To be clear, I begin by computing the four-year (non-overlapping) return for each of the eight periods in our 32-year data period; this annualized (geometric) mean return is then used to compute the average and volatility of such returns\(^{34}\) as well as other statistics (e.g., serial correlation

\[
\bar{r}_T = \frac{\sum_{t=1}^{N/T} \bar{r}_{t,t}}{N/T}, \quad \bar{r}_{t,t} = \sqrt{T} \left( \prod_{i=1}^{T} (1 + r_i) \right) - 1 \text{ and}\\
\sigma_T = \sqrt{\frac{\sum_{t=1}^{N/T} \left( r_{t,t} - \bar{r}_{t,t} \right)^2}{N/T - 1}} ; \text{ where: } N = \text{ number of total observations.}
\]
coefficients) over eight 4-year horizons. These, of course, are sample statistics and, as such, the earlier caveats apply – e.g., see Engle (2009) and Pástor and Stambaugh (2011).

IV.A. Summary Statistics for Long-Horizon Returns

Recasting our earlier Table 2 such that we can directly compare asset-class performance using one-year holding-period returns to that using, say, four-year holding-period returns produces the results displayed in Table 5:
### Table 5
Summary Statistics of One- and Four-Year Returns for Selected Asset Classes

Summary statistics of returns for the years ended 1978-2009 for the selected asset classes shown in Table 2. Panel A computes the summary statistics using one-year returns. Panel B computes the summary statistics using four-year (non-overlapping) returns. Source: Morningstar and author's calculations.

<table>
<thead>
<tr>
<th>Summary Statistics</th>
<th>Geometric Mean</th>
<th>Arithmetic Mean</th>
<th>Standard Deviation</th>
<th>Sharpe Ratio</th>
<th>Serial Correlation</th>
<th>Highest Return</th>
<th>Lowest Return</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Using One-Year Investment Horizons</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>12.78%</td>
<td>16.20%</td>
<td>17.37%</td>
<td>0.346</td>
<td>-0.01%</td>
<td>37.58%</td>
<td>-37.00%</td>
</tr>
<tr>
<td>U.S. Small Stocks</td>
<td>13.52%</td>
<td>15.56%</td>
<td>21.14%</td>
<td>0.350</td>
<td>-17.12%</td>
<td>60.70%</td>
<td>-36.72%</td>
</tr>
<tr>
<td>MSCI EAFE</td>
<td>8.02%</td>
<td>10.44%</td>
<td>22.67%</td>
<td>0.153</td>
<td>15.17%</td>
<td>66.80%</td>
<td>-45.09%</td>
</tr>
<tr>
<td>U.S. LT Government Bonds</td>
<td>8.96%</td>
<td>9.69%</td>
<td>12.91%</td>
<td>0.520</td>
<td>-31.96%</td>
<td>40.36%</td>
<td>-14.90%</td>
</tr>
<tr>
<td>U.S. LT Corporate Bonds</td>
<td>8.87%</td>
<td>9.37%</td>
<td>10.83%</td>
<td>0.508</td>
<td>-7.61%</td>
<td>42.56%</td>
<td>-7.45%</td>
</tr>
<tr>
<td>Domestic High-Yield Corporate Bonds</td>
<td>9.64%</td>
<td>10.73%</td>
<td>16.10%</td>
<td>0.323</td>
<td>-20.26%</td>
<td>58.21%</td>
<td>-26.16%</td>
</tr>
<tr>
<td>U.S. 30-Day Treasury Bills</td>
<td>5.69%</td>
<td>5.74%</td>
<td>3.31%</td>
<td>84.13%</td>
<td>14.71%</td>
<td>0.10%</td>
<td>-0.10%</td>
</tr>
<tr>
<td>NAREIT-Equity</td>
<td>12.41%</td>
<td>13.99%</td>
<td>18.01%</td>
<td>0.375</td>
<td>2.58%</td>
<td>37.13%</td>
<td>-37.33%</td>
</tr>
<tr>
<td>NCREIF Property</td>
<td>8.77%</td>
<td>9.10%</td>
<td>8.31%</td>
<td>53.36%</td>
<td>73.17%</td>
<td>20.46%</td>
<td>-16.86%</td>
</tr>
<tr>
<td>U.S. Inflation</td>
<td>3.97%</td>
<td>4.01%</td>
<td>2.98%</td>
<td>76.96%</td>
<td>13.31%</td>
<td>0.09%</td>
<td>-0.09%</td>
</tr>
<tr>
<td><strong>Panel B: Using (Scaled) Four-Year Investment Horizons</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>11.31%</td>
<td>11.58%</td>
<td>8.33%</td>
<td>0.702</td>
<td>29.57%</td>
<td>22.96%</td>
<td>-0.68%</td>
</tr>
<tr>
<td>U.S. Small Stocks</td>
<td>13.52%</td>
<td>13.89%</td>
<td>9.71%</td>
<td>0.840</td>
<td>13.00%</td>
<td>29.60%</td>
<td>-2.80%</td>
</tr>
<tr>
<td>MSCI EAFE</td>
<td>8.02%</td>
<td>8.48%</td>
<td>10.89%</td>
<td>0.252</td>
<td>2.16%</td>
<td>29.84%</td>
<td>-1.74%</td>
</tr>
<tr>
<td>U.S. LT Government Bonds</td>
<td>8.96%</td>
<td>9.13%</td>
<td>6.47%</td>
<td>0.525</td>
<td>-38.35%</td>
<td>20.90%</td>
<td>-1.15%</td>
</tr>
<tr>
<td>U.S. LT Corporate Bonds</td>
<td>8.87%</td>
<td>9.08%</td>
<td>7.26%</td>
<td>0.461</td>
<td>-44.84%</td>
<td>23.19%</td>
<td>-2.07%</td>
</tr>
<tr>
<td>Domestic High-Yield Corporate Bonds</td>
<td>9.64%</td>
<td>9.81%</td>
<td>6.69%</td>
<td>0.610</td>
<td>-54.55%</td>
<td>22.15%</td>
<td>0.83%</td>
</tr>
<tr>
<td>U.S. 30-Day Treasury Bills</td>
<td>5.69%</td>
<td>5.73%</td>
<td>3.07%</td>
<td>87.08%</td>
<td>10.85%</td>
<td>1.71%</td>
<td>-1.71%</td>
</tr>
<tr>
<td>NAREIT-Equity</td>
<td>12.41%</td>
<td>12.73%</td>
<td>8.94%</td>
<td>0.783</td>
<td>-52.02%</td>
<td>22.99%</td>
<td>-2.40%</td>
</tr>
<tr>
<td>NCREIF Property</td>
<td>8.77%</td>
<td>8.93%</td>
<td>6.31%</td>
<td>50.67%</td>
<td>9.20%</td>
<td>17.81%</td>
<td>-1.60%</td>
</tr>
<tr>
<td>U.S. Inflation</td>
<td>3.97%</td>
<td>4.00%</td>
<td>2.86%</td>
<td>64.26%</td>
<td>10.90%</td>
<td>2.30%</td>
<td>-2.30%</td>
</tr>
</tbody>
</table>
The sometimes vast differences in the summary statistics using one- as compared to (annualized) four-year returns highlight the importance of understanding the effects of autocorrelation upon observed returns. Accordingly, some further comments about Table 5 are merited:

The (annualized) geometric returns are unchanged (as between one- and four-year returns) – since both the one- and four-year returns employ equivalent methodologies.\(^{35}\)

The arithmetic means of the (annualized) four-year returns are, of course, all lower than the arithmetic mean of the one-year returns – with largest disparity (as compared to the one-year returns) found with those asset classes with the highest volatility (of one-year returns). This is not surprising; see Equation (2).

The standard deviations of (annualized) four-year returns displays considerable variability – as one should expect given the wide range of autocorrelation found in the one-year returns. By way of comparison, note that the volatility of the S&P 500 dropped by more than 50\%, while the volatility of private real estate dropped by less than 20\%. These results are consistent with our earlier analyses. Due to the near-zero serial correlation of the S&P 500, its long-term volatility should drop by approximately the square root of time: \(\frac{\sigma_T}{T} \approx \frac{1.737}{\sqrt{4}}\); but, because its serial correlation is not exactly zero and the average volatility is not realized every period, there is not a

\(^{35}\) In our specific case, \(\sqrt[32]{\prod_{t=1}^{32}(1 + r_t)} - 1 = \sqrt[8]{\prod_{t=1}^{8}(1 + y_t^\ast)} - 1\).
Due to the high serial correlation of private real estate, the long-run (annualized) volatility numbers decay much more slowly. See Figure 10 for a horizon-varying perspective on the same matter.

As shown in Table 6, the Sharpe ratios represent the interaction of changing means and volatilities as the holding period lengthens. The general effect of using longer investment horizons was to increase the average Sharpe ratio – as volatilities tend to drop faster than mean returns. Perhaps more interesting is the movement in rankings37 as our prism changes from using one- to four-year returns:

---

36 Had the S&P 500’s serial correlation been exactly zero and its average annual volatility had been realized in every period, then the (scaled) four-year volatility figure would have been 8.69%. As things were, the observed (scaled) four-year volatility figure was 8.33% – see Table 6.

37 The 30-day Treasury bill return was excluded from this exercise, as it acts as the risk-free rate and, accordingly, its Sharpe ratio is always zero.
Table 6
Comparison of Sharpe-Ratio Rankings of Selected Asset Classes

Sharpe ratios of returns for the years ended 1978-2009 for the selected asset classes shown in Table 2. Panel A computes the summary statistics using one-year returns. Panel B computes the summary statistics using four-year (non-overlapping) returns. Source: Morningstar and author’s calculations.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Panel A: Ranking using One-Year Returns:</th>
<th>Rank</th>
<th>Panel B: Ranking using Four-Year Returns:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>NCREIF Property</td>
<td>1.</td>
<td>U.S. Small Stocks</td>
</tr>
<tr>
<td>2.</td>
<td>U.S. LT Government Bonds</td>
<td>2.</td>
<td>NAREIT-Equity</td>
</tr>
<tr>
<td>4.</td>
<td>NAREIT-Equity</td>
<td>4.</td>
<td>Domestic High-Yield Corporate Bonds</td>
</tr>
<tr>
<td>5.</td>
<td>U.S. Small Stocks</td>
<td>5.</td>
<td>U.S. LT Government Bonds</td>
</tr>
<tr>
<td>7.</td>
<td>Domestic High-Yield Corporate Bonds</td>
<td>7.</td>
<td>U.S. LT Corporate Bonds</td>
</tr>
<tr>
<td>8.</td>
<td>MSCI EAFE</td>
<td>8.</td>
<td>MSCI EAFE</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Panel A</th>
<th></th>
<th>Panel B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>0.388</td>
<td>Average</td>
<td>0.585</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.129</td>
<td>Standard Deviation</td>
<td>0.191</td>
</tr>
<tr>
<td>Coefficient of Variation</td>
<td>0.331</td>
<td>Coefficient of Variation</td>
<td>0.326</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.534</td>
<td>Maximum</td>
<td>0.840</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.153</td>
<td>Minimum</td>
<td>0.252</td>
</tr>
<tr>
<td>Range</td>
<td>0.381</td>
<td>Range</td>
<td>0.588</td>
</tr>
</tbody>
</table>

Note the fall of private real estate (from first to sixth) and of long-term corporate bonds (from third to seventh); in the opposite direction, US small stocks jumped (from fifth to first) as did (equity) REITs (from fourth to second). The EAFE (Europe, Australia and the Far East) index was the only sector to have its rank (eighth) unchanged.

Finally, there are instances of dramatic changes in the level of serial correlation in the observed one- and four-year returns. For example, the two real estate return series (NAREIT and NCREIF) saw their serial correlation fall by more than 50 percentage points while U.S. equities (S&P 500 and small stocks) saw their serial correlation rise by approximately 30 percentage points. (In a stationary AR(1) setting, it can be shown that $\phi_{\pi T} = \phi_T^T$.) Interestingly,
the high level of serial correlation found in one-year Treasury-bill returns is essentially unchanged when four-year returns are examined.\(^{38}\)

The correlation across four-year asset-class returns is the other statistical perspective needed to optimize mixed-asset portfolios – see Table 7:

**Table 7**

**Correlation Matrix for Selected Asset Classes – Using Four-Year Returns**

Correlation matrix of four-year (non-overlapping) returns for selected asset classes (see Table 2) for the period 1978-2009. Source: Morningstar and author’s calculations.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>1.00</td>
<td>0.54</td>
<td>0.53</td>
<td>0.47</td>
<td>0.45</td>
<td>0.46</td>
<td>0.58</td>
<td>0.60</td>
<td>0.26</td>
<td>0.15</td>
</tr>
<tr>
<td>U.S. Small Stocks</td>
<td>0.54</td>
<td>1.00</td>
<td>0.17</td>
<td>-0.05</td>
<td>-0.03</td>
<td>0.16</td>
<td>0.68</td>
<td>0.84</td>
<td>0.66</td>
<td>0.70</td>
</tr>
<tr>
<td>MSCI EAFE</td>
<td>0.53</td>
<td>0.17</td>
<td>1.00</td>
<td>0.34</td>
<td>0.34</td>
<td>0.22</td>
<td>0.43</td>
<td>0.38</td>
<td>0.40</td>
<td>0.20</td>
</tr>
<tr>
<td>U.S. LT Government Bonds</td>
<td>0.47</td>
<td>-0.05</td>
<td>0.34</td>
<td>1.00</td>
<td>1.00</td>
<td>0.86</td>
<td>0.04</td>
<td>0.34</td>
<td>-0.24</td>
<td>-0.48</td>
</tr>
<tr>
<td>U.S. LT Corporate Bonds</td>
<td>0.45</td>
<td>-0.03</td>
<td>0.34</td>
<td>1.00</td>
<td>1.00</td>
<td>0.87</td>
<td>0.06</td>
<td>0.35</td>
<td>-0.20</td>
<td>-0.46</td>
</tr>
<tr>
<td>Domestic High-Yield Corporate Bonds</td>
<td>0.46</td>
<td>0.16</td>
<td>0.22</td>
<td>0.86</td>
<td>0.87</td>
<td>1.00</td>
<td>0.14</td>
<td>0.51</td>
<td>-0.29</td>
<td>-0.21</td>
</tr>
<tr>
<td>U.S. 30-Day Treasury Bills</td>
<td>0.58</td>
<td>0.68</td>
<td>0.43</td>
<td>0.04</td>
<td>0.06</td>
<td>0.14</td>
<td>1.00</td>
<td>0.43</td>
<td>0.51</td>
<td>0.77</td>
</tr>
<tr>
<td>NAREIT-Equity</td>
<td>0.60</td>
<td>0.84</td>
<td>0.38</td>
<td>0.34</td>
<td>0.35</td>
<td>0.51</td>
<td>0.43</td>
<td>1.00</td>
<td>0.53</td>
<td>0.36</td>
</tr>
<tr>
<td>NCREIF Property</td>
<td>0.26</td>
<td>0.66</td>
<td>0.40</td>
<td>-0.24</td>
<td>-0.20</td>
<td>-0.29</td>
<td>0.51</td>
<td>0.53</td>
<td>1.00</td>
<td>0.53</td>
</tr>
</tbody>
</table>

Average Coefficient * 0.49 0.37 0.35 0.34 0.35 0.36 0.36 0.50 0.20 1.00

* excluding inflation and itself.

As compared to Table 3 (using one-year returns), the data shown in Table 7 generally indicate that the average correlation coefficient (excluding itself and inflation) – as a crude measure of an asset class’ ability to help diversify the portfolio – increased by approximately ten percentage points when four-year returns are examined. Given our earlier analysis, this is not surprising and suggests – as does Figure 11 – that diversification benefits may be less helpful, as a means of reducing portfolio volatility, as the investment horizons lengthen.

\(...^{38}\) It could be argued that the average four-year Treasury-bond return should be utilized when optimizing the portfolio. I decided against this approach in order to illustrate the point that the serial correlation in Treasury-bill returns adversely affects their long-horizon volatility.
Those asset classes which saw their average correlation rise the most were both short- and long-term U.S. government bonds, while the only asset class to run counter to this trend were “junk” (i.e., high-yield corporate) bonds.

Among those pairs of asset classes that showed the biggest increase in correlation: small-capitalization stocks with both T-bills and with NCREIF and “junk” bonds with both long-term government and with corporate bonds. Among those pairs of asset classes that showed the biggest decrease: “junk” bonds with both small stocks and with EAFE. As an aside, the correlation between NAREIT and NCREIF returns increased to .53 from .25 – as the investment horizon increased to four years from one – lending further credence to the earlier assertion concerning the long-term relationship between these two forms of holding commercial real estate.

IV.B. Optimization Using Long-Horizon Returns

The next step is to place these four-year risk/return characteristics into an MPT-based optimization. In so doing, the efficient frontier (Figure 12) looks like the following:

**Figure 12. Efficient Frontier based on Selected Asset Classes – Using 4-Year Returns.** Plots the (arithmetic) average four-year (non-overlapping) return and standard deviation and represents the (constrained) optimized **ex-post** efficient frontier for the selected asset classes shown in Table 5 for the period 1978-2009 (where the constraint confines each asset’s weight ($w$) to [0, 1]) – using the arithmetic mean and standard deviation shown in Panel B of Table 5 and the correlation matrix shown in Table 7.

---

39 This approach is somewhat akin to Gunthorpe and Levy (1994) who examined the optimized portfolio composition (and the betas) of aggressive and defensive stocks as the holding period increased.
Figure 13 displays the *ex post* allocations to the various asset classes as one moves from the efficient lowest-risk/lowest-return portfolio to the efficient highest-risk/highest-return portfolio, using four-year returns. At the low end of the efficient risk/return spectrum, Treasury bills and private-market real estate again dominate the portfolio allocations. Towards the middle of the efficient risk/return spectrum, long-term government and high-yield corporate bonds dominate the portfolio allocations. Thereafter, small stocks again quickly begin to dominate the portfolio allocations.
Figure 13. Components of the Efficient Frontier – Using 4-Year Returns. Represents the optimal (ex-post) portfolio allocations among the selected asset classes – using four-year (non-overlapping) returns – as one moves from the efficient low-risk/low-return portfolios (shown in Figure 12) to the efficient high-risk/high-return portfolios for the selected asset classes shown in Panel B of Table 5 for the period 1978-2009. The white dashed line represents the average (efficient) portfolio allocation to both public and private real estate (as proxied by the NAREIT and NCREIF indices).

In this view, real estate no longer looks quite as appealing (as compared to Figure 2): the average real estate allocation approaches 10% of the portfolio – and all of this allocation is due to the private-market allocation. Moreover, the private-market real estate allocation is 75-80% of the allocation using one-year returns. In many ways, this real estate allocation is far more consistent with the actual
allocation practices of the large defined-benefit pension plans (i.e., less than a 10% allocation to real estate and most of which is to private real estate).

IV.C. A Note on Real Estate Allocations

Regarding Figure 13 (and Figure 2), most prudent investors would think the extreme allocations to small stocks to be unwise – no matter how impressive their past performance. In fact, one of the potential downfalls of a strict application of portfolio optimization is that small differences in risk/return characteristics may result in unjustifiably large differences in optimized portfolio weights – see, for example, Green and Hollifield (1992). One approach to circumventing this problem is to use a Bayesian, equilibrium approach – as advocated by Black and Litterman (1992) and others, including Ennis and Burik (1991a) who use a CAPM-based equilibrium model to estimate real estate’s role in mixed-asset portfolios. Another approach is to place “floors and ceilings” (or “collars”) around the allocations to each of the asset classes. This latter approach reduces the allocation to small stocks and largely increases the allocation to public-market real estate – which, even though it has (using four-year returns) the second-highest Sharpe ratio, is largely nudged out by small stocks (which have the highest Sharpe ratio), as indicated in Table 8:
Changing Allocations to Real Estate – One- v. Four-Year Holding Periods

Average allocation to public and private real estate for optimized \( \text{ex post} \) efficient frontiers based on one-year (Figure 2) and four-year (non-overlapping) (Figure 12) returns for selected asset classes (see Tables 2 and Panel B of Table 5) for the period 1978 - 2009. The optimized \( \text{ex post} \) efficient frontiers are initially constrained such that each asset’s weight \( (w) \) to are confined to \([0, 1]\) - see the top row. Subsequently, the constraints are tightened such that the maximum allocation to any one asset declines by ten percentage points \( (i.e., \) the second row from the top confines each asset’s weight \( (w) \) to are confined to \([0, .9]\), the third row confines each asset’s weight \( (w) \) to are confined to \([0, .8]\), etc.).

<table>
<thead>
<tr>
<th>Maximum Weight to Any Asset Class</th>
<th>Using One-Year Returns</th>
<th>Using Four-Year Returns</th>
<th>Differences based on Using One- v. Four-Year Returns:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Public</td>
<td>Private</td>
<td>Total</td>
</tr>
<tr>
<td>100%</td>
<td>13.5%</td>
<td>14.6%</td>
<td>27.4%</td>
</tr>
<tr>
<td>90%</td>
<td>13.6%</td>
<td>15.2%</td>
<td>28.4%</td>
</tr>
<tr>
<td>80%</td>
<td>13.6%</td>
<td>15.4%</td>
<td>29.1%</td>
</tr>
<tr>
<td>70%</td>
<td>14.1%</td>
<td>16.1%</td>
<td>30.2%</td>
</tr>
<tr>
<td>60%</td>
<td>15.9%</td>
<td>16.6%</td>
<td>32.5%</td>
</tr>
<tr>
<td>50%</td>
<td>19.1%</td>
<td>16.9%</td>
<td>36.0%</td>
</tr>
</tbody>
</table>

| Average                          | 15.0%  | 15.8%   | 30.6%  | 3.1%   | 12.0%   | 15.1%  | -11.9% | -3.8%   | -15.5% |
| Standard Deviation               | 2.2%   | 0.9%    | 3.2%   | 3.6%   | 0.6%    | 4.1%   | 1.5%   | 0.3%    | 0.9%   |
| Coefficient of Variation         | 14.7%  | 5.5%    | 10.4%  | 114.8% | 4.6%    | 26.8%  | -12.4% | -8.7%   | -5.9%  |

When using four-year returns, the allocation to private-market real estate is fairly stable (averaging around 12%) across the spectrum of efficient portfolio combinations; in contrast, the allocation to public-market real estate shows considerable fluctuation. Essentially, there is an increased allocation to public-market real estate as the constrained allocation to small stocks retreats.

Why does public-market real estate act a substitute for small stocks? Statistically, the answers lie in Table 6, which shows public-market real estate with the second-highest Sharpe ratio, and Table 7, which reveals the high correlation (.84) of returns between the two. As noted elsewhere, REITs are often thought to behave like small-capitalization stocks\(^{40}\) and, in fact, some investors include REITs

\(^{40}\) However, the market capitalization values of several REITs are quite significant. As of year-end 2009, fourteen REITs (with an average market capitalization of approximately $8.8 billion) were included in the S&P 500 large-cap index, twenty-five REITs (with an average market capitalization of approximately $2.7 billion) were included in the S&P 400 mid-cap index and twenty-seven REITs (with an average market
in their small-stock allocations (while separately holding private-market real estate elsewhere in their portfolios).

In this view, real estate no longer looks quite as appealing, but it is still higher than average institutional allocations to real estate (see Table 1) which have less than 10% of their portfolios dedicated to real estate. Naturally, there are other competing reasons as to why this difference still might exist. For example, investors may be considering investment horizons of something longer than four years – to which we next turn our attention.

V. Portfolio Optimizations using Infinite-Horizon Returns

The natural extension is to revisit our portfolio-optimization analysis and now assume that the investment horizon is infinite. In so doing, this is no longer an empirical exercise – as we obviously do not have an infinite data series – and all the earlier caveats about changing distributions, sampling error, etc. apply here in the extreme. Nevertheless, this exercise provides another perspective on the long-run portfolio-optimization process as we push (perhaps unfoundedly so) the data to their limit.

V.A. Summary Statistics for Infinite-Horizon Returns

For purposes of the infinite-horizon portfolio optimization, our three summary statistics are produced from the existing data using the earlier relationships (or well known variations thereof):

\[ \lim_{T \to \infty} (\bar{r}) = \bar{r} \]  

(2)

capitalization of approximately $1.0 billion) were included in the S&P 600 small-cap index – source: NAREIT (2010).
\[
\lim_{T \to \infty} \left( \frac{\sigma^2_T}{T} \right) = \sigma^2 \left( \frac{1 + \varphi}{1 - \varphi} \right) \tag{7}
\]

\[
\lim_{T \to \infty} \left( \rho_{x,y} \right) = \rho_{x,y} \frac{1 - \varphi_x \varphi_y}{\sqrt{1 - \varphi_x^2} \sqrt{1 - \varphi_y^2}} \tag{10}
\]

In turn, the following summary statistics\(^\text{41}\) are produced in Table 9:

\(^{41}\) The volatility of 4-year returns (Table 5) is not directly comparable to the volatility of the infinite-horizon returns (Table 9); the former utilizes equation (8) – an equation of scaled standard deviation: \(\frac{\sigma_T}{T}\) – whereas the latter utilizes a form of equation (7) – an equation of scaled variance: \(\sqrt{\frac{\sigma^2_T}{T}}\). As also discussed in footnote 32, the use of the latter avoids the ratio trivially approaching zero for all \(T\) of substantial length – irrespective of \(\varphi\).
### Table 9

**Annualized Summary Statistics for Selected Asset Classes – Assuming Infinite Holding Periods**

Summary statistics of one-year returns from Table 2 and the correlation matrix from Table 3 are extrapolated to annualized returns assuming an infinite holding period.\(^\text{42}\)

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>11.31%</td>
<td>17.37%</td>
<td>1.00</td>
<td>0.70</td>
<td>0.64</td>
<td>0.05</td>
<td>0.21</td>
<td>0.73</td>
<td>0.47</td>
<td>0.51</td>
<td>0.40</td>
</tr>
<tr>
<td>U.S. Small Stocks</td>
<td>13.52%</td>
<td>17.78%</td>
<td>0.70</td>
<td>1.00</td>
<td>0.47</td>
<td>-0.17</td>
<td>0.03</td>
<td>0.83</td>
<td>0.38</td>
<td>0.36</td>
<td>0.31</td>
</tr>
<tr>
<td>MSCI EAFE</td>
<td>8.02%</td>
<td>26.41%</td>
<td>0.64</td>
<td>0.47</td>
<td>1.00</td>
<td>-0.08</td>
<td>0.04</td>
<td>0.65</td>
<td>0.11</td>
<td>0.38</td>
<td>0.46</td>
</tr>
<tr>
<td>U.S. LT Government Bonds</td>
<td>8.96%</td>
<td>9.27%</td>
<td>0.05</td>
<td>-0.17</td>
<td>-0.08</td>
<td>1.00</td>
<td>0.96</td>
<td>0.10</td>
<td>-0.02</td>
<td>-0.05</td>
<td>-0.62</td>
</tr>
<tr>
<td>U.S. LT Corporate Bonds</td>
<td>8.87%</td>
<td>10.03%</td>
<td>0.21</td>
<td>0.03</td>
<td>0.04</td>
<td>0.96</td>
<td>1.00</td>
<td>0.38</td>
<td>0.10</td>
<td>0.14</td>
<td>-0.44</td>
</tr>
<tr>
<td>Domestic High-Yield Corporate Bonds</td>
<td>9.64%</td>
<td>13.11%</td>
<td>0.73</td>
<td>0.83</td>
<td>0.65</td>
<td>0.10</td>
<td>0.38</td>
<td>1.00</td>
<td>0.29</td>
<td>0.76</td>
<td>-0.05</td>
</tr>
<tr>
<td>U.S. 30-Day Treasury Bills</td>
<td>5.69%</td>
<td>11.27%</td>
<td>0.47</td>
<td>0.38</td>
<td>0.11</td>
<td>-0.02</td>
<td>0.10</td>
<td>0.29</td>
<td>1.00</td>
<td>0.27</td>
<td>0.39</td>
</tr>
<tr>
<td>NAREIT-Equity</td>
<td>12.41%</td>
<td>18.48%</td>
<td>0.51</td>
<td>0.76</td>
<td>0.38</td>
<td>-0.05</td>
<td>0.14</td>
<td>0.76</td>
<td>0.27</td>
<td>1.00</td>
<td>0.36</td>
</tr>
<tr>
<td>NCREIF Property</td>
<td>8.77%</td>
<td>21.11%</td>
<td>0.40</td>
<td>0.31</td>
<td>0.46</td>
<td>-0.62</td>
<td>-0.44</td>
<td>-0.05</td>
<td>0.39</td>
<td>0.36</td>
<td>1.00</td>
</tr>
<tr>
<td>U.S. Inflation</td>
<td>3.97%</td>
<td>8.26%</td>
<td>0.21</td>
<td>0.50</td>
<td>0.08</td>
<td>-0.78</td>
<td>-0.62</td>
<td>0.06</td>
<td>0.71</td>
<td>0.29</td>
<td>0.47</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Average Coefficient *</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.46</td>
<td>0.41</td>
<td>0.33</td>
<td>0.02</td>
<td>0.18</td>
<td>0.46</td>
<td>0.25</td>
<td>0.39</td>
<td>0.10</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*excluding inflation and itself.

In addition to the earlier caveats about possibly pushing the data too far, some commentary about Table 9 is in order: Because of the high serial correlation associated with Treasury bills and private real estate (NCREIF) returns, their infinite-horizon volatilities explode as compared to their one-year volatilities. So much so, that the former is no longer the lowest-risk asset class and the latter is now the second-most-volatile asset class. Moreover, unlevered private real estate returns (NCREIF) are now seen as more volatile than levered (~40-50%) public real estate (NAREIT). How can this be? Two explanations come to mind: 1) certain infirmities in the observed data series may cause this seemingly inconsistent result (*i.e.*, the returns of the levered asset are less volatile than that of the unlevered asset) and/or 2) the leverage employed by the REITs is highly correlated with real

\(^{42}\) In order to do so, the average return is given by equation (2), the variance by equation (7) and the correlation coefficient by equation (10).
estate returns and, accordingly, the leveraged equity position (which effectively “shorts” the indebtedness) acts like a hedged position producing less volatility than the underlying real estate.\footnote{In other words, the essence of arbitrage-free option pricing includes the notion of identifying the perfect hedge held in the right ratio such that volatility is eliminated. Of course, it is unlikely that the perfect hedge is found in the leverage market (and, as a result, the volatility is not eliminated). However, it does imply that returns on the real estate assets and on the real estate loans are highly correlated.}

Finally, the two asset classes which provide the least correlation, on average with other asset classes are long-term government debt and private real estate.

Notwithstanding these caveats and comments, we can construct an efficient frontier assuming these infinite-horizon return characteristics – see Figure 14:
Figure 14. Efficient Frontier based on Selected Asset Classes – Assuming Infinite Horizon. Plots the (geometric) average return and standard deviation and represents the (constrained) optimized ex-post efficient frontier for the selected asset classes – assuming (annualized) infinite holding-period returns (using equations (2), (7) and (10)) – as shown in Table 9 for the period 1978-2009, where the constraint confines each asset’s weight ($w_i$) to $[0, 1]$.

![Efficient Frontier Graph]

When examining the portfolio allocations – see Figure 15 – which produce the infinite-horizon efficient frontier, we again see that small stocks dominate the efficient high-risk/high-return portfolios. From a private real estate perspective, the NCREIF allocation averages approximately 4% across all efficient combinations, while the public real estate (NAREIT) fails to garner any efficient allocations. As noted previously, this level of real estate allocation is more consistent with the observed allocations of U.S. (defined-benefit) pension plans and endowment funds. But, are the majority of the plans and funds using this infinite-horizon approach to make their portfolio...
allocations? The answer would surely seem: No! So, is this happenstance? Perhaps. Or, could it be as described by a familiar analogy in another context? The baseball fielder may not know the physics governing the flight of the ball, yet somehow manages to catch the ball.

However, in addition to the aforementioned concerns about extrapolating the data too far, there may be other reasons as to why actual (private) real estate allocations diverge from those indicated by an ex post optimization process. For example, investors may not believe that ex post results represent long-run ex ante equilibrium characteristics. Alternatively, Chun, Ciochetti and Shilling (2000) criticize the implementation of portfolio optimization when the investor’s liabilities are ignored – as is the case here. Yet another possibility is the differential costs (e.g., investment management fees, transactions costs, search and monitoring costs, etc.) for (particularly, private-market) real estate vis-à-vis the other asset classes. Yet further still, Ennis and Burik (1991b) cite other factors (e.g., illiquidity, information costs, ownership involvement, conflicts of interest, inability to exploit certain tax benefits, etc.) which may be impediments to pension fund allocations to private real estate. In the other direction, Chun, Sa-Aadu and Shilling (2004) examine other reasons (e.g., real estate diversification delivers when most needed, return predictability, inflation-hedging effectiveness, improbability of large losses, etc.) as to why real estate allocations might be higher than that suggested by simple ex post portfolio optimization perspectives.

My objective is not to weigh in on the strengths and weaknesses of these sometimes competing arguments for or against real estate’s allocation in the context of mixed-asset portfolios;

44 Not only does such a comparison require an asset class-by-asset class comparison, the comparison also depends on the particular investor’s preference for active v. passive management vehicles. However, such comparisons are beyond the scope of this paper.
instead, my objective is to simply illustrate that the length of the investment horizon influences risk/return characteristics when asset classes display differing levels of autocorrelation (which, in turn, influences optimized portfolio allocations) and that this long-horizon perspective is yet another helpful prism through which the art of portfolio management can be viewed.

**Figure 15. Components of the Efficient Frontier – Assuming Infinite Horizons.** Represents the optimal portfolio allocations among the selected asset classes – using infinite-horizon returns – as one moves from the efficient low-risk/low-return portfolios (shown in Figure 14) to the efficient high-risk/high-return portfolios for the selected asset classes shown in Table 9. The white dashed line represents the average (efficient) portfolio allocation to both public and private real estate (as proxied by the NAREIT and NCREIF indices).
As before, the extremely high allocations to small stocks do not withstand a sense of investing prudence. In fact, many investors limit their allocation to one or more asset classes on an *ad hoc* basis. In that spirit, Table 10 compares the (public and private) real estate allocations under one-year horizons to those under infinite horizons, as the maximum allocation to any one asset class declines:

**Table 10**

**Changing Allocations to Real Estate – One-Year v. Infinite Holding Periods**

Average allocation to public and private real estate for optimized efficient frontiers based on one-year (Figure 2) and infinite holding-period (Figure 14) returns for selected asset classes (see Tables 2 and 9). The optimized efficient frontiers are initially constrained such that each asset’s weight ($w$) to are confined to [0, 1] - see the top row. Subsequently, the constraints are tightened such that the maximum allocation to any one asset declines by ten percentage points (*i.e.*, the second row from the top confines each asset’s weight ($w$) to are confined to [0, .9], the third row confines each asset’s weight ($w$) to are confined to [0, .8], etc.).

<table>
<thead>
<tr>
<th>Maximum Weight to Any Asset Class</th>
<th>Using One-Year Returns</th>
<th>Using Infinite-Horizon Returns</th>
<th>Differences based on Using One- v. ∞-Year Returns:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Public</td>
<td>Private</td>
<td>Total</td>
</tr>
<tr>
<td>100%</td>
<td>13.5%</td>
<td>14.6%</td>
<td>27.4%</td>
</tr>
<tr>
<td>90%</td>
<td>13.6%</td>
<td>15.2%</td>
<td>28.4%</td>
</tr>
<tr>
<td>80%</td>
<td>13.6%</td>
<td>15.4%</td>
<td>29.1%</td>
</tr>
<tr>
<td>70%</td>
<td>14.1%</td>
<td>16.1%</td>
<td>30.2%</td>
</tr>
<tr>
<td>60%</td>
<td>15.9%</td>
<td>16.6%</td>
<td>32.5%</td>
</tr>
<tr>
<td>50%</td>
<td>19.1%</td>
<td>16.9%</td>
<td>36.0%</td>
</tr>
<tr>
<td>Average</td>
<td>15.0%</td>
<td>15.8%</td>
<td>30.6%</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>2.2%</td>
<td>0.9%</td>
<td>3.2%</td>
</tr>
<tr>
<td>Coefficient of Variation</td>
<td>14.7%</td>
<td>5.5%</td>
<td>10.4%</td>
</tr>
</tbody>
</table>

Using infinite-horizon returns, we see that the allocations to private real estate are quite stable across all maximum-weighting schemes; however, the average allocation to private real estate is quite low compared to those allocations generated when using one-year returns. On the other hand, allocations to public real estate are zero in an unconstrained optimization, yet rise quickly to 18% when the maximum allocation to any one asset class is 50% – and, at this constraint level, look quite like the allocation when using one-year returns (*i.e.*, 18% v. 19%).

63
VI. Conclusions

The increasing popularity of alternative assets among institutional investors forces a more thorough examination of the portfolio-enhancing characteristics of these investments. It is well known that the return series for private-market assets generally display significant serial correlation. And, accordingly, the one-year returns as typically used in mixed-asset portfolio optimizations often fail to fully capture these effects – even when some form of de-smoothing is used to restate the private-market returns.

So, in the alternative, this paper suggests that examining long-horizon measures of risk and return may be preferable, as the illiquidity and transaction costs of such investments often render frequent portfolio rebalancing impractical. The use of long horizons allows the data to “speak for themselves” (as compared to the various de-smoothing models which – despite their elegance – often rely upon unknowable parameters and/or difficult-to-ascertain observations). The longer holding periods also captures the serial correlation observed in Treasury bills – which the various de-smoothing models are not designed to address.

In turn, this paper focuses on how the three inputs (average, volatility and correlation) needed for the standard portfolio-optimization technique change as the investment horizon lengthens: The arithmetic average is replaced with the geometric average. Long-run volatility decays more slowly when asset classes display significant serial correlation. And, long-run correlation is significantly increased when two asset classes have very different levels of serial correlation in their respective return series.
Given the well-documented problems of returns series for private equity, this paper uses (direct) real estate to illustrate the changing risk/return parameters of alternative investments as the holding period lengthens. In so doing, the allocations to private real estate are reduced from approximately 16% when using one-year investment horizons to approximately 12% when using four-year investment horizons and to approximately 4% when using infinite horizons. While the real estate allocations using infinite horizons more closely resemble the actual allocations of (large defined-benefit) pension plans and endowment funds, there is any number of other explanations as to why actual allocations might differ from those using one-year returns: inclusion of investor’s liabilities, differences in investment-management fees and costs and/or other factors (e.g., illiquidity, information costs, ownership involvement, conflicts of interest, etc.) which may be impediments to pension fund allocations to private real estate.

Interestingly, the public real estate return series (REITs) has significantly more volatile portfolio allocations as the investment horizon lengthens. The volatility of these allocations is not due to the serial correlation in REIT returns; instead, it reflects the improved risk/return performance of small-cap stocks as the investment horizon lengthens.
Appendix #1 – Appraisal De-smoothing Techniques

VII.A. Appraisers’ Partial-Adjustment Approach

The appraisal “de-smoothing” advocates – as perhaps best characterized by Geltner (1993) and Fisher and Geltner (2000) – assert that, due to the infrequent trading of private real estate, the lack of publicly available information and the imprecision with which the “true” prices are observed, appraisers anchor on past appraised values and only partially update their valuation estimates. Quan and Quigley (1991) and Childs, Ott and Riddiough (2002) argue the rationality of the appraiser’s partial updating (or rational-expectations adjustment) function in light of incomplete information, high search costs and heterogeneous expectations. Diaz and Wolverton (1998) provide a behavioral perspective, by using a longitudinal experiment to examine the tendency of appraisers to “anchor” on their past appraisal estimates. Fisher, Miles and Webb (1999) empirically support these assertions by documenting the lagging nature (in both rising and falling markets) of appraised values to recently sold properties.

Because of these imperfections in observed market pricing, appraisers are thought to employ a partial-adjustments approach45 to generating their valuation estimates. Consequently, the de-smoothing approaches attempt to recover true market prices from the observed appraisal-based return series and, in so doing, typically inject additional volatility in the return series while reducing

45 More extensive models can be postulated; they would include effects for seasonality, “serious” appraisals, etc. – for example, see Geltner (1993). The seasonality of private real estate returns is attributed to the appraisal process in which it is often the case that appraisals are externally prepared once a year (most typically at the end of the fourth quarter) with internal updates performed at the end of the other three quarters (i.e., private real estate’s version of the non-synchronous trading problem). This problem is often referred to as temporal aggregation.
(if not eliminating) autocorrelation (without biasing the first moment of the return distribution). These approaches attempts to “reverse engineer” the presumed infinite-order moving-average appraisal process such that the de-smoothing of the time series of private real estate returns displays volatility and autocorrelation characteristics that are more in keeping with the a priori intuition of many practitioners and academics.

VII.B. Enhanced De-Smoothing Techniques

The theoretical foundation for using those properties that have experienced the “full-cash” cycle (i.e., have been bought and sold) to interpolate periodic returns and to create a transaction-based index harkens back to Bailey, Muth and Nourse (1963), who advocated such an approach for residential properties. More recently, the concept has been extended to commercial properties – see for example Fisher (2000). The technique includes identifying all (relevant) sold properties and their intervening cash flows. Using this data, the cash flows and disposition values are regressed (constrained so as without an intercept) against the initial investment values. The inverse of the regression coefficients are then used to compute an estimate of the periodic returns. The avoidance upon appraisals in estimating the time series of private real estate returns is certainly an intriguing feature of this methodology. Unfortunately, the number of sold properties within the NCREIF index is fairly “thin” in the initial years (as well as severe market downturns), is not necessarily robust in terms of property-type and/or geographic dispersion and may differ in composition from the overall NCREIF index.
In part because of the thin sales data, other researchers – notably including Fisher and Geltner (2000) and Geltner and Goetzmann (2000) – substitute “serious” appraisals for property sales. Such appraisals are generally those that were either performed by an external third-party appraiser or indicated a significant change from the previous appraised value. This repeat-measure approach using pairs of serious appraisals (along with the intervening cash flows) involves a tradeoff: a substantially enlarged data set $v.$ reliance on appraised values – where presumably appraiser behavior suffers from the same confidence/inertia problem as earlier described (see §II.B).

Fischer, et al. (2003) propose controlling for the time-varying liquidity of private-market transactions, suggesting that liquidity – or the ease of selling – improves in periods of rising prices (and, conversely, deteriorates in periods of falling prices). Their approach defines a conceptual construct, the “constant-liquidity value,” to reflect prices that would have been observed had the ease of selling been constant across time. (In some sense, this is in keeping with the notion of “time on the market” as often used in the residential real estate market – see Cheng, et al. (2010).) Instead, we observe transaction prices with variable liquidity.

Borrowing from the housing literature, they postulate a double-sided search model in which the distributions $^{47}$ of market-wide buyers and sellers vary inter-temporally. The variations in the

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$^{46}$ By way of comparison, Fisher (2000) identifies approximately 2,800 repeat-sale pairs, whereas Geltner and Goetzmann (2000) identify approximately 20,700 serious repeat-appraisal pairs – yet, they eliminate roughly 75% of all quarterly NCREIF valuation estimates (because of “staleness”).

$^{47}$ For analytical convenience, they assume the reservation prices at time $t$ of aggregate sellers ($\bar{S}_t$) and buyers ($\bar{B}_t$) are normally distributed: $\bar{S}_t \sim N(\bar{P}_S, \sigma_S^2)$ and $\bar{B}_t \sim N(\bar{P}_B, \sigma_B^2)$. Over time (and given $P_s > P_B$), it is
buyers’ distribution (or the demand side) of the joint supply-and-demand process reflect changes in
the ease of selling commercial property. The variations in demand are estimated from a probit
model, designed to model the probability of a property being sold and to control for potential
sample-selection bias. The appropriately modified coefficients from such a model are then integrated
with a variable-liquidity index of hedonically modeled transaction prices to produce the constant-
liquidity value.

Their model is then applied to the NCREIF data. Their results for the transaction-based
hedonic index reflecting constant liquidity and with correction for sample-selection bias (as
compared to the appraisal-based NCREIF index) are intuitively pleasing: greater volatility, near-zero
autocorrelation and greater amplitude of the real estate cycle. Furthermore, the authors find that the
use of the volatility estimate from the transaction-based hedonic index reflecting constant liquidity
and with correction for sample-selection bias results in a much reduced portfolio allocation to
commercial real estate – and, as described herein, more consistent with that observed in practice.

However, nettlesome questions remain about their methodology: First, commercial real
estate sales are more generally characterized by modified sealed-bid auctions – not a dual-search
model. And, accordingly, their implications about the price-discovery process differ from the

the movement in buyers’ aggregate demand – as captured by movements in their average reservation price
\( \bar{P}_{B,t} \) – that reflects the ease of selling (or liquidity).

More specifically, they compute a mixed-asset efficient frontier for the period 1978-2001. When the
volatility estimate for private real estate uses the NCREIF return series, the optimal portfolio allocation
(evaluated at the portfolio producing the maximum Sharpe ratio) to real estate is over 50%; when the
volatility estimate for private real estate uses the hedonic series adjusted for sample-selection bias and
liquidity, the optimal portfolio allocation to real estate is under 10%.
housing market’s conventions. Second, the authors suggest that pro-cyclical trading volume derives from the buyers’ distribution leading the sellers’ (or, alternatively stated, the sellers’ distribution lags the buyers’ distribution). Yet, the very same institutions (e.g., REITS and pension funds) are often both buyers and sellers at the same time (but obviously of differing assets). How can the same institutions lead the market as buyers, but lag as sellers? Third, the authors’ calibration of their hedonic index (with and without error correction and liquidity adjustments) is subject to the usual potential criticisms: a) correctly specifying the model, b) omitting key variables, and/or c) mis-measuring variables. Of course, the explanatory power of the estimated regression equation(s) ultimately helps address the importance of these problems. Fourth, if detailed asset-level cash flows are not captured and evaluated, then any repeat-sales/-measure approach is subject to criticism concerning the treatment of renovation expenditures over the asset’s holding period: What is the “normal” level of such expenditures? More than normal leads to an over-estimated appreciation rate, while less than normal leads to an under-estimated rate.

Notwithstanding these criticisms, the popularity of such an approach has led to commercially available products that apply such techniques to a broader data set (i.e., beyond NCREIF alone) and, accordingly, has mitigated some of the criticism above. Moreover and as Geltner and Ling (2006) point out, there are two types of uses for theses modified real estate indices: within the asset class (e.g., benchmarking and performance attribution) and outside the asset class

49 See Moodys/REAL Commercial Property Price Index: http://web.mit.edu/cre/research/credl/rca.html. The popularity has been challenged, however, by competing indices – see Pruitt (2010).
50 Other techniques have also been proposed. Two deserving attention are: Ling, Naranjo and Nimalendran (2000) apply a latent-variables technique to identifying an unobserved commercial real estate return and Peng (2010) estimates factor loadings using detailed, property-level cash flows as well as acquisition and disposition values.
(e.g., asset-class research). It is not necessarily the case that these modified indices serve each use equally well.

VII.C. An Approach Spanning Real Estate and Private Equity

Budhraja and de Figueiredo (2005) begin by removing the autocorrelation in returns by employing a simple transformation (using the notation and the AR(1) process of this paper):

\[ \tilde{r}_t = \frac{r_t - \varphi(r_{t-1})}{1 - \varphi} \]

[as earlier proposed by Georgiev (2002) and Desouza and Gokcan (2004)]. (This approach is consistent with many of the early papers dealing with autocorrelation in the real estate return series – for example, see Firstenberg, Ross and Zisler (1988).) They go on to propose a two-stage least-squares approach, using instrumental variables, to simultaneously model “true” market returns for (private) real estate, leveraged buyouts and venture capital; in so doing, they hope to model private-market returns which are correlated with public-market counterparts and private-market shocks which affect all private markets in a similar manner. Their approach produces estimates of the “true” volatility of returns that is approximately 1.85 – 2.50 times the observed standard deviation of private-equity returns and that is approximately 3.30 times the observed standard deviation of (private) real estate returns. The authors extend their analysis to the correlation of these private-market returns with the S&P 500 returns; their approach produces estimates of the “true” correlation of returns that is approximately 1.25 – 1.45 times the observed correlation of private-equity returns (with the S&P 500), but is essentially unchanged for (private) real estate returns. In general, these outcomes tend to understate the findings of this paper. Nevertheless, Budhraja and de Figueiredo (2005) ought to be commended for attempting to collectively examine private-market assets (i.e., leveraged buyouts, real estate and venture capital).
Appendix #2 – Model of Auto-Correlated Returns

As noted earlier, let’s define our investments as $x$ and $y$ and model the nature of their return-generating processes as:

$$ r_{x,t} = \alpha_x + \varphi_x r_{x,t-1} + \varepsilon_{x,t} \quad \text{where}: \quad |\varphi_x| < 1 \quad \text{and} \quad \varepsilon_{x,t} \overset{iid}{\sim} N(0, \sigma_{\varepsilon_x}^2) $$

$$ r_{y,t} = \alpha_y + \varphi_y r_{y,t-1} + \varepsilon_{y,t} \quad \text{where}: \quad |\varphi_y| < 1 \quad \text{and} \quad \varepsilon_{y,t} \overset{iid}{\sim} N(0, \sigma_{\varepsilon_y}^2) $$

Equation (1) is an autoregressive model of order 1 [or, AR(1)]. From which, it is well known that the average return ($\bar{r}$), the variance of that return ($\sigma^2$) and the covariance of returns ($\sigma_{x,y}$) can be written, respectively, as:

$$ \bar{r}_x = \frac{\alpha_x}{1-\varphi_x} \quad \text{and} \quad \bar{r}_y = \frac{\alpha_y}{1-\varphi_y} $$

$$ \sigma_x^2 = \frac{\sigma_{\varepsilon_x}^2}{1-\varphi_x^2} \quad \text{and} \quad \sigma_y^2 = \frac{\sigma_{\varepsilon_y}^2}{1-\varphi_y^2} $$

$$ \sigma_{x,y} = \frac{\sigma_{\varepsilon_x,\varepsilon_y}}{1-\varphi_x \varphi_y} $$

From the preceding equations, the correlation of returns ($\rho_{x,y}$) can also be written:

$$ \rho_{x,y} = \frac{\sigma_{x,y}}{\sigma_x \sigma_y} = \frac{\sigma_{\varepsilon_x,\varepsilon_y}}{\sqrt{1-\varphi_x^2} \sqrt{1-\varphi_y^2}} = \rho_{\varepsilon_x,\varepsilon_y} \frac{\sqrt{1-\varphi_x^2} \sqrt{1-\varphi_y^2}}{1-\varphi_x \varphi_y} $$
While not a condition necessary for the case made here regarding long-horizon volatility and correlation, note that when $\varphi_x = \varphi_y (\equiv \varphi)$, then $\rho_{x,y} = \rho_{\varepsilon_x, \varepsilon_y}$ – otherwise, $\rho_{x,y} \neq \rho_{\varepsilon_x, \varepsilon_y}$.

Naturally, this begs the question of how well this simple model fits the pattern of observed returns. Table 11 provides the partial autocorrelation factors observed at four (annual) lags for each of the asset classes (plus inflation) shown earlier:

**Table 11**

Partial Autocorrelation Factors for Selected Asset Classes

The partial autocorrelation factors of annual returns for the selected asset classes shown in Table 2 (over the 1978-2009 time period) are shown below at up to four lags ($k$). Those instances in which the partial autocorrelation is statistically significant, at the 99% confidence level, are denoted by *.

The standard errors, used to compute the confidence levels, assume an AR[$k$-1] structure.) All other partial autocorrelation factors are statistically insignificant at conventional confidence levels.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>-0.013</td>
<td>-0.176</td>
<td>0.145</td>
<td>-0.325</td>
<td>-0.096</td>
<td>-0.175</td>
<td>0.803 *</td>
<td>0.029</td>
<td>0.591 *</td>
<td>0.727 *</td>
</tr>
<tr>
<td>2</td>
<td>0.020</td>
<td>0.028</td>
<td>-0.008</td>
<td>-0.050</td>
<td>0.067</td>
<td>-0.102</td>
<td>-0.225</td>
<td>-0.291</td>
<td>-0.333</td>
<td>-0.219</td>
</tr>
<tr>
<td>3</td>
<td>0.031</td>
<td>0.086</td>
<td>-0.100</td>
<td>0.073</td>
<td>0.059</td>
<td>-0.045</td>
<td>0.137</td>
<td>0.122</td>
<td>0.041</td>
<td>-0.083</td>
</tr>
<tr>
<td>4</td>
<td>-0.113</td>
<td>0.064</td>
<td>-0.271</td>
<td>0.054</td>
<td>-0.045</td>
<td>-0.123</td>
<td>0.058</td>
<td>-0.243</td>
<td>-0.123</td>
<td>0.057</td>
</tr>
</tbody>
</table>

As earlier noted, (private) real estate and Treasury bills are the only asset classes to display significant autocorrelation at a one-period lag; additionally, none of the indices show significant (at conventional confidence levels) partial autocorrelation at lags of two to four periods. Consequently, it seems plausible to assert that the first-order [AR(1)] process modeled in Equation (1) is a reasonable characterization of the asset classes’ return-generating processes (and, in turn, that higher-order processes are unneeded).

Finally, Equations (2.1) and (2.2) can be used to restate the approximation of the geometric mean (for either $x$ or $y$) found in Equation (2):
\[ \bar{r} \approx \bar{\sigma} - \frac{\sigma^2}{2} \quad (2) \]

\[ \bar{r} \approx \frac{\alpha}{1 - \varphi} \frac{\sigma^2_{\epsilon}}{2} \quad (2.5) \]

And when \( \varphi = 0 \), Equation (2.5) then obviously simplifies to:

\[ \bar{r} \approx \alpha - \frac{\sigma^2_{\epsilon}}{2} \quad (2.6) \]
Appendix #3 – Proof of Long-Horizon Volatility

We are interested in simplifying the following expression for long-horizon (\( i.e., T \geq 2 \)) variance:

\[
\sigma^2_T = T \sigma^2 + 2 \sigma^2 \sum_{t=1}^{T-1} \phi'(T-t)
\]  

(5)

To do so, let’s first consider a simple manipulation of the summation on the right-hand side of Equation (5) which produces:

\[
\sigma^2_T = T \sigma^2 + 2T \sigma^2 \sum_{t=1}^{T-1} \frac{\phi'(T-t)}{T}
\]

(3.1)

Second, let’s rewrite the summation on the right-hand side of Equation (3.1) as follows:

\[
\sum_{t=1}^{T-1} \frac{\phi'(T-t)}{T} = \sum_{t=1}^{T-1} \phi' - \frac{1}{T} \sum_{t=1}^{T-1} \phi' t
\]

(3.2)

Then, using one of the well-known rules of geometric sums, we can rewrite the first term on the right-hand side of Equation (3.2) as:

\[
\sum_{t=1}^{T-1} \phi' = \frac{\phi - \phi^{T+1}}{1-\phi}
\]

(3.3)

By differentiating both sides of Equation (3.3) with respect to \( \phi \), we have:

\[
\sum_{t=1}^{T-1} t \phi' = \frac{(1-\phi)(1-T \phi^T)+\phi - \phi^T}{(1-\phi)^2}
\]

(3.4)

Multiplying both sides of Equation (3.4) by \( \phi \), we have:

\[
\sum_{t=1}^{T-1} t \phi' = \frac{(1-\phi)(\phi - T \phi^{T+1})+\phi^2 - \phi^{T+1}}{(1-\phi)^2}
\]

(3.5)

We insert the expression found in Equation (3.5) into our expanded version of long-term
variance as found in the far-right summation of Equation (3.2) and then simplify as follows:

\[ \sigma^2_T = T\sigma^2 + 2T\sigma^2 \sum_{t=1}^{T-1} \varphi^t \left( \frac{T-t}{T} \right) \]

\[ = T\sigma^2 + 2T\sigma^2 \left[ \sum_{t=1}^{T-1} \varphi^t - \frac{1}{T} \sum_{t=1}^{T-1} \varphi^t t \right] \]

\[ = T\sigma^2 + 2T\sigma^2 \left[ \frac{\varphi - \varphi^{T+1}}{1-\varphi} - \frac{(1-\varphi)(\varphi-T\varphi^{T+1}) + \varphi^2 - \varphi^{T+1}}{T(1-\varphi)^2} \right] \]

\[ = T\sigma^2 + 2T\sigma^2 \left[ \frac{\varphi}{1-\varphi} - \frac{\varphi(1-\varphi^{T+1})}{T(1-\varphi)^2} \right] \]

\[ = T\sigma^2 \left[ 1 + 2 \left( \frac{\varphi}{1-\varphi} - \frac{\varphi(1-\varphi^{T+1})}{T(1-\varphi)^2} \right) \right] \]  \quad (3.6)

In order to scale long-horizon variance, we divide through by \( T \); consequently, Equation (3.6) becomes:

\[ \frac{\sigma^2_T}{T} = \sigma^2 + 2\sigma^2 \left[ \frac{\varphi}{1-\varphi} - \frac{\varphi(1-\varphi^{T+1})}{T(1-\varphi)^2} \right] \]  \quad (3.7)

And as \( T \to \infty \), the term \( \frac{\varphi(1-\varphi^{T+1})}{T(1-\varphi)^2} \to 0 \) [given \( |\varphi| < 1 \)]. Therefore, this expression (3.7) of scaled long-horizon variance further simplifies to:
$$\lim_{T \to \infty} \left( \frac{\sigma^2}{T} \right) = \sigma^2 + 2\sigma^2 \left( \frac{\varphi}{1 - \varphi} \right)$$

$$= \sigma^2 \left( \frac{1 + \varphi}{1 - \varphi} \right)$$

(7)
Appendix #4 – Proof of Long-Horizon Correlation

We are interested in the expression for long-horizon (i.e., $T \geq 2$) co-variance:

$$\sigma_{x,y} = w' \Sigma_{x,y} | w_y,$$

where: $w$ is a vector of ones (for both $x$ and $y$), and $\Sigma_{x,y}$ is the auto-


covariance matrix of $x$ and $y$ of length $T$. In a general sense, $\Sigma_{x,y}$ can be written as:

$$\Sigma_{x,y} = \begin{bmatrix}
\sigma_{x_1,y_1} & \sigma_{x_1,y_2} & \sigma_{x_1,y_3} & \sigma_{x_1,y_4} & \cdots & \sigma_{x_1,y_T} \\
\sigma_{x_2,y_1} & \sigma_{x_2,y_2} & \sigma_{x_2,y_3} & \sigma_{x_2,y_4} & \cdots & \sigma_{x_2,y_T} \\
\sigma_{x_3,y_1} & \sigma_{x_3,y_2} & \sigma_{x_3,y_3} & \sigma_{x_3,y_4} & \cdots & \sigma_{x_3,y_T} \\
\sigma_{x_4,y_1} & \sigma_{x_4,y_2} & \sigma_{x_4,y_3} & \sigma_{x_4,y_4} & \cdots & \sigma_{x_4,y_T} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{x_T,y_1} & \sigma_{x_T,y_2} & \sigma_{x_T,y_3} & \sigma_{x_T,y_4} & \cdots & \sigma_{x_T,y_T}
\end{bmatrix}$$  (9)

Because of the AR(1) assumptions made earlier, $\sigma_{x_1,y_1} = \rho_{x_1,y_1}, \sigma_{x_1,y_2} = \rho_{x_1,y_2} \sigma_{x_1,y_1}$ and

similarly for $\sigma_{x_1,y_T} = \rho_{x_1,y_T} \sigma_{x_1,y_1}, \sigma_{x_1,y_2} = \rho_{x_1,y_2} \sigma_{x_1,y_1}$; then, equation (9) can be rewritten as:

$$\Sigma_{x,y} = \begin{bmatrix}
\rho_{x_1,y_1} \sigma_{x_1,y_1} & \rho_{x_1,y_1} \sigma_{x_1,y_1} \phi y & \rho_{x_1,y_1} \sigma_{x_1,y_1} \phi y^2 & \rho_{x_1,y_1} \sigma_{x_1,y_1} \phi y^3 & \cdots & \rho_{x_1,y_1} \sigma_{x_1,y_1} \phi y^{T-1} \\
\rho_{x_1,y_1} \phi y \sigma_{x_1,y_1} & \rho_{x_1,y_1} \phi y \sigma_{x_1,y_1} \phi y & \rho_{x_1,y_1} \phi y \sigma_{x_1,y_1} \phi y^2 & \rho_{x_1,y_1} \phi y \sigma_{x_1,y_1} \phi y^3 & \cdots & \rho_{x_1,y_1} \phi y \sigma_{x_1,y_1} \phi y^{T-2} \\
\rho_{x_1,y_1} \phi y^2 \sigma_{x_1,y_1} & \rho_{x_1,y_1} \phi y^2 \sigma_{x_1,y_1} \phi y & \rho_{x_1,y_1} \phi y^2 \sigma_{x_1,y_1} \phi y^2 & \rho_{x_1,y_1} \phi y^2 \sigma_{x_1,y_1} \phi y^3 & \cdots & \rho_{x_1,y_1} \phi y^2 \sigma_{x_1,y_1} \phi y^{T-3} \\
\rho_{x_1,y_1} \phi y^3 \sigma_{x_1,y_1} & \rho_{x_1,y_1} \phi y^3 \sigma_{x_1,y_1} \phi y & \rho_{x_1,y_1} \phi y^3 \sigma_{x_1,y_1} \phi y^2 & \rho_{x_1,y_1} \phi y^3 \sigma_{x_1,y_1} \phi y^3 & \cdots & \rho_{x_1,y_1} \phi y^3 \sigma_{x_1,y_1} \phi y^{T-4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{x_1,y_1} \phi y^{T-1} \sigma_{x_1,y_1} & \rho_{x_1,y_1} \phi y^{T-1} \sigma_{x_1,y_1} \phi y & \rho_{x_1,y_1} \phi y^{T-1} \sigma_{x_1,y_1} \phi y^2 & \rho_{x_1,y_1} \phi y^{T-1} \sigma_{x_1,y_1} \phi y^3 & \cdots & \rho_{x_1,y_1} \phi y^{T-1} \sigma_{x_1,y_1} \phi y^{T-1}
\end{bmatrix}$$  (4.1)

$$= \rho_{x,y} \sigma_{x,y} \begin{bmatrix}
1 & \phi_y & \phi_y^2 & \phi_y^3 & \cdots & \phi_y^{T-1} \\
\phi_y & 1 & \phi_y & \phi_y^2 & \cdots & \phi_y^{T-2} \\
\phi_y^2 & \phi_y & 1 & \phi_y & \cdots & \phi_y^{T-3} \\
\phi_y^3 & \phi_y^2 & \phi_y & 1 & \cdots & \phi_y^{T-4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_y^{T-1} & \phi_y^{T-2} & \phi_y^{T-3} & \phi_y^{T-4} & \cdots & 1
\end{bmatrix}$$
Inserting Equation (4.1) into long-horizon (i.e., \( T \geq 2 \)) co-variance 

\[
\sigma_{\text{x,y}}[T] = w_{x}^{\prime} \Sigma_{\text{x,y}}[T] w_{y}
\]

produces the following result:

\[
\sigma_{\text{x,y}}[T] = \rho_{\text{x,y}} \sigma_{\text{x}} \sigma_{\text{y}} \left[ T + (T-1) \phi_{x} + (T-2) \phi_{x}^{2} + (T-3) \phi_{x}^{3} + \ldots + \phi_{x}^{T-1} \right] + (T-1) \phi_{y} + (T-2) \phi_{y}^{2} + (T-3) \phi_{y}^{3} + \ldots + \phi_{y}^{T-1}\n\]

(4.2)

As before with long-horizon variance, we can scale this notion of long-term co-variance by \( T \) to produce:

\[
\frac{\sigma_{\text{x,y}}[T]}{T} = \rho_{\text{x,y}} \sigma_{\text{x}} \sigma_{\text{y}} \left[ 1 + \sum_{i=1}^{T-1} \phi_{x}^{i} (T-i) + \sum_{i=1}^{T-1} \phi_{y}^{i} (T-i) \right]
\]

(4.3)

For reasoning identical to Appendix 3, equation (4.3) simplifies to:

\[
\frac{\sigma_{\text{x,y}}[T]}{T} = \rho_{\text{x,y}} \sigma_{\text{x}} \sigma_{\text{y}} \left[ 1 + \left( \frac{\phi_{x}}{1 - \phi_{x}} - \frac{\phi_{x} (1 - \phi_{x}^{T})}{T (1 - \phi_{x}^{2})} \right) + \left( \frac{\phi_{y}}{1 - \phi_{y}} - \frac{\phi_{y} (1 - \phi_{y}^{T})}{T (1 - \phi_{y}^{2})} \right) \right]
\]

(4.4)

And as \( T \to \infty \) [given \( |\phi_{x}|, |\phi_{y}| < 1\)], Equation (4.4) further simplifies to:

\[
\frac{\sigma_{\text{x,y}}[T]}{T} = \rho_{\text{x,y}} \sigma_{\text{x}} \sigma_{\text{y}} \left[ 1 + \frac{\phi_{x}}{1 - \phi_{x}} + \frac{\phi_{y}}{1 - \phi_{y}} \right]
\]

(4.5)

The scaled long-term correlation between \( x \) and \( y \) of any length \( T \geq 2 \) is as follows (with the numerator provided by Equation (4.4) and the denominators by Equation (6)): 
\[ \rho_{x,y|T} = \frac{\sigma_{x,y|T}}{\sigma_{x|x|T} \sigma_{y|y|T}} \]

\[
= \frac{T \left[ \rho_{x,y} \sigma_x \sigma_y \left[ 1 + \frac{\varphi_x}{1-\varphi_x} \left( 1 - \varphi_x^{T+1} \right) \right] + \frac{\varphi_y}{1-\varphi_y} \left( 1 - \varphi_y^{T+1} \right) \right] }{\sqrt{T} \left( \sigma_x^2 + 2\sigma_x^2 \frac{\varphi_x}{1-\varphi_x} \left( 1 - \varphi_x^{T+1} \right) \right) \sqrt{T} \left( \sigma_y^2 + 2\sigma_y^2 \frac{\varphi_y}{1-\varphi_y} \left( 1 - \varphi_y^{T+1} \right) \right) } \]

\[ \rho_{x,y} \left[ 1 + \frac{\varphi_x}{1-\varphi_x} \left( 1 - \varphi_x^{T+1} \right) \right] + \frac{\varphi_y}{1-\varphi_y} \left( 1 - \varphi_y^{T+1} \right) \]

\[
= \sqrt{1+2 \frac{\varphi_x}{1-\varphi_x} \left( 1 - \varphi_x^{T+1} \right) } \sqrt{1+2 \frac{\varphi_y}{1-\varphi_y} \left( 1 - \varphi_y^{T+1} \right) } \]

\[ (4.6) \]

Therefore, ratio of the long-horizon correlation to the single-period correlation is given by:

\[
\rho_{x,y|T} = \frac{1 + \frac{\varphi_x}{1-\varphi_x} \left( 1 - \varphi_x^{T+1} \right) + \frac{\varphi_y}{1-\varphi_y} \left( 1 - \varphi_y^{T+1} \right) }{\sqrt{1+2 \frac{\varphi_x}{1-\varphi_x} \left( 1 - \varphi_x^{T+1} \right) } \sqrt{1+2 \frac{\varphi_y}{1-\varphi_y} \left( 1 - \varphi_y^{T+1} \right) } } \]

\[ (4.7) \]

This ratio (or multiple) reaches its minimum, unity, when \( \varphi_x = \varphi_y = \varphi \) – regardless of their level.
And as $T \to \infty$, Equation (4.7) further simplifies as follows:

\[
\lim_{T \to \infty} \left( \rho_{\xi,\eta|T} \right) = \frac{\rho_{\xi,\eta} \sigma_x \sigma_y \left[ 1 + \frac{\varphi_x}{1 - \varphi_x} + \frac{\varphi_y}{1 - \varphi_y} \right]}{\sigma_x \sqrt{1 + \varphi_x} \left( \sigma_y \sqrt{1 + \varphi_y} \right)}
\]

\[
= \frac{\rho_{\xi,\eta} \left[ 1 + \frac{\varphi_x}{1 - \varphi_x} + \frac{\varphi_y}{1 - \varphi_y} \right]}{\sqrt{1 - \varphi_x^2} \sqrt{1 - \varphi_y^2}}
\]

Therefore, the ratio of the infinite-horizon correlation to the single-period correlation is:

\[
\lim_{T \to \infty} \left( \frac{\rho_{\xi,\eta|T}}{\rho_{\xi,\eta}} \right) = \frac{1 - \varphi_x \varphi_y}{\sqrt{1 - \varphi_x^2} \sqrt{1 - \varphi_y^2}}
\]

Finally, we can use equation (2.4) to restate equation (10) as follows:

\[
\lim_{T \to \infty} \left( \rho_{\xi,\eta|T} \right) = \frac{\rho_{\xi,\eta} \left[ 1 - \varphi_x \varphi_y \right]}{\sqrt{1 - \varphi_x^2} \sqrt{1 - \varphi_y^2}}
\]

\[
= \rho_{\xi,\xi} \sigma_x \sigma_y \left[ 1 - \varphi_x \varphi_y \right] \left( \sqrt{1 - \varphi_x^2} \sqrt{1 - \varphi_y^2} \right)
\]

\[
= \rho_{\xi,\xi}
\]
References


