Linear Programming
Projection Theory: Part 2
Chapter 2 (57-80)

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Outline

Duality Theory
Dirty Variables
From Here to Infinity
Complementary Slackness
Sensitivity Analysis
Degeneracy
Consistency Testing
Optimal Value Function
Summary: Key Results
Duality Theory

We are solving the linear programming problem using projection.

\[
\min\{c^\top x \mid Ax \geq b\}
\]

Rewrite as the system:

\[
\begin{align*}
    z_0 - c^\top x &\geq 0 \\
    Ax &\geq b
\end{align*}
\]

Project out the \(x\) variables and get

\[
\begin{align*}
    z_0 &\geq d_k, \quad k = 1, \ldots, q \\
    0 &\geq d_k, \quad k = q + 1, \ldots, r
\end{align*}
\]
Duality Theory

Since

\[ z_0 \geq d_k, \quad k = 1, \ldots, q \]
\[ 0 \geq d_k, \quad k = q + 1, \ldots, r \]

the \textbf{optimal value} of the objective function is

\[ z_0 = \max\{d_k \mid k = 1, \ldots, q\} \]

\textbf{Key Idea}: Any of the \( d_k \) provide a \textbf{lower bound} on the optimal objective function value since the optimal value is the maximum over all values.
Duality Theory

**Observation:** We projected out the $x$ variables from the system

\[
\begin{align*}
  z_0 - c^T x & \geq 0 \\
  Ax & \geq b
\end{align*}
\]

and got

\[
\begin{align*}
  z_0 & \geq d_k, \quad k = 1, \ldots, q \\
  0 & \geq d_k, \quad k = q + 1, \ldots, r
\end{align*}
\]

If we let $u$ be the dual multipliers on $Ax \geq b$ and $u_0$ the multiplier on $z_0 - c^T x \geq 0$ we have $(u^k)^T b = d_k$ and

\[
(u^k)^T A - u_0^k c = 0
\]

We take without loss $u_0^k = 1$
Duality Theory

Let me say this again because it is so important. Each of the

\[ z_0 \geq d_k, \quad k = 1, \ldots, q \]

constraints in the projection result from aggregating \( c^\top x \) with \( Ax \geq b \) where \( u^k \) is a vector of multipliers on \( Ax \geq b \) so that

\[ (u^k)^\top A = c \]
Thus any nonnegative $u^k$ with $A^\top u^k = c$ provides a lower bound value of

$$d_k = b^\top u^k$$

on the optimal objective function value. This is known as weak duality.

**Lemma 2.28 (Weak Duality)** If $\bar{x}$ is a solution to the system $Ax \geq b$ and $\bar{u} \geq 0$ is a solution to $A^\top u = c$, then $c^\top \bar{x} \geq b^\top \bar{u}$. 

Duality Theory

Since the minimum value of the linear program is given by

\[ z_0 = \max \{ d_k \mid k = 1, \ldots, q \} \]

it is necessary to find a nonnegative \( u \) such that \( A^\top u = c \) and \( b^\top u \) is as large as possible.

Finding the largest value of \( b^\top u \) is also a linear program:

\[
\max \{ b^\top u \mid A^\top u = c, \ u \geq 0 \}.
\]

The linear program \( \min \{ c^\top x \mid Ax \geq b \} \) is the primal linear program.

The linear program \( \max \{ b^\top u \mid A^\top u = c, \ u \geq 0 \} \) that generates the multipliers for the lower bounds is the dual linear program.

The \( x \) variables are the primal variables and the \( u \) variables the dual variables or dual multipliers.
Duality Theory

**Theorem 2.29 (Strong Duality)** If there is an optimal solution to the primal linear program

\[
\min \{c^\top x \mid Ax \geq b\},
\]

then there is an optimal solution to the dual linear program

\[
\max \{b^\top u \mid A^\top u = c, \ u \geq 0\}
\]

and the optimal objective function values are equal.

**Corollary 2.30** If the primal linear program \(\min\{c^\top x \mid Ax \geq b\}\) is unbounded then \(\max\{b^\top u \mid A^\top u = c, \ u \geq 0\}\) the dual linear program is infeasible.
Duality Theory

If the primal problem is

\[
\begin{align*}
\text{min} & \quad c^\top x \\
A x & \geq b
\end{align*}
\]

the dual problem is

\[
\begin{align*}
\text{max} & \quad b^\top u \\
A^\top u & = c \\
u & \geq 0
\end{align*}
\]

What is the dual of the dual?
Duality Theory

If the primal is

\[
\begin{align*}
\min & \quad c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

What is the dual?

What is the dual of the dual?
Duality Theory

If the primal is

\[
\begin{align*}
\min & \quad c^\top x \\
Ax &= b \\
x &\geq 0
\end{align*}
\]

What is the dual?

What is the dual of the dual?
Duality Theory

**DO NOT MEMORIZE PRIMAL DUAL PAIRS!**

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You get the idea.
## Duality Theory

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>inequality primal constraint</td>
<td>nonnegative dual variable</td>
</tr>
<tr>
<td>equality primal constraint</td>
<td>unrestricted dual variable</td>
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<tr>
<td>nonnegative primal variable</td>
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</tr>
<tr>
<td>unrestricted primal variable</td>
<td>equality dual constraint</td>
</tr>
<tr>
<td>min primal objective function</td>
<td>max dual objective function</td>
</tr>
</tbody>
</table>
Duality Theory

**Proposition 2.31** If the linear program \( \{ c^\top x \mid Ax \geq b \} \) has an optimal solution, then applying projection to the system \( z_0 - c^\top x \geq 0, \ Ax \geq b \) gives all of the extreme points of the dual polyhedron \( \{ u \mid A^\top u = c, \ u \geq 0 \} \) and all of the extreme rays of the dual recession cone \( \{ u \mid A^\top u = 0, \ u \geq 0 \} \).

You are going to basically prove this for homework.
Where We Are Headed

We want to solve problems with special structure! Real problems have special structure! One such structure is

$$\min \ c^T x + f^T y$$
$$\text{s.t.} \quad Ax + By \geq b$$

Later we will take advantage of special structure in the $A$ matrix and **project out** the $x$ variables and solve a problem in only the $y$ variables.
Where We Are Headed

\[ z_0 - c^\top x - f^\top y \geq 0 \]
\[ Ax + By \geq b \]

Rewrite this as

\[ z_0 - c^\top x \geq f^\top y \]
\[ Ax \geq b - By \]

Now project out \( x \)

\[ z_0 \geq f^\top y + (u^k)^\top (b - By), \ k = 1, \ldots, q \]
\[ 0 \geq (u^k)^\top (b - By), \ k = q + 1, \ldots, r \]
Where We Are Headed

Here is the new formulation

\[ z_0 \geq f^\top y + (u^k)^\top (b - By), \ k = 1, \ldots, q \]
\[ 0 \geq (u^k)^\top (b - By), \ k = q + 1, \ldots, r \]

Any problems with this?

What is the fix?

See the homework for a premonition.
Where We Are Headed

We will also use duality to generate bounds in enumeration algorithms.
Dirty Variables (See pages 43-46 in the text.)

Consider the system

\[
\begin{align*}
    x_1 - x_2 & \geq 1 \\
    x_1 + x_2 & \geq 1
\end{align*}
\]

▶ Can we eliminate $x_1$ using Fourier-Motzkin elimination?

▶ What is the projection of the polyhedron onto the $x_2$ space?
Dirty Variables (See pages 43-46 in the text.)

Recall

\[ \mathcal{H}_+(k) := \{ i \in I \mid a^k(i) > 0 \} \]

\[ \mathcal{H}_-(k) := \{ i \in I \mid a^k(i) < 0 \} \]  \hspace{1cm} (1)

\[ \mathcal{H}_0(k) := \{ i \in I \mid a^k(i) = 0 \} \]

Variable \( k \) is a **dirty variable** if \( \mathcal{H}_+(k) \) is empty and \( \mathcal{H}_-(k) \) is not empty, or \( \mathcal{H}_+(k) \) is not empty and \( \mathcal{H}_-(k) \) is empty,
Dirty Variables

In general, what is the projection of the system

\[ x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i, \quad i = 1, \ldots, m, \]

into \( \mathbb{R}^{n-1} \)?

What is the projection of the system

\[
\begin{align*}
    x_1 + a_{i2}x_2 + \cdots + a_{in}x_n & \geq b_i, \quad i = 1, \ldots, m_1 \\
    a_{i2}x_2 + \cdots + a_{in}x_n & \geq b_i, \quad i = m_1 + 1, \ldots, m
\end{align*}
\]

into \( \mathbb{R}^{n-1} \)?
Dirty Variables

If, at some point:

\[ x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i, \quad i = 1, \ldots, m, \]

Two implications:

- The projection of \( \mathbb{R}^n \) in \( \mathbb{R}^{n-1} \) is \( \mathbb{R}^{n-1} \) (the entire subspace)
- The only solution to the corresponding \( u^\top A = 0 \), for \( u \geq 0 \) is \( u = 0 \)

The above is only true if and only if there is a strictly positive (negative) coefficient in every row at some point in the projection process.
Dirty Variables

If there are finite number of constraints, dirty variables are not too bad, we just drop the constraints with the dirty variables.

However, if there are an infinite number of constraints, dirty variables hide dirty secrets and can be obscene.
Theorem: If

$$\Gamma = \{ x : a^1(i)x_1 + a^2(i)x_2 + \cdots + a^n(i)x_n \geq b(i) \text{ for } i \in I = \{1, 2, \ldots, m\} \}$$

then Fourier-Motzkin elimination gives

$$P(\Gamma; x_1) := \{(x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1} : \exists x_1 \in \mathbb{R} \text{ s.t. } (x_1, x_2, \ldots, x_n) \in \Gamma \}. \quad (2)$$

even when $x_1$ is dirty.

What happens when $I$ is not a finite index set? That is, when $\Gamma$ is a closed convex set, but not necessarily a polyhedron.
From Here to Infinity

Polyhedron

\((0,1)\)
\((-1,0)\)
\((0,-1)\)
\((1,0)\)

Closed convex set

\(x_1 \cdot x_2 \geq 1\)

(Not closed) convex set

Polyhedron

\((-1,0)\)
\((1,0)\)
Consider the system

\[ \begin{align*}
ix_1 + \frac{x_2}{i} & \geq 1, \quad i = 1, \ldots, N \\
x_1 & \geq -1 \\
x_2 & \geq -1
\end{align*} \]  

(3)

Project out \( x_2 \) (it is a dirty variable) and get

\[ x_1 \geq -1 \]
From Here to Infinity

\[ N = 2 \]
The Road to Infinity

Now what about (note the $\infty$)

\[ \begin{align*}
    ix_1 + \frac{x_2}{i} & \geq 1, \quad i = 1, \ldots, \infty \\
    x_1 & \geq -1 \\
    x_2 & \geq 0
\end{align*} \] (4)

Project out $x_2$ (it is a dirty variable) and get

\[ x_1 \geq -1 \]

Is this correct?
From Here to Infinity

\[ i x_1 + \frac{x_z}{i} \geq 1 \quad i=1, \ldots, \infty \]

\[ x_1 \geq -1 \]

\[ x_2 \geq 0 \]

\[ x_1 \geq 0 \]

Typo: \( x_1 > 0 \) not \( x_1 \geq 0 \).
dirty variables
"hide"
dirty secrets
Brief Review: Solve the finite linear programming problem using projection.

\[
\min \{ c^\top x \mid Ax \geq b \}
\]

Rewrite as the system:

\[
\begin{align*}
z_0 - c^\top x & \geq 0 \\
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\]

Project out the \(x\) variables and get

\[
\begin{align*}
z_0 & \geq d_k, \quad k = 1, \ldots, q \\
0 & \geq d_k, \quad k = q + 1, \ldots, r
\end{align*}
\]

The optimal value of the objective function is

\[
v(LP) = V(LP) = \max \{ d_k \mid k = 1, \ldots, q \}\]
From Here to Infinity

Fourier-Motzkin algorithm modification.

**Do Not** drop dirty variables.

Pass over them and eliminate only the **clean** variables.

For a formal statement see pages 6-7 in:

Uncovering dirty secrets in SILP

\[
\inf z = \begin{align*}
&z - c_1 x_1 - c_2 x_2 - \cdots - c_n x_n \geq 0 \\
a^1(i)x_1 - a^2(i)x_2 - \cdots - a^n(i)x_n \geq b(i)
\end{align*}
\]

eliminate all "clean" variables

\[
\inf z = \begin{align*}
&0 \geq \tilde{b}(h) \quad h \in I_1 \\
&\tilde{c}^\ell(h)x_\ell + \cdots + \tilde{c}^n(h)x_n \geq \tilde{b}(h) \quad h \in I_2 \\
z \geq \tilde{b}(h) \quad h \in I_3 \\
z + \tilde{c}^\ell(h)x_\ell + \cdots + \tilde{c}^n(h)x_n \geq \tilde{b}(h) \quad h \in I_4
\end{align*}
\]
Consider

\[ 0 \geq \tilde{b}(h), \quad h \in I_1 \]

\[ \tilde{a}^\ell(h)x_\ell + \cdots + \tilde{a}^n(h)x_n \geq \tilde{b}(h), \quad h \in I_2 \]

\[ z \geq \tilde{b}(h), \quad h \in I_3 \]

\[ z + \tilde{a}^\ell(h)x_\ell + \cdots + \tilde{a}^n(h)x_n \geq \tilde{b}(h), \quad h \in I_4 \]  \hspace{1cm} (5)

Assume \((\tilde{z}, \tilde{x}_\ell, \ldots, \tilde{x}_n)\) is a **feasible** solution to the projected system above.
**Observation:** For the feasible \((\bar{z}, \bar{x}_\ell, \ldots, \bar{x}_n)\), we must have

\[
\bar{z} \geq \sup \{ \tilde{b}(h) - \tilde{a}_\ell(h)\bar{x}_\ell - \cdots - \tilde{a}_n(h)\bar{x}_n : h \in l_4 \}
\]

and

\[
\bar{z} \geq \sup \{ \tilde{b}(h) : h \in l_3 \}
\]
Lemma: If $z$ is an optimal solution value to the linear program then

$$z \geq \lim_{\delta \to \infty} \sup \{ \tilde{b}(h) - \delta \sum_{k=\ell}^{n} |\tilde{a}^k(h)| : h \in I_4 \}$$

Proof: the key idea is that if $(\overline{x}_\ell, \ldots, \overline{x}_n)$ is a feasible point in the projected space then

$$\hat{x}_k = \begin{cases} 
\delta, & \text{if } \overline{x}_k > 0 \\
-\delta, & \text{if } \overline{x}_k < 0
\end{cases} \quad k = \ell, \ldots, n$$

where

$$\delta = \max \{|x_k| : k = \ell, \ldots, n\}$$

is a feasible point in the projected space.
Theorem: If ($SILP$) is feasible, then the optimal solution value is

$$v(SILP) = \max\{S, L\}$$

where

$$S = \sup\{\tilde{b}(h) : h \in I_3\}$$

and

$$L = \lim_{\delta \to \infty} \sup\{\tilde{b}(h) - \delta \sum_{k=\ell}^{n} |\tilde{a}^k(h)| : h \in I_4\}$$

Corollary When the cardinality of $I$ is finite, $L = -\infty$ and $v(SILP) = S$. 
From Here to Infinity

\[ \inf z \]

\[ \sum_{i \in I} b(i) y(i) \leq c_k \]

\[ \sum_{i \in I} a^\ell(i) y(i) + \cdots + a^x(i) y(i) \geq b(h) \quad h \in I \]

\[ z \leq a^\ell(h) x_\ell + \cdots + a^x(h) x \geq b(h) \quad h \in I \]

Observe: As in the finite case, FM multiplier in \( I_3 \) give feasible dual solutions.

Key technical lemma: For every dual feasible solution, there exists an FM multiplier with larger dual value.

Theorem: When (SILP) is feasible: \( v(FDSILP) = s \)
Complementary Slackness

**Motivation:** Most work algorithms work by moving from point to point until a set of *optimality conditions* are satisfied.

**Generic Algorithm:**

**Initialization:** Start with a point that satisfies a subset of the optimality conditions.

**Iterative Step:** Move to a better point.

**Termination:** Stop when you have satisfied (to numerical tolerances) the optimality conditions.
Complementary Slackness

**Theorem 2.33** If \( \bar{x} \) is a feasible solution to the primal problem \( \min \{ c^\top x \mid Ax \geq b \} \) and \( \bar{u} \) is a feasible solution to the dual problem \( \max \{ b^\top u \mid A^\top u = c, \ u \geq 0 \} \), then \( \bar{x}, \bar{u} \) are primal-dual optimal if and only if

\[
(b - A\bar{x})^\top \bar{u} = 0. \quad \text{primal complementary slackness}
\]

**Corollary 2.34** If

\[
\begin{align*}
A\bar{x} & \geq b \quad (6) \\
A^\top \bar{u} &= c \quad \bar{u} \geq 0 \quad (7) \\
(b - A\bar{x})^\top \bar{u} &= 0 \quad (8)
\end{align*}
\]

then \( \bar{x} \) is optimal to the primal problem \( \min \{ c^\top x \mid Ax \geq b \} \) and \( \bar{u} \) is optimal to the dual problem \( \max \{ b^\top u \mid A^\top u = c, \ u \geq 0 \} \).
Complementary Slackness

Condition (8) is

\[(b - A\bar{x})^T \bar{u} = 0\]

What is the economic interpretation?

What is the interpretation in terms of projection?
Complementary Slackness

**Theorem 2.35 (Strict Complementary Slackness)** If there is an optimal solution to the linear program

$$\min \{ c^\top x \mid (a^i)^\top x \geq b_i, \ i = 1, \ldots, m \},$$

then there is an optimal primal-dual pair \((\bar{x}, \bar{u})\) such that

$$(b_i - (a^i)^\top \bar{x}) < 0 \Rightarrow \bar{u}_i = 0 \text{ and } (b_i - (a^i)^\top \bar{x}) = 0 \Rightarrow \bar{u}_i > 0, \ i = 1, \ldots, m. \quad (9)$$

**Corollary 2.37** If \(\bar{x}, \bar{u}\) is an optimal primal-dual pair for the linear program \(\min \{ c^\top x \mid Ax \geq b\}\), but is not strictly complementary, then there are either alternative primal optima or alternative dual optima.

This is not good. More on this later.
The linear program is \( \min \{ x_1 + x_2 \mid x_1 + x_2 \geq 1, x_1, x_2 \geq 0 \} \). Solve the linear program using projection.

\[
\begin{align*}
\begin{cases}
-z_0 - x_1 - x_2 & \geq 0 \quad (E0) \\
x_1 + x_2 & \geq 1 \quad (E1) \\
x_1 & \geq 0 \quad (E2) \\
x_2 & \geq 0 \quad (E3)
\end{cases} \Rightarrow \begin{cases}
z_0 & \geq 1 \quad (E0) + (E1) \\
z_0 & \geq 0 \quad (E0) + (E2) + (E3)
\end{cases}
\end{align*}
\]
Complementary Slackness

Proof Motivation:

Idea One: It is sufficient to show strict complementary slackness must hold for the first constraint.

Why is showing the result for one constraint sufficient?

Idea Two: Apply Corollary 2.21 on page 52 of the text.
Complementary Slackness

Corollary (2.21)

If there is no solution to the system

\[ A_1 x > b^1, \ A_2 x \geq b^2 \]

then there are nonnegative \( u^1, u^2 \) such that

\[ A_1^\top u^1 + A_2^\top u^2 = 0, \ (b^1)^\top u^1 + (b^2)^\top u^2 \geq 0, \ u^1 \neq 0 \] (10)

has a solution, or

\[ A_1^\top u^1 + A_2^\top u^2 = 0, \ (b^1)^\top u^1 + (b^2)^\top u^2 > 0, \] (11)

has a solution. Conversely, if there is no solution to either (10) or (11), there is a solution to \( A_1 x > b^1, \ A_2 x \geq b^2 \).
Complementary Slackness

The complementary slackness results of this section are stated in terms of the symmetric primal-dual pair.

If

\[
\begin{align*}
A\bar{x} & \geq b, \quad \bar{x} \geq 0 \\
A^\top \bar{u} & \leq c, \quad \bar{u} \geq 0 \\
(b - A\bar{x})^\top \bar{u} & = 0 \\
(c - A^\top \bar{u})^\top \bar{x} & = 0
\end{align*}
\]

then \( \bar{x} \) is optimal to the primal problem \( \min \{ c^\top x \mid Ax \geq b, \ x \geq 0 \} \) and \( \bar{u} \) is optimal to the dual problem \( \max \{ b^\top u \mid A^\top u \leq c, \ u \geq 0 \} \).
Complementary Slackness

Conditions (12) - (15) are called **optimality conditions**.

- Condition (12) – primal feasibility
- Condition (13) – dual feasibility
- Conditions (14)-(15) – complementary slackness

**Simplex Algorithm:** Enforce conditions (12), (14), and (15) and iterate to satisfy (13) and then stop.

Nonlinear Programming: – Karush-Kuhn-Tucker conditions generalize this to **nonlinear programming**.

Constructing a primal dual-pair with equal objective function values is a very useful proof technique.
Sensitivity Analysis

People are interested not only in primal solutions, but also dual information.

What is the practical significance (or utility) of knowing the optimal values of the dual variables?

**Solvers** not only give the optimal dual information, but they also provide *range analysis*.

Let’s look at some actual solver output.
Sensitivity Analysis

Consider the linear program we have been working with

\[
\begin{align*}
\text{MIN} & \quad 2 \: X_1 - 3 \: X_2 \\
\text{SUBJECT TO} & \\
2) & \quad X_2 \geq 2 \\
3) & \quad 0.5 \: X_1 - X_2 \geq -8 \\
4) & \quad -0.5 \: X_1 + X_2 \geq -3 \\
5) & \quad X_1 - X_2 \geq -6
\end{align*}
\]
Sensitivity Analysis (LINDO/LINGO)

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>VALUE</th>
<th>REDUCED COST</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>4.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>X2</td>
<td>10.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ROW</th>
<th>SLACK OR SURPLUS</th>
<th>DUAL PRICES</th>
</tr>
</thead>
<tbody>
<tr>
<td>2)</td>
<td>8.000000</td>
<td>0.000000</td>
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<tr>
<td>3)</td>
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<tr>
<td>4)</td>
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<td>0.000000</td>
</tr>
<tr>
<td>5)</td>
<td>0.000000</td>
<td>-1.000000</td>
</tr>
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</table>
Sensitivity Analysis (LINDO/LINGO)

**Terminology:** You will see

- Dual price
- Dual variable/value
- Shadow price
### Sensitivity Analysis (LINDO/LINGO)

**Ranges in which the basis is unchanged:**

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>CURRENT COEF</th>
<th>ALLOWABLE INCREASE</th>
<th>ALLOWABLE DECREASE</th>
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</thead>
<tbody>
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<td>0.500000</td>
</tr>
<tr>
<td>X2</td>
<td>-3.000000</td>
<td>1.000000</td>
<td>1.000000</td>
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</table>

**Objective Coefficient Ranges**

<table>
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<tr>
<th>VARIABLE</th>
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<th>ALLOWABLE INCREASE</th>
<th>ALLOWABLE DECREASE</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>2.000000</td>
<td>1.000000</td>
<td>0.500000</td>
</tr>
<tr>
<td>X2</td>
<td>-3.000000</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
</tbody>
</table>

**Right Hand Side Ranges**

<table>
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<tr>
<th>ROW</th>
<th>CURRENT RHS</th>
<th>ALLOWABLE INCREASE</th>
<th>ALLOWABLE DECREASE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.000000</td>
<td>8.000000</td>
<td>INFINITY</td>
</tr>
<tr>
<td>3</td>
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<td>2.000000</td>
<td>INFINITY</td>
</tr>
<tr>
<td>4</td>
<td>-3.000000</td>
<td>11.000000</td>
<td>INFINITY</td>
</tr>
<tr>
<td>5</td>
<td>-6.000000</td>
<td>INFINITY</td>
<td>2.000000</td>
</tr>
</tbody>
</table>
Sensitivity Analysis (Excel Solver)

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [projectionExample.xls]Sheet1
Report Created: 9/29/2010 1:45:51 AM

### Adjustable Cells

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Value</th>
<th>Reduced Cost</th>
<th>Objective Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$2</td>
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<td>4</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
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<tr>
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<td>0</td>
<td>-3</td>
<td>1</td>
<td>1</td>
</tr>
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</table>

### Constraints

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Shadow Price</th>
<th>R.H. Side</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$5</td>
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<td>10</td>
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<td>8</td>
<td>$1E+30</td>
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<td>-8</td>
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<tr>
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<td>-6</td>
<td>1</td>
<td>-6</td>
<td>$1E+30</td>
<td>2</td>
</tr>
</tbody>
</table>
Sensitivity Analysis: Allowable Increase/Decrease

Given an optimal dual solution, $\bar{u}$, the **allowable increase** (decrease) on the right hand side of the constraint

$$(a^k)^T x \geq b_k$$

is the maximum increase (decrease) on the right hand side $b_k$ such that $\bar{u}$ is still an optimal dual solution (or the primal becomes infeasible).

The dual values and the allowable/increase is a natural by-product of projection.
Sensitivity Analysis

The result of projection on the linear program is

\[
\begin{align*}
    z_0 & \geq -22 \quad (E0) + 2(E2) + (E4) & (16) \\
    z_0 & \geq -24 \quad (E0) + 3(E2) + 0.5(E5) & (17) \\
    z_0 & \geq -44 \quad (E0) + 4(E2) + 2(E3) + (E4) & (18) \\
    z_0 & \geq -35 \quad (E0) + 4(E2) + (E3) + 0.5(E5) & (19) \\
    z_0 & \geq -30 \quad (E0) + (E1) + 4(E2) & (20) \\
    z_0 & \geq -32 \quad (E0) + 4(E2) + (E6) & (21) \\
    0 & \geq -11 \quad (E2) + (E3) & (22)
\end{align*}
\]
Sensitivity Analysis

Questions:

- Why is the dual price on row (E2) -2?
- Why is the allowable increase on row (E1) two 8?
- Why is the allowable decrease on row (E4) 2?
Sensitivity Analysis – 100% Rule

In calculating the allowable/increase for the linear programming right hand sides we assume that only one right hand side changes while the others are held constant.

What if we change more than one \( b \)?

If the sum of the percentage increase (decrease) of all the right hand sides in the linear program, \( \min \{ c^\top x \mid Ax \geq b \} \), does not exceed 100% then the current optimal dual solution remains optimal.

**Proof:** Homework!
Degeneracy

People are interested in knowing if there are alternative optima. Why?

If an optimal solution is NOT strictly complementary, then we have alternative primal or dual solutions. Why?

If an optimal solution is NOT strictly complementary, then the range analysis may not be valid. Argle Bargle!

Can we tell from a solver output if we have primal or dual alternative optima?

It depends. Degenerate solutions mess things up.
Closely related to alternative dual optima is the concept of primal degeneracy.

A solution \( \bar{x} \) of the primal linear program \( \min \{ c^\top x \mid Ax \geq b \} \) where \( A \) is an \( m \times n \) matrix is **primal degenerate** if the submatrix of \( A \) defined by the binding constraints (\( (a^i)^\top \bar{x} = b_i \) where \( a^i \) is row \( i \) of \( A \)) has rank strictly less than the number of binding constraints.

Primal degeneracy can also lead to incorrect values of the allowable increase/decrease. Argh!!!!

Some authors define **primal degeneracy** to be alternative dual optima, but this is not correct.
Degeneracy

Consider the linear program

\[
\begin{align*}
\text{min} & \quad -x_2 & \quad (E0) \\
& x_1 & -x_2 & \geq -3 & \quad (E1) \\
& -x_1 & -x_2 & \geq -7 & \quad (E2) \\
& & -x_2 & \geq -2 & \quad (E3) \\
& -2x_1 & +x_2 & \geq -2 & \quad (E4) \\
& x_1 & \geq 0 & \quad (E5) \\
& x_2 & \geq 0 & \quad (E6)
\end{align*}
\]
Degeneracy

Applying projection to this linear program and eliminating the $x$ variables gives

<table>
<thead>
<tr>
<th></th>
<th>$z_0$</th>
<th>$z_0$</th>
<th>$z_0$</th>
<th>$z_0$</th>
<th>$z_0$</th>
</tr>
</thead>
<tbody>
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<td>$-7$</td>
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</tr>
<tr>
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<td>$(E0)$</td>
<td>$(E0)$</td>
<td>$(E0)$</td>
<td>$(E0)$</td>
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<tr>
<td></td>
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<td>$+(E2)$</td>
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<tr>
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<td>$+(E3)$</td>
<td>$+(E4)$</td>
<td>$+(E5)$</td>
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<td>$+3(E5)$</td>
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<table>
<thead>
<tr>
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<td>$(E1)$</td>
<td>$(E2)$</td>
<td>$(E1)$</td>
<td>$(E2)$</td>
</tr>
<tr>
<td></td>
<td>$2(E1)$</td>
<td>$(E2)$</td>
<td>$(E3)$</td>
<td>$(E4)$</td>
<td>$(E3)$</td>
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<td></td>
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<td>$(E6)$</td>
<td>$(E6)$</td>
<td>$(E6)$</td>
<td>$(E6)$</td>
</tr>
</tbody>
</table>
Degeneracy

**Question:** What is the allowable decrease on constraint (E3)?

**Answer:** ?????????

Let’s look at what we get from an LP code.
Degeneracy

The LINGO solution (RHS of E3 is -2)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>X2</td>
<td>2.000000</td>
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</tr>
<tr>
<td>X1</td>
<td>0.000000</td>
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</table>

<table>
<thead>
<tr>
<th>Row</th>
<th>Slack or Surplus</th>
<th>Dual Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2.000000</td>
<td>-1.000000</td>
</tr>
<tr>
<td>2</td>
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</tr>
<tr>
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<tr>
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<td>-1.000000</td>
</tr>
<tr>
<td>5</td>
<td>4.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
Degeneracy

The LINGO Range analysis (RHS of E3 is -2)
Degeneracy

What is the allowable decrease on (E3) according to LINGO?

What is the allowable decrease on (E3) according to projection?

Argle bargle!!!!!!!!! What happened? Do we have strict complementary slackness?

Let’s look at a picture. In the picture we have decreased the RHS to -4 from -2, that is the constraint is $x_2 \leq 4$. 
Degeneracy
Degeneracy

The LINGO solution (RHS of E3 is -4)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Cost</th>
</tr>
</thead>
<tbody>
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<td>X2</td>
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</tr>
<tr>
<td>X1</td>
<td>3.000000</td>
<td>0.000000</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>Row</th>
<th>Slack or Surplus</th>
<th>Dual Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-4.000000</td>
<td>-1.000000</td>
</tr>
<tr>
<td>2</td>
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<tr>
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<td>-1.000000</td>
</tr>
<tr>
<td>5</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
Degeneracy

The LINGO Range analysis (RHS of E3 is -4)
Degeneracy Range analysis (RHS of E3 is -4)

The primal system (tight constraints):

\[-x_1 - x_2 = -7 \quad (E2)\]
\[-x_2 = -4 \quad (E3)\]
\[-2x_1 + x_2 = -2 \quad (E4)\]

The dual system:

\[-u_2 - 2u_4 = 0\]
\[-u_2 - u_3 + u_4 = -1\]
\[u_1, u_2, u_3 \geq 0\]

A unique dual solution, but alternative primal optima.

\[u_2 = 0 \quad u_3 = 1 \quad u_4 = 0\]
Degeneracy

The LINGO solution (RHS of E3 is -5)

Solution Report - degeneracy

Total solver iterations: 5
Model Class: LP
Total variables: 2
Nonlinear variables: 0
Integer variables: 0
Total constraints: 5
Nonlinear constraints: 0
Total nonzeros: 8
Nonlinear nonzeros: 0

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>X2</td>
<td>5.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>X1</td>
<td>2.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Row</th>
<th>Slack or Surplus</th>
<th>Dual Price</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>5</td>
<td>3.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
Degeneracy

The LINGO Range analysis (RHS of E3 is -5)

Ranges in which the basis is unchanged:

**Objective Coefficient Ranges:**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Current Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>X2</td>
<td>-1.000000</td>
<td>1.000000</td>
<td>INFINITY</td>
</tr>
<tr>
<td>X1</td>
<td>0.000000</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
</tbody>
</table>

**Righthand Side Ranges:**

<table>
<thead>
<tr>
<th>Row</th>
<th>Current RHS</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-3.000000</td>
<td>2.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>3</td>
<td>-7.000000</td>
<td>4.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>4</td>
<td>-5.000000</td>
<td>0.000000</td>
<td>INFINITY</td>
</tr>
<tr>
<td>5</td>
<td>-2.000000</td>
<td>3.000000</td>
<td>INFINITY</td>
</tr>
</tbody>
</table>
Degeneracy Range analysis (RHS of E3 is -5)

The primal system (tight constraints):

\[ x_1 - x_2 = -3 \quad (E1) \]
\[ -x_1 - x_2 = -7 \quad (E2) \]
\[ -x_2 = -5 \quad (E3) \]

The dual system:

\[ u_1 - u_2 = 0 \]
\[ -u_1 - u_2 - u_3 = -1 \]
\[ u_1, u_2, u_3 \geq 0 \]

**Alternative** dual solution, but **unique** primal optima \((x_1 = 2, \ x_2 = 5)\).

\[ -2u_1 - u_3 = -1 \]
Range Analysis Summary

The diagram illustrates the range analysis with the following key elements:

- **Axes:**
  - $x_1$ axis
  - $x_2$ axis

- **Lines:**
  - Lines labeled $(E1)$, $(E2)$, $(E3)$, and $(E4)$

- **Points:**
  - Intersection point at $(3.0, 4.0)$

- **Region:**
  - The shaded region indicates the range analysis output.
## Range Analysis Summary

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Solution</th>
<th>Degenerate</th>
<th>Alternate Primal</th>
<th>Alternate Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-x_2 \geq -2$</td>
<td>$x_1 = 0, x_2 = 2$</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$-x_2 \geq -3$</td>
<td>$x_1 = 0, x_2 = 3$</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$-x_2 \geq -4$</td>
<td>$x_1 = 3, x_2 = 4$</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$-x_2 \geq -5$</td>
<td>$x_1 = 2, x_2 = 5$</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Here is something to think about while lying in bed at night.

Assume there is an optimal solution to the linear program such that

- the solution is not strictly complementary
- the solution is not degenerate

What can we conclude about alternative optimal primal and alternative optimal dual solutions?

How can Simplex spot a degenerate solution?
Degeneracy – Key Take Aways

Here is the problem: Simplex codes calculate the allowable increase/decrease by how much a right hand side can change before the set of positive primal variables and constraints with positive slack change.

Stated another way: Simplex codes calculate the allowable increase/decrease by how much a right hand side can change before there is a change in the basis.

This may not actually be equal to the true allowable increase (i.e. before the value of the optimal dual variables change.)

There may be several basis changes before the dual solution changes.
Degeneracy – Key Take Aways

- If there is NOT strict complementary slackness we have alternative primal optima or dual optima.

- If there is NOT strict complementary slackness range analysis may be misleading.

- If there is primal degeneracy knowing which kind of alternative optima (primal or dual) we have is difficult.
Objective: It is desirable to characterize for which $b \in \mathbb{R}^m$ the LP set $\{x \in \mathbb{R}^n \mid Ax \geq b\}$ is nonempty or consistent.

Let $A$ be an $m$ by $n$ matrix.

We call $V : \mathbb{R}^m \to \mathbb{R}$ an LP-consistency tester for $A$ if for any $b \in \mathbb{R}^m$, $V(b) \leq 0$ if and only if $\{x \in \mathbb{R}^n : Ax \geq b\}$ is nonempty.
Consistency Testing

The beauty of projection is that it provides an LP-consistency tester.

Projecting out the $x$ variables in the system $Ax \geq b$ gives

$$0 \geq b^\top u^k, \ k = 1, \ldots, q$$

Define $V(b)$ by

$$V(b) := \max\{b^\top u^k, \ k = 1, \ldots, q\}$$

We have seen that $Ax \geq b$ is consistent if and only if $V(b) \leq 0$. 
Optimal Value Function

What is the efficient frontier in finance?
Increasing the right hand side $b_k$, of constraint $k$, in the linear program

$$\min \{ c^T x \mid Ax \geq b \},$$

will result in infeasibility or a new optimal dual solution. If a new optimal dual solution is the result then the new dual solution will have a strictly greater value for component $k$.

*Hurting hurts more and more.*
If $\bar{u}$ is an optimal dual solution to min $\{c^\top x \mid Ax \geq b\}$ and $\theta_k$ is the allowable increase (decrease) on the right hand side $b_k$ then increasing (decreasing) $b_k$ by more than $\theta_k$ will either result in an infeasible primal or a new optimal dual solution $\hat{u}$ where $\hat{u}_k > \bar{u}_k$ ($\hat{u}_k < \bar{u}_k$).

Why does this follow from projection?
Proposition 2.48: If the projection of the system $z_0 - c^\top x, Ax \geq b$ into the $z_0$ variable space is not $\mathbb{R}^1$, and is not null, then there exists a set of vectors, $u^1, \ldots, u^q$ such that

$$z_0(b) = \min \{ c^\top x \mid Ax \geq b \} = \max \{ b^\top u^k \mid k = 1, \ldots, q \}$$

for all $b$ such that $\{ x \mid Ax \geq b \} \neq 0$.

Corollary 2.49: The optimal value function $z_0(b) = \min \{ c^\top x \mid Ax \geq b \}$ is a piecewise linear convex function over the domain for which the linear program is feasible.
Key Results

- Projection
  - Projection does not create or destroy solutions
  - Projection yields dual multipliers

- Farkas’ Lemma – Theorems of the Alternative

- Weyl’s Theorem

- Solve a linear program with projection

- Weak and Strong Duality

- Complementary Slackness

- Sensitivity Analysis

- Optimal value function