Outline

Key Concepts

MILP Set

Monoids

LP set Relaxation of MILP Set

Formulation Quality

LP-R Sets

LP-R and Finite Basis Theorems
Key Concepts

- MILP set
- Monoid
- LP set Relaxation of MILP Set
- Formulation Quality
- MILP-R set

\[(LP) = (LP-R) \subset (MILP) \subset (MILP-R)\]

- Finite basis theorem and reformulation as an LP-R.
MILP Set

The set of points defined by $\Gamma$

$$\Gamma = \{ x \in \mathbb{R}^n : Ax \geq b, x_j \in \mathbb{Z}, j \in I \}$$  \hspace{1cm} (1)

where $A, B$ are rational matrices, $b$ is a rational vector, and $I \subseteq \{1, 2, \ldots, n\}$ is called an MILP (Mixed Integer Linear Program) set.

This is the most generic formulation.
Sometimes we write the problem as

\[ \begin{align*}
\text{min} & \quad c^\top x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0 \\
& \quad x_j \in \{0, 1\}, \ j \in B \\
& \quad x_j \in \mathbb{Z}, \ j \in I
\end{align*} \] 

In (2)-(4) we have a linear program in standard form. In (5) we add binary restrictions. In (6) we add general integer restrictions.
MILP Set

A few classic applications that involve MILP sets.

- fixed charge constraints
- minimum batch size
- logical restrictions
- piecewise linear functions
- \( k \) of \( n \) alternative
- disjunctive constraints

*Interfaces* articles – the vast majority are mixed integer programming models
A **monoid** $M \subseteq \mathbb{R}^n$ is a set of vectors such that $0 \in M$ and if $x, y \in M$, then $x + y \in M$.

A monoid is a set of vectors in $\mathbb{R}^n$ that contains the zero vector and is closed under addition.

The monoid $M$ is an **integral monoid** if $M \subseteq \mathbb{Z}^n$.

The monoid $M = \{x \in \mathbb{Z}^n \mid Ax \geq 0\}$ is a **polyhedral monoid**.

Note the relationship between a polyhedral cone and a polyhedral monoid. The polyhedral monoid is an example of an MILP set.
The monoid

\[ M = \{ x \in \mathbb{Z}^n \mid x = \sum_{i=1}^{t} \mu_i r^i, \mu_i \in \mathbb{Z}_+^1, i = 1, \ldots, t \} \]

where \( \{ r^1, \ldots, r^t \} \) is a set of integer vectors in \( \mathbb{Z}^n \) is called an **integer cone** and is often written

\[ M = \text{intcone}\{ r^1, \ldots, r^t \}. \]

Note the similarity between a finitely generated cone and an integer cone.
Monoids

We know a polyhedral cone is finitely generated and a finitely generated cone is a polyhedral cone.

If life were fair, a polyhedral monoid would be finitely generated and an integer cone would be a polyhedral monoid.

So what do you think?

Unfortunately, fair is the place where they display animals in summer and has absolutely nothing to do with the way life actually works!

https://en.wikipedia.org/wiki/Fair
Monoids

It turns out that a polyhedral monoid is finitely generated (a result due to Hilbert).

However, an integer cone is not necessarily a polyhedral monoid.

Consider the set $M = \{x \in \mathbb{Z}^1 \mid x = 2\mu, \mu \in \mathbb{Z}^1_+\}$

1. Is $M$ an integer cone?
2. Is $M$ a polyhedral monoid?
Here is a very interesting result due to Bob Jeroslow.

**Theorem (Jeroslow):** Let \( \Gamma \) be a nonempty set of integer vectors that lie in \( \text{cone}(E) \) where \( E \) is a finite subset of \( \Gamma \). Then there exists a finite subset \( V \) of \( \Gamma \) such that \( \Gamma \subseteq V + \text{intcone} \ E \).
Monoids

**Danger Ahead:** Life is more difficult when integer variables are involved.
LP set Relaxation of MILP Set

Given the MILP set

\[ \Gamma = \{ x \in \mathbb{R}^n : Ax \geq b, x_j \in \mathbb{Z}, j \in I \} \]

where \( I \subseteq \{1, 2, \ldots, n\} \).

The LP set relaxation of the MILP set is

\[ \bar{\Gamma} = \{ x \in \mathbb{R}^n : Ax \geq b \}. \]

I dropped the \( x_j \in \mathbb{Z}, j \in I \) in the MILP to get the LP relaxation of the MILP.
LP set Relaxation of MILP Set:

\[
\begin{align*}
\text{min} & \quad -x_1 - x_2 \\
\text{s.t.} & \quad -x_1 + 2x_2 \leq 5 \\
& \quad 9x_1 + 4x_2 \leq 18 \\
& \quad 4x_1 - 2x_2 \leq 4 \\
& \quad x_1, x_2 \geq 0 \\
& \quad x_1, x_2 \in \mathbb{Z}
\end{align*}
\]

and replace with

\[
\begin{align*}
\text{min} & \quad -x_1 - x_2 \\
\text{s.t.} & \quad -x_1 + 2x_2 \leq 5 \\
& \quad 9x_1 + 4x_2 \leq 18 \\
& \quad 4x_1 - 2x_2 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
LP set Relaxation of MILP Set

Note: Dots show the location of feasible integer solutions.
Consider the following two sets of integer points.

\[ \Gamma = \{ (x_1, x_2, y) \mid x_1 \leq y, \ x_2 \leq y, \ x_1, x_2, y \in \{0, 1\} \}. \]

\[ \Gamma' = \{ (x_1, x_2, y) \mid x_1 + x_2 \leq 2y, \ x_1, x_2, y \in \{0, 1\} \}. \]

What are the points in each set and how do they differ?
Consider the following two sets of integer points.

\[ \Gamma = \{(x_1, x_2, y) \mid x_1 \leq y, \ x_2 \leq y, \ x_1, x_2, y \in \{0, 1\}\}. \]

<table>
<thead>
<tr>
<th>y</th>
<th>x_1</th>
<th>x_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\(\Gamma'\) has **exactly the same** set of points!
Formulation Quality:

Consider the following linear relaxations.

\[
\Gamma = \{ (x_1, x_2, y) \mid x_1 \leq y, \ x_2 \leq y, \ x_1, x_2, y \geq 0, \ x_1, x_2, y \leq 1 \}.
\]

\[
\Gamma' = \{ (x_1, x_2, y) \mid x_1 + x_2 \leq 2y, \ x_1, x_2, y \geq 0, \ x_1, x_2, y \leq 1 \}.
\]

What are the points in each set and how do they differ? What about extreme points?
Formulation Quality:

\( \bar{\Gamma} \) is the convex hull of integer points in \( \Gamma \). Not so for \( \Gamma' \)

<table>
<thead>
<tr>
<th></th>
<th>( y )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Important:** \( \bar{\Gamma} \) is a proper subset of \( \Gamma' \). It is a **better relaxation**. Why?
THE BIG TAKE AWAY: Different polyhedra may contain exactly same set of integer points!

WHAT IS THE BIG TAKE AWAY?
Formulation Quality:

**QUESTION:** If different polyhedra may contain exactly same set of integer points, which polyhedron should we pick?
MILP-R Sets

The (LP) sets:

\[ \{ S \subset \mathbb{R}^n : S = \{ x \in \mathbb{R}^n : Ax \geq b \} \} \]

The (LP-R) sets:

\[ \{ S \subset \mathbb{R}^n : S = \text{proj}_x \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Bx + Cy \geq d \} \} \]

The (MILP) sets:

\[ \{ S \subset \mathbb{R}^n : S = \{ x \in \mathbb{R}^n : Ax \geq b, x_j \in \mathbb{Z}, j \in I \} \} \]

The (MILP-R) sets:

\[ \{ S \subset \mathbb{R}^n : S = \text{proj}_x \{ (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{Z}^q : Bx + Cy + Dz \geq d \} \} \]
MILP-R Sets

We know \((LP) = (LP-R)\) and \((LP) \subset (MILP)\) and therefore

\[(LP) = (LP-R) \subset (MILP).\]

We show that unlike \((LP)\), where \((LP) = (LP-R)\), we have

\[(MILP) \subset (MILP-R).\]
MILP-R Sets

This example is due to Paul Williams. Consider the following MILP set (which is actually a polyhedral monoid)

\[
\begin{align*}
7x - 2y & \geq 0 \\
-3x + y & \geq 0 \\
x, y & \in \mathbb{Z}
\end{align*}
\]

Projecting out \(x\) (more on this later) gives

\[
0 \geq \left\lceil \frac{2}{7}y \right\rceil + \left\lceil -\frac{1}{3}y \right\rceil
\]

which is the set of points 0, 3, 6, 7, 9, 10, 12, 13, \ldots,. Is this a MILP set?
MILP-R Sets

We now have:

\[(LP) = (LP-R) \subset (MILP) \subset (MILP-R)\]

Unlike polyhedral cones and polyhedra, the projection of polyhedral monoids and MILP sets does not result in a polyhedral monoid or a MILP set.

We now examine some MILP-R sets that do project down to MILP sets and are very useful for computational purposes.
LP-R and Finite Basis Theorems:

Assume $A$ is a rational matrix, $b$ a rational vector, $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ and $\Gamma = P \cap \mathbb{Z}^n$.

**Finite Basis Theorem:** There exists a finite set $V \subseteq \Gamma$ and a finite set of integer vectors $E$ such that

$$\Gamma = V + \text{intcone}(E). \quad (7)$$

**Proof:** Homework!

Note the relationship to the result of Jeroslow. The fact that $\Gamma$ consists of the integer vectors in a polyhedron allows for equality in 7.

**Key Theme:** Replace an infinite set of integer vectors with the Minkowski sum of a finite set of integer vectors and an integer cone (monoid).
LP-R and Finite Basis Theorems:

An equivalent statement is

$$\Gamma = V + M$$  \hspace{1cm} (8)

where $M$ is the polyhedral monoid $\{x \in \mathbb{Z}^n \mid Ax \geq 0\}$.

Another result that will be very useful from a computational standpoint is

$$\text{conv}(\Gamma) = \text{conv}(V) + \text{cone}(E).$$  \hspace{1cm} (9)
LP-R and Finite Basis Theorems:

When \( P = \{ x \in \mathbb{R}^n \mid Ax \geq 0 \} \) then \( \Gamma = P \cap \mathbb{Z}^n \) is a polyhedral monoid. Then the finite basis theorem is a finite basis result for polyhedral monoids, i.e. we have a finite sets of vectors \( V \) and \( E \) such that

\[
\Gamma = V + \text{intcone}(E)
\]

It follows that for \( E' = V + E \),

\[
\Gamma = \text{intcone}(E').
\]

**Question:** Assume \( A \) is a rational matrix. Can we take the elements of \( E' \) above to be the extreme rays of \( P \) scaled to be integer vectors? Homework!
LP-R and Finite Basis Theorems:

Mix together integer and continuous variables:

**Mixed Integer Finite Basis Theorem:** If

\[ \Gamma = \{(x, y) | Ax + By \geq b, y \in \mathbb{Z}_+^p \}, \tag{10} \]

where \( A, B \) are rational matrices and \( b \) is a rational vector, then \( \text{conv}(\Gamma) \) is a rational polyhedron.
LP-R and Finite Basis Theorems:

Variation on a Theme

Mixed Integer Finite Basis Theorem: If $\Gamma = \{x \in \mathbb{R}^n \mid Bx \geq d, x_j \in \mathbb{Z}, j \in I\}$, then $\text{conv}(\Gamma)$ is a polyhedron. Then there exist $x^1, \ldots, x^r \in \Gamma$ such that

$$\text{conv}(\Gamma) = \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{q} z_ix^i + \sum_{i=q+1}^{r} z_ix^i, \sum_{i=1}^{q} z_i = 1, z_i \geq 0, i = 1, \ldots, r\}$$  \hspace{1cm} (11)
LP-R and Finite Basis Theorems:

**Critical Observation:** $\text{conv}(\Gamma)$ is an LP set! Why?

Here is what we have done.

1. Take a $\Gamma \in \text{MILP}$ where $\Gamma \neq \bar{\Gamma}$

2. Found an LP-R set $S$ where $S = \text{conv}(\Gamma)$

Why is this significant?
LP-R and Finite Basis Theorems:

Absolutely Critical: Understand the differences between

1. $\Gamma$ – this is an MILP

2. $\bar{\Gamma}$ – this is an LP

3. $\text{conv}(\Gamma)$ this is an LP-R

In most interesting cases $\text{conv}(\Gamma) \subset \bar{\Gamma}$.

We want to find $\text{conv}(\Gamma)$ or a close approximation.
LP-R and Finite Basis Theorems:

The mixed integer linear program under consideration is now

\[
\min c^\top x \\
\text{s.t. } Ax \geq b \\
\quad Bx \geq d \\
\quad x \geq 0 \\
\quad x_j \in \mathbb{Z}, j \in I
\]

(MILP)

Objective: identify a set of constraints with special structure

\[
\Gamma = \{ x \in \mathbb{R}^n | Bx \geq d, x \geq 0, x_j \in \mathbb{Z}, j \in I \}
\]

and then use the special structure to improve the formulation.
LP-R and Finite Basis Theorems:

Apply finite basis theorem to $\Gamma$

\[
\begin{align*}
\min_{x} & \quad c^\top x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad x \in \text{conv}(\Gamma)
\end{align*}
\]

(conv(MIP))

\[
\begin{align*}
\min_{x} & \quad c^\top x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad \sum_{i=1}^{q} z_i x^i + \sum_{i=q+1}^{r} z_i x^i = x \\
& \quad \sum_{i=1}^{q} z_i = 1 \\
& \quad z_i \geq 0, \quad i = 1, \ldots, r \\
& \quad x_j \in \mathbb{Z}, \quad j \in I
\end{align*}
\]

(MILPFB)
LP-R and Finite Basis Theorems:

Options:

- Apply column generation to (MILPFB).

- Characterize $\text{conv}(\Gamma)$ with cutting planes.

- Lagrangian Approaches
LP-R and Finite Basis Theorems:

\[
\begin{align*}
\min & \quad -x_1 - x_2 \\
(\text{MILP}) & \quad \text{s.t.} \quad x_1 - x_2 \leq 2 \\
& \quad (x_1, x_2) \in \Gamma
\end{align*}
\]

where $\Gamma$ is defined by

\[
\Gamma = \{(x_1, x_2) \mid 4x_1 + 9x_2 \leq 18, \ -2x_1 + 4x_2 \leq 4, \ x_1, x_2 \in \mathbb{Z}_+\}.
\]
LP-R and Finite Basis Theorems:

Clearly $\text{conv}(\Gamma)$ is a polytope and is generated by taking the convex hull of the four points (0, 0), (4,0), (2,1) and (0, 1).

An alternative formulation of ($\text{MILP}$) using the mixed integer finite basis theorem is

$$
\begin{align*}
\min & \quad -x_1 - x_2 \\
\text{s.t.} & \quad x_1 - x_2 \leq 2 \\
& \quad 4z_2 + 2z_3 = x_1 \\
& \quad z_3 + z_4 = x_2 \\
& \quad z_1 + z_2 + z_3 + z_4 = 1 \\
& \quad z_i \geq 0, \quad i = 1, \ldots, 4 \\
& \quad x_1, x_2 \in \mathbb{Z}_+
\end{align*}
$$
If the auxiliary $z$ variables are projected out, the resulting system of nonredundant constraints in the original $(x_1, x_2)$ variables is

\[
\begin{align*}
    x_1 - x_2 & \leq 2 \\
    (1/2)x_1 + x_2 & \leq 2 \\
    x_2 & \leq 1 \\
    x_1, x_2 & \in \mathbb{Z}_+
\end{align*}
\]
LP-R and Finite Basis Theorems: