Optimization in Algebraic and Topological Vector Spaces

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# Contents

<table>
<thead>
<tr>
<th>Preface</th>
<th>vii</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Two Key Themes</td>
<td>1</td>
</tr>
<tr>
<td>1.2 An invitation to abstraction</td>
<td>1</td>
</tr>
<tr>
<td>1.3 Key Theme One: Duality</td>
<td>3</td>
</tr>
<tr>
<td>1.3.1 A Duality Trailer</td>
<td>3</td>
</tr>
<tr>
<td>1.3.2 Linear Programming</td>
<td>7</td>
</tr>
<tr>
<td>1.3.3 A New Level of Abstraction</td>
<td>8</td>
</tr>
<tr>
<td>1.4 Key Theme Two: Generalized Constraints</td>
<td>11</td>
</tr>
<tr>
<td>1.5 Notation</td>
<td>14</td>
</tr>
<tr>
<td>1.6 Notes</td>
<td>15</td>
</tr>
<tr>
<td>1.7 Exercises</td>
<td>16</td>
</tr>
<tr>
<td>2 Duality and Convexity in Vector Spaces</td>
<td>17</td>
</tr>
<tr>
<td>2.1 Vector spaces</td>
<td>17</td>
</tr>
<tr>
<td>2.2 Algebraic duality</td>
<td>21</td>
</tr>
<tr>
<td>2.3 Hyperplanes</td>
<td>25</td>
</tr>
<tr>
<td>2.4 Convexity and algebraic topological notions</td>
<td>33</td>
</tr>
<tr>
<td>2.4.1 Convexity</td>
<td>33</td>
</tr>
<tr>
<td>2.4.2 Core and intrinsic core</td>
<td>34</td>
</tr>
<tr>
<td>2.4.3 Algebraic closure</td>
<td>40</td>
</tr>
<tr>
<td>2.4.4 Connections between the core and algebraic closure</td>
<td>41</td>
</tr>
<tr>
<td>2.5 Separating hyperplanes</td>
<td>46</td>
</tr>
<tr>
<td>2.5.1 Types of separation</td>
<td>46</td>
</tr>
<tr>
<td>2.5.2 Stone’s Lemma and basic separation</td>
<td>48</td>
</tr>
<tr>
<td>2.5.3 Improvements on basic separation</td>
<td>54</td>
</tr>
<tr>
<td>2.6 Cones and orderings</td>
<td>62</td>
</tr>
<tr>
<td>2.7 Convex mappings</td>
<td>69</td>
</tr>
<tr>
<td>2.8 Notes</td>
<td>69</td>
</tr>
</tbody>
</table>
2.9 Exercises ......................................................... 70

3 Algebraic Lagrangian Duality .................................... 75
  3.1 Motivation: Penalty Functionals ............................. 76
  3.2 Lower Bounds and Weak Duality ............................ 77
  3.3 The Lagrangian dual problem ............................... 78
  3.4 Zero duality gap and strong duality ....................... 78
    3.4.1 Characterizing the primal and dual optimal values .... 79
    3.4.2 Sufficient conditions for no duality gap ............. 83
    3.4.3 Strong duality ........................................ 85
  3.5 Optimal value functional .................................. 87
  3.6 An Extension with Affine Mappings ....................... 91
  3.7 Linear Programming ....................................... 96
  3.8 Notes ...................................................... 104
  3.9 Exercises .................................................. 104

4 Duality in Topological Vector Spaces .......................... 109
  4.1 Motivation #1: Algebraic duals are nasty! ............... 109
  4.2 Topological spaces ....................................... 115
  4.3 Topological vector spaces ................................ 118
  4.4 Continuity of linear functionals .......................... 123
  4.5 Interior and closure ...................................... 126
  4.6 Hyperplanes in topological vector spaces ............... 131
  4.7 Motivation #2: Improved separation ....................... 135
    4.8.1 Separation is algebraic .............................. 136
    4.8.2 Hierarchy of sufficient conditions ................. 139
    4.8.3 The Kozbur convex set .............................. 141
    4.8.4 Some implications .................................... 144
  4.9 The Barvinok topology .................................... 144
    4.9.1 Connections to core and algebraic closure ........ 149
    4.9.2 Equivalence with separation ........................ 150
  4.10 Notes ...................................................... 151
  4.11 Exercises .................................................. 152

5 Topological Langrangian Duality ................................ 157
  5.1 Motivation .................................................. 157
  5.2 No duality gap and strong duality in topological vector spaces .... 158
  5.3 A brief summary of strong duality ....................... 161
  5.4 Notes ...................................................... 163
  5.5 Exercises .................................................. 164
CONTENTS

6 Locally Convex Spaces 165
   6.1 Paired vector spaces ............................................. 165
   6.2 Linear programs with paired spaces .......................... 170
   6.3 Countably Infinite Linear Programming ....................... 173
   6.4 Countable state Markov Decision Processes .................. 177
   6.5 Notes ........................................................... 179
   6.6 Exercises ....................................................... 179

7 Semidefinite Programming 181
   7.1 Motivation ........................................................ 181
   7.2 The Semidefinite Programming Primal and Dual ............... 182
   7.3 Dual of the Dual ................................................. 188
   7.4 Semidefinite Relaxations of 0/1 Quadratic Programs ........ 190
      7.4.1 A Simple Unconstrained Quadratic Program ............ 190
      7.4.2 Semidefinite Relaxations for Quadratic Programs ..... 192
   7.5 The Lagrangian Dual and the SDP Relaxation ................. 197
      7.5.1 Quadratic programming properties ..................... 198
      7.5.2 Lagrangian dual and SDP dual equivalence ............. 199
   7.6 Lower bound value comparisons ................................ 202
   7.7 A quadratic problem reformulation ............................ 203
   7.8 Combinatorial Optimization Applications .................... 209
      7.8.1 Maximum cut ............................................... 209
      7.8.2 Quadratic assignment problem ......................... 212
      7.8.3 Stable set problem ...................................... 213
   7.9 The tres quadratics ............................................. 215
      7.9.1 Quadratic uno ............................................. 215
      7.9.2 Quadratic dos ............................................. 215
      7.9.3 Quadratic tres ........................................... 217
   7.10 The tres relaxations ........................................... 218
   7.11 Exercises ....................................................... 219

A Appendices 221
   A.1 Partially ordered sets and Zorn’s Lemma .................... 221
   A.2 Matrix algebra .................................................. 223
      A.2.1 Vector basics ............................................. 223
      A.2.2 Matrix basics ............................................. 224
      A.2.3 Quadratic forms and definiteness ....................... 225
      A.2.4 Trace ...................................................... 226
      A.2.5 The operations diag and Diag ............................ 228
      A.2.6 Eigenvalues and Spectral decomposition ............... 229
      A.2.7 Schur’s Lemma ............................................ 234
CONTENTS

A.3 Appendix: Miscellaneous Results ........................................... 236
Bibliography ................................................................. 237
List of Figures

1.1 The dual function. ................................................. 5

2.1 A hyperplane $H$ in $\mathbb{R}^2$. ................................. 26
2.2 An affine subspace ................................................. 27
2.3 Proof of Proposition 2.3.5 ......................................... 31
2.4 A convex set in $\mathbb{R}^2$ .......................................... 33
2.5 A nonconvex set in $\mathbb{R}^2$. The line segment between $x$ and $y$ is not contained in the set. ................................................. 33
2.6 We can draw a line to separate our set from any point outside of it. ...... 34
2.7 Can’t separate our set from the point $w$! Arghh. ......................... 34
2.8 Why do we need affinely independent points for Lemma 2.4.8? Part One. 38
2.9 Why do we need affinely independent points for Lemma 2.4.8? Part Two. 38
2.10 Proof of Theorem 2.4.16 ............................................ 42
2.11 Proof of Theorem 2.4.20 ............................................ 45
2.12 Weak separation of two non-disjoint sets. Let $A = \{x \in \mathbb{R}^2 : 0 \leq x_i \leq 1, \text{ for } i = 1,2\}$ and $B = \{x \in \mathbb{R}^2 : x_1 = 0, 0 \leq x_2 \leq 1\}$ with separating hyperplane $H = \{x \in \mathbb{R}^2 : x_1 = 0\}$. ................................. 47
2.13 An illustration of Stone’s Lemma in $\mathbb{R}^2$. In this figure $C$ is not maximal. This leads to $D$ being nonconvex. ................................. 50
2.14 Notation and geometric intuition for Stone’s Lemma ...................... 51
2.15 An example where we cannot apply Basic Separation Theorem but sets can still be separated. Disjoint sets $A$ and $B$ in $\mathbb{R}^3$ are drawn so that their affine hulls are parallel two dimensional affine subspaces of $\mathbb{R}^2$. Since their set has $\text{aff}(A) \neq \mathbb{R}^3$ it is immediate that $\text{cor}(A) = \text{cor}(B) = \emptyset$ so the Basic Separation Theorem does not apply. However, it is clear that $A$ and $B$ can be separated by a hyperplane. Geometrically, simply take a hyperplane parallel to $\text{aff}(A)$ and $\text{aff}(B)$ positioned, say, half way in between them. ................................. 57
2.16 An example where we cannot apply Basic Separation Theorem but sets can still be separated. Assume sets $A$ and $B$ lie in $\mathbb{R}^2$ and hence have non-empty core, but we have $A \cap B = \{x\}$ and so fail to be disjoint. The Basic Separation Theorem does not apply. However, there is a hyperplane through $x$ which separates $A$ and $B$.

2.17 Figure illustrating Theorem 2.5.16.

2.18 Geometric interpretation of a set-based ordering. On the left is a set $S$, on right its translate to the vector $x$. In the situation shown at the right we have $x \preceq_S y$.

2.19 A nonconvex cone.

2.20 A non-pointed convex cone.

3.1 A visualization of $\Gamma$ and the values $\mu$ and $\nu$ defined in (3.4.1), (3.4.2) and (3.4.3) respectively.

3.2 Proof of Corollary 3.4.8: Illustrates how if $\nu < \mu$ then we can derive a contradiction.

3.3 A summary of conditions for no duality gap and strong duality.

3.4 A visualization of the adjoint map $A'$. 

4.1 Taking the dual of the dual.

4.2 Proof of Lemma 4.5.5.

4.3 Proof of Theorem 4.7.2.

4.4 An hierarchy of containments. The separating hyperplanes are guaranteed to exist for separating from $A$ under the hypotheses of the theorems indicated in the figure.

4.5 The set (4.9.1) (shaded in blue) which is algebraically open in $\mathbb{R}^2$ but not open in the usual topology.

5.1 A summary of conditions for no duality gap and strong duality in the topological case.

5.2 Strong duality theorems.

6.1 The paired adjoint $A^*$. The mapping $J_W : W \to Y'$ and $J_Z : Z \to X'$ are as defined in Theorem 6.1.4. That is, $J_W(w) = \psi_w = \langle \cdot | w \rangle$. We require that $A$ is continuous so that $A'(\psi_w)$ is a continuous linear functional and thus maps under $J_{Z}^{-1}$ to an element of $Z$.

7.1 Illustrating a Stable Set.

A.1 Angle $\delta$ between vectors $x$ and $y$.

A.2 The projection of a vector onto $\mathcal{R}(Q)$.
Preface

Chris and Kipp are deeply indebted to Tom Best, Hoguen Jang, Damian Kozbur, Angelo Mancini, Matt Stern, Zhiyoung Wang, Jing Wu, Payman Yadollahpour, and Jian Yao. These students were subjected to the first iteration of these notes. They deserve the Croix de Guerre, Medal of Honor, Distinguished Flying Cross, Purple Heart and all sorts of other medals for all the pain and suffering they endured. Their corrections, comments, examples, and patience are greatly appreciated and have benefited this effort enormously. In addition, Amitabh Basu has provided incredibly valuable feedback and ideas. We are truly grateful.
Chapter 1

Introduction

1.1 Two Key Themes

There are two key themes in this course. The first is that there are important optimization models where the variable space may not be the usual vector space $\mathbb{R}^n$, or the constraint space may not be the usual vector space $\mathbb{R}^m$, or there may be inequality constraints that are more general than the usual inequality constraints requiring a function to be nonnegative or nonpositive. The second theme is that duality is critically important for these problems. Now, onto the details.

1.2 An invitation to abstraction

These notes discuss convex optimization problems that lie outside the usual world of $\mathbb{R}^n$, and instead treat more general problems where the decision variables lie in arbitrary vector spaces. Why do this? The reasons are many.

Theoretically, by developing the theory in a more general setting, we get a unified view of seemingly disparate forms of optimization. For instance, if we can manage to prove things often desired in an optimization problem – existence of solutions, duality theory, optimality conditions, etc. – in an abstract setting then the result applies to a large class of problems.

Algorithmically, we can leverage ideas developed in one area of optimization (for instance, linear programming) and use them to solve other related problems connected through the more abstract setting. Without an abstract underpinning, the connections, or in particular the right type of connections, would not be as apparent.

In terms of applications, it turns out that not all optimization problems involve choosing an optimal vector (such as in LP); sometimes the choice may be to decide an optimal matrix with some properties, or a continuous function with certain characteristics. Even if the original problem may be most naturally stated in $\mathbb{R}^n$ we still might want to consider a
related problem in some other vector space where its structure might be more apparent. As an example, integer programming researchers have become interested in “semi-definite relaxations” of classic integer programs, where the original problem is approximated or rephrased in terms of optimizing over the vector space of symmetric matrices.

To study optimization problems over general vector spaces some new definitions, notions and vocabulary are required. Expanding one’s vision from \( \mathbb{R}^n \) to include a wider panorama can be both challenging and exhilarating. The challenge comes in thinking more abstractly about spaces that share some, but not all, of the niceties we take granted in the reals. The exhilaration comes in realizing that most of the tools we already know, with the right care in translation, can be used to solve and understand problems which at first seem quite exotic. As with all good adventures, there are a few surprises and the potential for both crisis and victory.

What are some of the niceties we take for granted due to our upbringing in \( \mathbb{R}^n \)? We have nice notions of length (our standard Euclidean norm), distance (the metric defined by that norm), angle (relating to the so-called “dot product” of vectors), ordering (the usual component-wise partial order familiar from LP), convergence and limits useful in describing the behavior of algorithms, this lists goes on. In our flight towards more abstract realms we can take some of these familiar concepts and friends with us, even if this sometimes requires some mental gymnastics, whilst others we might have to leave behind and do without.

The ultimate goal of these notes, to develop a deep enough understanding of the general setting to fully leverage our knowledge and intuition from basic LP to grasp a panorama of new optimization problems, including semidefinite programming (SDP) and infinite-dimensional linear programming (IDLP). One bias will be on understanding SDP, and in particular, on how to use SDP to approach problems in integer programming. One reason for this bias is the freshness of this research direction, and another is the personal taste of the authors. We hope to convince the reader it is a worthwhile endeavor.

We start with stating the most general problem we will discuss. In so doing we will introduce some new concepts and notions that we will define and discuss over the course of these notes. We introduce the problem now as a touchstone and organizing principle:

\[
\begin{align*}
\text{inf } & f(x) \\
\text{s.t. } & G(x) \preceq_P \theta_Y \\
x & \in \Omega
\end{align*}
\]

where \( \Omega \) is a convex set contained in vector space \( X \), \( f : X \to \mathbb{R} \) is a convex functional defined on vector space \( X \), \( G : X \to Y \) is a convex function from \( X \) into a vector space \( Y \), whose zero element we denote \( \theta_Y \). This vector space comes equipped with a partial order \( \preceq \) which is defined by a convex pointed cone in \( Y \). We make no assumption that either \( X \) or \( Y \) is finite dimensional.

At this point, some things should be a bit mysterious about this general set up. First, we have not defined carefully what we mean by a topological vector space or a pointed
1.3. **Key Theme One: Duality**

It was most definitely not love at first sight. Rather, it was more like a perplexing indifference that has somehow matured into a deep love. Understanding duality comes in layers. Much like peeling an onion. It may also make you want to cry!

### 1.3.1 A Duality Trailer

In this section we give what amounts to a movie “trailer” for duality. Consider the simple example

\[
\min x_1^2 + x_1x_2 + 4x_2^2 \quad (1.3.1)
\]
\[
x_1 + 2x_2 = 4 \quad (1.3.2)
\]

A standard way to handle this problem is through the use of a Lagrange multiplier. The Lagrange multiplier is used to convert the “difficult” constrained optimization problem into an “easier” unconstrained optimization problem. Denote by \( \psi \) the Lagrange multiplier and form the Lagrangian problem

\[
L(x_1, x_2, \psi) = x_1^2 + x_1x_2 + 4x_2^2 + \psi(4 - x_1 - 2x_2). \quad (1.3.3)
\]

In the optimization vernacular, forming the Lagrangian function \( L(x_1, x_2, \psi) \) in Equation (1.3.3), is called “dualizing the constraint”. Indeed, here we see one of the first and most important uses of duality – making an optimization problem easier to solve. Next, the Lagrangian function \( L(x_1, x_2, \psi) \) is minimized. This is easy to do since we have a convex optimization problem. Take the partial derivatives with respect to \( x_1, x_2, \) and \( \psi \).

\[
\frac{\partial L(x_1, x_2, \psi)}{\partial x_1} = 2x_1 + x_2 - \psi \quad (1.3.4)
\]
\[
\frac{\partial L(x_1, x_2, \psi)}{\partial x_2} = x_1 + 8x_2 - 2\psi \quad (1.3.5)
\]
\[
\frac{\partial L(x_1, x_2, \psi)}{\partial \psi} = 4 - x_1 - 2x_2 \quad (1.3.6)
\]

Setting the partial derivatives to zero and solving gives \( x_1 = 2, x_2 = 1, \psi = 5 \). Due to convexity, this solution minimizes \( L(x_1, x_2, \psi) \) and gives an value of 10 for the objective function defined in (1.3.1). It turns out that \( x_1 = 2, x_2 = 1 \) is the optimal solution for the optimization problem given in (1.3.1)-(1.3.2).
CHAPTER 1. INTRODUCTION

Proposition 1.3.1. If \((\overline{x}_1, \overline{x}_2, \overline{\psi})\) minimizes the Lagrangian function \(L(x_1, x_2, \psi)\) defined in (1.3.3) then \((\overline{x}_1, \overline{x}_2)\) is an optimal solution to the optimization problem defined in (1.3.1)-(1.3.2).

Critical Observations:

1. When we optimize \(L(x_1, x_2, \psi)\) using simple Calculus we get an \((\overline{x}_1, \overline{x}_2, \overline{\psi})\) and \((\overline{x}_1, \overline{x}_2)\) that satisfies the primal constraint. Why is this so? Because we take the partial with respect to \(\psi\) and set it equal to zero. This is then a primal feasibility constraint.

2. For the given \(\overline{\psi}\), we solve \(\min L(x_1, x_2, \overline{\psi})\) (1.3.7)

Why? Because for any given \(\overline{\psi}\), we take the partials with respect to \(x_1\) and \(x_2\) and set to zero so we are optimizing \(L(x_1, x_2, \overline{\psi})\) assuming this is convex for fixed \(\overline{\psi}\).

The logic behind Proposition 1.3.1 illuminates many of the critical ideas we study in this course. We know that the “optimal” value of \(\psi\) is 5. Let’s experiment a bit and minimize the Lagrangian function \(L(x_1, x_2, \psi)\) for some sample values of \(\psi\). We try the sample values of -4, 0, and 6. For each of these values we find the minimum value of \(L(x_1, x_2, \psi)\) given the fixed value of \(\psi\). The \(\psi\) values of -4, 0, and 6, give respectively,

\[
\begin{align*}
\psi &= -4 & x_1 &= -8/5 & x_2 &= -4/5 & L(-8/5, -4/5, -4) &= -22.4 < 10 = L(2, 1, 5) \\
\psi &= 0 & x_1 &= 0 & x_2 &= 0 & L(0, 0, 0) &= 0 < 10 = L(2, 1, 5) \\
\psi &= 6 & x_1 &= 12/5 & x_2 &= 6/5 & L(12/5, 6/5, 6) &= 9.6 < 10 = L(2, 1, 5)
\end{align*}
\]

Table 1.1: Illustrating Weak Duality

A sample size of three is pretty small, but we are starting to see a trend. If we minimize \(L(x_1, x_2, \psi)\) for a value of \(\psi \neq 5\), then the minimum value of \(L(x_1, x_2, \psi)\) will be less than the minimum value of \(L(x_1, x_2, 5)\). That is, given any \(\overline{\psi} \in \mathbb{R}\)

\[
\min\{L(x_1, x_2, \overline{\psi}) \mid (x_1, x_2) \in \mathbb{R}^2\} \leq \min\{L(x_1, x_2, 5) \mid (x_1, x_2) \in \mathbb{R}^2\}
\] (1.3.8)

There are two ways to see that (1.3.8) must hold. Here is the first way. Solve the partial derivative equations in (1.3.4) and (1.3.5) for \(x_1\) and \(x_2\) in terms of \(\psi\). This gives

\[
x_1(\psi) = 2\psi/5 \quad x_2(\psi) = \psi/5
\]

Then

\[
L(x_1(\psi), x_2(\psi), \psi) = -(2\psi(\psi - 10))/5
\] (1.3.9)

The function \(L(x_1(\psi), x_2(\psi), \psi)\) is obviously concave in \(\psi\) and is plotted in Figure 1.1. Taking the derivative of \(L(x_1(\psi), x_2(\psi), \psi)\) with respect to \(\psi\), setting the derivative equal
1.3. KEY THEME ONE: DUALITY

Figure 1.1: The dual function.

to zero, and solving gives a value of $\psi = 5$ and function value of 10. Therefore, (1.3.8) is valid.

The concavity of $L(x_1(\psi), x_2(\psi), \psi)$ is not a fluke. Later, in Chapter 3, in Proposition 3.5.3, we show that a Lagrangian dual function based on $L(x_1, x_2, \psi)$ is always concave. Now show (1.3.8) is valid without using an argument based on the concavity of $L(x_1(\psi), x_2(\psi), \psi)$. This argument is based on the fundamental concept called weak duality.

**Proposition 1.3.2.** (Weak Duality) If $(\bar{x}_1, \bar{x}_2)$ be a feasible solution to the constraint (1.3.2). Then for any $\bar{\psi}$,

$$\min\{L(x_1, x_2, \bar{\psi}) \mid (x_1, x_2) \in \mathbb{R}^2\} \leq f(\bar{x}_1, \bar{x}_2) \tag{1.3.10}$$

where $f(x_1, x_2) = x_1^2 + x_1x_2 + 4x_2^2$ is the objective function for our optimization problem given in (1.3.1).

**Proof.** For any given $(\bar{x}_1, \bar{x}_2)$ and $\bar{\psi}$,

$$\min\{L(x_1, x_2, \bar{\psi}) \mid (x_1, x_2) \in \mathbb{R}^2\} \leq L(\bar{x}_1, \bar{x}_2, \bar{\psi}) \tag{1.3.11}$$

But if $(\bar{x}_1, \bar{x}_2)$ also are feasible to the constraint (1.3.2) as stated in the hypothesis, then

$$L(\bar{x}_1, \bar{x}_2, \bar{\psi}) = \bar{x}_1^2 + \bar{x}_1\bar{x}_2 + 4\bar{x}_2^2 + \psi(4 - \bar{x}_1 - 2\bar{x}_2)$$

$$= \bar{x}_1^2 + \bar{x}_1\bar{x}_2 + 4\bar{x}_2^2$$

$$= f(\bar{x}_1, \bar{x}_2)$$
Then from (1.3.11)
\[
\min \{ L(x_1, x_2, \psi) \mid (x_1, x_2) \in \mathbb{R}^2 \} \leq f(\bar{x}_1, \bar{x}_2)
\]

Equation (1.3.8) now follows easily from Proposition 1.3.2. When \( \psi = 5 \), the solution to (1.3.4)-(1.3.5) is \( x_1 = 2 \) and \( x_2 = 1 \). This is feasible to the constraint (1.3.2) so \( L(2, 1, 5) = f(2, 1) \) and we have the desired result since
\[
\min \{ L(x_1, x_2, \psi) \mid (x_1, x_2) \in \mathbb{R}^2 \} \leq f(2, 1) = \min \{ L(x_1, x_2, 5) \mid (x_1, x_2) \in \mathbb{R}^2 \}.
\]

This kind of logic is used a lot in this course. The result in Proposition 1.3.2 is a weak duality result and we give a very general weak duality results in Chapter 3. Before moving on, lets take a closer look at what just happened. We formed the Lagrangian function
\[
L(x_1, x_2, \psi) = x_1^2 + x_1 x_2 + 4x_2^2 + \psi(4 - x_1 - 2x_2).
\]

By weak duality, we know the minimum value of \( L(x_1, x_2, \psi) \) will never exceed the primal objective function evaluated at a primal feasible solution. Not only that, but in this specific example, we found a value of \( \psi \) (in this case 5) such that (i) the minimum value \( L(x_1, x_2, 5) \) was equal to \( f(2, 1) \), and (ii) \( x_1 = 2 \) and \( x_2 = 1 \) was the optimal primal solution. This follows immediately from weak duality and for this example we have

\textbf{Proposition 1.3.3.} If
\[
\varphi(\psi) := \min \{ L(x_1, x_2, \psi) \mid (x_1, x_2) \in \mathbb{R}^2 \}
\]

then
\[
\max_{\psi} \varphi(\psi) = \min \{ x_1^2 + x_1 x_2 + 4x_2^2 \mid x_1 + 2x_2 = 4, (x_1, x_2) \in \mathbb{R}^2 \}
\]

\textit{Proof.} Observe \( \varphi(5) = f(2, 1) \) and \((2, 1)\) is primal feasible. The result follows by weak duality. \( \Box \)

In (1.3.12) \( \varphi(\psi) \) is called the \textit{Lagrangian dual function}. When the primal and dual problem have optimal solutions and the optimal value of the Lagrangian dual and the original primal problem are equal we have \textit{strong duality}. In this little example we have strong duality. Finding sufficient conditions for when strong duality holds is a key theme in these notes. Chapters 3 and 5 focus on sufficient conditions for strong duality to hold in algebraic and topological vector spaces.
1.3. KEY THEME ONE: DUALITY

1.3.2 Linear Programming

The primal linear program in standard form in

$$
\begin{align*}
\min & \quad \phi^T x \\
\text{s.t.} \quad Ax &= b \\
\quad & x \geq 0
\end{align*}
$$

(P)

where $x, \phi \in \mathbb{R}^n$ (technically $\phi$ is in $(\mathbb{R}^n)'$, but more on that later), $b \in \mathbb{R}^m$ and $A$ is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ (which we also think of, somewhat abusively, as an $m$ by $n$ real matrix). Following what we did in Section 1.3.1, define the Lagrangian dual function

$$
\varphi(\psi) = \min_{x \geq 0} \{ \phi^T x + (b - Ax)^T \psi \} = b^T \psi + \min_{x \geq 0} \{ \phi^T x - (Ax)^T \psi \} = b^T \psi + \min_{x \geq 0} \{ x^T \phi - x^T A^T \psi \} = b^T \psi + \min_{x \geq 0} \{ x^T (\phi - A^T \psi) \}
$$

In the last step in this derivation we rewrite $x^T \phi - x^T A^T \psi$ as $x^T (\phi - A^T \psi)$ by simply factoring out $x^T$. Factoring out the $x$ from the objective function and the constraints is something we could not do in the example in Section 1.3.1 because the objective function is nonlinear. Whenever the objective function or the constraints contain nonlinear terms we cannot factor out the $x$. However, in the linear case we can, and this is significant. Consider

$$
\min_{x \geq 0} \{ x^T (\phi - A^T \psi) \}
$$

Now for a “trick” we will use throughout the course. We are trying to minimize the term $x^T (\phi - A^T \psi)$. If we pick a $\psi$ that makes any component of $(\phi - A^T \psi)$ negative, say component $j$, then we will let the corresponding $x_j$ go to $\infty$ since there is no upper bound on $x$. Then, in this case $\min_{x \geq 0} \{ x^T (\phi - A^T \psi) \} = -\infty$ which is not very useful. Therefore, we should pick a $\psi$ so that $(\phi - A^T \psi)$ is nonnegative. The objective is to maximize the dual function $\varphi(\psi)$ and this entails selecting a $\psi$ so that $(\phi - A^T \psi)$ is nonnegative. Therefore, the Lagrangian dual of (P) is equivalent to

$$
\begin{align*}
\max \quad & b^T \psi \\
\text{s.t.} \quad & (\phi - A^T \psi) \geq 0.
\end{align*}
$$

(D)

Of course (D) is the well-known dual of the linear program (P) in standard form. What utility does the dual provide? Well in Section 1.3.1 we illustrated how forming the
Lagrangian dual made the solution of problem easier because we were able to convert a constrained problem into an unconstrained problem. But there is much more. Duality is used to provide a certificate of optimality. How does the simplex algorithm work? At each iteration of simplex we have a primal feasible solution and a primal-dual complementary solution. When dual feasibility, \((\phi - A^\top \psi) \geq 0\) is achieved we are guaranteed to be optimal. Even more generally, if we can construct a primal feasible solution and dual feasible solution such that the primal and dual objectives are equal, then by weak duality we are primal-dual optimal. The importance of duality cannot be overstated.

1.3.3 A New Level of Abstraction

The purpose of this introductory chapter is to give an overview of many of the important concepts we study this course. We are going to study duality in a more abstract setting than in the previous setting. In the primal problem \((P)\) introduced in Section 1.3.2 think of the primal variables as being in the space \(\mathbb{R}^n\). In Section 2.1 of Chapter 2, the concept of a vector space is introduced and \(\mathbb{R}^n\) is an example of a simple vector space. Assume that in \((P)\) \(A\) is an \(m \times n\) matrix. Then \(A\) takes a vector \(x \in \mathbb{R}^n\) and maps it into a vector in \(\mathbb{R}^m\). That is \(A : \mathbb{R}^n \rightarrow \mathbb{R}^m\). So \(A\) is a linear mapping from the vector space \(\mathbb{R}^n\) to the vector space \(\mathbb{R}^m\).

Next, look at the primal objective function, \(\phi^\top x\). What does \(\phi\) do? It takes a vector in \(\mathbb{R}^n\) and maps it into the real numbers. In other words \(\phi : \mathbb{R}^n \rightarrow \mathbb{R}\). It is also a linear mapping. Hopefully at some point in a linear algebra or abstract algebra class you studied these linear mappings and were exposed to the idea of a linear functional. The space of linear functionals is also called the dual space. If \(X\) is a vector space its dual space is often denoted by \(X'\). If you have not seen this idea before, don’t worry! It is covered in Chapter 2 in Section 2.2.

So let’s step back and see where we are. We have a primal problem where the variables are in the vector space \(X = \mathbb{R}^n\) and we have a linear functional \(\phi\) which is dual space of the primal vector space. We also have a linear mapping \(A\) that maps elements from the \(\mathbb{R}^n\) vector space into the \(\mathbb{R}^m\) vector space. Well if the vector space \(\mathbb{R}^n\) has a linear functional, isn’t it only fair and equitable that \(\mathbb{R}^m\) should have one also? Aha, it does! Remember we defined the Lagrangian function

\[
\varphi(\psi) = \min_{x \geq 0} \{\phi^\top x + (b - Ax)^\top \psi\}
\]

What space is \(\psi\) in? Yes, you guessed it, the dual space to \(\mathbb{R}^m\). And just what does \(\psi\) do in the dual space of \(\mathbb{R}^m\)? It takes a vector such as \((b - Ax)^\top\) in \(\mathbb{R}^m\) and maps it into the real numbers through the operation \((b - Ax)^\top \psi\). Remember the onion analogy to duality. Am I making you cry yet? For ease of exposition we write out the primal and dual problems
1.3. KEY THEME ONE: DUALITY

again

\[
\begin{align*}
\text{min} & \quad \phi^\top x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

(P)

\[
\begin{align*}
\text{max} & \quad b^\top \psi \\
\text{s.t.} & \quad (\phi - A^\top \psi) \geq 0
\end{align*}
\]

(D)

If we let \( X \) denote the vector space \( \mathbb{R}^n \) and \( X' \) the space of linear functionals on defined on \( X \), \( Y \) denote the vector space \( \mathbb{R}^m \) with dual space \( Y' \) then we have:

\[
P : \begin{bmatrix} X \\ X' \end{bmatrix} \rightarrow \begin{bmatrix} Y \\ Y' \end{bmatrix} : \text{D}
\]

(1.3.13)

What we are saying in (1.3.13) is that the primal problem variables are in the vector space \( X \) and the primal constraints are in the vector space \( Y \). Likewise, for the dual problem, the dual variables are in the dual vector space \( Y' \) and dual constraints are in the dual vector space \( X' \). Throughout the notes we use a paradigm like (1.3.13) to make explicit which vector spaces constitute the primal and dual problems. It will prove to be incredibly useful. The notation used in (1.3.13) is taken from Anderson and Nash [3].

Now, just when you think it can’t get any worse, it does. Consider the dual constraints \((\phi - A^\top \psi) \geq 0\). In these constraints \(A^\top\) is the transpose of the matrix \(A\). Let’s see if we can take the simple concept of a matrix transpose and make it as complicated as is humanly possible. Consider \((\phi - A^\top \psi)\). The primal constraint space is \(X'\), the space of linear functionals for \(X\) so \(A^\top \psi\) is in \(X'\). But \(\psi\) is a linear functional in \(Y'\). Thus, \(A^\top\) is a mapping from \(Y'\) to \(X'\), that is \(A^\top : Y' \rightarrow X'\). Going back to the Lagrangian function there is a term

\[
(b - Ax)^\top \psi = b^\top \psi - x^\top A^\top \psi
\]

So how does \(A^\top\) take a linear functional in \(Y'\) and map it into a linear functional in \(X'\)? Like this:

\[
A^\top \psi = \psi \circ A
\]

So the linear functional \(A^\top \psi\) maps an element \(x \in X\) into \(\mathbb{R}\) by first taking \(Ax\) into \(Y\) and then applies \(\psi \in Y'\) to \(Ax\) to give a value in \(\mathbb{R}\). That is

\[
\psi^\top (Ax) = (A^\top \psi)^\top x.
\]

In abstractese, \(A^\top\) is the known as the adjoint of \(A\).
CHAPTER 1. INTRODUCTION

Let’s look at one more example. It is from Anderson and Nash [3]. Consider the following linear program.

\[
\begin{align*}
\min & \quad \phi^\top x \\
\text{subject to} & \quad a(s)^\top x - b(s) \geq 0, \quad s \in [0, 1]
\end{align*}
\]

In this linear program assume \( x \in \mathbb{R}^n \). But what about the constraints? There are an infinite number! Here is one way to think about the problem.

- Given any \( \vec{x} \in \mathbb{R}^n \), the function \( a(s)^\top x - b(s) \) is continuous for all \( s \in [0, 1] \). That is, given any \( \vec{x} \in \mathbb{R}^n \), \( a(s)^\top \vec{x} - b(s) \in C[0, 1] \).

- We want to find the \( x \in \mathbb{R}^n \) that minimizes \( \langle \phi, x \rangle \) and satisfies the condition that \( a(s)^\top \vec{x} - b(s) \) is nonnegative over its domain \([0, 1]\).

- The way we enforce the condition that the function we generate for a given \( x \), is nonnegative over its domain of \([0, 1]\) is to write the uncountably infinite constraint set

\[
a(s)^\top \vec{x} - b(s) \geq 0, \quad s \in [0, 1]
\]

Let’s consider an example. We are in the business of taking concrete, easy to understand models, and making them abstract and unintelligible (we are almost kidding of course). What better way to illustrate an infinite dimensional linear program than through the formerly simple EOQ. It will no longer be simple. This derivation is directly from Adelman and Klabjan [1]. The standard EOQ assumptions are made and we define \( \lambda \) to be the constant deterministic demand rate, \( h \) to be the per-time holding cost, \( a \) the order quantity, and \( C \) the fixed order cost. The decision variable is \( \rho \) which represents the long-run average cost per unit time. The fixed plus holding cost per cycle is \( C + (h/2\lambda)a^2 \) and the cycle length is \( a/\lambda \). Further assume that demand has been scaled so that we can assume \( a \in [0, 1] \). Then the optimization problem is

\[
\sup \rho \\
\rho a/\lambda \leq C + (h/2\lambda)a^2, \quad a \in [0, 1]
\]

In the standard EOQ model, \( a \), the order quantity is the decision variable. It is not here, rather \( a \) is the variable that defines a function in \( C[0, 1] \). The function in \( C[0, 1] \) has the form

\[
\rho a/\lambda - C + (h/2\lambda)a^2 \tag{1.3.14}
\]

and the optimization problem is to find the largest \( \rho \) such that the resulting function (1.3.14) in \( C[0, 1] \) is nonpositive over its domain.
1.4. KEY THEME TWO: GENERALIZED CONSTRAINTS

So how does one take the dual of a linear program with an uncountably infinite number of constraints? This will come later in Chapter ?? . Taking the dual for an infinite dimensional linear program is nontrivial. Typically, we define a topology on the primal constraint vector space, in this case $C[0, 1]$. Then, we restrict the dual variables to be the continuous linear functionals where continuity is defined based on the open sets in the topology. The material on topology and continuous linear functionals for infinite dimensional vector spaces is in Chapters 4 and 6. For those who cannot wait, if we define a norm (which implies a topology) on $C[0, 1]$ by the usual sup or infinity norm, then the dual space is $M_r[0, 1]$ the space of regular Borel measures on $[0, 1]$. See below.

$$P : \left[ \mathbb{R}^n \rightarrow \mathbb{R}^n \right] \rightarrow \begin{bmatrix} C[0, 1] \\ M_r[0, 1] \end{bmatrix} : D$$

(1.3.15)

It will get easier. We promise.

1.4 Key Theme Two: Generalized Constraints

Interfaces is a journal published by the INFORMS (Institute for Operations Research and the Management Sciences) society. This journal is devoted to publishing Operations Research “success stories.” The articles in Interfaces describe very impressive actual model implementations in industry and government. Many of these implementations require solving a mixed integer programming optimization model. Numerous interesting problems can be formulated as a mixed integer program. Consider the NY Times article on December 21, 2011, “The Problem of the Traveling Politician.” This article (http://campaignstops.blogs.nytimes.com/2011/12/21/the-problem-of-the-traveling-politician/) was written by William Cook, a researcher who has made significant contributions to methodology for solving the classical traveling salesperson problem. In this article, Bill writes about the Iowa Republican primary and a significant problem facing each primary candidate. In their whirlwind bus tours about the state of Iowa, how do they route their stops most efficiently. Michele Bachmann planned a 99-county tour, while Newt Gingrich announced a 44-city bus tour. Finding the shortest route through the cities and counties is a challenging problem. Here is an integer programming formulation for the traveling salesperson problem.

Let $x_{ij} = 1$ if city $i$ is the $jth$ city in the tour. For example $x_{73} = 1$ means that we visit the city indexed by 7 third on the tour. Let $c_{ik}$ be the distance between city $i$ and city $k$. Assume there are $n$ cities. The problem, simply stated, is to find a permutation of
\{1, 2, \ldots, n\} that minimizes total distance travelled.

\[
\min \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n-1} c_{ik}x_{ij}x_{k,j+1}
\]  

\[
\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, \ldots, n
\]  

\[
\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n
\]  

\[
x_{ij} \in \{0, 1\}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n
\]

In this formulation, constraint (1.4.2) requires exactly one city to be assigned to the \(j\)th position in the permutation. Constraint (1.4.3) requires that each city be assigned a position in the permutation. Now consider the objective function (1.4.1). If \(x_{ij} = 1\) then city \(i\) is the \(j\)th city in the tour. If \(x_{k,j+1} = 1\), then city \(k\) is the \((j + 1)\)th city in the tour. Therefore \(x_{ij}x_{k,j+1} = 1\) and we incur the cost, \(c_{ij}\), of traveling from city \(i\) to city \(k\).

Unfortunately, this is a nonlinear problem because of the \(x_{ij}x_{k,j+1}\) terms in the objective function. Even worse it is nonconvex. Even worse it is an integer program. Are you getting queasy? A standard approach to this problem is to convert it to a linear program by replacing each \(c_{ik}x_{ij}x_{k,j+1}\) with \(c_{ik}y_{ik}\) and adding the constraints

\[
x_{ij} + x_{k,j+1} \leq 1 - y_{ik}
\]

This appears to be a clever thing to do because now we can solve this as a linear integer programming problem. Unfortunately this reformulation is extremely ineffective. Why is this so? An alternative approach is to apply what we will learn in Chapter 7. An optimization problem with a generic quadratic objective function subject to linear constraints is

\[(QP) \quad \min x^\top Qx \quad \text{subject to} \quad Ax = b, \quad x_i \in \{0, 1\}, \quad i = 1, \ldots, n\]

In the objective function, \(Q\) is an \(n \times n\) symmetric matrix. Throughout the notes, we denote this space of matrices by \(\mathcal{S}^n\). See Appendix A.2 for results on eigenvalues, matrix trace, and semidefinite matrices. In that appendix we show that

\[x^\top Qx = \langle Q, xx^\top \rangle\]  

(1.4.5)

where \(\langle Q, xx^\top \rangle\) denote the trace of the matrix \(Qxx^\top\). Using this trace result in (1.4.5), we create an equivalent problem to (QP) by introducing an \(X \in \mathcal{S}^n\) where \(X = xx^\top\). Using
1.4. KEY THEME TWO: GENERALIZED CONSTRAINTS

the $X = xx^\top$ substitution gives the equivalent problem $(QP)$.

$$(QP) \quad \min \langle Q, X \rangle$$

$X = xx^\top$

$Ax = b$

$x_i - x_i^2 = 0, \quad i = 1, \ldots, n$

We next replace the constraints $x_i - x_i^2 = 0, \quad i = 1, \ldots, n$ with the equivalent $\text{diag}(X) = x$. Problem $(QP)$ is then

$$(QP) \quad \min \langle Q, X \rangle$$

$X = xx^\top$

$Ax = b$

$\text{diag}(X) = x$

Replacing binary constraints $x_i \in \{0, 1\}$ is a standard modeling trick. It converts an integer program into a quadratic program. Unfortunately, the constraints $X = xx^\top$ are nonconvex. In order to make the constraint set convex, we replace $X = xx^\top$ with $X \succeq xx^\top$ and replace problem $(QP)$ with a relaxation $(QPR)$.

$$(QPR) \quad \min \langle Q, xx^\top \rangle$$

$\text{diag}(X) = x$

$Ax = b$

$X - xx^\top \succeq 0, \quad i = 1, \ldots, n$

From Corollary A.2.10, $X - xx^\top \succeq 0$ is a convex constraint. Again, by Corollary A.2.10, we can rewrite $(QPR)$ as

$$(QPR) \quad \min \langle Q, xx^\top \rangle$$

$\text{diag}(X) = x$

$Ax = b$

$$\begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0$$

In problem $(QPR)$ look at the constraint

$$\begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0$$

This constraint requires picking an $x \in \mathbb{R}^n$ such that the matrix

$$\begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix}$$
is positive semidefinite. We have moved from our safe and secure world of nonnegativity to requiring a matrix be positive semidefinite. We will see Chapter 7 that the $\succeq$ is a generalization of $\geq$.

Alas, we are not yet done in our quest for convexity. The objective function remains nonconvex. Not to worry, we create another relaxation

$$(QP RR) \quad \min \langle Q + D, xx^\top \rangle - \text{diag}(D) \top x$$

$$\text{diag}(X) = x$$

$$Ax = b$$

$$\begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0$$

where $D$ is a diagonal matrix with every diagonal element equal to the negative of the minimum eigenvalue of $Q$. This idea is due to Hammer and Rubin [14]. It is left as Exercise 1.1 to show that $f(x) = \langle Q + D, xx^\top \rangle$ is a convex function. It left as Exercise 1.2 to show that $\langle Q + D, xx^\top \rangle - \text{diag}(D) \top x = \langle Q, xx^\top \rangle$ when $x$ is a binary vector. Why is this significant? Finally, in Chapter 7, we show that there are better convex relaxations of $(QP)$ than $(QRR)$. Finding good convex relaxations to nonconvex optimization problems is a worthwhile research task.

In case you are wondering, Michelle Bachmann can do her 99-county tour and travel only 2,739 miles. Barack Obama with seemingly infinite dollars does not have to worry about optimization.

1.5 Notation

We try to be consistent in naming and notation throughout the notes. This is a difficult task, but we make every effort. Part of your job is sorting out our typos!

First, we set some conventions for labeling scalars and vectors. If we are going to give a list of indexed vectors we prefer the indices to be superscripts. For example, a set of four vectors in $\mathbb{R}^n$ we would write as $\{x^1, x^2, x^3, x^4\}$. We use subscripts to index scalars. For instance, a vector $x$ in $\mathbb{R}^n$ can be thought of as a list of $n$-real numbers, which we denote $x = (x_1, \ldots, x_n)$. This helps keep track of vectors versus scalars.

Now some naming conventions of elements in the spaces given in (1.3.13), reproduced here for convenience.

$$P : \begin{bmatrix} X \\ X' \end{bmatrix} \rightarrow \begin{bmatrix} Y \\ Y' \end{bmatrix} : D$$

A variable in $X$ is typically denoted $x$, and a variable in $Y$ is typically denoted $y$. When in need of extra variables we often use $z$ and $w$ and which space it lies in ($X$ or $Y$) will come from the specific context. A parameter or constant in $X$ are typically denoted $c$, and parameters in $Y$ we denote by $b$. Extra parameters include $d$ (denoting a direction),
\[ a, p, \text{ and } q \] are used. Whether they are parameters in \( X \) or \( Y \) should also be clear from the context. We consistently attempt to use latin alphabet letters to denote vectors in the primal spaces.

Linear functionals are typically denoted by Greek letters. For elements of \( X' \) we typically use \( \phi \), whereas in \( Y \) we favor \( \psi \).

Throughout the notes in the course of the proofs you will often see the three letters WTS, which represents the phrase “want to show”. It typically helps focus the reader on where the proof is heading next and what the argument will focus on. We sometimes use the phrase as a noun: “the WTS is property \( P \).”

1.6 Notes

The main source of inspiration for these notes is Geometric Functional Analysis and Its Applications by Richard Holmes. See [16]. This is a book on functional analysis written from the standpoint of someone interested in optimization. It is quite different than most books you will find on functional analysis. Holmes provides key results, for example about separating hyperplanes, without making any assumption about the existence of an underlying topology. More importantly, Holmes makes it crystal clear why certain hypothesis are required in infinite dimensional spaces. We cannot recommend this book highly enough. If you can master the first 100 or so pages of the book you will have an understanding of optimization at a deeper level than most of your Ph.D. student peers at top institutions. Indeed, you may well have a deeper understanding than many of the faculty at those institutions. However, the book is pretty tough to read. Holmes writes well, but his use of the word clearly throughout his proofs does not exactly coincide with how Chris and Kipp would define clearly. Suffice to say, that if you can digest a single page of Holmes in several hours of intense reading and study you have done well. Indeed, a major objective of our notes is to clearly present these results where the word clearly is used in the Chris and Kipp sense. However, these notes are more than a rehash of Holmes, there are new proofs and results that do not appear Holmes or elsewhere. Unfortunately the Holmes book is no longer being published and only used copies are available for purchase. But you better be quick, whenever one becomes available for less than $100, Chris or Kipp snatch it up. Kipp scored most recently at an unbelievable $9.91 (it is actually in pretty good shape, but one might classify its aroma as, ummm – odd?).

Another excellent text is Optimization by Vector Space Methods by David Luenberger. See [20]. This is more accessible than Holmes, a truly great book, but assumes a norm topology for all of the key results. Students wishing to brush up on topology should consult [21]. Two excellent texts with material on measure, integration, sigma algebras and Borel sets are Capiński and Kopp [7] and Rudin [26]. The book on functional analysis by Rudin [27] is also outstanding. If the reader is interested in results for infinite dimensional linear programs the book by Anderson and Nash is excellent. See [3]. For some unfathomable
reason, Anderson and Nash is also no longer published. Sadly, one can only surmise that people would rather watch *Dancing with the Stars* than spend time contemplating infinite dimensional linear programs. Speaking of infinite dimensions, the text *Infinite Dimensional Analysis: a Hitchhiker’s Guide* by Aliprantis and Border is an extremely well written and comprehensive text that provides background material on virtually every topic in these notes. See [2]. For undergraduate level textbooks on analysis, linear algebra, and abstract algebra, see respectively Rudin [25], Horn and Johnson [17], and Herstein [15]. Finally, it is not necessary to acquire any of the these references, as we have made every effort to make these notes self contained.

1.7 Exercises

**Exercise 1.1.** Assume $Q \in S^n$. Prove that

$$f(x) = (Q + D, xx^\top) - \text{diag}(D)^\top x$$

is a convex function if $D$ is equal to the negative of the smallest eigenvalue of $Q$.

**Exercise 1.2.** If $x$ is binary vector show that

$$(Q + D, xx^\top) - \text{diag}(D)^\top x = (Q, xx^\top).$$

**Exercise 1.3.** Problem (*QPRR*) is a continuous relaxation of the original discrete problem (*QR*). In practice, the problem (*QPRR*) would be embedded in a branch and bound algorithm. In a fair world, the optimal value of (*QPRR*) would be a good approximation of (*QP*). However, “fair” is where they display animals in the summer and has absolutely nothing to do with reality. In general, we can expect the continuous relaxation given by the optimal solution of (*QPRR*) to be a poor lower bound on the optimal value of (*QP*).

- Show that the optimal value of (*QPRR*) is a lower bound on the optimal value of (*QP*).
- Explain why you can expect the optimal value of (*QPRR*) to be a poor lower bound on the optimal value of (*QP*).
Chapter 2

Duality and Convexity in Vector Spaces

2.1 Vector spaces

A vector space is the fundamental algebraic construction used in these notes. This is where we optimize, theorize, and situate our applications. So why not start at the beginning and get a sense of the natural habitat? We assume familiarity with the basic notions of finite dimensional vectors spaces, including the ideas of subspaces, linear independence, basis and dimension. What we discuss here applies to general vector spaces; that is, those which may not have a finite basis.

A vector space \( X \) is a set of elements (called vectors) equipped with two operations: 1) vector addition, denoted by “+” that combines two vectors, say \( x^1 \) and \( x^2 \), into a new vector \( x^1 + x^2 \), and 2) scalar multiplication, that combines a “scalar” \( \alpha \) from some underlying field (for us this is always \( \mathbb{R} \)) with a vector \( x \) to return a new vector \( \alpha x \), appropriately “scaled” by \( \alpha \in \mathbb{R} \). Sometimes people adorn vectors with underlining, boldface or arrows to distinguish them from scalars, we are lazy here and ask the reader to keep track. Formally, a nonempty set \( X \) over the field \( \mathbb{R} \) is a vector space if \( X \) is an abelian group under the “+” operation and

1. \( \alpha (x + y) = \alpha x + \alpha y \)
2. \( (\alpha + \beta) x = \alpha x + \beta x \)
3. \( \alpha (\beta x) = (\alpha \beta) x \)
4. \( 1 x = x \)

There is one important element which is inside every vector space, its origin, or zero element, denoted \( \theta \). When we want to specify the origin of a particular vector space \( X \) we write \( \theta_X \).
We distinguish two classes of vector spaces: finite-dimensional vector spaces and infinite dimensional vector spaces. The most familiar example of a finite dimensional vector space is $\mathbb{R}^n$ to which most of our natural and mathematical intuition is attached. We refer to $\mathbb{R}^n$ often throughout the notes to lend familiarity to some of the more exotic ideas we introduce in the study of optimization over abstract vector spaces. The vector space structure of every finite dimensional vector space essentially boils down to that of $\mathbb{R}^n$ (see Proposition 2.2.4).

The second class is infinite-dimensional vector spaces, those which cannot be spanned by a finite set of linearly independent vectors. Within infinite dimensional vector spaces, there are those with a countably infinite dimension (such as $\mathbb{R}^\infty$ as defined in Example 2.2.7) and those with an uncountably infinite dimension (such as $C[0,1]$ defined in Section 1.3.3). Much of the same reasoning we are familiar with applies here, but with subtleties we discover over the course of notes. There are, however, some surprising counter-examples to facts we take for granted in finite dimensions which you will become well acquainted with.

The operations of vector spaces allow us to consider combinations of vectors using vector addition and scalar multiplication. We say $x$ is a linear combination of the vectors in a set $S$ (which is potentially infinite, even uncountably so) if there exists a finite set of vectors $w_1, \ldots, w_k \in S$, and scalars $\alpha_1, \ldots, \alpha_k$, such that $x = \sum_{i=1}^k \alpha_i w_i$. We could also write $x = \sum_{w \in S} \alpha_w w$, but this should make you a bit nervous. Sums of an infinite number of vectors is something we always try to avoid, because such sums may not converge. This is why our theory assumes that only finitely many $\alpha_w \neq 0$ and thus the sum $\sum_{w \in S} \alpha_w w$ amounts to a finite sum.

The span of a set $S$, denoted $\text{span}(S)$, is the set of all linear combinations arising from the set $S$. We say a vector $x$ is linearly independent of the vectors in $S$ if it cannot be expressed as a linear combination of the vectors in $S$. The set $S$ is said to be linearly independent if no vector $x \in S$ is a linear combination of the vectors in $S \setminus \{x\}$. A Hamel basis (or simply a basis) of a vector space $X$ is an independent subset $\mathcal{H} \subseteq X$ of vectors with $\text{span}(\mathcal{H}) = X$.

A key result in the study of vector spaces is the existence of a Hamel basis of every vector space. We assume familiarity with the result in finite dimension and prove the case for general (and hence possibly infinite dimensional) vector spaces. The result appeals to Zorn’s lemma, what one might consider a “necessary evil” when it comes to life in infinite dimensions. See the appendix for a complete statement of Zorn’s lemma. The lemma is useful here since it gives us the notion of a “biggest” thing in a set that seems to go on forever, in this case a basis (a “biggest” independent set which spans the whole space) in an infinite dimensional vector space (which “goes on forever” in the sense that you never run out of new “dimensions” to consider).

**Theorem 2.1.1.** Every nontrivial vector space has a Hamel basis.

---

1 we will define what we mean by the dimension of a vector space below.
Proof. Let $X$ be a nontrivial vector space, that is $X \neq \{\theta\}$. Our goal is to apply Zorn's lemma; hence we need a partially ordered set (poset), then show that all chains in that set are bounded, and finally conclude the existence of maximal elements in that poset set which hopefully mean something to us. Intuitively, a basis is “maximal” in some sense, so the hope is that we can set things up in the right way that when we turn the Zorn crank a basis will pop out.

Enough big picture, let’s get to the details. The poset in the Zorn story is $\chi$, the collection of all linearly independent subsets of $X$ ordered by set inclusion. We first show that maximal elements in $\chi$, if they exist, correspond to Hamel bases. To see this, assume there is a Hamel basis $B$. We know that $B$ is a set of linearly independent vectors and thus in $\chi$. Every basis is maximal among independent sets and thus are clearly maximal elements of $\chi$. Conversely, consider a maximal element $S$ in $\chi$. Clearly $S$ is independent, we will show that span$(S) = X$. Suppose not, then there exists a vector $x \in X$ such that $x \notin$ span$(S)$. It then follows immediately that $S \cup \{x\}$ is an independent set, which violates the maximality of $S$.

So maximal elements are exactly what we want them to be – Hamel bases – now it is a matter of showing that they exist. By way of Zorn’s lemma, consider an arbitrary chain $C$ in $\chi$. We claim that $C$ is bounded in $\chi$. Indeed, consider the set $M = \bigcup_{C \in C} C$. It is easy to see that every element in the chain is contained in $M$. Thus, $M$ would be an upper bound if we can show that $M \in \chi$; that is, that $M$ is an independent set. Suppose that this is not the case, then we would have an element of $M$, say $x$, which is a linear combination of a finite number of elements $x^1, \ldots, x^n$ in $M$. Since $x \in M$, it lies in some set $C_0$ in the chain $C$. Similarly, each of the $x^i$ lie in some independent set $C_i$ in the chain. Since $C$ is a chain, there exists a largest element (in the containment ordering) $\bar{C}$ from among the sets $C_0, C_1, \ldots, C_n$. Hence, since we are in a chain, each $C_i \subseteq \bar{C}$ and thus $x, x^1, \ldots, x^n$ are all elements of the independent set $\bar{C}$. This is a contradiction, since we assumed that $x$ was a linear combination of the $x^i$ which would mean that $\bar{C}$ is not an independent set of vectors.

Thus we may apply Zorn’s and establish the existence of a maximal element and we have shown that maximal elements correspond to Hamel bases. 

Let $X$ be a vector space, the previous theorem guarantees it has a Hamel basis $\mathcal{H}$. It is certainly possible for a vector space to have more than one Hamel basis. Consider even $\mathbb{R}^2$. The set $\{(1, 0), (0, 1)\}$ forms a basis of $\mathbb{R}^2$ and so does $\{(1, 1), (1, 0)\}$. However, one thing in common to all bases is that they have the same cardinality. This is true of general vector spaces, and is a deep result (also using Zorn’s lemma) which we do not cover in these notes (for a proof sketch see [2] page 195.) Thus, we can define the dimension of a vector space as the cardinality of one its Hamel bases. The dimension is well-defined since all bases have the same cardinality.

The following result is used throughout the notes and is true in both finite and infinite dimensions:
**Proposition 2.1.2.** Let $H$ be a Hamel basis of vector space $X$. Then every vector $x \in X$ can be written uniquely as a linear combination of vectors arising from $H$.

**Proof.** See Exercise 2.1. □

We also state another useful result, which is well known in finite dimensions, and also can be extended to the infinite dimensional case. First two definitions:

**Definition.** If $X$ is a vector space, and $M \subset X$ is also a vector space under the vector space operations of $X$, then $M$ is a subspace of $X$.

**Definition.** Let $M$ be a subspace of a vector space $X$. We call a subspace $N$ a complementary subspace of $M$ in $X$ if:

$$X = M \oplus N$$

where $\oplus$ denotes a direct sum of vector spaces, meaning that every vector in $x$ can be written uniquely as the sum of a vector in $M$ and a vector in $N$.

**Example 2.1.3.** Consider the vector space $\mathbb{R}^3$ under the usual vector addition and scalar multiplication. Define the subspace $M \subseteq \mathbb{R}^3$ to be the vector space spanned by the vector $(1, 2, 0)$. We show how to construct a complementary subspace $N$. Initialize a set of vectors $B$ by $B = \{(1, 2, 0)\}$. Next, start adding vectors to the set $B$ with the requirement that if we add a vector to $B$, the resulting set of vectors in the updated $B$ must be linearly independent. Keep adding vectors until we span the space $\mathbb{R}^3$. Which vectors should we add? Well the standard basis vectors for $\mathbb{R}^3$ are good candidates. Consider $(1, 0, 0)$. If we add $(1, 0, 0)$ to $B$, then the two vectors in $B$ will be linearly independent so add $(1, 0, 0)$ and the updated $B$ is

$$B = \{(1, 2, 0), (1, 0, 0)\}$$

Next, consider $(0, 1, 0)$. If we were to add $(0, 1, 0)$ to $B$ then the resulting three vectors would be linearly dependent. Do not add $(0, 1, 0)$. What about $(0, 0, 1)$? This is fine since if we add $(0, 0, 1)$ to $B$ we have a set of linearly independent vectors. Thus,

$$B = \{(1, 2, 0), (1, 0, 0), (0, 0, 1)\}$$

and $B$ is a set of three linearly independent vectors in $\mathbb{R}^3$. Therefore $B$ spans $\mathbb{R}^3$. Now define $N$ to be the subspace spanned by the vectors $(1, 0, 0)$ and $(0, 0, 1)$. Notice that any vector in $x \in \mathbb{R}^3$ can be uniquely written as $y + z$ where $y \in M$ and $z \in N$. However, it is not the case that given $M$, the complementary space $N$, is unique. For example,

$$N = \{(0, 1, 0), (0, 0, 1)\}$$

is also a complementary space to $M$. 
In Example 2.1.3 our procedure for generating the complementary subspace \( N \) relied on the fact that we had a finite dimensional vector space. However, this result still holds for infinite dimensional vector spaces.

**Proposition 2.1.4.** Let \( M \) be a subspace of vector space \( X \). Then \( M \) has a complementary subspace.

**Proof.** See Exercise 2.4.

Vector spaces, without any additional structure, are in some ways too sparse for a complete theory of optimization. In particular, at some point it would be nice to consider notions of openness and closedness to discuss interiors of sets and continuity of functions. In addition, some optimization algorithms produce a sequence of iterates that approach optimal solutions requiring some notion of convergence, and thus at least a metric to describe distances.

However, a main theme in these notes is that the core of the underlying theory can be developed in a fully abstract setting with no topologies or norms assumed. We call this approach, in the tradition of the field, *algebraic*. This name underscores that only linear algebra, with no additional notions from analysis, is needed to drive key results in optimization, particularly duality theory.

### 2.2 Algebraic duality

The words “dual” and “duality” are thrown around often in optimization and related fields. Simply do a search of any modern paper that uses optimization and the word is sure to show up at least a couple times. The word will come up a few times in a few different contexts. The most common usage in optimization is in defining a related problem to \((1.2.1)\) called its *dual*. Understanding the interplay between an optimization problem and its dual is of critical importance in optimization theory and applications. The concept should be familiar from our knowledge of linear programming, where duality played a key role in gaining a deep understanding of the nature of linear programming, and in devising algorithms to solve large scale instances. The same is true in this more abstract setting.

However, we begin with a slightly different meaning of the word “dual”, here using it define the dual of a vector space. In fact, there is more than one dual of a vector space, but we begin with the simplest notion, the algebraic dual. The algebraic dual is based on the concept of a linear functional.

**Definition.** Let \( X \) be a vector space. A **linear functional** on \( X \) is a function \( \phi : X \to \mathbb{R} \) with the following properties: for all \( x, y \in X \) and \( \alpha \in \mathbb{R} \)

\[
\begin{align*}
(L1) \quad & \phi(x + y) = \phi(x) + \phi(y), \\
(L2) \quad & \phi(\alpha x) = \alpha \phi(x).
\end{align*}
\]
In other words, \( \phi \) is a mapping that preserves the operations of the vector space. Indeed, a linear functional is a homomorphism from the vector space \( X \) to the vector space \( \mathbb{R} \). See the outstanding text [15] by University of Chicago algebraist Israel Herstein (who’s advisor was our good friend Zorn) for more on homomorphisms and vector spaces.

**Remark 2.2.1.** In the above definition, we introduced the term linear functional. In these notes we work with vector spaces over the reals. We will use the term functional for a map (possibly nonliner) from a vector space into its underlying scalar field, i.e. reals. This is standard terminology.

**Example 2.2.2.** Let’s start with tried and true \( \mathbb{R}^n \). We show that linear functionals of \( \mathbb{R}^n \) are precisely of the form

\[
\phi_c(x) = \sum_{i=1}^{n} c_i x_i
\]

(2.2.1)

where \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). To emphasize the role of \( c \) in the linear functional in (2.2.1) we use the notation \( \phi_c \).

It is straightforward to check that \( \phi_c \) satisfies (L1) and (L2) and thus \( \phi_c \) is a linear functional. We prove that any linear functional on \( \mathbb{R}^n \) can be expressed in the form \( \phi_c \) for some \( c \in \mathbb{R}^n \). Let \( \phi \) be an arbitrary linear functional on \( X \), that is not, a priori, of the form (2.2.1). Consider the standard basis vectors \( \{e^1, \ldots, e^n\} \) of \( \mathbb{R}^n \) where \( e^i_j = 1 \) for \( i = j \) and 0 otherwise. An arbitrary \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) can be expressed as a sum \( x = \sum_{i=1}^{n} x_i e^i \) of the basis vectors. Using (L1) and (L2) we have

\[
\phi(x) = \sum_{i=1}^{n} x_i \phi(e^i).
\]

Taking \( c_i = \phi(e^i) \) shows that \( \phi \) can be written in form (2.2.1) with \( c = (\phi(e^1), \ldots, \phi(e^n)) \).

**Definition.** Let \( X \) be a vector space. The set of all linear functionals on \( X \) is called the algebraic dual of \( X \) and is denoted \( X' \).

It is straightforward to show that \( X' \) is also a vector space (see Exercise 2.2). Its “vectors” are functionals on \( X \).

**Example 2.2.3.** We just established that in \( \mathbb{R}^n \), all linear functionals had the form \( \phi_c(x) = \sum_{i=1}^{n} c_i x_i \). Thus, we have

\[
(\mathbb{R}^n)' = \{ \phi_c : c \in \mathbb{R}^n \}.
\]

It is clear that every element of \( \mathbb{R}^n \) corresponds to an element in \( (\mathbb{R}^n)' \), since any \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \) defines the map \( \phi_c \). Thus, we see \( (\mathbb{R}^n)' \) is isomorphic to \( \mathbb{R}^n \). Recall that an isomorphism is a homomorphism that is one-to-on and onto. Two vector spaces are isomorphic if there is an isomorphism from one onto the other. See Herstein [15].
2.2. ALGEBRAIC DUALITY

Inspired by this example, when \( X \) is isomorphic to its algebraic dual, we say \( X \) is self-dual. If \( X \) and \( Y \) are isomorphic vector spaces then we write \( X \cong Y \). In this notation, a vector space is self-dual if \( X \cong X' \). From the above see that \( \mathbb{R}^n \) is self-dual. We now carefully show that every finite dimensional vector space (including \( \mathbb{R}^n \)) is self-dual. The result requires several steps. First, we show that every \( n \)-dimensional vector space is isomorphic to \( \mathbb{R}^n \).

**Proposition 2.2.4.** If \( X \) is a vector space over \( \mathbb{R} \) with finite dimension \( n \), then \( X \) is isomorphic to \( \mathbb{R}^n \).

**Proof.** See Exercise 2.3. \( \square \)

**Proposition 2.2.5.** Every finite dimensional vector space is self-dual.

**Proof.** Let \( X \) be an \( n \)-dimensional vector space. Our approach is to show that \( X' \) is also an \( n \)-dimensional vector space. Then we apply Proposition 2.2.4 to conclude that it is also isomorphic to \( \mathbb{R}^n \), implying immediately that \( X \cong X' \).

Since \( X \) has finite dimension \( n \), by Proposition 2.2.4 it is isomorphic to \( \mathbb{R}^n \), and we can take the basis of \( X \) to be \( e^i, i = 1, \ldots, n \). We construct a finite basis for \( X' \). For each \( e^i, i = 1, \ldots, n \) define a linear functional \( \phi_{e^i} \) in \( X' \) by

\[
\phi_{e^i}(x) = x_i, \quad \forall x \in X
\]

The \( \phi_{e^i} \) give a set \( n \) linearly independent vectors in \( X' \). **WTS:** show that the linear functionals \( \phi_{e^i}, i = 1, \ldots, n \), span \( X' \). Let \( \phi \) be an arbitrary element of \( X' \). Construct a linear functional in \( X' \) by

\[
\phi' = \sum_{i=1}^{n} \phi(e^i)\phi_{e^i}
\]

Show \( \phi' = \phi \). By definition \( \phi_{e^i}(x) = x_i \) so,

\[
\phi(x) = \phi \left( \sum_{i=1}^{n} x_i e^i \right) = \sum_{i=1}^{n} x_i \phi(e^i) = \sum_{i=1}^{n} \phi(e^i)x_i = \sum_{i=1}^{n} \phi(e^i)\phi_{e^i}(x) = \phi'(x)
\]

for all \( x \in X \) which implies

\[
\phi'(x) - \phi(x) = (\phi' - \phi)(x) = 0, \quad \forall x \in X
\]

which implies \( \phi = \phi' \) and therefore the \( \phi_{e^i}, i = 1, \ldots, n \), span \( X' \). \( \square \)

**Remark 2.2.6.** Now for a brief interlude on notation. It is common to express \( \phi(x) \) in the symmetric notation \( \langle x, \phi \rangle \) we call “angle brace” notation. If we want to emphasize the underlying vector space, in this case \( X \), we will sometime subscript the braces as \( \langle x, \phi \rangle_X \).
This is convenient for many reasons that will become apparent over time, particularly in Chapters 4 and 6. This notation has the potential, however, to introduce some confusion. You have probably seen the sum $\sum_{i=1}^{n} c_i x_i$ where $c$ and $x$ are vectors in $\mathbb{R}^n$ sometimes written as $\langle c, x \rangle$ as opposed to our favored notation $c^\top x$ (which we espouse in Appendix A.2). Note that $c$ and $x$ are both elements of $\mathbb{R}^n$ and thus it does not fit the angle brace notation as used for linear functionals. Of course, in light of Proposition 2.2.5 the space of linear functionals over $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$. Thus, although to be 100% kosher, we should write $\langle x, \phi c \rangle = \sum_{i=1}^{n} c_i x_i = c^\top x$ (where $\phi c$ is as defined in (2.2.1)), we sometimes write $\langle x, c \rangle = \sum_{i=1}^{n} c_i x_i$, implicitly evoking the isomorphism which maps $c$ to $\phi c$.

We promise you will get used to our laziness and lack of clarity as times wears on when we start mixing up notation. If not, ask us to clarify!

Let’s take on our first example of an infinite dimensional vector space and possibly our first surprise, a vector space which is not self-dual.

**Example 2.2.7.** Consider the space of real sequences with a finite number of nonzero terms, which we denote $\mathbb{R}^\infty$. More precisely, $$\mathbb{R}^\infty = \{ (x_1, x_2, \ldots) : \text{ each } x_i \in \mathbb{R}, \text{ and finitely many } x_i \neq 0 \}.$$ One can think of the elements as “infinitely” long vectors, but with only a finite number of elements which are nonzero. Or, alternatively, each element of $\mathbb{R}^\infty$ corresponds to a function whose domain is the set of positive integers, and the function can evaluate to a nonzero number for only a finite number of points in its domain. For instance, the sequence $(1, 1, \ldots)$ of all ones is not in $\mathbb{R}^\infty$ since an infinite number of its entries (in fact all of them) are nonzero.

We turn to finding a Hamel basis of $\mathbb{R}^\infty$. Let $e^i$ denote the sequence whose $i$th term in the sequence is 1 and the rest are 0. We claim that $\mathcal{H} = \{ e^1, e^2, \ldots \}$ forms a Hamel basis. It is clear that each of these vectors is independent from the others, since no two vectors have nonzero entries in any common term. It is also clear that these vectors span the space. The fact the sequences in $\mathbb{R}^\infty$ are finitely nonzero is critical in establishing this result, since in the span of the vectors in $\mathcal{H}$ we are only allowed to use only finitely many vectors from $\mathcal{H}$ at a time. In particular, $(1, 1, \ldots)$ is not in $\mathbb{R}^\infty$ since an infinite number of its entries (in fact all of them) are nonzero.

What do linear functionals look like on $\mathbb{R}^\infty$? As above, what matters is how they are defined on the basis elements $e^i$. We can choose a real sequence $c = (c_1, c_2, \ldots)$ and define the following linear functional: $$\phi_c(x) = \sum_{i=1}^{\infty} c_i x_i \quad (2.2.2)$$ where $x$ is written in terms of the Hamel basis as $x = \sum_{i=1}^{\infty} x_i e^i$. The sum in (2.2.2) is well-defined since only finitely many of the $x_i$ are nonzero, and so the seemingly infinite sum (scary) is a finite sum of nonzero terms (nice). Note also that the sequence $c = (c_1, c_2, \ldots)$
need not lie in $\mathbb{R}^\infty$, in fact any real sequence can be used for form a linear functional of the form (2.2.2) over $\mathbb{R}^\infty$. We denote the set of arbitrary real sequences by $\mathbb{R}^N$. Thus every element $c \in \mathbb{R}^N$ gives rise to a linear functional on $\mathbb{R}^\infty$. Just as in Example 2.2.2 the opposite is also true, every linear functional over $\mathbb{R}^\infty$ can be expressed in the form (2.2.2) and is thus an element of $\mathbb{R}^N$. In other words, we have shown is that $(\mathbb{R}^\infty)' = \mathbb{R}^N$. In particular, $\mathbb{R}^\infty$ is not self-dual.

The idea in this last example and Example 2.2.2 inspires one to give the following representation of linear functionals. Let $\mathcal{H}$ be a Hamel basis of the space $X$. Since any $x \in X$ can be written as $x = \sum_{h \in \mathcal{H}} x_h h$, if $f$ is any mapping from $\mathcal{H}$ into the reals, then a corresponding $\phi_f$ in $X'$ is defined through the mapping

$$\langle x, \phi_f \rangle = \sum_{h \in \mathcal{H}} x_h \langle h, \phi_f \rangle = \sum_{h \in \mathcal{H}} x_h f(h)$$

This observation, and our experience with $\mathbb{R}^n$ and gives rise to the following proposition.

**Proposition 2.2.8.** If $\mathcal{H}$ is a Hamel basis of $X$, then $X' \cong \mathbb{R}^\mathcal{H}$.

This is a canonical representation of linear functionals, mappings of the Hamel basis vectors to the reals. We shall see in later chapters that under more restricted conditions we can represent linear functionals much more compactly.

### 2.3 Hyperplanes

Having introduced linear functionals we are led to consider one the workhorses of convex analysis: separating hyperplanes. The main idea behind this theory is simple, but like most simple and powerful ideas, getting the details right leads to challenges and additional insights.

We start by thinking about hyperplanes. Most of us have had some exposure to hyperplanes. If you have a piece of paper and are asked to draw a hyperplane you would likely produce something like Figure 2.1; that is, a “line” (not necessarily through the origin) in $\mathbb{R}^2$. How does some of this intuition carry over to general vector spaces? We start by generalizing the idea of a “line”:

**Definition** (Affine subspace). The translation of a subspace in a vector space $X$ is an **affine subspace**.

**Remark 2.3.1.** We call a regular old subspace of $X$ a **linear subspace** to distinguish it from an affine subspace.

---

\[\text{We use the following notation throughout these notes: } A^B \text{ consists of all functions with domain } B \text{ and codomain (range) } A; \text{ that is, maps from } B \text{ to } A. \text{ In particular, the notation } \mathbb{R}^N \text{ denotes maps from the the integers } \mathbb{N} \text{ into the the reals } \mathbb{R}, \text{ which is precisely what a sequence is.}\]
CHAPTER 2. DUALITY AND CONVEXITY IN VECTOR SPACES

An affine subspace $A$ can be written as $A = x^0 + M$ where $M$ is a subspace. In this representation $M$ is unique but any vector in $A$ can serve as $x^0$ (see Exercise 2.8). This is illustrated in Figure 2.2.

When we are in an affine subspace of a vector space, it is useful to talk about vectors that “span” the affine space. However, the usual vector addition is not closed in affine subspaces. Consider a linear subspace $M$. Then we know that $M$ is closed under linear combinations. This need not be the case for an affine subspace. Consider adding two vectors $x$ and $y$ from the affine subspace $A = x^0 + M$ where $x^0 \neq 0$. Notice that $x + y \notin A$, indeed it lies in the parallel affine subspace $2x^0 + M$. However, $A$ is closed under “affine” combinations, which we define now.

**Definition.** An affine combination $x$ of a set of vectors $S$ is a linear combination where the coefficients sum to 1; that is,

$$x = \sum_{w \in S} \alpha_w w, \quad \text{where} \quad \sum_{w \in S} \alpha_w = 1.$$ 

Recall, that in a linear combination, and thus an affine combination, only finitely many of the $\alpha_w$ are nonnegative.

Every set $A$ in a vector space has an associated affine subspace called its affine hull, denoted $\text{aff}(A)$, which is the smallest affine subspace that contains $A$. The concept of affine hull plays a key role in the theory to come. There are two alternate definitions of affine hull which turn out to be useful, one related to the idea of affine combinations, which we capture in the following proposition. First, a useful definition:
Definition (Minkowski sum). Let $A$ and $B$ be two sets in a vector space $X$. The Minkowski sum $A + B$ is the algebraic sum of its elements; that is,

$$A + B = \{a + b : a \in A, b \in B\}.$$ 

More generally, we can take linear combinations of sets in a vector space as follows: for $A, B \subseteq X$ and $\alpha, \beta \in \mathbb{R}$:

$$\alpha A + \beta B = \{\alpha a + \beta b : a \in A, b \in B\}.$$ 

Proposition 2.3.2. Let $A$ be a set in a vector space $X$. The following are two alternate characterizations of the affine hull of $A$:

1. $\text{aff}(A)$ is the set of all affine combinations of the elements of $A$

$$\text{aff}(A) = \left\{ \sum_{x \in A} \alpha_x x : \sum_x \alpha_x = 1, \text{ finitely many } \alpha_x \neq 0 \right\}.$$

2. $\text{aff}(A) = x + \text{span}(A - A)$ for any fixed $x \in A$, where $A - A = \{x - y : x, y \in A\}$ is the Minkowski difference of $A$ with itself.
CHAPTER 2. DUALITY AND CONVEXITY IN VECTOR SPACES

Proof. See Exercise 2.9

We now observe that an affine subspace is closed under affine combinations. Let \( x, y \in A = x^0 + M \) where \( A \) is an affine subspace, that is \( x = x^0 + m^1 \) and \( y = x^0 + m^2 \) for some \( m^1, m^2 \in M \). Consider the affine combination of \( x \) and \( y \):

\[
\alpha x + (1 - \alpha)y = \alpha(x^0 + m^1) + (1 - \alpha)(x^0 + m^2) = x^0 + (\alpha m^1 + (1 - \alpha)m^2) \in A.
\]

It is straightforward then to see that if \( A \) is an affine subspace it follows that \( A = \text{aff}(A) \).

We are particularly interested in a special class of affine subspaces: hyperplanes.

**Definition** (Hyperplane). A hyperplane \( H \) in vector space \( X \) is a maximal proper affine subspace of \( X \). That is, \( H \) is an affine subspace such that \( H \neq X \) and if \( A \) is an affine subspace containing \( H \), then either \( A = X \) or \( A = H \).

This “maximal” notion can sometimes be a bit slippery, so let’s anchor it to a more familiar and constructive notion: dimension. The dimension of an affine subspace \( A \) is the dimension of its associated linear subspace \( M \). Recall that the dimension of a linear subspace corresponds to the maximum number of linearly independent vectors which span the set. There is a related notion of affine independence: a set of vectors \( \{x^0, x^1, \ldots, x^n\} \) are affinely independent if no one of the vectors can be written as an affine combination of the others. Naturally, the concepts of linear and affine independence are related:

**Proposition 2.3.3.** A set of vectors \( \{x^0, x^1, \ldots, x^n\} \) are affinely independent if and only if the set of vectors \( \{x^1 - x^0, \ldots, x^n - x^0\} \) are linearly independent.

**Proof.** See Exercise 2.10.

A further exercise is to use this result to conclude that the affine hull of an affinely independent set \( \{x^0, x^1, \ldots, x^n\} \) with \( n + 1 \) elements has dimension \( n \).

In finite dimensional spaces we can directly use dimension to generalize our intuitive notion in Figure 2.1 of a hyperplane as a “line” (a one-dimensional affine subspace) in \( \mathbb{R}^2 \) (a two-dimensional vector space):

**Proposition 2.3.4 (Hyperplane in a finite dimensional vector space).** Let \( X \) be a finite dimensional vector space of dimension \( n \) and let \( A \) be an affine subspace of \( X \). Then, \( A \) is a hyperplane if and only if \( A \) has dimension \( n - 1 \).

**Proof.** (\( \Rightarrow \)) Write the hyperplane \( A = x^0 + M \) where \( x^0 \in X \) and \( M \) a subspace of \( X \). We need to show \( M \) has dimension \( n - 1 \). Since \( A \) is a proper subset of \( X \), \( \dim(M) < n \). Suppose \( \dim(M) \leq n - 2 \). There exists a vector \( x \) in \( X \) not in \( M \) and not equal to \( x^0 \). Then the subspace \( M' = \text{span}(M \cup x) \) has dimension \( n - 1 \) or less, and certainly \( M \subset M' \). This contradicts the fact \( A \) is maximal since \( A \subset x^0 + M' \subset X \). We conclude \( \dim(M) = n - 1 \).
Write $A = x^0 + M$ where $M$ is a subspace of $X$ of dimension $n - 1$. Let $A'$ be an affine subspace containing $A$. If $A' = A$ we are done, so we may assume $A'$ strictly contains $A$. Since $A'$ is an affine subspace it can be written as $A' = y^0 + P$. We must show that $\dim(P) = n$, and hence $A' = X$. Our approach is as follows. First, show that $M \subseteq P$ to establish $\dim(P) \geq n - 1$. Second, argue that since $A'$ contains a point outside of $A$ this requires $\dim(P) > \dim(M)$. From this we can conclude that $\dim(P) = n$.

To place $M$ inside of $P$ things are greatly simplified if we take $y^0 = x^0$. This is valid by Exercise 2.8 since $x^0 \in A'$. Let $m_1, \ldots, m_{n-1}$ be a basis of $M$, then $x^0 + m_1, \ldots x^0 + m_{n-1}$ lie in $A'$. This in turn implies $m_1, \ldots, m_{n-1}$ lie in $P$ and hence $M \subseteq P$ and $\dim(P) \geq n - 1$.

Let $x \in A' \setminus A$. Since $x \in A'$, we have $x = x^0 + p$ for some $p \in P$. However, since $x \notin A$ there does not exist an $m \in M$ such that $x = x^0 + m$. We conclude that $p \notin M$, which implies $\dim(P) > \dim(M)$. Since $\dim(M) = n - 1$, $\dim(P) = n$ and $A' = X$ which implies $A$ is a hyperplane.

In infinite dimensional vector spaces, the world is less kind. Can we really speak about a set with dimension one less than the space by saying it has dimension “$\infty - 1$”? Isn’t “$\infty - 1$” just $\infty$? To better understand the dimension of hyperplanes in infinite dimensions we need another notion: codimension. We start by defining the quotient space $X/M$ of a subspace $M$ in vector space $X$. The elements of $X/M$ are the affine subspaces “parallel” to $M$; that is,

$X/M = \{ x + M | x \in X \}$

The cool fact is that $X/M$ is itself a real vector space with operations inherited from $X$. The operations work as follows:

$$(x + M) + (y + M) = (x + y) + M$$

for all $x, y \in X$; and

$$\alpha(x + M) = (\alpha x) + M.$$ 

for $\alpha \in \mathbb{R}, x \in X$. We leave it to the reader (Exercise 2.11) to verify that $X/M$ is indeed a vector space under these operations.

The dimension of $X/M$ (which is well-defined since it is a vector space) is called the codimension of $M$, denoted $\text{codim}(M)$. The codimension of an affine subspace $A = x^0 + M$ is the codimension of $M$. In an $n$-dimensional vector spaces we have the useful relation:

$$\dim(M) + \text{codim}(M) = n \quad (2.3.1)$$

which you are asked to verify as Exercise 2.12.

The equivalent statement for infinite dimensional spaces, however, is less useful. Saying that $\dim(M)$ and $\text{codim}(M)$ sum to infinity means that at least one is infinity, but maybe both are! Nonetheless, we can make use of codimension when it comes to characterizing hyperplanes in abstract spaces.
CHAPTER 2. DUALITY AND CONVEXITY IN VECTOR SPACES

Proposition 2.3.5 (Hyperplanes have codimension one). Let $X$ be a vector space and let $A$ be an affine subspace of $X$. Then, $A$ is a hyperplane if and only if $A$ has codimension one.

Proof. Let $A = x^0 + M$ where $M$ is a linear subspace of $X$.

($\Leftarrow$) (By contrapositive) Show if $A$ is not a hyperplane, then the codimension of $A$ is not one. If $A$ is an affine subspace that is not a hyperplane then there exists a proper linear subspace $M'$ which strictly contains $M$. Take $x \in M' \setminus M$ and $y \in X \setminus M'$ (we know that this $y$ exists since $M'$ is a proper linear subspace). See Figure 2.3 for geometric intuition.

We claim that $x + M$ and $y + M$ are linearly independent in $X/M$ and hence the dimension of $X/M$ is greater than one. Suppose they were linearly dependent. If $x + M$ and $y + M$ are linearly dependent, then one is a scalar multiple of the other. That is, $x + M = \alpha(y + M) = \alpha y + M$ for some $\alpha \neq 0$. However, $x + M \subseteq M'$ since $x \in M'$ and $M \subseteq M'$. However, $\alpha y \notin M'$ (otherwise $y \in M'$ since $M'$ is a subspace) and this implies $\alpha y + \theta \notin M'$ which in turn means $\alpha y + M \not\subseteq M'$ since $\theta \in M$. Thus $x + M$ and $\alpha y + M$ cannot be equal as sets, since one is contained entirely in $M'$ whereas the other is not. Therefore we must have that $x + M$ and $y + M$ are linearly independent in $X/M$ and therefore the dimension of $X/M$ is greater than two.

($\Rightarrow$) (By contrapositive) Suppose $A$ has codimension 2; that is, it has at least two linearly independent elements $x^1 + M$ and $x^2 + M$ where $x^1, x^2 \notin M$. Since they are linearly independent it means, for instance, that $x^2 \notin \alpha x^1 + M$ for any $\alpha$ and hence $x^2 \notin \text{span}(M \cup \{x^1\})$. This implies that $\text{span}(M \cup \{x^1\})$ is a proper linear subspace of $X$ which strictly contains $M$. This implies that $M$ is not maximal and thus $A$ is not a hyperplane.

Remark 2.3.6. Note that Proposition 2.3.4 is an immediate corollary of this result in view of (2.3.1). An alternate proof was included to underscore some of the differences (and commonalities) between reasoning in finite and infinite dimensional spaces.

The other piece of intuition we live and breathe when we think about hyperplanes in $\mathbb{R}^n$ is that they can be formed as sets of the form

$$\{x \in \mathbb{R}^n : a^\top x = \alpha\}$$

for some $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. It is common to call the vector $a$ the “normal” vector of the hyperplane and $\alpha$ the “right-hand side”. The normal vector gives us some notion of what “direction” the hyperplane is facing whereas, $\alpha$ tells us something about its position. We know that $a$ and $\alpha$ are not unique: for instance, the exact same hyperplane as above can be written as: $H = \{x \in \mathbb{R}^n : (2a)^\top x = 2\alpha\}$. However, if $\alpha \neq 0$ (that is, $H$ does not contain the origin) we can normalize the right-hand side to one, and thus a unique normal vector $a'$ defines $H$. 

2.3. HYPERPLANES

How can we bootstrap this intuition into abstract vector spaces? First of all, note that $a^\top x$ is a linear functional in $\mathbb{R}^n$. This inspires a characterization of hyperplanes in a general vector space. Before proceeding we need a definition that is used in the proof of Proposition 2.3.7.

**Definition.** Let $M$ be a subspace of $X$. The annihilator of $M$ denoted by $M^\circ$ is the set of all functionals $\phi \in X'$ such that $\langle x, \phi \rangle = 0$ for all $x \in M$. In other words, the annihilator of $M$ is that subset of linear functionals on $X$ which “annihilate” every element of $M$, in the sense of evaluating to zero at those points.

**Proposition 2.3.7** (Hyperplanes and linear functionals). Let $X$ be a vector space and let $H$ be an affine subspace of $X$. Then, $H$ is a hyperplane if and only if $H = \{x \in X : \phi(x) = \alpha\}$ for some nonzero element $\phi$ of the algebraic dual space $X'$ and scalar $\alpha$.

**Proof.** ($\Rightarrow$) Let $H$ be a hyperplane in $X$. Then $H = x^0 + M$ for some linear subspace $M$. There are two cases to consider: $x^0 \notin M$ and $x^0 \in M$. In the first case, since $H$ is maximal we have $\text{span}(M \cup \{x^0\}) = X$. In other words, for every vector $x \in X$ we can express it as $x = \alpha x^0 + m$ for some $\alpha \in \mathbb{R}$ and $m \in M$. Define a linear functional $\phi : X \to \mathbb{R}$ by $\phi(x) = \alpha$ when $x$ is expressed as in the previous sentence. It is straightforward to see that $\phi$ is a linear functional. From the definition of $\phi$, $H = \{x \in X : \phi(x) = 1\}$, and we are done.
In the second case, \( x^0 \in M \). Since \( M \) is a subspace, and therefore closed under vector addition, this implies \( H = M \) and the hyperplane is a subspace. In this case we can take \( \phi \) to be any nonzero element in \( M^\circ \). We leave it as Exercise 2.14 to show that \( M^\circ \) has nonzero elements. Then \( M = H = \{ x \in X \mid \phi(x) = 0 \} \).

(\( \Leftarrow \)) Let \( \phi \) be a nonzero linear functional and let \( M = \{ x \in X : \phi(x) = 0 \} \). It is straightforward to check that \( M \) is a linear subspace. It remains to show that \( M \) is maximal. Since \( \phi \) is nonzero we may assume that there exists an \( x^0 \) with \( \phi(x^0) = 1 \). The reasoning is simple, we know that there exists an \( x \) with \( \phi(x) = \alpha \neq 0 \). Hence by linearity \( \phi(\frac{1}{\alpha} x) = 1 \). Now consider an arbitrary \( x \in X \), and note that

\[
\phi(x - \phi(x)x^0) = \phi(x) - \phi(x)\phi(x^0) = \phi(x) - \phi(x) \cdot 1 = 0
\]

and thus we can conclude that \( x - \phi(x)x^0 \in M \). In particular, this means every element of \( X \) is in \( \text{span}(M \cup \{x^0\}) \) and thus we conclude \( M \) is a maximal proper subspace. Note that, for any \( \alpha \) there exists an \( x^1 \) such that \( \phi(x^1) = \alpha \). Then \( \{ x \in X : \phi(x) = \alpha \} = x^1 + M \) and is thus a hyperplane.

\[\ pregnancies \] 

Remark 2.3.8. We often use Proposition 2.3.7 implicitly to define a hyperplane in terms of the \( \phi \) and \( \alpha \) given in statements of results and in the course of proofs. In fact, it is so common that use the following shorthand: we write \( H = [\phi, \alpha] \) when \( H = \{ x \in X : \phi(x) = \alpha \} \).

Note that the first part of the proof guarantees that given a hyperplane \( H \) which does not contain the origin we may write \( H = \{ x \in X : \phi(x) = 1 \} \) for some \( \phi \in X' \), in other words we may assume that the right-hand side is 1. Note that this last proposition gives us a powerful interpretation of the algebraic dual \( X' \) of \( X \); that is, it corresponds to the set of hyperplanes in \( X \). This represents a key theme of “duality”, the interchange of points with lines, variables for constraints, or vectors with hyperplanes.

We now have three ways to think about a hyperplane, which are canonized in the following theorem:

**Theorem 2.3.9.** Let \( H \) be an affine subspace of \( A \). Then the following are equivalent (each saying that \( H \) is a hyperplane):

1. \( H \) is a maximal affine subspace of \( A \);
2. \( H \) has codimension 1;
3. and, \( H = \{ x \in X : \phi(x) = \alpha \} \) for some nonzero linear functional \( \phi \) on \( X \) and scalar \( \alpha \in \mathbb{R} \).
2.4 Convexity and algebraic topological notions

2.4.1 Convexity

The next key idea is convexity. Let $x, y \in X$, then the set $[x, y] = \{ \alpha x + (1 - \alpha) y : 0 \leq \alpha \leq 1 \}$ is called the line segment joining $x$ and $y$. The set $A$ is convex if for every $x, y \in A$, $[x, y] \subseteq A$. A singleton is trivially a convex set. Convex sets play a pivotal role in optimization theory, and despite their simple definition they exhibit many powerful analytical features. One reason for the power of convexity, is that convex sets “fit” human intuition, since everything boils down to understanding what is happening along line segments between points. Line segments between points are “one-dimensional”, something humans understand pretty well. In this section we develop a fairly sophisticated theory of convex sets and their properties, but all definitions and results rely on the simple notion of line segments. This is why in many proofs in this section we are able to give “line drawings” which capture the essence of the argument.

Let’s build some intuition for convexity in $\mathbb{R}^2$, everybody’s favorite mathematical playground. Figures 2.4 and 2.5 are some example doodles of convex and nonconvex sets in $\mathbb{R}^2$.

![Figure 2.4: A convex set in $\mathbb{R}^2$.](image)

![Figure 2.5: A nonconvex set in $\mathbb{R}^2$. The line segment between $x$ and $y$ is not contained in the set.](image)

One way to construct a convex set, is to “convexify” a set of points $A$. This is done by taking convex combinations of the elements in the set. A convex combination of a set of points $S$ is a special kind of affine combination of those points, where the scalars are all nonnegative. That is, $x$ is a convex combination of vectors in $S$ if $x = \sum_{w \in S} \alpha_w w$ where: (i) $\sum_{w \in S} \alpha_w = 1$, (ii) $\alpha_w \geq 0$ and (iii) finitely many $\alpha_w$ are nonzero. The set of all convex combinations of the points in a set $A$ is called the convex hull of $A$ is denoted $\text{conv}(A)$. As with the span and affine hull defined before it, in the convex hull we take only combinations of a finite number of elements of $A$ at a time, even when $A$ is an infinite set.

A fundamental question is whether we can “separate” disjoint convex sets using a
A special case of this is “isolation”, where we look to isolate a given set from a single point in the vector space using a hyperplane. We make this precise later, but in \( \mathbb{R}^2 \) this amounts to asking whether I can draw a line so that one of the convex set lies on one side of the line, and the point lies on the other. Your own doodling and Figures 2.6 and 2.7 demonstrate that this always seems possible for a convex set in \( \mathbb{R}^2 \) and not always possible for a non-convex set.

To a large extent, our intuition in \( \mathbb{R}^2 \) gives us a good understanding of how convex sets, particularly in finite dimensional spaces, behave in the wild. However, diabolical cases do exist, particularly in infinite dimensional spaces. We will meet some of these unhappy characters later in this section.

### 2.4.2 Core and intrinsic core

In general, more than convexity is needed to guarantee separation of convex sets. We often need additional properties about the nature of the “inside” of the convex sets and the behavior of their “boundaries”. We should be familiar with concepts from point-set topology about the interior and closure of sets, but these concepts assume the space has a mathematical structure called a topology. Here we are purists and focus instead on notions of “inside” and “boundary” that rely only on the underlying basic structure of the vector space.

We first talk about a notion of “inside” called the core of a set. Assume \( A \) and \( B \) are in \( X \). The core of \( A \) relative to \( B \), denoted by \( \text{cor}_B(A) \), is the set of all \( a \in A \) such that
for every \(b \in B\) there is an \(x \in (a, b)\) such that \([a, x] \subseteq A\). If \(B\) is the entire space \(X\), we denote the core relative to \(X\) by \(\text{cor}(A)\) and simply call it the core of \(A\). A useful way to think about the core of a set \(A\) is that it consists of those points in \(A\) from which you can move at least a "small amount" in the direction of any point in \(X\) and still remain in the set \(A\). If \(A\) has \(\text{cor}(A) = A\) then we say \(A\) is algebraically open.

**Lemma 2.4.1.**

\[
\text{cor}(A) = \{ a \in A : \text{ for all } z \in X \text{ there exists } \lambda > 0 \text{ such that } [a, a + \lambda z] \subseteq A \}
\]

**Proof.** See Exercise 2.16.

Based on Lemma 2.4.1, there are three ways to think of a core point. If \(a\) is a core point of set \(A\), then for every \(z \in X\):

1. There is an \(x \in (a, z)\) such that \([a, x] \subseteq A\). This is the original definition.

2. There is an \(\lambda > 0\) such that \([a, a + \lambda(z - a)] \subseteq A\). This is an equivalent statement of the definition.

3. There is an \(\lambda > 0\) such that \([a, a + \lambda z] \subseteq A\). This is Lemma 2.4.1.

In the first two definitions we are moving on the line segment \((a, z)\) in the direction of \(z - a\) towards the point \(z\). In Lemma 2.4.1 we are moving in direction \(z\). In our proofs we use whichever definition we find most convenient. Picture a point \(a \in \mathbb{R}^2\) inside of an open set in the usual topology. You will see that the definitions above are equivalent.

**Remark 2.4.2.** In our definitions of core point, instead of requiring \([a, x] \subseteq A\) we could equivalently require \([a, x) \subseteq A\).

It is important to note that the \(\lambda\) in Lemma 2.4.1 is a function of the target vector \(z\). One might wonder whether we can find a smallest scalar \(\lambda_{\text{min}}\) in which we can move towards every point some minimal amount \(\lambda_{\text{min}}\). A moment of thought reveals that this "smallest" scalar cannot be guaranteed to exist in general, indeed if for a given point \(z\) we have \(x + \lambda z \in A\), then will the same \(\lambda\) work for \(\alpha z \in X\) as the real scalar \(\alpha\) diverges to infinity? There are certainly cases where this does not hold. We return to discussion of this and related issues when we introduce topological vector spaces.

The core of a set inherits some properties of that set, as exemplified in the following result:

**Lemma 2.4.3.** If \(A\) is a convex set then \(\text{cor}(A)\) is convex.

**Proof.** See Exercise 2.18.
The intrinsic core of \( A \), denoted \( \text{icr}(A) \), is the core of \( A \) relative to the affine hull of \( A \). It consists of those points in \( A \) whereby we can move towards every point in the affine hull of \( A \) and still remain in the set \( A \). Here is another way to look at the intrinsic core when \( A \) is convex:

**Lemma 2.4.4.** If \( A \) is convex, then \( a \in \text{icr}(A) \) if and only if for each \( x \in A \), there exists \( y \in A \) such that \( a \in (x, y) \).

**Proof.** \((\implies)\) Assume \( a \in \text{icr}(A) \) and \( x \) is an arbitrary element in \( A \). Consider the line

\[
L = \{ \lambda a + (1 - \lambda)x = x + \lambda(a - x) \mid \lambda \in \mathbb{R} \}.
\]

The line \( L \) is in entirely in \( \text{aff}(A) \) since we are taking affine combinations of \( a \) and \( x \) which are in \( A \). Take a point \( z \in L \) where \( z = x + \lambda(a - x) \) with \( \lambda > 1 \). Since \( a \in \text{icr}(A) \) we have \([a, y] \in A \) for some \( y \in [a, z] \). That is, \( y = x + \lambda(a - x) \) where \( 1 < \lambda < \lambda \). Since \( a \in L \) for \( \lambda = 1, x \in L \) for \( \lambda = 0 \), and \( y \in L \) for \( \lambda = 1 > 1 \), it follows for \( a \in (x, y) \).

\((\iff)\) Assume for each \( x \in A \), there exists \( y \in A \) such that \( a \in (x, y) \), show \( a \) is in the intrinsic core of \( A \). Let \( z \) be an arbitrary point in the affine hull of \( A \). Consider the line

\[
L = \{ (1 - \lambda)a + \lambda z = a + \lambda(z - a) \mid \lambda \in \mathbb{R} \}.
\]

Since \( a \) is \( A \) and \( z \) is in the affine hull of \( A \), there must be another point \( x \in L \) (see Exercise 2.19 for justification) such that \( x \in A \). By hypothesis, there is a \( y \in A \) such that \( a \in (x, y) \subseteq A \). Then there exists \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) such that

\[ [a, a + \lambda_1(y - x)] \subseteq A \quad \text{and} \quad [a, a + \lambda_2(x - y)] \subseteq A \]

Since \( a, x, y, \) and \( z \) are all on the same line,

\[ y - x = \hat{\lambda}_1(z - a) \quad \text{and} \quad (x - y) = \hat{\lambda}_2(z - a) \]

It must the case that either \( \hat{\lambda}_1 > 0 \) or \( \hat{\lambda}_2 > 0 \). Assume without loss of generality, \( \hat{\lambda}_1 > 0 \). Then

\[ [a, a + \lambda_1\hat{\lambda}_1(z - a)] \subseteq A \]

and \( a \) is in the intrinsic core. \( \square \)

**Remark 2.4.5.** Another way to state Lemma 2.4.4 is that given a convex set \( A \subseteq X \), \( a \in \text{icr}(A) \) if and only if \( a \) is an affine combination of two other points in \( A \).

**Example 2.4.6.** Show \( \text{icr}(A) \neq \text{cor}(A) \). Consider the line \( L = \{ x : x_1 + x_2 = 1 \} \) in \( \mathbb{R}^2 \). Note that \( L \) is an affine subspace of \( \mathbb{R}^2 \) and hence \( \text{aff}(L) = L \). We can conclude that \( \text{icr}(L) = L \) using the definition of intrinsic core in Lemma 2.4.4 and the remark that followed. Note however, that \( \text{cor}(L) = \emptyset \). Indeed, take any \( y \in L \) then \( y + \lambda(1, 1) \notin L \) for any \( \lambda > 0 \) since \( (y_1 + \lambda) + (y_2 + \lambda) = (y_1 + y_2) + 2\lambda = 1 + 2\lambda \neq 1 \).
For any set $A$, regardless of whether or not it is convex, if $A$ has a nonempty core then the intrinsic core is equal to the core\(^3\). This is stated formally in Lemma 2.4.7.

**Lemma 2.4.7.** If $A$ is a set with a nonempty core, then $\text{icr}(A) = \text{cor}(A)$.

**Proof.** See Exercise 2.27. \qed

A fundamental question which drives later results (particularly the existence of separating hyperplanes and zero duality gaps) is whether a set has a non-empty core or intrinsic core. This question is particularly tricky in infinite dimensions. In Corollary 2.4.9 we show that in a finite dimensional vector space, every convex set has a non-empty intrinsic core. However, in Example 2.4.11 we show that in an infinite dimensional space we can have convex sets with an empty intrinsic core.

**Lemma 2.4.8.** Let $A$ be the convex hull of an affinely independent set of points $\{x^0, x^1, \ldots, x^n\}$ in $X$. Then,

$$\text{icr}(A) = \left\{ \sum_{i=0}^{n} \alpha_i x^i : \sum_{i=1}^{n} \alpha_i = 1, \alpha_i > 0, i = 0, \ldots, n \right\}$$

(2.4.1)

and hence $\text{icr}(A) \neq \emptyset$.

**Proof.** ($\supseteq$) Let $a = \sum_{i=0}^{n} \alpha_i x^i$, $\sum_{i=0}^{n} \alpha_i = 1$, and $\alpha_i > 0$, for $i = 0, \ldots, n$. Show $a$ is in the intrinsic core. Let $z$ be an arbitrary element in the affine hull of $A$. Then $z$ can be written as an affine combination of the $x^i$, $i = 1, \ldots, n$. That is,

$$z = \sum_{i=0}^{n} \beta_i x^i, \quad \sum_{i=0}^{n} \beta_i = 1$$

Consider the line segment $[a, a + \lambda(z - a)]$. Show for a sufficiently small $\lambda > 0$ that this line segment is contained in $A$.

$$a + \lambda(z - a) = \sum_{i=0}^{n} \alpha_i x^i + \lambda \left( \sum_{i=0}^{n} \beta_i x^i - \sum_{i=0}^{n} \alpha_i x^i \right) = \sum_{i=0}^{n} \left( \alpha_i - \lambda(\alpha_i - \beta_i) \right) x^i$$

$$\sum_{i=0}^{n} \left( \alpha_i - \lambda(\alpha_i - \beta_i) \right) = \sum_{i=0}^{n} \alpha_i - \lambda \sum_{i=0}^{n} (\alpha_i - \beta_i) = \sum_{i=0}^{n} \alpha_i = 1.$$ 

There is a sufficiently small $\lambda > 0$, such that $(\alpha_i - \lambda(\alpha_i - \beta_i)) > 0$ for all $i = 0, \ldots, n$ since $\alpha_i > 0$ for all $i = 1, \ldots, n$. Thus all points on the line segment $[a, a + \lambda(z - a)]$ are convex combinations of the $x^i$ for $i = 0, \ldots, n$ so it follows that $[a, a + \lambda(z - a)] \subseteq A$.

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\(^3\)The authors thank Matthew Stern for this nice observation.
We prove this direction by contrapositive. Assume there is a point \( a \in A \) that written as a convex combination \( a = \sum_{i=0}^{n} \alpha_i x^i \) has at least one \( \alpha_k = 0 \). Show that this implies \( a \notin \text{icr}(A) \). If \( a \) were in the intrinsic core of \( A \), by Lemma 2.4.4, there is a \( y \in A \) such that \( a \in (x^k, y) \). This implies \( y = x^k + \lambda (a - x^k) \) for some \( \lambda > 1 \) and rewriting gives

\[
a = \frac{1}{\lambda} y - \frac{1-\lambda}{\lambda} x^k.
\]

But \( \lambda > 1 \) implies \( a \) is a convex combination of \( y \) and \( x^k \) and the coefficient on \( x^k \) is not zero. Thus \( a \) does not have a unique representation as a convex combination of \( \{x^0, \ldots, x^n\} \) which contradicts the fact that the \( \{x^0, \ldots, x^n\} \) are affinely independent (see Exercise 2.10).

A key aspect of the proof of Lemma 2.4.8 is illustrated in Figure 2.8 where \( A = \text{conv}(x^1, \ldots, x^5) \) is a full-dimensional convex set in \( \mathbb{R}^2 \). Observe that the five vertices \( \{x^1, \ldots, x^5\} \) are not affinely independent. The point \( a \) is in the intrinsic core of the set, however, it can be written as a convex combination of \( \{x^1, \ldots, x^5\} \) with an \( \alpha_4 = 0 \) and \( \alpha_5 = 0 \). Hence we cannot characterize \( \text{icr}(A) \) as we have done in equation (2.4.1) of Lemma 2.4.8 when the set is not the convex hull of affinely independent vectors, that is, a simplex.

![Figure 2.8: Why do we need affinely independent points for Lemma 2.4.8? Part One.](image1)

![Figure 2.9: Why do we need affinely independent points for Lemma 2.4.8? Part Two.](image2)

Now look at Figure 2.9. If we have any \( \alpha_k = 0 \) in the convex combination \( a = \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3 \) it is clear we are not in the \( \text{icr}(A) \). In the figure we have \( \alpha_1 = 0 \) and we see that \( a \) lies on the boundary of the set.

Lemma 2.4.8 allows us to establish that every convex set in a finite dimensional vector space has a nonempty intrinsic core.
Corollary 2.4.9. If \( A \subseteq X \) is a convex set with finite dimension \( n \geq 1 \), then \( A \) has a nonempty intrinsic core, i.e. \( \text{icr}(A) \neq \emptyset \).

Proof. If \( A \) has finite dimension \( n \), then \( A \) contains \( n+1 \) affinely independent points \( x^0, \ldots, x^n \). Let \( B \) be the convex hull of these points. By Lemma 2.4.8, \( \text{icr}(B) \neq \emptyset \). Since \( \text{aff}(B) = \text{aff}(A) \), it follows that \( \text{icr}(B) \subseteq \text{icr}(A) \). Therefore \( \text{icr}(A) \) is not empty.

It is then immediate to characterize convex sets in finite dimensional space which have a non-empty core:

Theorem 2.4.10. If \( X \) is a finite dimensional vector space, and \( A \subseteq X \) is convex, then \( \text{cor}(A) \neq \emptyset \) if and only if \( \text{aff}(A) = X \).

Proof. Assume \( \text{aff}(A) = X \) and show \( \text{cor}(A) \neq \emptyset \). Since \( X \) is finite dimensional, \( A \) has finite dimension. Since \( A \) is convex, by Corollary 2.4.9, \( \text{icr}(A) \neq \emptyset \). The intrinsic core of \( A \), is the core relative to the affine hull of \( A \). In this case \( \text{aff}(A) = X \), so the intrinsic core is the core. Since the intrinsic core is nonempty, the core is nonempty.

Now assume \( \text{cor}(A) \neq \emptyset \) and show \( \text{aff}(A) = X \). If \( \text{cor}(A) \neq \emptyset \), there exists an \( a \in \text{cor}(A) \). Take an arbitrary \( x \in X \). We show \( x \in \text{aff}(X) \). Since \( a \) is in the core of \( A \), given the arbitrary \( x \in X \), there is a \( y \in (a,x) \) such that \([a,y] \subseteq A\). Thus

\[
y = a + \epsilon(x - a)
\]

where \( \epsilon \in (0,1) \). Then rearranging terms gives

\[
x = \frac{1}{\epsilon}y - \frac{1 - \epsilon}{\epsilon}a
\]

but

\[
\frac{1}{\epsilon} + \left(-\frac{1 - \epsilon}{\epsilon}\right) = 1
\]

so \( x \) is an affine combination of \( y \) and \( a \). But \( y \) and \( a \) are in \( A \) so \( x \in \text{aff}(X) \) from part 1) of Proposition 2.3.2.

Theorem 2.4.10 does not hold for an infinite dimensional vector space. In an infinite dimensional vector space, it is possible to have a convex set \( A \) where \( \text{aff}(A) = X \) but \( \text{cor}(A) = \text{icr}(A) = \emptyset \). The following example demonstrates this peculiar fact and is the first illustration of how things can go really haywire in infinity.

Example 2.4.11. Here we give an example of a convex set \( A \) in a vector space \( X \) where:

- the intrinsic core of the set \( A \) is empty
- the affine hull of the set \( A \) is \( X \) but \( A \) has an empty core
Let $X = \mathbb{R}^\infty$ (see Example 2.2.7). Let $A$ be the subset of $X$ where the last nonzero component is strictly positive. It is straightforward to see that $A$ is convex. We will show that $\text{aff}(A) = \mathbb{R}^\infty$ and $\text{icr}(A) = \text{cor}(A) = \emptyset$.

First show that $\text{aff}(A) = \mathbb{R}^\infty$. If $x \in A$, then $y = 2x \in A$ since the last nonzero component of $y$ is positive. Then $2x - y = \theta \in \text{aff}(A)$ since we took an affine combination of $x$ and $y$. Given that $\theta$ is in the affine hull, it follows $2\theta - x$ is in the affine hull of $A$ for all $x \in A$. Thus all points with the last nonzero negative are also in the affine hull of $A$. Therefore $\text{aff}(A) = X$. Since $\text{aff}(A) = X$, $\text{icr}(A) = \text{cor}(A)$. We now show the core is empty. Take any $x \in A$ and let $N = \arg\max \{n \in \mathbb{N} : x_n \neq 0\}$. Now consider the direction $-e^{N+1} = (0, \ldots, 0, -1, 0, \ldots, 0)$ where the $-1$ is in the $(N+1)$-st position. Note that there does not exist a $\lambda > 0$ so that $x + \lambda e^{N+1} \in A$ since the last nonzero component of this vector is always negative. This implies that $x \notin \text{cor}(A)$ and we conclude that $\text{cor}(A) = \emptyset$.

### 2.4.3 Algebraic closure

A point $x \in X$ is linearly accessible from $A$ if there exists an $a \in A$, $a \neq x$ such that $[a, x] \subseteq A$. The set of all points linearly accessible from $A$ is called $\text{lina}(A)$. Define $\text{lin}(A) := A \cup \text{lina}(A)$. We call $\text{lin}(A)$ the algebraic closure of $A$. The set $A$ is algebraically closed if $A = \text{lin}(A)$. One might ask what is the difference between $\text{lin}(A)$ and $\text{lina}(A)$. The following example demonstrates that they can indeed differ:

**Example 2.4.12** (Example where $\text{lina}(A) \neq \text{lin}(A)$). Consider the open unit disc $D$ in $\mathbb{R}^2$; that is, $D = \{x : ||x|| < 1\}$ where $|| \cdot ||$ is the usual Euclidean distance. Then $\text{lina}(D)$ is the closed unit disc $\overline{D} = \{x : ||x|| \leq 1\}$. Now, consider the set $D' = D \cup \{(2, 2)\}$. Here, $\text{lina}(D') = \overline{D} \neq \text{lin}(D')$ since the point $(2, 2)$ is not linearly accessible from $D$.

The driver in Example 2.4.12 is the fact that $D'$ is not a convex set. There is an “isolated point” $(2, 2)$ that is not linearly accessible from other points in $D'$. However, as you might have suspected, for nontrivial convex sets $\text{lin}(A)$ and $\text{lina}(A)$ are identical.

**Lemma 2.4.13.** If $A$ is convex and $|A| > 1$, then $\text{lin}(A) = \text{lina}(A)$.

**Proof.** See Exercise 2.20.

As with the core, the algebraic closure inherits some properties of the set $A$, including convexity:

**Lemma 2.4.14.** If $A$ is a convex set then $\text{lin}(A)$ is a convex set.

**Proof.** See Exercise 2.21
2.4.4 Connections between the core and algebraic closure

We develop more intuition regarding the connection between the algebraic closure and some of the other objects we have introduced thus far. First, we relate \( \text{lin}(A) \) and \( \text{aff}(A) \).

**Lemma 2.4.15.** The algebraic closure of a set is contained in its affine hull; that is for \( A \subseteq X \), \( \text{lin}(A) \subseteq \text{aff}(A) \)

**Proof.** See Exercise 2.22. \qed

The following relationship between \( \text{icr}(A) \) and \( \text{lin}(A) \) turns out to be useful in proving the major results of this section:

**Theorem 2.4.16.** Let \( A \) be a convex set. If \( x \in \text{lin}(A) \) then \( [p, x] \subseteq A \) for all \( p \in \text{icr}(A) \).

**Proof.** Assume \( p \) is arbitrary element in the intrinsic core and let \( q \in [p, x] \). We want to show \( q \in A \). We may write \( q = \mu p + (1 - \mu)x \) where \( 0 \leq \mu < 1 \). Since \( x \in \text{lin}(A) \) there exists \( y \in A \) such that \( [y, x] \subseteq A \). Consider the direction \( x - y \). By construction, \( x, y, \) and \( p \) are all points in \( \text{lin}(A) \). By Lemma 2.4.15 \( \text{lin}(A) \subseteq \text{aff}(A) \). Then \( x, y, \) and \( p \) are points in \( \text{aff}(A) \), and it follows that \( p + (x - y) \) is in \( \text{aff}(A) \) since \( p + (x - y) \) is an affine combination of \( p, x, \) and \( y \). Then by definition of intrinsic core, we can move a small distance from \( p \) to the point \( p + (x - y) \) and remain in \( A \). When we move from \( p \in \text{icr}(A) \) to \( p + (x - y) \) we are moving in the direction \( (x - y) \) from \( p \). In other words, there is a \( \tau > 0 \) such that \( z = p + t(x - y) \in A \) for \( 0 < t \leq \tau \). Our goal is to show \( q \in A \), and our approach is to write \( q \) as the convex combination of other points in \( A \). Our candidates for these two other points are \( z \) and some point \( w \in [y, x] \) which we also know is in \( A \). The idea is captured in Figure 2.10, and we give the justification that this is indeed possible below.

Let \( w \) denote an arbitrary element in the half-line \( L = \{ \lambda q + (1 - \lambda)z : \lambda \geq 0 \} \) which starts at \( z \) and contains the point \( q \). Then we may write:

\[
\begin{align*}
w &= \lambda q + (1 - \lambda)z \\
&= \lambda(\mu p + (1 - \mu)x) + (1 - \lambda)(p + t(x - y)) \\
&= (\lambda \mu + (1 - \lambda))p + (\lambda(1 - \mu) + (1 - \lambda)t)x - t(1 - \lambda)y. \quad (2.4.2)
\end{align*}
\]

We claim for \( t \) sufficiently small one such \( w \in L \) lies in the interval \( [y, x] \). We simply need to show that we may choose \( \lambda \) so that \( w \) is convex combination of \( x \) and \( y \) which entails (via (2.4.2)) the feasibility of the following system:

\[
\begin{align*}
\lambda \mu + (1 - \lambda) &= 0 \\
\lambda(1 - \mu) + (1 - \lambda)t &\geq 0 \\
-t(1 - \lambda) &\geq 0 \\
(\lambda(1 - \mu) + (1 - \lambda)t) - t(1 - \lambda) &= 1
\end{align*}
\]
The last equation simplifies to
\[(\lambda(1 - \mu) + (1 - \lambda)t) - t(1 - \lambda) = \lambda(1 - \mu) = 1\]
which implies \(\lambda = \frac{1}{1 - \mu}\) is a solution to this system. Note that since \(0 \leq \mu < 1\) we have \(\lambda > 1\). Hence we can find a \(w \in [y, x] \cap L\) and in particular, this \(w\) is in \(A\).

The final step is to show that \(q\) can be written as a convex combination of \(z\) and \(w\). But this is straightforward since \(w = \lambda q + (1 - \lambda)z\) for some \(\lambda > 1\) and solving for \(q\) yields:
\[q = \frac{1}{\lambda}w + \frac{\lambda - 1}{\lambda}z\]
which is clearly a convex combination when \(\lambda > 1\). Therefore, \(q\) is in \(A\) since \(w\) and \(z\) are in \(A\) and \(A\) is convex.

\[\square\]

**Figure 2.10:** Proof of Theorem 2.4.16

**Remark 2.4.17.** Theorem 2.4.16 will be very useful for us. Indeed, if \(p \in \text{icr}(A)\) and \(x \in \text{aff}(A)\) I know that there exists \(y \in (p, x)\) such that \([p, y] \in A\). Theorem 2.4.16 says something stronger for \(x \in \text{lin}(A) \subseteq \text{aff}(A)\). When \(x \in \text{lin}(A)\) we have \([p, x] \subseteq A\); in other words, we can take \(y\) in the previous sentence arbitrarily close to \(x\). On the other hand, if \(x \in \text{lin}(A)\) we know there exists some \(y \in A\) such that \([y, x] \subseteq A\). With Theorem 2.4.16 we have specific choices for \(y\) that always work, namely any point \(p\) in the intrinsic core.
We use Theorem 2.4.16 to give a surprising characterization of infinite dimensional vector spaces. First, we need one more concept. We say a set $A$ is ubiquitous if $\text{lin}(A) = X$. Intuitively, an ubiquitous set is “close” to every point in the vector space $X$. Right now it might be hard to imagine, but there are proper convex sets which are ubiquitous. More than this, every infinite dimensional space contains at least one proper convex ubiquitous set:

**Theorem 2.4.18.** The vector space $X$ is infinite dimensional if and only if $X$ contains a proper convex ubiquitous set.

**Proof.** ($\Leftarrow$) If $X$ contains a proper convex ubiquitous set, show $X$ is infinite dimensional. Prove the contrapositive. Assume $X$ is finite dimensional and show $X$ cannot contain proper convex ubiquitous set. Let $A$ be a convex ubiquitous set in $X$. We show that $A$ must be all of $X$ and hence not a proper convex set.

Since $A$ is ubiquitous, $\text{lin}(A) = X$ by definition of ubiquitous. By Lemma 2.4.15, $\text{lin}(A) \subseteq \text{aff}(A)$ so $\text{aff}(A) = X$. Since we are in a finite dimensional space, by Theorem 2.4.10, $\text{cor}(A)$ is nonempty. Without loss of generality, assume that $\theta \in \text{cor}(A)$.

Let $x$ be an arbitrary vector in $X$. We show $x \in A$. Note that since $A$ is ubiquitous $\alpha x \in \text{lin}(A)$ for any $\alpha > 1$ (since $A$ is ubiquitous it works for any $\alpha$, but we need $\alpha > 1$ for our result). By Theorem 2.4.16, $[\theta, \alpha x] \subseteq A$. But $\alpha > 1$, so $[\theta, x] \subseteq A$ and therefore $x \in A$. Thus we have $A = X$ and thus $A$ is not proper.

($\Rightarrow$) Let $X$ be an infinite dimensional vector space. We construct a proper ubiquitous convex set in $X$ as follows. We invoke Hamel’s theorem to conclude that $X$ has a basis $B$. Every vector $x \in X$ can be uniquely written as a linear combination of a finite number of basis vectors. Index basis vectors with nonzero multipliers by $B(x)$ and write

$$x = \sum_{y \in B(x)} \alpha_y y$$

where we may assume the $\alpha_y \neq 0$.

Now, by Zermelo’s Theorem (Theorem A.1.8) the basis $B$ can be well-ordered. In other words, every subset in $B$ has a least element in the ordering. This implies that for a finite subset we can find its largest element (iteratively, say, through finding its smallest element, throwing it out, and so on). In particular, we can find the largest element $y(x)$ of the set $B(x)$. Let $L(x) = \alpha_{y(x)}$ be the coefficient of the largest basis vector in the linear combination which defines $x$.

Consider the set $A = \{x \in X : L(x) > 0\}$. Clearly, $A$ is a a proper subset of $X$. Notice also that $A$ is convex. Take $x, x' \in A$, then:

$$L(\lambda x + (1 - \lambda)x') = \lambda L(x) + (1 - \lambda)L(x'), \quad \text{when } y(x) = y(x')$$
$$L(\lambda x + (1 - \lambda)x') = \lambda L(x), \quad \text{when } y(x) > y(x')$$
$$L(\lambda x + (1 - \lambda)x') = (1 - \lambda)L(x'), \quad \text{when } y(x) < y(x')$$
We now show that $A$ is ubiquitous. Take an arbitrary $x \in X$, and show that $x \in \text{lin}(A)$. Consider the set $C(x) = \{b \in B : b \succ y, \text{ for every } y \in B(x)\}$ and let $w$ be the minimal element of $C(x)$ in the $\preceq$ ordering (which is well defined, since $C(x) \subseteq B$ and $B(x)$ finite – if $B(x)$ were infinite we could not claim the existence of a minimal element). For any basis element $b \in B$, $L(b) = 1$ and in particular, $L(w) = 1$. Therefore, $w \in A$. We show that the interval $[w, x) \in A$ which establishes that $x \in \text{lin}(A)$. Observe,

$$[w, x) = \{w + t(x - w) : 0 \leq t < 1\}$$

and $L(w + t(x - w)) = (1 - t) > 0$. Thus $x$ is in $\text{lin}(A)$ and it follows $A$ is ubiquitous since $x$ was an arbitrary element in $X$.

**Remark 2.4.19.** Note that the set discussed in Example 2.4.11 is a special case of the construction in the second half of the proof. This was not a fluke example, in every infinite dimensional vector space there is a ubiquitous proper subset of $X$ with an empty core (the proof that the core of set $A$ is empty in Theorem 2.4.18 follows the line of logic as used in Example 2.4.11). The set is, in some sense, spread over the entire vector space, but has no inside. By Theorem 2.4.10, in a finite dimensional vector space, a ubiquitous convex set must have a nonempty core. So Theorem 2.4.18 provides a nice characterization of infinite dimensional vector spaces.

We end this section with some additional properties of the core which will be useful when we consider constructing separating hyperplanes.

**Theorem 2.4.20.** Let $A$ be a convex subset of the vector space $X$ and let $p \in \text{cor}(A)$. Then for any $x \in A$ the interval $[p, x)$ is contained in $\text{cor}(A)$ and

$$\text{cor}(A) = \bigcup \{[p, x) : x \in A\}$$

**Proof.** (\(\supseteq\)) Let $y \in \bigcup \{[p, x) : x \in A\}$. Then there is an $x \in A$, such that $y \in [p, x)$. We show $y \in \text{cor}(A)$. Since $y$ is on the line segment $[p, x)$, we can write $y = tx + (1 - t)p$ for $0 \leq t < 1$. See Figure 2.11. From Lemma 2.4.1, if $p \in \text{cor}(A)$, then for any direction $z \in A$, there exists some $\lambda > 0$ so that $p + \lambda z \in A$. We show that $y + \mu z \in A$ for some $\mu > 0$ to conclude that $y \in \text{cor}(A)$. Since $p + \lambda z$ and $x$ are both in $A$, the point $w = tx + (1 - t)(p + \lambda z)$ is also in $A$ since it is a convex combination of $x$ and $p + \lambda z$. Now for a bit of algebra.

$$w = tx + (1 - t)(p + \lambda z) = tx + (1 - t)p + (1 - t)\lambda z \tag{2.4.3}$$

But $y = tx + (1 - t)p$ so $tx = y - (1 - t)p$ and substituting into (2.4.3), gives

$$w = tx + (1 - t)(p + \lambda z) = y - (1 - t)p + (1 - t)p + (1 - t)\lambda z = y + (1 - t)\lambda z$$

Both $y$ and $w = y + (1 - t)\lambda z$ are in the convex set $A$, so the interval $[y, y + (1 - t)\lambda z]$ is in $A$ where $(1 - t) > 0$ and $\lambda > 0$ so $y$ is a core point of $A$ by Lemma 2.4.1.
(⊆) Let \( q \in \text{cor}(A) \setminus p \). We construct an \( x \) such that \( q \in [p, x) \). Since \( q \in \text{cor}(A) \) we have \( x = q + \delta(q - p) \in A \) for some \( \delta > 0 \) (again using Lemma 2.4.1). It follows that

\[
q = \frac{\delta p + x}{1 + \delta} = \frac{\delta}{1 + \delta} p + \frac{1}{1 + \delta} x \in [p, x)
\]

(2.4.4)
since \( \delta > 0 \).

![Figure 2.11: Proof of Theorem 2.4.20.](image)

Corollary 2.4.21. Let \( A \) be a convex subset of the vector space \( X \) and let \( p \in \text{cor}(A) \). Then for any \( x \in \text{lina}(A) \) the interval \([p, x)\) is contained in \( \text{cor}(A) \).

\textbf{Proof.} Let \( x \in \text{lina}(A) \). Since \( p \) is in the core of \( A \) it is in the intrinsic core of \( A \), so from Theorem 2.4.16, \([p, x) \subseteq A\). Let \( y \) be any element in \([p, x)\), then \( y \in A \). Then by Theorem 2.4.20, \([p, y) \subseteq \text{cor}(A)\). This implies \([p, x) \subseteq \text{cor}(A)\). \(\square\)

A similar statement can be made for intrinsic cores, the proof of which is left as an exercise:

Corollary 2.4.22. Let \( A \) be a convex subset of the vector space \( X \) and assume \( p \) is in the intrinsic core of \( A \). If \( x \in A \), then \([p, x) \subseteq \text{icr}(A)\).

\textbf{Proof.} See Exercise 2.23 \(\square\)

A final implications of Theorem 2.4.20 is used later in this chapter.

Corollary 2.4.23. If \( A \) is a convex set then \( \text{cor}(A) = \text{cor}(\text{cor}(A)) \).

\textbf{Proof.} See Exercise 2.25. \(\square\)
2.5 Separating hyperplanes

Separating disjoint convex sets with a hyperplane is a phenomenally important problem in optimization. Interestingly enough, it is also a critical idea in functional analysis. Indeed, there are even variations of the famous Hahn-Banach theorem proved using separating hyperplane theorems. See, for example, Holmes [16]. We present very general separation results in this section for infinite dimensional vector spaces. We make no assumptions about a topology.

2.5.1 Types of separation

First, some definitions:

Definition. Let $A$ and $B$ be convex sets in vector space space $X$. Then $A$ and $B$ are:

1. separated if there exists a hyperplane $H = [\phi, \alpha]$ with $\phi \in X'$ and $\alpha \in \mathbb{R}$ such that $\phi(x) \geq \alpha$ for all $x \in A$ and $\phi(x) \leq \alpha$ for all $x \in B$.

2. strictly separated if they are separated by hyperplane $H = [\phi, \alpha]$ and additionally $\phi(x) > \phi(y)$ for all $x \in A$ and for all $y \in B$.

3. strongly separated if they are separated by a hyperplane $H = [\phi, \alpha]$ and additionally $\phi(x) > \phi(y) + \delta$ for some $\delta > 0$.

When we separate a convex set $A$ from a point $x \in X$ (which is trivially a convex set) we sometimes call this isolation of set $A$ from point $x$. Note that separation allows for the possibility that there are points in $A$, and points in $B$, that lie in the hyperplane $H = [\phi, \alpha]$; that is, have $\phi(x) = \alpha$. It is possible that both $A$ and $B$ lie entirely in $H$. Indeed, we need not have $A$ and $B$ disjoint to have separation. See Figure 2.12.

Clearly the sets $A$ and $B$ must be disjoint in order to allow for strict and strong separation.

It turns out that having (intrinsic) core points is essential to guarantee separation. The following example demonstrates why core points are so important:

Example 2.5.1. [Example 2.4.11 cont’d. Convex set which cannot be isolated from a point.] Consider the set $A$ defined in Example 2.4.11. Show that $A$ cannot be isolated from the origin. If $x \in A$, then the last nonzero component of $x$ is a positive number. This implies $\theta \notin A$. So $A$ is convex, $\theta \notin A$, yet $\theta$ cannot be separated from $A$. Let

$$H = \{x \in X \mid \langle x, \phi \rangle = 0\}$$

be an arbitrary hyperplane through the origin $\theta \in X$. We show that there are points in $A$ on either side of $H$ and hence $A$ cannot be separated from $\theta$. 

2.5. SEPARATING HYPERPLANES

Figure 2.12: Weak separation of two non-disjoint sets. Let $A = \{x \in \mathbb{R}^2 : 0 \leq x_i \leq 1, \text{ for } i = 1, 2\}$ and $B = \{x \in \mathbb{R}^2 : x_1 = 0, 0 \leq x_2 \leq 1\}$ with separating hyperplane $H = \{x \in \mathbb{R}^2 : x_1 = 0\}$.

Consider the quotient space $X/H$. Since $H$ is a hyperplane, by Theorem 2.3.9 $X/H$ has dimension one. Let $v$ be a nonzero element of $X/H$. We can write

$$v = x + H, \quad x \notin H$$

Let $i$ be the last nonzero index of $x$. Define

$$y^+ = x + e_i^{i+1}$$
$$y^- = -x + e_i^{i+1}$$

Show $e_i^{i+1} \in H$. Assume not and show that this implies that the dimension of $X/H$ exceeds one. We know $x \notin H$ so if $e_i^{i+1} \notin H$ then either $y^+$ is not in $H$ or $y^-$ is not in $H$. If both $y^+$ and $y^-$ were in $H$, then $y^+ + y^- = 2e_i^{i+1}$ would be in $H$. This follows because $H$ is a hyperplane through the origin and therefore a subspace of $X$. Assume without loss, $y^+ \notin H$. If $y^+ = x + e_i^{i+1}$ is not in $H$, then $x + H$ and $(x + e_i^{i+1}) + H$ are two nonzero vectors in $X/H$. Since $X/H$ has dimension one, this implies $x + H$ and $(x + e_i^{i+1}) + H$ are linearly dependent. Thus, there exists $\alpha_1$ and $\alpha_2$ both nonzero, such that

$$\alpha_1 x + \alpha_2 (x + e_i^{i+1}) = 0$$
However, $x$ has a zero in component $i+1$ and $x + e^{i+1}$ does not. This implies $\alpha_1 = \alpha_2 = 0$ which implies the dimension of $X/H$ is greater than one. Therefore, $e^{i+1}$ must be in $H$.

Both $y^+$ and $y^-$ are in $A$ since their last nonzero component is $e^{i+1}$. However $e^{i+1}$ in $H$ and $x \notin H$ implies that $\langle y^+, \phi \rangle$ and $\langle y^-, \phi \rangle$ are nonzero and opposite in sign. This completes the argument.

2.5.2 Stone’s Lemma and basic separation

Marshall Stone was a famous mathematician and head of the mathematics department at the University of Chicago in the late 1940s. He recruited a number of brilliant mathematicians to the department (see http://www.math.uchicago.edu/about/history.shtml).

Below we prove an incredibly powerful result due to him. It is very general, and unlike many results on separating convex sets, makes no assumption that the vector space $X$ is finite.

Stone’s result applies to convex sets that are complementary; that is, sets whose union comprises the whole space but have an empty intersection. If $C$ and $D$ are complementary then $C \cap D = \emptyset$ and $C \cup D = X$. For brevity we write $X = C \cup D$ to means disjoint union; that is, $C \cup D = X$ and $C \cap D = \emptyset$. Lemma 2.5.2 follows the proof given by Holmes [16] on page 7, but with far more detail.

Lemma 2.5.2. (Stone) Let $A$ and $B$ be disjoint convex subsets of $X$. Then there exist complementary convex sets $C$ and $D$ in $X$ such that $A \subseteq C$ and $B \subseteq D$.

Proof. To find $C$ we apply Zorn’s lemma (Lemma A.1.7). Let $\mathcal{C}$ be the family of convex sets that contain $A$ but do not intersect $B$. This family is not empty since $A \in \mathcal{C}$. Define a partial order on $\mathcal{C}$ by set inclusion. Let $\mathcal{K}$ be a chain in $\mathcal{C}$ and let $S = \bigcup_{K \in \mathcal{K}} K$. Then $S$ is an upper bound for the chain $\mathcal{K}$ and $S \in \mathcal{C}$. In order to show $S \in \mathcal{C}$, it is sufficient to show $S$ is a convex set. Let $x, y \in S$. Then $x \in K$ and $y \in K'$ for some $K, K' \in \mathcal{K}$. But $\mathcal{K}$ is a chain with respect to set inclusion, so either $K \subseteq K'$ or $K' \subseteq K$. Assume without loss, $K' \subseteq K$. It follows that $x, y \in K$ and so every convex combination of $x$ and $y$ lies in $K$. Hence every convex combination of $x$ and $y$ is in $S$, which implies its convexity. Thus, by Zorn’s Lemma there exits a maximal element in $\mathcal{C}$, let’s call it $C$. Let $D = X \setminus C$.

Since $A \subseteq C$, and $B \cap C = \emptyset$ we know $B \subseteq D$. We must show $D$ is convex. If $D = B$ we are done, since $B$ is convex by assumption. If $D$ is not convex, then $D$ must contain two points $x$ and $z$ such that $(x, z)$ intersects $C$. In the current proof we assume that we can take $x, z \in D \setminus B$. Proving that the result holds under other possibilities (for instance, if one of $x$ or $z$ in $B$) is left as an exercise (see Exercise 2.26).

See Figure 2.13 for an illustration of why we want $C$ to be maximal. If $C$ is not maximal it is possible for $D$ to be nonconvex. If $D$ is not convex and $D \cup C = X$, there is a $y \in C$ with $y \in (x, z)$. We are going to show later in the proof the following fact (we call it (*)).
2.5. SEPARATING HYPERPLANES

that there exists $p, q \in C$ such that there exist:

$$u \in (p, x) \cap B \neq \emptyset$$
and

$$v \in (q, z) \cap B \neq \emptyset. (*)$$

See Figure 2.14 for a sense of the relationship between $x, y, z$ and $p, q, u, v$. For now we take (*) as given. By (*) we have:

$$u = (1 - \epsilon_u)p + \epsilon_u x$$ (2.5.1)
$$v = (1 - \epsilon_v)q + \epsilon_v z$$ (2.5.2)

and

$$y = (1 - \epsilon_y)x + \epsilon_y z$$ (2.5.3)

where $\epsilon_u, \epsilon_v, \epsilon_y \in (0, 1)$. We now show $x$ is an affine combination of $u$ and $p$ and $z$ is an affine combination of $v$ and $q$. Rewrite (2.5.1) and (2.5.2) as

$$x = \frac{u}{\epsilon_u} - \frac{(1 - \epsilon_u)}{\epsilon_u} p$$ (2.5.4)
$$z = \frac{v}{\epsilon_v} - \frac{(1 - \epsilon_v)}{\epsilon_v} q$$ (2.5.5)

and therefore $x$ is an affine combination of $u$ and $p$ and $z$ is an affine combination of $v$ and $q$. Rewrite (2.5.3) as

$$(1 - \epsilon_y)x + \epsilon_y z - y = 0$$ (2.5.6)

and observe that the sum of the multipliers on $x, y, z$ is zero. Next substitute in (2.5.4) and (2.5.5) for $x$ and $z$, respectively, and get

$$(1 - \epsilon_y)\left(\frac{u}{\epsilon_u} - \frac{(1 - \epsilon_u)}{\epsilon_u} p\right) + \epsilon_y \left(\frac{v}{\epsilon_v} - \frac{(1 - \epsilon_v)}{\epsilon_v} q\right) - y = 0$$ (2.5.7)

since

$$\left(\frac{1}{\epsilon_u} - \frac{(1 - \epsilon_u)}{\epsilon_u}\right) = 1, \quad \left(\frac{1}{\epsilon_v} - \frac{(1 - \epsilon_v)}{\epsilon_v}\right) = 1$$

and $(1 - \epsilon_y) + \epsilon_y = 1$ it follows that the sum of all the multipliers in (2.5.7) is zero. Therefore the sum of the positive multiplies is equal to the sum of the negative multipliers. Collecting the points with positive multipliers on one side and negative multipliers on the and rearranging (2.5.7) gives

$$(1 - \epsilon_y)\frac{u}{\epsilon_u} + \epsilon_y \frac{v}{\epsilon_v} = y + (1 - \epsilon_y)\frac{(1 - \epsilon_u)}{\epsilon_u} p + \epsilon_y \frac{(1 - \epsilon_v)}{\epsilon_v} q$$ (2.5.8)
But since the sum of the positive multipliers is equal to the sum of the negative multipliers in equation (2.5.7), it follows that the sum of the multipliers on the left in equation (2.5.8) is equal to the sum of the multipliers on the right in this equation. Also, all multipliers on both sides of (2.5.8) are nonnegative. Therefore, if we divide both sides of (2.5.8) by the sum of the multipliers on the left, we have a convex combination of $u$ and $v$ equal to a convex combination of $y, p, q$. The geometric intuition for this result is captured in Figure 2.14, but of course a proof is not a picture, hence the tedious details. Thus $[u, v] \cap \text{conv} \{p, q, y\} \neq \emptyset$ which contradicts the fact $B$ and $C$ are disjoint. Therefore, assuming $D$ is not convex is false, and $D$ is convex. All that remains is to show (*).

Now for the proof of (*). Consider the statement: “If $C$ is a maximal convex set containing $A$ and not intersecting $B$, then there must be a point $p \in C$ such that $(p, x) \cap B \neq \emptyset$ and a point $q \in C$ such that $(q, z) \cap B \neq \emptyset.” The contrapositive of this is: “for all $p \in C$, $(p, x) \cap B = \emptyset$, or for all $q \in C$ $(q, z) \cap B = \emptyset$ then $C$ is not a maximal convex set containing $A$ and not intersecting $B.” First assume that for all $p \in C$, $(p, x) \cap B = \emptyset$. If for all $p \in C$, $(p, x) \cap B = \emptyset$, then $C \subset \text{co}(\{x, C\})$. By assumption $x \notin B$, so $\text{co}(\{x, C\})$ does not intersect $B$. This contradicts the fact that $C$ is the maximal convex set containing $A$ but not intersecting $B$. Therefore, there is a $p \in B$ with $(p, x) \cap B \neq \emptyset$. By the exact same logic, there must be a $q \in C$ with $(q, z) \cap B \neq \emptyset$.

Figure 2.13: An illustration of Stone’s Lemma in $\mathbb{R}^2$. In this figure $C$ is not maximal. This leads to $D$ being nonconvex.

The rest of the section brings home the intuition captured in Figure 2.13. Of course some work is required to make things precise. We begin with an observation regarding the
core and algebraic close of complementary convex sets:

**Lemma 2.5.3.** Let \( C \) and \( D \) be complementary convex sets. Then, \( \text{lin}(C) = X \setminus \text{cor}(D) \).

*Proof.* This is a special case of Exercise 2.28. \( \square \)

**Lemma 2.5.4.** Let \( C \) and \( D \) be complementary convex sets. Then we can partition \( X \) as:

\[
X = (\text{lin}(C) \cap \text{lin}(D)) \cup \text{cor}(C) \cup \text{cor}(D)
\]

*Proof.* Note from Lemma 2.5.3 we have:

\[
X = \text{lin}(C) \cup \text{cor}(D)
= \text{lin}(D) \cup \text{cor}(C).
\]

Note that \( X \cap X = X \) so by De Morgan’s laws and using the two expressions for \( X \) given above yields the following characterization of \( X \):

\[
X \cap X = (\text{lin}(C) \cup \text{cor}(D)) \cap (\text{lin}(D) \cup \text{cor}(C))
= (\text{lin}(C) \cap \text{lin}(D)) \cup (\text{lin}(C) \cap \text{cor}(C)) \cup (\text{lin}(D) \cap \text{cor}(C)) \cup (\text{cor}(D) \cap \text{cor}(C)).
\]

However, observe that \( \text{cor}(D) \cap \text{cor}(C) = \emptyset \) since \( C \) and \( D \) are complementary and \( \text{lin}(C) \cap \text{cor}(C) = \text{cor}(C) \) since \( \text{cor}(C) \subseteq C \subseteq \text{lin}(C) \). Thus,

\[
X = (\text{lin}(C) \cap \text{lin}(D)) \cup \text{cor}(C) \cup \text{cor}(D)
\]

\( \square \)
Now for the key result of this subsection. It basically says only two things can happen on the boundary between complementary convex sets: something we expect from Figure 2.13 and something crazy. That is, the boundary between $C$ and $D$ (technically $\text{lin}(C) \cap \text{lin}(D)$) is either a hyperplane (we like!) or the entire vector space (we don’t like!):

**Lemma 2.5.5.** Let $C$ and $D$ be nonempty complementary convex sets in $X$. Let $M := \text{lin}(C) \cap \text{lin}(D)$. Then either $M$ is a hyperplane in $X$ (we like), or $M = X$ (we don’t like).

**Proof.** Both $C$ and $D$ are convex so by Lemma 2.4.14, $\text{lin}(C)$ and $\text{lin}(D)$ are convex sets. The intersection of two convex sets is convex so $M$ is a convex set (it is easy to see that $M$ is not empty). By Lemma 2.5.3 and the fact that $C$ and $D$ are complementary in $X$ gives,

$$C = X \setminus D$$

$$\text{lin}(C) = \text{lin}(X \setminus D) = X \setminus \text{cor}(D)$$

$$D = X \setminus C$$

$$\text{lin}(D) = \text{lin}(X \setminus C) = X \setminus \text{cor}(C)$$

and this implies

$$M = \text{lin}(C) \cap \text{lin}(D) = X \setminus \text{cor}(D) \cap X \setminus \text{cor}(C)$$

**WTS 1:** Show $M$ is an affine subspace of $X$. Let $x$ and $y$ be distinct points in $M$ and $z$ a point on the line that is generated by taking all affine combinations of $x$ and $y$. If we can show $z \in M$ then it follows that $M$ is an affine subspace of $X$. So assume $z \notin M$ and show that this leads to a contradiction. Since $M$ is convex, and $x$ and $y$ are in $M$, we know $z$ is not in the line segment $(x,y)$. We also know $z \notin M$ so we can assume without loss that $z \in \text{cor}(C)$. Since $z$ is not in the line segment $(x,y)$, we have $y$ is in the line segment $(z,x)$ or $x$ is in the line segment $(z,y)$. Assume without loss, that $y$ is in the line segment $(z,x)$. We know $x \in \text{lin}(C)$. But $x \notin \text{cor}(C)$ so $x \in \text{lin}(C)$. Then by Corollary 2.4.21 all of the points $(z,x)$ are in $\text{cor}(C)$. However, $y$ is in the line segment $(z,x)$ and this contradicts the fact that $y \notin \text{cor}(C)$. Hence $z \in M$ and $M$ is an affine subspace of $X$.

**WTS 2:** Show if $M \neq X$ then $M$ is a hyperplane. The theorem claims that $M$ is either all of $X$ or a hyperplane. From WTS 1 we know $M$ is an affine subspace and now we show that if $M \neq X$ then $M$ is a hyperplane. Assume without loss, that $M$ is a linear subspace. Since $M \neq X$ there exists $p \in X \setminus M$. If $p$ is not in $M$ then $p$ is in the core of $C$ or the core of $D$. Assume without loss, $p \in \text{cor}(C)$. Since $p \notin M$, and $M$ is a linear subspace, it follows that $-p \notin M$. Then $-p \in \text{cor}(C)$ or $-p \in \text{cor}(D)$. By Lemma 2.4.3, $\text{cor}(C)$ is convex, so if $p$ and $-p$ are in $\text{cor}(C)$ then $\theta \in \text{cor}(C)$. But $\theta \in \text{cor}(C)$ would imply $\theta \notin M$ which contradicts the fact the $M$ is a vector space. Therefore, $-p \in \text{cor}(D)$. If we can show $X = \text{span}(M, \{p\})$ we will be done because this implies $M$ is a maximal proper subspace. But first prove the following intermediate result.

$$\forall x \in C, \quad \{-p,x\} \cap M \neq \emptyset \quad (2.5.9)$$

$$\forall y \in D, \quad \{p,y\} \cap M \neq \emptyset \quad (2.5.10)$$
2.5. SEPARATING HYPERPLANES

and this will help us establish later that \( X = \text{span}(M, \{p\}) \). It suffices to show one of these conditions since the proof is identical for both cases. We choose to show that for all \( y \in D \) that \( [p, y] \) has a nonempty intersection with \( M \). Since \( y \in D \), we cannot have \( y \in \text{cor}(C) \) so by Lemma 2.5.4 we know \( y \) is in \( M \) or \( y \in \text{cor}(D) \). If \( y \in M \) then \( y \in \text{span}(M, \{p\}) \) and we are done. So assume \( y \in \text{cor}(D) \) and define \( \lambda_1 \) as follows:

\[
\lambda_1 = \sup\{ \lambda : 0 < \lambda \leq 1 \text{ and } [p, p + \lambda(y - p)] \subseteq C \}
\]

The implication of \( \lambda_1 = 1 \) is that the interval \( [p, y] \) lies in \( C \). We argue that this is not possible. Since \( y \) is a core point of \( D \), we cannot move an epsilon amount from \( y \) in the direction of \( p \) and remain in \( D \). This implies there points in the interval \( [p, y] \) that are in both \( C \) and \( D \) which contradicts the fact that \( C \) and \( D \) are complementary.

Now assume \( \lambda_1 < 1 \). Consider the point \( z = p + \lambda_1(y - p) \). By Lemma 2.5.4 either \( z \in \text{cor}(C) \), \( z \in \text{cor}(D) \), or \( z \in M \). If \( z \in \text{cor}(C) \) along the line \( [p, y] \) in the direction of \( y \) from \( p \) while remaining in \( C \). This contradicts the fact the \( \lambda_1 \) is a sup. If \( z \in \text{cor}(D) \) then we can move an epsilon distance in the direction of \( p \) and remain in \( D \). Then we have points in the intersection of \( C \) and \( D \) which cannot happen. Therefore \( z \in M \) and we have satisfied our condition.

WTS 3: Show (2.5.9)-(2.5.10) implies \( X = \text{span}(M, \{p\}) \). Consider (2.5.10). Since \( p \in \text{cor}(C) \) and \( p \notin M \), if \( [p, y] \cap M \neq \emptyset \) there is a \( \lambda \in (0, 1) \) such that \( m \in M \) where

\[
m = \lambda p + (1 - \lambda)y
\]

Since \( \lambda \neq 1 \) we have

\[
y = \frac{m}{1 - \lambda} - \frac{\lambda p}{1 - \lambda}
\]

and this implies \( y \in \text{span}(M, \{p\}) \). Likewise for (2.5.9). Since \( X = C \cup D \) we have shown for any \( x \in X \), \( x \in \text{span}(M, \{p\}) \). This implies that \( M \) is a maximal linear subspace and thus a hyperplane.

\[\square\]

Theorem 2.5.6 (Basic Separation Theorem). Let \( A \) and \( B \) be disjoint non-empty convex subsets in \( X \). Assume that either \( X \) is finite dimensional or else that \( \text{cor}(A) \cup \text{cor}(B) \neq \emptyset \). Then \( A \) and \( B \) can be separated by a hyperplane.

Proof. By Stone’s result (Lemma 2.5.2), we can find complementary sets \( C \) and \( D \) in \( X \) such that \( A \subseteq C \) and \( B \subseteq D \). As in Lemma 2.5.5, define \( M = \text{lin}(C) \cap \text{lin}(D) \). By the Lemma, either \( M \) is a hyperplane, or \( M = X \). If \( M \) is a hyperplane, then \( M \) separates \( A \) and \( B \). The reasoning is as follows. Assume without loss that \( M \) goes through the origin, and let \( \phi \) be the element in the dual of \( X \) which defines \( M \); that is, \( M = [\phi, 0] \). We now
argue that for all \( x \in C \), we have \( \langle x, \phi \rangle \geq 0 \) (we choose \( \geq \) without loss of generality). If \( x \in C \setminus \text{cor}(C) \) then \( x \in M \) and hence \( \langle x, \phi \rangle = 0 \) and we’re done. Suppose for contradiction that there exists \( x, y \in \text{cor}(C) \) such that \( \phi(x) = \alpha > 0 \) and \( \phi(y) = \beta < 0 \). Consider the point:

\[
z = \frac{-\beta}{\alpha - \beta} x + \frac{\alpha}{\alpha - \beta} y
\]

which is a convex combination of \( x \) and \( y \) and hence remains in \( \text{cor}(C) \). By definition of hyperplane, \( \phi \) is a linear functional so

\[
\phi(z) = \phi \left( \frac{-\beta}{\alpha - \beta} x + \frac{\alpha}{\alpha - \beta} y \right) = \frac{-\beta}{\alpha - \beta} \phi(x) + \frac{\alpha}{\alpha - \beta} \phi(y)
\]

\[
= \frac{-\beta}{\alpha - \beta} \alpha + \frac{\alpha}{\alpha - \beta} \beta = 0
\]

and implies \( z \in M \) which cannot be since it is a convex combination of points in \( \text{cor}(C) \). Thus we have \( \phi(x) > 0 \) for all \( x \in \text{cor}(C) \). We use similar logic to conclude \( \phi(y) < 0 \) for all \( y \in \text{cor}(D) \). We conclude \( M \) separates \( A \) and \( B \).

Suppose \( M \) is not a hyperplane. By Lemma 2.5.5 \( M = \text{lin}(C) \cap \text{lin}(D) = X \), which in turn implies \( \text{cor}(C) = \text{cor}(D) = \emptyset \) by Lemma 2.5.4. However, this contradicts the hypothesis that \( \text{cor}(A) \cup \text{cor}(B) \neq \emptyset \). It remains to show that \( X \) cannot be finite dimensional. Now, \( M = \text{lin}(C) \cap \text{lin}(D) = X \) this implies \( \text{lin}(C) = X \) and \( \text{lin}(D) = X \) and thus \( C \) and \( D \) are both ubiquitous. Both \( C \) and \( D \) are proper sets since \( A \) and \( B \) are not void. Then by Theorem 2.4.18, \( X \) has infinite dimension, a contradiction. We finally conclude that, under the hypotheses, \( M \) must be a hyperplane and thus separates \( A \) and \( B \).

2.5.3 Improvements on basic separation

The Basic Separation Theorem will be a workhorse result for us in the remainder of the notes. It proof is revealing in that it tells us precisely where and why core points are needed to turn Stone’s Lemma into a separation result. However, the result is called “Basic” because it can immediately be made more general with a few tweaks. The need for a more general result is immediately apparent from simple examples in \( \mathbb{R}^2 \) which violate the conditions of the Basic Separation Theorem but can clearly be separated. Figures 2.15 and 2.16 provide a couple of examples of this.

The conditions of the Basic theorem can fail in two ways: (i) the sets are not disjoint, and (ii) neither set may have a non-empty core (see Figure 2.15). We first address issue (i).

Both directions use the following powerful result which speaks to relationships between \( A \) and \( \text{icr}(A) \) – for convex \( A \) with non-empty intrinsic cores – in terms of separation.

**Lemma 2.5.7.** Let \( A \) be a convex set with a non-empty intrinsic core.
(i) Suppose $\langle a, \phi \rangle \leq \alpha$ for all $a \in \text{icr}(A)$. Then we have $\langle a, \phi \rangle \leq \alpha$ for all $a \in A$. In particular, if you can separate a set $B$ from $\text{icr}(A)$, you can separate the set from $A$ using the same hyperplane.

(ii) Suppose $\langle a, \phi \rangle \leq \alpha$ for all $a \in A$. Then either $\langle a, \phi \rangle = \alpha$ for all $a \in A$ or $\langle a, \phi \rangle < \alpha$ for all $a \in \text{icr}(A)$. In particular, if you can (weakly) separate a set $B$ from $A$, then you can strictly separate that set from $\text{icr}(A)$ using the same hyperplane assuming the hyperplane does not contain all of $A$.

**Proof.** (i) Assume there is a $z \in A$ such that $\langle z, \phi \rangle > \alpha$. By hypothesis the intrinsic core of $A$ is not empty so let $p \in \text{icr}(A)$. Since $z \in A$ we have by Corollary 2.4.22 that $[p, z) \subset \text{icr}(A)$. For sufficiently small $\epsilon > 0$, $z + \epsilon(p - z)$ is a point on the line segment $[p, z)$ and therefore in the intrinsic core. Then

$$\langle z + \epsilon(p - z), \phi \rangle = \langle z, \phi \rangle + \epsilon((p - z), \phi)$$

If $\langle z, \phi \rangle > \alpha$, then for sufficiently small $\epsilon > 0$, $\langle z + \epsilon(p - z), \phi \rangle > \alpha$. However, this would imply that for the point $z + \epsilon(p - z)$ in the intrinsic core which does not satisfy $\langle z + \epsilon(p - z), \phi \rangle \leq \alpha$. This violates our hypothesis and therefore we must have $\langle z, \phi \rangle \leq \alpha$, implying the desired result.

The fact that if you can separate a set $B$ from $\text{icr}(A)$, you can separate that set from $A$ using the same hyperplane now follows immediately. Suppose the hyperplane that separates $B$ from $\text{icr}(A)$ has the form $H = [\phi, \alpha]$. Then we have $\langle a, \phi \rangle \leq \alpha \leq \langle b, \phi \rangle$ for all $a \in \text{icr}(A)$ and $b \in B$. We have just shown that we have $\langle a, \phi \rangle \leq \alpha$ for all $a \in A$. This implies the desired separation.

(ii) If $\langle a, \phi \rangle \leq \alpha$ for all $a \in A$, and there is an $a \in A$ such that $\langle a, \phi \rangle < \alpha$, then $\langle z, \phi \rangle < \alpha$ for all $z$ in the intrinsic core of $A$. Proof by contradiction. Assume there exists a $z \in \text{icr}(A)$ with $\langle z, \phi \rangle = \alpha$. By Lemma 2.4.4 there exists an $x \in A$ such that $z \in (a, x)$. By Exercise 2.33 we have

$$\langle a, \phi \rangle < \langle z, \phi \rangle < \langle x, \phi \rangle$$

But if $\langle z, \phi \rangle = \alpha$, then $\langle z, \phi \rangle < \langle x, \phi \rangle$ implies $\langle x, \phi \rangle > \alpha$. This is a contradiction since $\langle a, \phi \rangle \leq \alpha$ for all $a \in A$. Therefore $\langle a, \phi \rangle < \alpha$ for all $a \in \text{icr}(A)$.

**Remark 2.5.8.** In Theorem 2.5.13 we show that when a convex set $A$ has a nonempty intrinsic core, then there always exists a hyperplane defined by a linear functional $\phi$ such that $\langle a, \phi \rangle < \alpha$ for all $a$ in the intrinsic core of $A$.

Now for a basic strengthening of the Basic Separation Theorem which no longer requires disjointness of the sets and gives a characterization of separation:
Theorem 2.5.9. Let $A$ and $B$ be convex sets in the vector space $X$. Assume that $\text{cor}(A) \neq \emptyset$. Then $A$ and $B$ can be separated by a hyperplane if and only if $\text{cor}(A) \cap B = \emptyset$.

Proof. ($\Rightarrow$) Suppose $A$ and $B$ can be separated by a hyperplane $H = [\phi, \alpha]$, with $\phi(a) \leq \alpha$ for all $a \in A$, and $p$ be a core point of $A$. If $\phi(p) = \alpha$ then we have points in $A$ on either side of $H$: take a $y \in X$ with $\phi(y) > 0$. Then since $p \in \text{cor}(A)$ we have $x + \lambda_1y \in A$ for some $\lambda_1 > 0$ and $x - \lambda_2y \in A$ for some $\lambda_2 > 0$. But $\phi(x + \lambda_1y) = \alpha + \lambda_1\phi(y) > \alpha$ and $\phi(x - \lambda_2y) = \alpha - \lambda_2\phi(y) < \alpha$ and we have points in $A$ (namely, $x + \lambda_1y$ and $x - \lambda_2y$) which lie on either side of $H$, contradicting the fact that $H$ separates $A$ and $B$. Thus we may assume that $\phi(p) < \alpha$ for all $p \in \text{cor}(A)$. Since $\phi(b) \geq \alpha$ for all $b \in B$ this implies that $\text{cor}(A)$ and $B$ are disjoint.

($\Leftarrow$) Since $\text{cor}(A)$ and $B$ are disjoint – and by Corollary 2.4.23 the core of $\text{cor}(A)$ is nonempty – we may apply the Basic Separation Theorem to separate these sets by a hyperplane $H$. Now leverage Lemma 2.5.7(i). Since $\text{cor}(A)$ is nonempty, by Lemma 2.4.7, $\text{icr}(A) = \text{cor}(A)$. So the hyperplane $H$ separates $\text{icr}(A)$ and $B$. Thus by Lemma 2.5.7(i), the hyperplane $H$ also separates $A$ and $B$.

To address (ii) we develop a series of results to demonstrate how to relax the core condition with an intrinsic core condition. Here is a building block in that theory. Its main function is to translate the question of separating convex sets to that of isolating convex sets from points:

Lemma 2.5.10. The nonempty convex sets $A$ and $B$ can be separated if and only if $\theta$ can be separated from $A - B$.

Proof. ($\Leftarrow$) Suppose $\theta$ can be separated from $A - B$; that is, there exists a hyperplane $H = [\phi, \alpha]$ such that $\phi(c) \leq \alpha \leq \phi(c)$ for all $c \in A - B$. Note that we may assume that $\alpha = 0$, since $\phi(\theta) = 0$ always and if $\phi(c) \geq \alpha$ for all $c \in A - B$ and $\alpha > 0$ then clearly $\phi(c) \geq 0$ and so we can take $\alpha = 0$. Now, note that $\phi(c) \geq 0$ for all $c \in A - B$ implies $\phi(a - b) \geq 0$ for $a \in A$ and $b \in B$ and thus by linearity we have $\phi(a) \geq \phi(b)$ for all $a \in A$ and $b \in B$. Letting $\beta = \inf \{\phi(a) : a \in A\}$ we can then conclude that $H' = [\phi, \beta]$ separates $A$ and $B$. Note that $\beta > -\infty$ since $\phi(a) \geq \phi(b)$ for all $a \in A$ and $b \in B$ and since $B$ is nonempty we get a finite lower bound on $\beta$.

($\Rightarrow$) Suppose $H = [\phi, \beta]$ separates $A$ and $B$; that is $\phi(a) \geq \beta \geq \phi(b)$ for all $a \in A$ and $b \in B$. Subtracting $\phi(b)$ from all sides we have: $\phi(a) - \phi(b) \geq \beta - \phi(b) \geq 0$ for all $a \in A$ and $b \in B$. Then, clearly $H = [\phi, 0]$ separates $A - B$ and the origin, since for all $a \in A$ and $b \in B$

$$\phi(a - b) = \phi(a) - \phi(b) \geq 0 = \phi(\theta).$$

This completes the proof.
2.5. SEPARATING HYPERPLANES

Figure 2.15: An example where we cannot apply Basic Separation Theorem but sets can still be separated. Disjoint sets $A$ and $B$ in $\mathbb{R}^3$ are drawn so that their affine hulls are parallel two dimensional affine subspaces of $\mathbb{R}^2$. Since their set has $\text{aff}(A) \neq \mathbb{R}^3$ it is immediate that $\text{cor}(A) = \text{cor}(B) = \emptyset$ so the Basic Separation Theorem does not apply. However, it is clear that $A$ and $B$ can be separated by a hyperplane. Geometrically, simply take a hyperplane parallel to $\text{aff}(A)$ and $\text{aff}(B)$ positioned, say, half way in between them.

We now show how to separate points from the intrinsic core. The proof uses a powerful result from linear algebra called the Fredholm alternative. The result in finite dimensions should be familiar:

**Proposition 2.5.11.** Let $X$ and $Y$ be finite dimensional vector spaces and let $A$ be a matrix describing a linear map between $X$ and $Y$. Either 1) the system $Ax = b$ has a solution or 2) there is a solution to $y^\top A = 0$, $y^\top b \neq 0$ but not both 1) and 2).

Proposition 2.5.11 is easily proved by pivoting on the system $Ax = b$. But what about infinite dimensional vector spaces. What is the analog? We need the following concept:

**Definition.** Let $M$ be a subspace of a vector space $X$. Then the annihilator of $M$, denoted $M^\circ$, is that subset of $X'$ which consists of the linear functions which map every point in $M$ to zero. In other words:

$$M^\circ = \{ \phi \in X' : \phi(m) = 0 \text{ for all } m \in M \}.$$
Figure 2.16: An example where we cannot apply Basic Separation Theorem but sets can still be separated. Assume sets $A$ and $B$ lie in $\mathbb{R}^2$ and hence have non-empty core, but we have $A \cap B = \{x\}$ and so fail to be disjoint. The Basic Separation Theorem does not apply. However, there is a hyperplane through $x$ which separates $A$ and $B$.

The annihilator is analogous to the perpendicular subspace $M^\perp$ in $\mathbb{R}^n$. We can state a general Fredholm result:

**Proposition 2.5.12** (Fredholm Alternative). Let $M$ be a proper subspace of $X$. If $x \in X$, then either: 1) $x \in M$, or 2) $x \notin M$ and there exists a $\phi \in M^0$ such that $\langle x, \phi \rangle \neq 0$, but not both 1) and 2).

**Proof.** Either $x$ is in $M$ or it is not. If $x$ is in $M$, then by definition of $M^0$, $\langle x, \phi \rangle = 0$, for all $\phi \in M^0$ so we have 1) and not 2).

Now assume $x \notin M$. Since $x \notin M$ we know $x \neq \theta$. Also, since $M$ is a subspace of $X$ there exists another subspace $N$ of $X$ such that $X = M \oplus N$ via Proposition 2.1.4. Thus any $y \in X$ can be written as $y = m_y + n_y$ with $m_y \in M$ and $n_y \in N$. Since $x \notin M$, this implies $n_x \neq \theta$.

We invoke Hamel’s theorem and note that $N$ has a basis $B$. Recall that this means that every $n \in N$ can be uniquely written as a linear combination of a finite number of basis vectors. Call that finite set $B(n)$. Thus, we write

$$n = \sum_{b \in B(n)} \alpha^n_b b$$

(2.5.11)

where we may assume the $\alpha^n_b \neq 0$. Note that since $n_x \neq \theta$, $B(n_x) \neq \emptyset$.

Now, by Zermelo’s Theorem (Theorem A.1.8) the basis $B$ can be well-ordered. Thus, every nonempty subset in $B$ has a least element in the ordering. This allows us to define a
smallest basis vector \( s(n) \) in the set \( B(n) \) when it is nonempty. In particular, basis element \( s(n_x) \) is well-defined. Now, our given \( x = m_x + n_x \) in the statement of the theorem allows us to define a linear functional \( f : X \to \mathbb{R} \) defined as follows: for \( y = m_y + n_y \in X \):

\[
f(y) = a_{s(n_x)}^{n_y}.
\]

In other words, \( f(y) \) gives the coefficient on basis element \( s(n_x) \) in the expansion of \( n_y \) described in (2.5.11). When \( s(n_x) \notin B(n_y) \) we simply take \( a_{s(n_x)}^{n_y} = 0 \). Clearly, such a function is linear, \( f(y + z) \) is nothing other than sum of the coefficients on basis element \( s(n_x) \) in the expansions of \( n_y \) and \( n_z \). Seeing that \( f(\alpha y) = \alpha f(y) \) for \( \alpha \in \mathbb{R} \) is equally as straightforward.

Next, let \( P \) be the projection mapping from \( X \) onto \( N \). That is, \( P(n + m) = n \). Define \( \phi \in X' \) by \( \phi = f \circ P \). Then for \( y \in X \), \( \phi(y) = f(n_y) \), and in particular when \( y \in M \) we have \( \phi(y) = f(\theta) = 0 \). Therefore \( \phi \in M^\circ \) and we have 2) and not 1).

Now for our separation result:

**Theorem 2.5.13.** (Strictly separating a point from the intrinsic core of a convex set \( A \)) Let \( A \) be a convex set in the vector space \( X \). If \( \text{icr}(A) \neq \emptyset \) and \( x \notin \text{icr}(A) \), then \( x \) can be strictly separated from \( \text{icr}(A) \).

**Proof.** Without loss, assume \( \theta \in A \). Let \( M = \text{span}(A) \). Since \( \text{icr}(A) \neq \emptyset \), and \( \theta \in A \), it follows that \( M \) is a proper subspace of \( X \). Either \( x \notin M \) or \( x \in M \). Consider first the case where \( x \notin M \). By the Fredholm Alternative (Proposition 2.5.12), there exits a \( \phi \in M^\circ \) such that \( \langle x, \phi \rangle \neq 0 \). Assume, without loss, \( \langle x, \phi \rangle > 0 \). Define the hyperplane \( H := [\phi, 0] \).

Since \( \phi \) is an annihilator of \( M \), and \( \text{icr}(A) \) is a subset of \( M \), \( \langle y, \phi \rangle = 0 < \langle x, \phi \rangle \) for all \( y \in \text{icr}(A) \). Therefore \( H \) strictly separates \( x \) from \( \text{icr}(A) \).

Now assume \( x \in M \). Since \( \theta \in A \), \( \text{aff}(A) = \text{span}(A) = M \) and so we can consider the setting as a separation problem in the vector space \( M \). Note that with respect to \( M \), \( A \) has a non-empty core (since \( \text{icr}(A) = \text{cor}_{\text{aff}(A)}(A) \)). Applying Theorem 2.5.6 yields a hyperplane \( H_M = [\phi_M; \alpha] \) such that for all \( a \in A \):

\[
\langle a, \phi_M \rangle \leq \alpha \leq \langle x, \phi_M \rangle
\]

Since all of the intrinsic core points are core points relative to \( M \), we apply Exercise 2.37 to all the intrinsic core points \( a \) and conclude that \( \langle a, \phi_M \rangle \neq \alpha \). This together with \( \langle a, \phi_M \rangle \leq \alpha \) for all points in the intrinsic core of \( A \) implies \( \langle a, \phi_M \rangle < \alpha \) for all points in the intrinsic core of \( A \).

Now extend \( \phi_M \) to \( X \) as follows. Let \( P \) be the standard projection map from \( X \) to \( M \). That is, \( P : X \to M \) is defined by \( P(m + n) = m \) for all \( m \in M \) and \( n \in N \) where \( X = M \oplus N \). Define \( \phi \in X' \) by \( \phi := \phi_M \circ P \), which is clearly linear, since it is the composition of linear maps. Then the hyperplane \( H := [\phi, \alpha] \) strictly separates \( x \) from the intrinsic core of \( A \).
Putting these together yields the following important result:

**Theorem 2.5.14. (Separating a point from a convex set)** Let $A$ be a convex set with nonempty intrinsic core. Then if $x \notin \text{icr}(A)$ it can be separated from $A$.

**Proof.** By Theorem 2.5.13 we can separate $x$ from $\text{icr}(A)$. Applying Lemma 2.5.7(i) we immediately conclude that $x$ can be separated from $A$. □

We are now ready to show how we can separate two convex sets with disjoint nonempty intrinsic cores. We can apply this theorem to the examples in Figures 2.15 and 2.16.

**Theorem 2.5.15.** Let $A$ and $B$ be convex subsets in $X$ with non-empty intrinsic cores. Then $A$ and $B$ can be separated by a hyperplane if $\text{icr}(A) \cap \text{icr}(B) = \emptyset$.

**Proof.** ($\Leftarrow$). First we show a weaker condition, namely $\text{icr}(A)$ and $\text{icr}(B)$ can be separated. Define $K = \text{icr}(A) - \text{icr}(B)$. Since $\text{icr}(A)$ and $\text{icr}(B)$ are convex sets it follows that $K$ is convex set. We first show $\text{icr}(K)$ not empty. By hypothesis the intrinsic cores of $A$ and $B$ are not empty, so there exists $a \in \text{icr}(A)$ and $b \in \text{icr}(B)$. We show $a - b \in \text{icr}(K)$. Let $z \in \text{aff}(K)$, our goal is to show that there is a $\lambda > 0$ such that $[(a-b), (a-b)+\lambda(z-(a-b))] \subset A$. It is easy to show

$$\text{aff}(K) = \text{aff}(\text{icr}(A) - \text{icr}(B)) \subseteq \text{aff}(\text{icr}(A)) - \text{aff}(\text{icr}(B))$$

Then $z \in \text{aff}(K)$ implies there are $z^1 \in \text{aff}(\text{icr}(A)) \subseteq \text{aff}(A)$ and $z^2 \in \text{aff}(\text{icr}(B)) \subseteq \text{aff}(B)$ such that $z = z^1 - z^2$. Since $a \in \text{icr}(A)$ and $b \in \text{icr}(B)$ there exists $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$[a, a + \lambda_1(z^1 - a)] \subset A \quad [b, b + \lambda_2(z^1 - b)) \subset B$$

Let $\overline{\lambda} = \min\{\lambda_1, \lambda_2\}$, then:

$$[a, a + \overline{\lambda}(z^1 - a)] \subset A \quad [b, b + \overline{\lambda}(z^1 - b)) \subset B$$

Next observe that for any $x \in [a, a + \overline{\lambda}(z^1 - a))$ we have $x \in A$ so by our Corollary 2.4.22:

$$[a, x) \subset \text{icr}(A).$$

This implies

$$[a, a + \overline{\lambda}(z^1 - a)) \subset \text{icr}(A)$$

By the same logic

$$[b, b + \overline{\lambda}(z^2 - b)) \subset \text{icr}(B)$$
Therefore
\[
[a, a + \lambda(z^1 - a)) - [b, b + \lambda(z^1 - b)) \in \text{icr}(A) - \text{icr}(B)
\]

Also, \(\theta \notin K\) since \(\text{icr}(A) \cap \text{icr}(B) = \emptyset\). By Theorem 2.5.14, \(\theta\) (which is also not in \(\text{icr}(K)\)) can be separated from \(K\). Then by Lemma 2.5.10, \(\text{icr}(A)\) and \(\text{icr}(B)\) can be separated. It is then straightforward by applying Lemma 2.5.7(i) (possibly twice) to conclude that \(A\) and \(B\) can be separated.

This final separating hyperplane results concerns \textit{strong separation}. This result underpins the Lagrangian duality theory of Chapter 3.

**Theorem 2.5.16. (Strongly separating a point from the algebraic closure of a convex set)\)** If \(A\) is a convex set in the vector space \(X\), and the intrinsic core of \(A\) is not empty, then any \(x \notin \text{lin}(A)\) can be \textit{strongly separated} from \(\text{lin}(A)\). That is, there exists a \(\phi \in X'\) and \(\epsilon > 0\) such that \(\langle x, \phi \rangle > \langle y, \phi \rangle + \epsilon\) for all \(y \in \text{lin}(A)\).

\textit{Proof.} We are given an \(x \notin \text{lin}(A)\). By hypothesis, there is a point \(a \in A\) that is in the intrinsic core of \(A\). Consider the line segment \([a, x]\). There must exist a \(z \in (a, x)\) such that \(z \notin A\) otherwise we contradict the fact that \(x \notin \text{lin}(A)\). Since \(z \notin A\), by Theorem 2.5.13 \(z\) can be strictly separated from the intrinsic core of \(A\). That is, there exists \(\phi \in X'\) such that \(\langle z, \phi \rangle > \langle y, \phi \rangle\) for all \(y \in \text{icr}(A)\). Then by the first part of Lemma 2.5.7, \(\langle z, \phi \rangle \geq \langle y, \phi \rangle\) for all \(y \in A\).

Since \(\langle z, \phi \rangle \geq \langle y, \phi \rangle\) for all \(y \in A\), it follows \(\langle z, \phi \rangle \geq \langle y, \phi \rangle\) for all \(y \in \text{lin}(A)\) (See Exercise 2.38). At this point, we know \(\langle z, \phi \rangle > \langle a, \phi \rangle\) since \(a\) is in the intrinsic core of \(A\), and that \(z \in (a, x)\). It now follows from Exercise 2.33 applied to the collinear points \(a, z,\) and \(x\) that
\[
\langle x, \phi \rangle > \langle z, \phi \rangle > \langle a, \phi \rangle.
\]

Now set
\[
\epsilon = \frac{\langle x, \phi \rangle - \langle z, \phi \rangle}{2}
\]
and observe that \(\langle x, \phi \rangle \geq \langle y, \phi \rangle + \epsilon\) for all \(y \in \text{lin}(A)\). This establishes strong separation.

This above result is of primary importance in the next chapter when we take up duality theory in optimization. The immediate corollary is also commonly invoked:

**Corollary 2.5.17.** Let \(A\) be an algebraically closed set with nonempty intrinsic core, and let \(x \notin A\). Then \(x\) can be strongly separated from \(A\).
2.6 Cones and orderings

Separating hyperplanes play a crucial role in convex analysis, and essentially drive the key results in optimization. As we have seen, however, we need intrinsic core points (or similar requirements) to guarantee such separating hyperplanes exist. In optimization, the convex sets of interest are derived from the feasible sets of our key problem (1.2.1):

\[
\inf f(x) \\
\text{s.t. } G(x) \preceq \theta_Y \\
x \in \Omega.
\]

which are of the form \( \{x \in X : G(x) \preceq \theta\} \) and \( \Omega \). Usually \( \Omega \) is something simple, such as the entire space or a cone (see this section for a definition). However, the notation for \( \{x \in X : G(x) \preceq \theta\} \) still needs to be unpacked. This is where we finally set up house: in the next two subsections we take up the question of what this notation precisely means in general vector spaces.

The first thing that stands out is that the notation \( \preceq \) invokes some notion of ordering. But what are we ordering? In this case we are ordering a vector space. The feasible region is cut out of the space by asking whether vectors are “bigger” or “smaller” than others. A Wikipedia search (and we always trust Wikipedia) for defining a well-behaved order on a vector space yields the following desiderata:

**Definition.** The pair \( (X, \preceq) \) where \( X \) is a vector space and \( \preceq \) a binary relation on \( X \) is
called an ordered vector space if the following five conditions hold for all \( x, y, z \in X \) and \( \lambda \in \mathbb{R} \):

(V1) \( x \preceq x \),

(V2) \( x \preceq y \) and \( y \preceq z \) implies \( x \preceq z \),

(V3) \( x \preceq y \) and \( y \preceq x \) implies \( x = y \),

(V4) \( x \preceq y \) implies \( x + z \preceq y + z \) and

(V5) \( x \preceq y \) implies \( \lambda x \preceq \lambda y \) for \( \lambda \geq 0 \).

Property (V1) is known as reflexivity, (V2) as transitivity, and (V3) as antisymmetry. Any binary relation that satisfies (V1) and (V2) on a set (it need not be a vector space) is called a pre-order. If additionally the ordering satisfies (V3) for all elements in the set it is called a partial order (see Appendix A.1 for more details on partial orders). Properties (V4) and (V5) are only relevant for vector spaces and ensure that the ordering respects vector addition and scalar multiplication respectively. Notice that we don’t have a requirement sometimes called completeness: that every two vectors are comparable, that is, for \( x, y \in X \) either \( x \preceq y \) or \( y \preceq x \).

So how do we get our hands on meaningful binary relations in general vector spaces? In linear programming the constraints are based around statements such as \( Ax \leq b \) where \( \leq \) is a particular vector ordering of \( \mathbb{R}^n \). Sometimes this fact is a bit obscured when one is first acquainted with linear programming. This “usual” ordering of \( \mathbb{R}^n \) works component-wise; that is, \( x \preceq y \) if \( x_i \leq y_i \) where this latter \( \leq \) is the standard order in \( \mathbb{R} \). Let’s interpret this a bit differently and note that it is equivalent to saying that \( x \preceq y \) if \( y - x \in \mathbb{R}^n_+ \) where \( \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_i \geq 0 \} \) is the nonnegative orthant of \( \mathbb{R}^n \).

We can generalize the idea that \( x \preceq y \) if and only if \( y - x \in \mathbb{R}^n_+ \) to more general vector spaces. Consider a binary relation \( \preceq \) for the ordered vector space \((X, \preceq)\). By (V4), \( x \preceq y \) if and only if \( x - x \preceq y - x \) if and only if \( \theta \preceq y - x \). In an ordered vector space \((X, \preceq)\), saying that \( x \preceq y \) is equivalent to saying \( \theta \preceq y - x \). This motivates us to associate with the binary relation \( \preceq \), a set \( S = \{ x \in X : \theta \preceq x \} \). In fact, we sometimes write \( \preceq_S \) to emphasize the fact that there is a set in \( S \subseteq X \) that corresponds to the binary relation and \( x \preceq_S y \) means \( y - x \in S \).

We know there is a set \( S \), that is associated with a given binary relation \( \preceq \) in the ordered vector space \((X, \preceq)\). Can we go in the reverse direction and associate a binary relation with a set? Assume we are given an arbitrary set \( S \) in \( X \). Define a binary relation \( x \preceq_S y \) if and only if \( y - x \in S \). Does \( \preceq_S \) define an order on \( X \) for \( S \)? In general, the answer is no! A major objective of this section is proving and understanding Theorem 2.6.3 that characterizes exactly which sets \( S \) define an order \( \preceq_S \). Yes, we know the suspense is too great to take – these sets turn out to be pointed, convex cones.
Remark 2.6.1. First a note on the geometry of our orderings. One way to see that \( x \preceq_S y \) is to reinterpret the statement that \( y - x \in S \) as \( y \in x + S \). See Figure 2.18 for an illustration.

![Figure 2.18: Geometric interpretation of a set-based ordering. On the left is a set \( S \), on right its translate to the vector \( x \). In the situation shown at the right we have \( x \preceq_S y \).](image)

Our motivating question then is which properties on the set \( S \) yield a binary relation \( \succeq_S \) with properties (V1)–(V5) and thus ensuring that \((X, \succeq_S)\) is an ordered vector space. Any old set \( S \) will not do, since any set that does not contain \( \theta \) will violate reflexivity (V1). Sets that contain \( \theta \) can still fail to satisfy other properties, for instance scalar multiplication (V5). Indeed, a simple consequence of (V5) is that for every \( s \in S \), \( \lambda s \in S \) for all \( \lambda \geq 0 \).

To see this, observe that if \( s \in S \), then \( \theta \preceq s \). But if (V5) holds, then \( \theta \leq s \) implies \( \theta \preceq_S \lambda s \) for all \( \lambda \geq 0 \) and hence \( \lambda s \in S \). A set with this property – that for all \( s \in S \), \( \lambda \geq 0 \), \( \lambda s \in S \) – is called a cone.

So will cones give us ordered vector spaces? Unhappily, we are not quite done. Consider the cone \( C = \mathbb{R}^2_+ \cup \mathbb{R}^2_- \) where again \( \mathbb{R}^2_+ \) is the nonnegative orthant and \( \mathbb{R}^2_- \) the nonpositive orthant (see Figure 2.19). Some playing around reveals that transitivity (V2) is violated. A simple example is as follows: Clearly \((0,0) \preceq_C (1,3) \). In addition, \((2,2) \preceq_C (0,0) \) since \((-2,-2) \) is in \( \mathbb{R}^2_- \). But \((2,2) \not\preceq_C (1,3) \) which violates transitivity. Observe also that \((0,0) \preceq_C (2,2) \) and \((2,2) \preceq_C (0,0) \), but \((0,0) \neq (2,2) \) so we also violate antisymmetry (V3).

So cones only get us so far, in particular we might not have (V2) or (V3). Looking at the cone \( C \) in Figure 2.19 a first idea is to consider convex cones. Indeed, we show that if a cone \( C \) is convex then the binary relation \( \preceq_C \) satisfies transitivity (V2).

**Lemma 2.6.2.** Let \( C \) be a convex cone in a vector space \( X \), then \((X, \preceq_C) \) is a pre-ordered vector space; that is, the relation \( \preceq_C \) satisfies (V1), (V2), (V4) and (V5).
2.6. CONES AND ORDERINGS

Proof. First show that the binary relation $\preceq_C$ satisfies (V4) which is the order preserves vector addition. If $x \preceq_C y$ then $x + z \preceq_C y + z$ for arbitrary $z \in X$ since $(y + z) - (x + z) = y - z \in C$. Since $C$ is a cone it contains $\theta$ which implies the reflexivity property (V1) holds: $x \preceq_C x$ since $x - x = \theta \in C$. We argued above property (V5), that the order preserves scalar multiplication, holds. This was true because $C$ is a cone. It remains to show that (V2), the transitivity property, holds. Suppose $x \preceq_C y$ and $y \preceq_C z$. Our goal is to establish $x \preceq_C z$; that is, $z - x \in C$. However, we know $y - x, z - y \in C$ and so by convexity of $C$:

$$\frac{1}{2}(y - x) + \frac{1}{2}(z - y) = \frac{1}{2}z - \frac{1}{2}x \in C.$$  

Finally, since $C$ is a cone we multiply the last expression by two (which allows it to remain in the cone) and hence $z - y \in C$. This establishes transitivity (V2). Convexity is required for transitivity.

Unfortunately, we still fall short of $(X, \preceq_C)$ being an ordered vector space for $C$ an arbitrary convex cone. Consider the following example. Let $C = \{x \in \mathbb{R}^2 : x_1 \geq 0\}$ (see Figure 2.20). Then $C$ is clearly a convex cone. Note that although $(0, 1) \preceq_C (0, -1)$ and $(0, -1) \preceq_C (0, 1)$ we have $(0, 1) \neq (0, -1)$ and thus (V3) is violated.

The essence of why anti-symmetry fails is that the cone is invariant under vertical translations. Any two points on the vertical axis are both bigger and smaller than each
other in the ordering, however they are not equal. Thus, we need an additional concept, that of a pointed cone. First we define the lineality space $\text{lineal}(C)$ of a convex cone $C$:

$\text{lineal}(C) = \{ x \in X : y + \lambda x \in C \text{ for all } y \in C, \lambda \in \mathbb{R} \}$.

If $x \in \text{lineal}(C)$, then we can take any point $y$ in the cone, and have the line $L = \{ y + \lambda x : \lambda \in \mathbb{R} \}$ contained in the cone. Yet another way to state this, is that if $x \in \text{lineal}(C)$ then $x + C = C$, that is $C$ is invariant under translations arising from $\text{lineal}(C)$ (see Exercise 2.39). A convex cone $C$ is pointed if $\text{lineal}(C) = \{ 0 \}$ (see Exercise 2.40 for an alternate definition). We now state the key result of this subsection, that pointed cones are precisely what is needed to define an order on a vector space.

**Theorem 2.6.3.** Let $X$ be a vector space.

(i) If $C$ is a pointed, convex cone, then $(X, \preceq_C)$ is an ordered vector space.

(ii) If $(X, \preceq)$ is an ordered vector space with binary relation $\preceq$, then $C = \{ x \in X : \theta \preceq x \}$ is a pointed, convex cone.

**Proof.** (part (i) ) Assume $C$ is a pointed, convex cone. Show that $\preceq_C$ defines an order on the vector space $X$. Since $C$ is a convex cone, by Lemma 2.6.2 the ordering $\preceq_C$ satisfies
2.6. CONES AND ORDERINGS

(V1), (V2), (V4) and (V5). It remains to show (V3). Prove the contrapositive and show that if antisymmetry does not hold, then C is not pointed. If \( \leq_C \) violates antisymmetry, there exists \( x \) and \( y \) in \( X \) such that \( x \leq y \), \( y \leq x \) but \( x \neq y \). Then \( x - y, y - x \in C \). Consider an arbitrary point \( w \in C \). We show that \( w + \lambda(x - y) \in C \) for all \( \lambda \in \mathbb{R} \), thus establishing \( x - y \in \text{lineal}(C) \).

This is proved using a useful result for convex cones, if \( a, b \in C \) then \( a + b \in C \). The reasoning is simple: if \( a, b \in C \) then by convexity of \( C \) we have \( \frac{1}{2}a + \frac{1}{2}b \in C \). But then since a cone is closed under nonnegative scalings we have \( 2(\frac{1}{2}a + \frac{1}{2}b) = a + b \in C \).

With this result in hand, we simply note that \( \lambda(x - y) \in C \) for all \( \lambda \in \mathbb{R} \) since both \( x - y \) and \( y - x \) are in \( C \). Then given our arbitrary \( w \in C \) it follows by the result in the previous paragraph that \( w + \lambda(x - y) \in C \) for all \( \lambda \in \mathbb{R} \). Thus, \( x - y \in \text{lineal}(C) \) which contradicts the fact that \( C \) is pointed.

(part (ii)) Assume \((X, \preceq)\) is an ordered vector space. Let \( C = \{x \in X : x \succeq \theta\} \). It suffices to show that \( C \) is a pointed, convex cone. By (V5), \( \alpha x \succeq \alpha \theta = \theta \) whenever \( x \succeq \theta \) and therefore \( C \) is a cone. To see that \( C \) is also convex, let \( x \in C \) and \( y \in C \) and let \( 0 \leq \lambda \leq 1 \). By (V5), we have \( \lambda x \succeq \theta \) and \( (1 - \lambda)y \succeq \theta \). Then (V4) implies

\[
\lambda x + (1 - \lambda)y \succeq \theta + \theta = \theta.
\]

It remains to show \( C \) is pointed. Prove the contrapositive and show that if \( C \) is not pointed, then \( \succeq \) is not a order on \( X \). If \( C \) is not pointed, there exists a nonzero \( x \) in \( \text{lineal}(C) \). This implies \( y + \lambda x \in C \) for all \( y \in C \) and \( \lambda \in \mathbb{R} \). By taking \( y = \theta \) and \( \lambda = \pm 1 \) we get \( x \in C \) and \( -x \in C \). This means that we have \( x \neq \theta \) such that \( x \preceq \theta \) and \( x \succeq \theta \) which contradicts (V3).

**Remark 2.6.4.** Given an ordered vector space \((X, \preceq)\), we associate a set \( C = \{x : \theta \preceq x\} \) with the binary relation \( \preceq \). In part (ii) of Theorem 2.6.3 we proved that \( C \) was a pointed, convex cone. Since \( C \) is a cone, we know for an arbitrary \( z \in C \), that \((1 + \epsilon)z \in C \) and \( \epsilon z \in C \) for all \( \epsilon > 0 \). Then \( z = (1 + \epsilon)z - \epsilon z \). Let \( x = \epsilon z \) and \( y = (1 + \epsilon)z \). Then \( x \preceq y \) since \( y - x = z \in C \). Our story is now complete! Not only does \( x \preceq y \) imply \( y - x \in C \), but for any \( z \in C \) there are corresponding vectors \( x \) and \( y \) that are related by \( \preceq \).

**Remark 2.6.5.** This proof gives a complete characterization of ordered vector spaces \((X, \preceq)\). The binary relation \( \preceq \) corresponds to the set \( C = \{x \in X : x \succeq \theta\} \). The binary relation \( \preceq \) yields is an ordered vector space \((X, \preceq)\) if and only if \( C \) is a pointed, convex cone. We call \( C \) the positive cone associated with that ordering \( \preceq \). Sometimes we use the notation \( P \) instead of \( C \) to emphasize we have a positive cone.

Recall our key problem (1.2.1):

\[
\inf f(x) \quad \text{s.t.} \quad G(x) \preceq \theta \nu, \quad x \in \Omega.
\]
CHAPTER 2. DUALITY AND CONVEXITY IN VECTOR SPACES

We assume that the binary relation \( \preceq \) in the constraint set \( G(x) \preceq \theta Y \) defines an ordering on the vector space \( Y \). By Theorem 2.6.3, there is a corresponding positive cone \( C_Y \) in the vector space \( Y \). Then \( G(x) \preceq \theta Y \) is equivalent to saying \( \theta Y - G(x) \in C_Y \). So a feasible \( x \in \Omega \) is one where \( -G(x) \in C_Y \).

Just as every vector space \( X \) comes with an associated dual \( X' \), cones also have duals. In fact, we use cones to generalize the notion of nonnegativity of linear functionals on a space. Let \( C \) be a cone, then a linear functional \( \phi \in X' \) is nonnegative on \( C \) if \( \langle x, \phi \rangle \geq 0 \) for all \( x \in C \). The set of all nonnegative linear functionals on \( C \), denoted \( C^+ \), is called the dual cone of \( C \). The dual cone \( C^+ \) is obviously a subset of \( X' \). Similar to the definition of self-duality of vector spaces, we say a cone \( C \) is self-dual if \( C \) is isomorphic to \( C^+ \) (we say isomorphic since \( C \subseteq X \) and \( C^+ \subseteq X' \) so they cannot be equal).

Example 2.6.6. A commonly seen cone in linear programming is the nonnegative orthant:

\[
\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i = 1, \ldots, n \}
\]

It is straightforward to see that \( \mathbb{R}^n_+ \) is a pointed, convex cone and thus orders \( \mathbb{R}^n \) as a vector space. The ordering, \( \preceq \mathbb{R}_+ \), is typically denoted \( \leq \). We show that \( \mathbb{R}^n_+ \) is self-dual. In Example 2.2.3 we proved that \( \mathbb{R}^n \) is self-dual (as a vector space) and in this discussion we use this fact. The dual cone to \( \mathbb{R}^n_+ \) is

\[
(\mathbb{R}^n_+)^+ = \{ \phi_c \in (\mathbb{R}^n)' : \phi_c(x) \geq 0 \text{ for all } x \in \mathbb{R}^n_+ \}.
\]

Doing a little work reveals the following string of implications:

\[
\phi_c(x) \geq 0 \text{ for all } x \in \mathbb{R}^n_+ \iff c^\top x \geq 0 \text{ for all } x \in \mathbb{R}^n_+ \iff c \geq 0
\]

where we use the fact \( \phi_c(x) = c^\top x \) to move the second step, and the fact that if \( c_i < 0 \) for some \( i = 1, \ldots, n \) then \( c^\top e^i = c_i < 0 \) (where \( e^i \) is the \( i \)th standard basis vector of \( \mathbb{R}^n \)). This gives

\[
(\mathbb{R}^n_+)^+ = \{ \phi_c \in (\mathbb{R}^n)' : c \geq 0 \}
\]

which is clearly isomorphic to \( \mathbb{R}^n_+ \).

The following result is used to establish dual feasibility at a later point in the notes.

Theorem 2.6.7. If \( C \) is a proper convex cone of \( X \) with nonempty intrinsic core, then \( C^+ \) contains non-zero elements.

Proof. Since \( C \) is proper there exists a point \( x \in X \setminus C \). Since \( C \) is convex with a nonempty intrinsic core, we can apply Theorem 2.5.16 and strongly separate \( x \) from \( C \). This implies there exists \( \phi \in X' \) and \( \epsilon > 0 \) such that

\[
\langle x, \phi \rangle > \langle y, \phi \rangle + \epsilon \quad \forall y \in C.
\]
2.7. CONVEX MAPPINGS

This implies $\phi \neq \theta$, otherwise we would have $0 > \epsilon$. Now show $-\phi \in C^+$. We need the following:

**WTS:** If a linear functional $\phi$, is bounded above on a cone $C$, then it must be nonpositive on that cone.

Assume this is not the case, i.e., there is a $y$ such that $\langle y, \phi \rangle > 0$. Since $C$ is a cone, $\lambda y \in C$ for all $\lambda \geq 0$. Then $\langle y, \phi \rangle > 0$ implies $\langle \lambda y, \phi \rangle = \lambda \langle y, \phi \rangle$ becomes arbitrarily large as $\lambda$ goes to infinity. This cannot be since $\langle y, \phi \rangle$ is bounded above by $\langle x, \phi \rangle$. Therefore $-\phi \in C^+$ and $\phi$ is nonzero. We have produced a nonzero linear functional in $C^+$.

2.7 Convex mappings

Our main optimization problem (1.2.1) is not confined to a single vector space, but instead involves relationships between three different vector spaces. Recall the key problem:

$$\inf f(x)
\text{s.t. } G(x) \preceq \theta_Y
\quad x \in \Omega.$$ 

The variable $x$ lies in a convex subset $\Omega$ of vector space $X$. Thus, $X$ is one of our spaces which we call the *primal variable space*. The second vector space is $Y$ which we call the *primal constraint space*. The third space, a little less obvious than the others, is $\mathbb{R}$ which one might call the *objective space*. The mappings $f$ and $G$ characterize the relationships between these spaces. The following definition makes this precise:

**Definition.** Let $X$ and $Y$ be vector spaces. A function $G : X \to Y$ is called a *mapping* of vector spaces. Suppose additionally, we are given a convex set $\Omega \subset X$ and a pointed convex cone $C$ in $Y$. The mapping $G$ is **convex** over $\Omega$ with respect to ordering $\preceq_C$ if

$$G(\alpha x^1 + (1 - \alpha)x^2) \preceq_C \alpha G(x^1) + (1 - \alpha)G(x^2)$$

for all $x^1, x^2 \in \Omega$ and all $\alpha \in (0, 1)$.

Note that a mapping $G$ might be convex over one set $\Omega$ and not another, and similarly may be convex over one ordering $\preceq_C$ but not another. Thinking about simple mappings will elucidate the various possibilities.

2.8 Notes

The development in Sections 2.4 and 2.5 follows closely that of Holmes [16], although Holmes usually states his results assuming the existence of core points, whereas we prefer to
state them in terms of intrinsic core points, which is slightly more general. The codimension approach to defining hyperplanes was taken from Barvinok [4] along with Example 2.4.11 of a convex set that cannot be separated from the origin. The material in the remaining sections is fairly standard, with Aliprantis and Border [2] being a great reference in this area. Some of the material in Section 2.5 can also be found in Luenberger [20].

Wow. This was lot of work. Chris and Kipp are really tired!

2.9 Exercises

Exercise 2.1. Prove Proposition 2.1.2: Let $B$ be a Hamel basis of vector space $X$. Show that every vector $x \in X$ can be written as uniquely as a linear combination of vectors from $B$.

Exercise 2.2. Let $X$ be a vector space and $X'$ its algebraic dual. Show that $X'$ is itself a vector space.

Exercise 2.3. Prove Proposition 2.2.4: If $X$ is a $n$-dimensional real vector space, then $X$ is isomorphic (there is a one-on-one and onto vector space homomorphism) to $\mathbb{R}^n$.

Exercise 2.4. Prove Proposition 2.1.4: Let $M$ be a subspace of vector space $X$. Then $M$ has a complementary subspace.

Exercise 2.5. Let $X$ be an arbitrary vector space. Let $X'$ denote the dual of $X$ and $X''$ denote the dual of the dual, that is the second dual.

a. Show that $X$ is isomorphic to a subspace of $X''$.

b. Show that if $X$ has finite dimension, then $X \cong X''$. In general, if $X \cong X''$, then $X$ is said to be reflexive.

Exercise 2.6. Assume $X$ is a vector space of dimension $n$ and $Y$ is a vector space of dimension $m$. Let $\text{hom}(X,Y)$ denote the set of all homomorphisms from $X$ into $Y$.

a. Prove that $\text{hom}(X,Y)$ is a vector space.

b. What is the dimension of $\text{hom}(X,Y)$? Provide a proof of your answer.

Exercise 2.7. Let $\mathcal{S}^n$ denote the set of all symmetric matrices. Then $\mathcal{S}^n$ is a vector space under matrix addition and scalar multiplication. What is the dimension of this vector space? Provide a proof of your answer.

Exercise 2.8. Let $A = x^0 + M$ be an affine subspace. Let $x^1 \in A$ then $A = x^1 + M$. 

2.9. EXERCISES

Exercise 2.9. Prove Proposition 2.3.2: Let $A$ be a set in a vector space $X$. The following are two alternate characterizations of the affine hull of $A$:

1. $\text{aff}(A) = \{ \sum_i \alpha_i x^i : x^i \in A : \sum_i \alpha_i = 1, x^i \in A \}$. That is, $\text{aff}(A)$ is the set of all affine combinations of the elements of $A$.

2. $\text{aff}(A) = x + \text{span}(A - A)$ for any fixed $x \in A$, where $A - A = \{ x - y : x, y \in A \}$ is the Minkowski difference of $A$ with itself.

Exercise 2.10. Prove Proposition 2.3.3: Show that a set of vectors $\{x^0, \ldots, x^n\}$ are affinely independent if and only if $\{x^1 - x^0, \ldots, x^n - x^0\}$ are linearly independent. Use this to conclude that the affine hull of a set of $n+1$ affinely independent points has dimension $n$.

Exercise 2.11. Show that the quotient space $X/M$ is a vector space.

Exercise 2.12. Show that for an $n$-dimensional vector space $X$ and $M$ a linear subspace that $\dim(M) + \text{codim}(M) = n$.

Exercise 2.13. Assume $\{x^0, x^1, \ldots, x^n\}$ is an affinely independent set of vectors. Let $A = \text{conv}(\{x^0, x^1, \ldots, x^n\})$ denote their convex hull. Show that every point in $A$ can be expressed as a unique convex combination of points in $\{x^0, x^1, \ldots, x^n\}$.

Exercise 2.14. Show that if $M$ is a proper subspace, then $M^\circ$ has a nonzero element.

Exercise 2.15. Let $V$ be the space of differentiable functions on the interval $[0,1]$. Let $H = \{ f : f'(1/2) = 0 \}$. Prove that $H$ is a hyperplane. This is from Barvinok [4], Exercise 5 on page 113.

Exercise 2.16. Prove Lemma 2.4.1: Show that $\text{cor}(A) = \{ a \in A : \text{ for all } z \in X \text{ there exists } \lambda > 0 \text{ such that } [a, a + \lambda z] \subseteq A \}$. (2.9.1)

Exercise 2.17. Consider the nonnegative cone in $\mathbb{R}^n$ defined by

$$ C = \{ x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n \} $$

What is the core of $C$? Justify your answer based on the definition of core. Next consider

$$ K = \{ x \in \mathbb{R}^N \mid x_i \geq 0, i = 1, \ldots \} $$

What is the core of $K$? Justify your answer based on the definition of core.

Exercise 2.18. Prove Lemma 2.4.3: Show that if $A$ is convex then $\text{cor}(A)$ is convex.

Exercise 2.19. Show that in the $\iff$ part of the proof of Lemma 2.4.4, it is indeed possible to find another point $x \in L$, in addition to $a$, such that $x \in A$ where

$$ L = \{(1 - \lambda)a + \lambda z = a + \lambda(z - a) \mid \lambda \in \mathbb{R} \}. $$
Exercise 2.20. Prove Lemma 2.4.13: Show that if $A$ is convex and $|A| > 1$ then $\text{lin}(A) = \text{lin}(A)$. What happens when $|A| = 1$?

Solutions to Homework due Week 5

Exercise 2.21. Prove Lemma 2.4.14: Show that if $A$ is convex then $\text{lin}(A) = \text{lin}(A)$. What happens when $|A| = 1$?

Exercise 2.22. Prove Lemma 2.4.15: Show $\text{lin}(A) \subseteq \text{aff}(A)$.

Exercise 2.23. Prove Corollary 2.4.22: Let $A$ be a convex subset of the vector space $X$ and assume $p$ is in the intrinsic core of $A$. Show that if $x \in \text{lin}(A)$, then $[p, x] \subset \text{icr}(A)$.

Exercise 2.24. Let $\mathbb{R}^N$ denote the set of all real sequences. Consider the set $P = \{x \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i \in \mathbb{N}\}$. Show that $P$ has no intrinsic core.

Exercise 2.25. Prove Corollary 2.4.23: Show that when $A$ is convex $\text{cor}(\text{cor}(A)) = \text{cor}(A)$. In other words, show that $\text{cor}(A)$ is algebraically open.

Exercise 2.26. Lemma 2.5.2 was proven assuming $x, z \in D \setminus B$. Show the result holds in general, by considering the possibility that one or both of $x, z \in B$.

Hint: You may need to reconsider condition (*).

Exercise 2.27. Prove Lemma 2.4.7: If $A$ is a set with a nonempty core, then $\text{icr}(A) = \text{cor}(A)$.

Exercise 2.28. Prove Lemma 2.5.3: For a convex set $A$ in $X$, $\text{lin}(X \setminus A) = X \setminus \text{cor}(A)$.

Exercise 2.29. Show that no hyperplane is ubiquitous.

Exercise 2.30. Let $A \subseteq \mathbb{R}^n$ for some $n$. In this question the topology we refer to is the usual one induced using the Euclidean norm.

a. If $A$ is convex show: (i) $\text{cor}(A) = \text{int}(A)$ (the topological interior of $A$) and (ii) $\text{lin}(A) = \text{cl}(A)$ (the topological closure of $A$).

b. Show by example that when $A$ is not convex there can be points in $\text{cor}(A)$ which are not interior points of $A$. ($n = 2$ suffices)

Exercise 2.31. Let $M = x^1, \ldots, x^n$ be a set of affinely dependent points. Then, there are subsets $M_1$ and $M_2$ of $M$ with $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M$ such that the convex hulls of $M_1$ and $M_2$ intersect.

Exercise 2.32. (Radon’s Theorem) Let $M = x^1, \ldots, x^n$ be a set of $n$ points in $\mathbb{R}^d$ with $n \geq d + 2$. Then, there are subsets $M_1$ and $M_2$ of $M$ with $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M$ such that the convex hulls of $M_1$ and $M_2$ intersect.
Exercise 2.33. Assume $a, x,$ and $z$ are three points in $X$ with $z \in (a, x)$. If $\phi \in X'$, and $\langle z, \phi \rangle > \langle a, \phi \rangle$ show

$$\langle x, \phi \rangle > \langle z, \phi \rangle > \langle a, \phi \rangle$$

Exercise 2.34. If $A$ is a convex set in the vector space $X$ with a nonempty intrinsic core then $\text{lin} (\text{lin} (A)) = \text{lin} (A)$. Is the statement true if $A$ has an empty intrinsic core? [To what we currently know, this is an open question.]

Exercise 2.35. Let $A$ be a convex set with nonempty intrinsic core in a vector space $X$. Suppose $x \notin \text{lin} (A)$, show it can be strongly separated from $\text{icr} (A)$.

Exercise 2.36. Let $x$ and $y$ be two points in $X$ with $x \neq y$. Show that $x$ and $y$ can be strictly separated.

Exercise 2.37. Not assigned.

Exercise 2.38. If $A$ is a convex set in the vector space $X$ with $\phi \in X'$ such that $\langle x, \phi \rangle \leq \alpha$ for all $x \in A$, then $\langle x, \phi \rangle \leq \alpha$ for all $x \in \text{lin} (A)$.

Exercise 2.39. Let $C$ be a convex cone, show that $C = x + C$ for all $x \in \text{lineal} (C)$.

Exercise 2.40. Show that a convex cone $C$ is pointed if and only if $C \cap (-C) = \{0\}$.

Exercise 2.41. Assume $P$ is a convex cone in vector space $X$ and that $P$ has a nonempty intrinsic core, and $P = \text{lin} (P)$. Show that

$$\langle x, \phi \rangle \geq 0 \text{ for all } \phi \in P^+$$

implies that $x \in P$. 
Chapter 3

Algebraic Lagrangian Duality

In this chapter we begin dealing seriously with optimization. A deep understanding of vector space duality and conic duality leads to us ponder how "duality" might arise in optimization. We take up one kind of duality in optimization, typically called Lagrangian duality. This flavor of duality refers to a strong relationship between two related optimization problems called "duals" of each other. The optimal values and set of optimal solutions of these "dual" problems share surprisingly strong relationships that are both useful in practice and powerful in theory. Our goal in this chapter is to describe these relationships.

Naturally, one optimization problem in our "dual" pair is problem (1.2.1) reproduced here for convenience:

\[
\begin{align*}
\inf \; f(x) \\
\text{s.t. } G(x) \preceq_P \theta_Y \\
x \in \Omega.
\end{align*}
\] (P)

where \(\Omega\) is a convex set contained in vector space \(X\), \(f : X \to \mathbb{R}\) is a convex functional defined on vector space \(X\), \(G : X \to Y\) is a convex map from \(X\) into a vector space \(Y\). This vector space comes equipped with an ordering \(\preceq_P\) which is defined by a convex pointed cone \(P_Y\) in \(Y\).

We introduce notation that is used in the rest of the notes. Denote the optimal value of a problem (*) by \(v(*)\). As an example, for problem (P) we have

\[
v(P) = \inf \{f(x) : G(x) \preceq_P \theta_Y, x \in \Omega\}.
\]

We call (P) the primal optimization problem. When \(x\) is a feasible solution to the primal (that is, \(x \in \Omega\) and \(G(x) \preceq_P \theta_Y\)) we call \(x\) primal feasible.

A close cousin to the primal is called the Lagrangian dual problem, or simply its dual. You will be introduced to it shortly. We start with some motivation which describes how we end up considering the dual.
Remark 3.0.1. In all of the following results we make the assumption that the primal problem (P) is feasible.

3.1 Motivation: Penalty Functionals

Our primal problem (P) is a constrained optimization problem. The constraints come in two flavors. One is the requirement that \( x \in \Omega \). In practice, this constraint is easy to describe and handle explicitly, \( \Omega \) is typically a cone or even the entire vector space. On the contrary, the requirement that \( G(x) \preceq \theta Y \) is often nasty. Wouldn’t it be nice if we could remove those constraints from the feasible region? The idea is not to remove them entirely, but penalize the objective for violations:

\[
\inf_{x} f(x) + \text{penalty}(G(x)) \quad \text{s.t.} \quad x \in \Omega.
\]

The penalty functional maps elements from \( Y \) into the reals, and is designed to “nudge” the optimizer solving (3.1.1) to stick to those \( x \) which are feasible. In particular, we would like the penalty to “penalize” infeasibility and “reward” feasibility. We get into how to achieve this later. By introducing the penalty functional we “simply” need to optimize a new penalized objective over the simple feasible region \( \Omega \). If the penalty function is well chosen, in particular, if it sufficiently penalizes violations from \( G(x) \preceq \theta Y \), then by solving (3.1.1) we hope to recover an optimal solution to (P). Of course, if the penalty function is complicated there may be nothing gained.

A natural and common approach is to restrict ourselves to linear penalty functionals. Since \( G(x) \) is an element of \( Y \), the set of linear functionals is precisely the dual space \( Y' \). That is, we restrict attention to linear penalty functionals of the form:

\[
\text{penalty}(y) = \langle y, \psi \rangle
\]

for some \( \psi \in Y' \). With this penalty, problem (3.1.1) is:

\[
\inf_{x} f(x) + \langle G(x), \psi \rangle \quad \text{s.t.} \quad x \in \Omega.
\]

for a given \( \psi \in Y' \).

The value of (3.1.2) for different choices of \( \psi \) is central to our story and we introduce a new functional to track how the value changes as a function of \( \psi \). We define \( L : Y' \to \mathbb{R} \) by

\[
L(\psi) := \inf_{x \in \Omega} f(x) + \langle G(x), \psi \rangle.
\]
and call this the dual functional of the primal (P). As we discussed above, an “ideal” choice of penalty functional \( \bar{\psi} \) has

\[
L(\bar{\psi}) = v(P)
\]

(3.1.4)

where \( v(P) \) is the optimal value of the primal problem (P). The \( \bar{\psi} \) is chosen to penalize deviations from feasibility so that the complicating constraints \( G(x) \leq \theta_Y \) can be ignored in finding an optimal \( x \). In other words, our goal is to “optimize” over \( \psi \), which lies in the dual space \( Y' \). Hopefully, it is beginning to make sense why this is called duality theory!

Under what conditions does there exist a \( \psi \in Y' \) that satisfies (3.1.4)? Answering this question, and describing how to find such a \( \psi \) when one does exist, are the motivating issues for the next two sections.

### 3.2 Lower Bounds and Weak Duality

Our search for a \( \psi \in Y' \) which satisfies (3.1.4) happens in two steps. First, we show for any \( \psi \in P_Y^+ \), that \( L(\psi) \) gives a lower bound on the optimal primal value, that is:

\[
L(\psi) \leq v(P).
\]

(3.2.1)

Condition (3.2.1) is so significant that it is given a special name: weak duality.

**Theorem 3.2.1 (Weak Duality Theorem).** If \( \bar{\psi} \in P_Y^+ \), then \( L(\bar{\psi}) \leq v(P) \).

**Proof.** If \( x \) is any primal feasible solution, then \( x \in \Omega \) and \( G(x) \leq_P \theta_Y \). However, \( G(x) \leq_P \theta_Y \) implies \( -G(x) \geq_P \theta_Y \). Then by definition of \( P_Y \), \( -G(x) \in P_Y \). Since \( \bar{\psi} \in P_Y^+ \) we have \( \langle -G(x), \bar{\psi} \rangle \geq 0 \) for any primal feasible \( x \). This implies \( \langle G(x), \bar{\psi} \rangle \leq 0 \) for any primal feasible \( x \). Thus,

\[
L(\bar{\psi}) = \inf_{x \in \Omega} \{ f(x) + \langle G(x), \bar{\psi} \rangle \} \\
\leq \inf_{x \in \Omega} \{ f(x) + \langle G(x), \bar{\psi} \rangle : G(x) \leq \theta_Y \} \\
\leq \inf_{x \in \Omega} \{ f(x) : G(x) \leq \theta_Y \} = v(P) \quad \text{since } \langle G(x), \bar{\psi} \rangle \leq 0
\]

and this completes the proof. \( \square \)

**Remark 3.2.2.** Note this result did not use the convexity properties of \( f \) and \( G \), and thus holds under even more general conditions.

The essence of weak duality is that when \( x \) is feasible the penalty function \( \text{penalty}(x) = \langle G(x), \psi \rangle \) is negative when \( \psi \) is in the positive cone \( P_Y^+ \). In other words, such penalty functions *rewards* feasibility. Clearly, in such setting you should always do better than the primal! Thus, the penalized problem (3.1.1) is a relaxation of the primal.
3.3 The Lagrangian dual problem

By weak duality, the dual functional \( L(\psi) \) for any \( \psi \in P_Y^+ \) provides a lower bound on the optimal primal value. A good choice of \( \psi \) is one that gives a lower bound close to the optimal primal value. The best lower bound we can hope for is the greatest lower bound, so we solve the optimization problem:

\[
\sup_{\psi \in P_Y^+} L(\psi). \tag{D}
\]

It is more common to see the dual with \( L(\psi) \) written out explicitly:

\[
\sup_{\psi \in P_Y^+} \left\{ \inf_{x \in \Omega} f(x) + \langle G(x), \psi \rangle \right\}. \tag{3.3.1}
\]

Problem (D) is called the Lagrangian dual, or simply the dual of the primal optimization problem (P). The “Lagrangian” part of the name comes from the fact that the function \( L : X \times Y' \rightarrow \mathbb{R} \) defined by

\[
L(x, \psi) = f(x) + \langle G(x), \psi \rangle
\]

is often called the Lagrangian of the primal problem (P). The “dual” part of the name may come from the fact that decision variable lies in the dual space \( Y' \). Any linear functional \( \psi \) which is feasible to the dual (that is, \( \psi \in P_Y^+ \)) is called dual feasible.

Remark 3.3.1. In all of the following results we make the assumption that the dual problem (D) is feasible. This amounts to assuming that \( P_Y^+ \) is nonempty. The dual is trivial when \( P_Y^+ = \{0\} \), however by Theorem 2.6.7 when \( P_Y \) is proper and has a non-empty intrinsic core then \( P_Y^+ \) has non-zero elements and the dual becomes more interesting.

The primal problem (P) and the dual problem (D) are called a primal-dual pair. The primal optimization problem is a minimization (inf) problem and the dual optimization problem given is a maximization (sup) problem. Many practical problems have interesting primal-dual pairings.

3.4 Zero duality gap and strong duality

Now that we have a primal-dual pair of problems, our main goal is to give sufficient conditions for when these two problems have equal optimal value; that is, for \( v(D) = v(P) \). The difference

\[
gap(P) = v(P) - v(D)
\]
3.4. ZERO DUALITY GAP AND STRONG DUALITY

is called the duality gap of the primal-dual pair. We say there is no duality gap if \( \text{gap}(P) = 0 \). This is, of course, just another way to say that our primal and dual problems have the same optimal value!

There are many flavors of sufficient conditions that ensure zero duality gap (see for instance, Chapter 3 of Anderson and Nash [3]). We focus on a very general approach using the concepts of the linear closure and intrinsic core from Chapter 2. The approach starts with defining some new objects. An important set used in characterizing sufficient conditions is:

\[
\Gamma := \{(r, y) \in \mathbb{R} \times Y : \exists x \in \Omega \text{ s.t. } f(x) \leq r, G(x) \preceq y\}. \tag{3.4.1}
\]

The set can be interpreted as follows, a pair \((r, y)\) is in the set if there is a feasible solution \(x\) to the perturbed version of \((P)\) where the right-hand side is now \(y\) whose objective value is less than or equal to \(r\).\footnote{This set has a geometric interpretation in terms of the optimal value function defined in Section 3.5: it is the epigraph of \(\omega\). See Exercise 3.10.}

**Lemma 3.4.1.** The set \(\Gamma\) defined in (3.4.1) is convex.

**Proof.** See Exercise 3.1.

We define two quantities related to \(\Gamma\):

\[
\mu := \inf \{r : \exists y \preceq \theta_Y \text{ where } (r, y) \in \Gamma\} \tag{3.4.2}
\]

\[
\nu := \inf \{r : \exists y \preceq \theta_Y \text{ where } (r, y) \in \text{lin}(\Gamma)\} \tag{3.4.3}
\]

These quantities are represented on Figure 3.1.

### 3.4.1 Characterizing the primal and dual optimal values

Clearly \(\nu \leq \mu\) (since \(\Gamma \subseteq \text{lin}(\Gamma)\)) and when \(\Gamma = \text{lin}(\Gamma)\) these two values are equal. Why is this significant in terms of no duality gap? The next two results relate \(\mu\) and \(\nu\) to the optimal values of our primal-dual pair.

**Theorem 3.4.2.** The value of \(\mu\) defined in (3.4.2) is equal to the optimal value of the primal problem \((P)\). That is, \(\mu = v(P)\).

**Proof.** (Show \(\mu \leq v(P)\)) Since the primal problem is a minimization problem, it is sufficient to show that \(\mu\) is a lower bound on any primal feasible solution value. Let \(\overline{x}\) be a primal feasible solution (we are assuming that the primal is always feasible in this chapter). Show \(\mu \leq f(\overline{x})\). Since \(\overline{x}\) is primal feasible, \(G(\overline{x}) \preceq \theta_Y\). Define \(\overline{r} = f(\overline{x})\) and \(\overline{y} = \theta_Y\). Then \((\overline{r}, \overline{y}) \in \Gamma\) by definition of \(\Gamma\). Then by definition of \(\mu\), we have:

\[
\mu = \inf \{r : \exists y \preceq \theta_Y \text{ where } (r, y) \in \Gamma\} \leq \overline{r} = f(\overline{x}).
\]
CHAPTER 3. ALGEBRAIC LAGRANGIAN DUALITY

Figure 3.1: A visualization of $\Gamma$ and the values $\mu$ and $\nu$ defined in (3.4.1), (3.4.2) and (3.4.3) respectively.

(Show $\mu \geq v(P)$) From the definition of $\mu$, it is sufficient to show that given an arbitrary $(\bar{r}, \bar{y}) \in \Gamma$, there is a primal feasible $x$ such that $\bar{r} \geq f(x)$. If $(\bar{r}, \bar{y}) \in \Gamma$, then by definition of $\Gamma$ there is a primal feasible $x$ such that $\bar{r} \geq f(x)$ and we are done.2

Remark 3.4.3. It is possible that the primal problem is unbounded in which case $v(P) = \mu = -\infty$.

No special structure is needed on the problem for Theorem 3.4.2 to hold. In our next result we show $\nu = v(D)$, but under the condition that $\Gamma$ has a nonempty intrinsic core. We first prove a useful lemma prior to stating and proving our main result.

Lemma 3.4.4. If $\nu$ is finite, and the intrinsic core of $\Gamma$ is not empty, then $(\nu, \theta_Y) \in \text{lin}(\Gamma)$.

Proof. See Exercise 3.2.  

\[\text{An alternate proof using the fact that } \Gamma \text{ is the epigraph of the optimal value functional (see Exercise 3.10) is as follows: we know that } v(P) = \omega(\theta_Y). \text{ Also, } \omega(y) \text{ is monotonically decreasing in } y, \text{ thus } \omega(y) \geq \omega(\theta_Y) \text{ for all } y \preceq \theta_Y. \text{ This implies } \mu = \omega(\theta_Y) = v(P).\]

This result is intuitively clear in view of Figure 3.1.
3.4. ZERO DUALITY GAP AND STRONG DUALITY

**Remark 3.4.5.** By the definition of $\nu$ and $\mu$, if $\nu$ is finite, then $\mu$ is finite since $\mu \geq \nu$. By primal feasibility $\mu$ is not $\infty$.

The following proof is a generalization of one in Anderson and Nash [3]:

**Theorem 3.4.6.** If $\nu$ is finite, and if the intrinsic core of $\Gamma$ is not empty, then $\nu = v(D)$.

In more detail,

$$\nu = \sup_{\psi \in P_Y^+} \left\{ \inf_{x \in \Omega} f(x) + \langle G(x), \psi \rangle \right\} = v(D).$$

**Proof. Basic Outline:** We show for any $\epsilon > 0$, that there is a $\overline{\psi} \in P_Y^+$ such that

$$\inf_{x \in \Omega} f(x) + \langle G(x), \overline{\psi} \rangle \geq \nu - \epsilon$$

This implies

$$v(D) = \sup_{\psi \in P_Y^+} \left\{ \inf_{x \in \Omega} f(x) + \langle G(x), \psi \rangle \right\} \geq \nu$$

We then show

$$v(D) = \sup_{\psi \in P_Y^+} \left\{ \inf_{x \in \Omega} f(x) + \langle G(x), \psi \rangle \right\} > \nu$$

leads to a contradiction.

Let $\epsilon > 0$. By definition of $\nu$, $(\nu - \epsilon, \theta_Y) \notin \text{lin}(\Gamma)$. Now $\Gamma$ is a convex set and by hypothesis, the intrinsic core of $\Gamma$ is not empty, so by Theorem 2.5.16, $(\nu - \epsilon, \theta_Y)$ can be strictly separated from $\text{lin}(\Gamma)$ (in fact the theorem guarantees strong separation). Then there exits $(\lambda, \psi) \in \mathbb{R} \times Y'$ such that

$$\lambda(\nu - \epsilon) + \langle \theta_Y, \psi \rangle < \lambda r + \langle y, \psi \rangle \quad \forall (r, y) \in \text{lin}(\Gamma) \quad (3.4.4)$$

We first show $v(D) \geq \nu$ and then $v(D) > \nu$ leads to a contradiction.

**WTS 1:** Show $v(D) \geq \nu$. To do this we first argue that $\psi \in P_Y^+$. Assume not, i.e. $\psi \notin P_Y^+$. If $\psi \notin P_Y^+$, then there exists $\hat{y} \in P_Y$ (i.e. $\hat{y} \succeq \theta_Y$) such that $\langle \hat{y}, \psi \rangle < 0$. Then $\langle \alpha \hat{y}, \psi \rangle < 0$ for all $\alpha > 0$. We always assume the primal problem has a feasible solution $\overline{x}$. Then by picking $\alpha > 0$, we have $\alpha \hat{y} \succeq G(\overline{x})$. Then

$$(f(\overline{x}), \alpha \hat{y}) \in \Gamma \subseteq \text{lin}(\Gamma)$$

By letting $\alpha$ go to $\infty$ we can make $\lambda f(\overline{x}) + \langle \alpha \hat{y}, \psi \rangle$ arbitrarily small and thus strictly less than $\lambda(\nu - \epsilon) + \langle \theta_Y, \psi \rangle$, which violates (3.4.4). Thus, we conclude $\psi \in P_Y^+$.\n
We argue next that \( \lambda > 0 \). Assume not, i.e. \( \lambda \leq 0 \). Since \( \epsilon > 0 \),

\[
\lambda \nu + \langle \theta_Y, \psi \rangle \leq \lambda (\nu - \epsilon) + \langle \theta_Y, \psi \rangle \leq \lambda r + \langle y, \psi \rangle, \quad \forall (r, y) \in \text{lin}(\Gamma)
\]

which implies there is a hyperplane defined using the linear function \((\lambda, \psi)\) that strictly separates \((\nu, \theta_Y)\) from \(\text{lin}(\Gamma)\) and this contradicts the Lemma 3.4.4. Hence \( \lambda > 0 \). Since \((f(x), G(x)) \in \Gamma \subseteq \text{lin}(\Gamma)\) for all \( x \in \Omega \) we have

\[
\lambda (\nu - \epsilon) + \langle \theta_Y, \psi \rangle \leq \langle G(x), \psi \rangle + \lambda f(x), \quad \forall x \in \Omega
\]

But \( P_Y^+ \) is a cone, so \( \psi \in P_Y^+ \) and \( \lambda > 0 \) implies \( \psi/\lambda \in P_Y^+ \) and hence dual feasible. But \( \epsilon > 0 \) is arbitrary so we have a feasible dual solution with value at least \( \nu \).

**WTS 2:** Show \( v(D) \leq \nu \). Show that \( v(D) > \nu \) leads to a contradiction. If \( v(D) > \nu \), then \( v(D) = \nu + \delta \) for some \( \delta > 0 \). Then there is a \( \psi \in P_Y^+ \) such that

\[
\inf_{x \in \Omega} f(x) + \langle G(x), \psi \rangle > \nu + \frac{\delta}{2}
\]

and hence

\[
\langle \theta_Y, \psi \rangle + \nu + \frac{\delta}{2} < f(x) + \langle G(x), \psi \rangle, \quad \forall x \in \Omega
\]

Since \( \psi \in P_Y^+ \), we have

\[
f(x) + \langle G(x), \psi \rangle \leq r + \langle y, \psi \rangle
\]

when \( y \geq G(x) \) and \( r \geq f(x) \). Then

\[
\langle \theta_Y, \psi \rangle + \nu + \frac{\delta}{2} < f(x) + \langle G(x), \psi \rangle \leq r + \langle y, \psi \rangle, \quad \forall (r, y) \in \Gamma
\]

Thus \((1, \psi)\) generates a hyperplane that strictly (actually strongly) separates \((\nu, \theta_Y)\) from \(\Gamma\). Then \((\nu, \theta_Y)\) can be strictly separated from \(\text{lin}(\Gamma)\) (see Exercise 3.3) which contradicts Lemma 3.4.4. \( \square \)
3.4. ZER0 DUALITY GAP AND STRONG DUALITY

3.4.2 Sufficient conditions for no duality gap

Theorems 3.4.2 and 3.4.6 show that the optimal value of the primal is the infimum over \( \Gamma \) and the optimal value Lagrangian dual is the infimum over the algebraic closure of \( \Gamma \). Neat! Therefore, if \( \mu = \nu \) there is no duality gap! We now give three sufficient conditions for no duality gap, all based around the definition of \( \mu \) and \( \nu \). The first is totally trivial and the second involves our old friend the intrinsic core. The last condition is a more readily checkable condition which is really an algebraic Slater condition.

**Corollary 3.4.7.** Let \( \Gamma, \mu \) and \( \nu \) be as defined in (3.4.1)–(3.4.3). If \( \nu \) is finite, \( \Gamma \) has a nonempty intrinsic core, and \( \Gamma = \text{lin}(\Gamma) \), then there is no duality gap.

**Proof.** If \( \Gamma = \text{lin}(\Gamma) \) then by definition \( \mu = \nu \) and thus in view of Theorems 3.4.2 and 3.4.6 (noting the hypothesis that \( \text{icr}(\Gamma) \neq \emptyset \)) we have no duality gap. \( \square \)

**Corollary 3.4.8.** Let \( \Gamma, \mu \) and \( \nu \) be as defined in (3.4.1)–(3.4.3). If \( \nu \) is finite, and there exists a point \((\bar{r}, \bar{y})\) in the intrinsic core of \( \Gamma \) with \( \bar{y} \preceq \theta_Y \), then there is no duality gap.

**Proof.** By hypothesis \((\bar{r}, \bar{y})\) is in the intrinsic core of \( \Gamma \), so from Theorem 3.4.6 \( v(D) = \nu \). Also, by Theorem 3.4.2, \( v(P) = \mu \). In light of these conditions we use \( v(D) \) and \( \nu \) interchangeably, and similarly for \( v(P) \) and \( \mu \).

By weak duality, \( \mu \geq \nu \). Assume there is a duality gap, i.e. \( \mu = \nu + \delta \) for \( \delta > 0 \). We show that this contradicts the optimality of \( \mu \). By definition of \( \nu \), it follows that there exists \( \tilde{y} \preceq \theta_Y \) such that \((\nu + \frac{\delta}{2}, \tilde{y}) \in \text{lin}(\Gamma)\). Then \((\nu + \frac{\delta}{2}, \tilde{y}) \) is either in \( \Gamma \) or in \( \text{lin}(\Gamma) \). It cannot be the case that \((\nu + \frac{\delta}{2}, \tilde{y}) \) is in \( \Gamma \) because this would contradict the fact the \( \mu \) is the optimal primal value. Indeed, if \((\nu + \frac{\delta}{2}, \tilde{y}) \in \Gamma \) then there exists an \( \tilde{x} \) such that \( f(\tilde{x}) \leq \nu + \frac{\delta}{2} \) and \( G(\tilde{x}) \preceq \tilde{y} \preceq \theta \). That is, \( \tilde{x} \) is primal feasible with an objective value less than \( \mu \). This cannot happen. Therefore we can conclude that \((\nu + \frac{\delta}{2}, \tilde{y}) \in \text{lin}(\Gamma)\). Consider the line segment

\[
\left[ (\bar{r}, \bar{y}), (\nu + \frac{\delta}{2}, \tilde{y}) \right]
\]

Since \( \bar{r}, \bar{y} \) is in the intrinsic core of \( \Gamma \), by Lemma 2.4.16 the entire line segment \( \left[ (\bar{r}, \bar{y}), (\nu + \frac{\delta}{2}, \tilde{y}) \right] \) is in \( \Gamma \). See Figure 3.2. Okay, pretty much a done deal. We now pick a sufficiently small \( \epsilon > 0 \) and move to the point \((r(\epsilon), y(\epsilon)) \in \Gamma \). That is,

\[
(r(\epsilon), y(\epsilon)) = (\bar{r}, \bar{y}) + (1 - \epsilon) \left( (\nu + \frac{\delta}{2}, \tilde{y}) - (\bar{r}, \bar{y}) \right) \in \Gamma
\]

Observe

\[
\begin{align*}
r(\epsilon) &= \epsilon \bar{r} + (1 - \epsilon)(\nu + \frac{\delta}{2}) \\
y(\epsilon) &= \epsilon \bar{y} + (1 - \epsilon)\tilde{y}
\end{align*}
\]
Figure 3.2: Proof of Corollary 3.4.8: Illustrates how if $\nu < \mu$ then we can derive a contradiction.

Since $\tilde{y} \preceq \theta_Y$ and $\hat{y} \preceq \theta_Y$ it follows that $y(\epsilon) \preceq \theta_Y$ since the positive cone $P_Y$ is convex. Now pick $\epsilon > 0$ small enough so that $\epsilon \bar{r} < \frac{\delta}{2}$. Then $r(\epsilon) < \nu + \delta$. But $(r(\epsilon), y(\epsilon)) \in \Gamma$ with $y(\epsilon) \preceq \theta_Y$. This means we have contradicted the optimality of $\mu$. \hfill \Box

Theorem 3.4.9. If there exists $\bar{x} \in \Omega$ such that $G(\bar{x})$ is in the core of $N_Y$ (called the negative cone in $Y$, which is equal to $-P_Y$) then there is no duality gap, i.e. $v(P) = v(D)$.

Proof. WTS: Let $\epsilon > 0$ be given. Show that the point $(f(\bar{x}) + \epsilon, \theta_Y)$ is a core point of $\Gamma$. Then the result is immediate from Corollary 3.4.8, since a core point is an intrinsic core point.

Let $(r, p)$ be an arbitrary point in $\mathbb{R} \times Y$. In order to prove $(f(\bar{x}) + \epsilon, \theta_Y)$ is a core point in $\Gamma$ we apply Lemma 2.4.1 and show there is a $\lambda > 0$ such that the line segment

$$[(f(\bar{x}) + \epsilon, \theta_Y), (f(\bar{x}) + \epsilon, \theta_Y) + \lambda(r, p)] \subseteq \Gamma$$

By hypothesis $G(\bar{x})$ is a core point in the negative cone $N_Y$. Apply the definition of core point to $G(\bar{x})$ and move in the direction $-p$. It follows from Lemma 2.4.1 that there is a $\lambda > 0$ such that:

$$[G(\bar{x}), G(\bar{x}) - \lambda p] \subseteq N_Y.$$
3.4. ZERO DUALITY GAP AND STRONG DUALITY

Then $G(\pi) - \lambda p$ is in $N_Y$ so by definition of $N_Y$, $G(\pi) - \lambda p \preceq \theta_Y$ and this implies $G(\pi) \preceq \theta_Y + \lambda p$. Then $G(\pi) \preceq \theta_Y + \lambda p$, and we can pick $\lambda$ small enough so that:

$$f(\pi) + \epsilon + \lambda r > f(\pi)$$

and this implies

$$[(f(\pi) + \epsilon, \theta_Y), (f(\pi) + \epsilon, \theta_Y) + \lambda(r, p)) \subseteq \Gamma$$

We now leverage Corollary 3.4.8 to establish no duality gap.

**Remark 3.4.10.** In Theorem 3.4.9, the condition that $G(\pi)$ is in the core of $N_Y$ is commonly referred to as a *Slater condition*. For more detail on this condition in finite dimensions see the book by Boyd [6]. In finite dimensions, the condition that $G(\pi)$ is in the core of $N_Y$ becomes $g_i(\pi) < 0, i = 1, \ldots, m$. One disturbing issue is that, unlike Slater, we do not impose any condition other than $\pi \in \Omega$. Slater’s condition imposes $\pi$ to be in the relative interior of $\Omega$.

For an example of a problem with a duality gap, see Exercise 3.4.

3.4.3 Strong duality

In the defining the primal problem in (P) we use inf rather than min. It may be the case that $v(P)$ is finite, but there does not exist an $\bar{x} \in \Omega$ such that $f(\bar{x}) = v(P)$. Likewise, in defining the dual problem in (D) we use sup rather than max, which in theory might also be unattained. Even when primal and dual values are finite and equal (no duality gap) we still need to worry about whether or not there exist $x$ and $\psi$ that give, respectively, the optimal primal and dual values. See Exercise 3.5 for an example where this is not the case. When $\mu = \nu$ there is no duality gap. If there is no duality gap and the primal problem and dual problem are solvable (that is, the inf and the sup are attained) strong duality holds. In many optimization problems, including all standard linear programs, if the duality gap is zero, then there are primal and dual solutions that give the optimal primal and dual value. It is common for people to use “strong duality” and “no duality gap” interchangeably. We use the naming convention we learned in Anderson and Nash [3] to differentiate the equality of optimal values (no duality gap) and existence of solutions which achieve those values (strong duality). In Corollary 3.4.11 we provide a sufficient condition for the existence of an optimal dual solution. In Corollary 3.4.12 and Corollary 3.4.14 below we provide sufficient conditions for strong duality to hold.

**Corollary 3.4.11.** Let $\Gamma$, $\mu$, and $\nu$ be as defined in (3.4.1)–(3.4.3). If $\nu$ is finite, and there exists a point $(\bar{r}, \bar{y})$ is in the intrinsic core of $\Gamma$ with $\bar{y} \preceq \theta_Y$, then there is a $\bar{\psi} \in P_Y^+$ such that

$$L(\bar{\psi}) = \inf_{x \in \Omega} f(x) + \langle G(x), \bar{\psi} \rangle = \nu$$
Proof. **WTS 0:** Show that that \((\nu, \theta_Y)\) cannot be in the intrinsic core of \(\Gamma\). If \((\nu, \theta_Y)\) is in the intrinsic core of \(\Gamma\), then for \(\epsilon > 0\), \((\nu + \epsilon, \theta_Y)\) is in \(\Gamma\). Then the point
\[
(\nu - \epsilon, \theta_Y) = 2(\nu, \theta_Y) - (\nu + \epsilon, \theta_Y)
\]
is in the affine hull of \(\Gamma\). If \((\nu - \epsilon, \theta_Y)\) is in the affine hull of \(\Gamma\), then by definition of intrinsic core, we can connect \((\nu, \theta_Y)\) and \((\nu - \epsilon, \theta_Y)\) with a line segment and there are points on this line segment in \(\Gamma\). This implies the existence of \((r, y)\) in \(\Gamma\) with \(f(x) = r < \nu\) and \(G(\pi) \leq \theta_Y\) and \(\pi \in \Omega\). This contradicts the definition of \(\nu\) and we conclude \((\nu, \theta_Y)\) is not in the intrinsic core of \(\Gamma\). Since \((\hat{r}, \hat{y})\) is in the intrinsic core of \(\Gamma\), it follows that \(\Gamma\) has a nonempty intrinsic core, and by Theorem 2.5.13 we can strictly separate \((\nu, \theta_Y)\) from the intrinsic core of \(\Gamma\). Then there exits \((\lambda, \psi) \in \mathbb{R} \times \mathbb{Y}'\) such that
\[
\lambda \nu + \langle \theta_Y, \psi \rangle < \lambda r + \langle y, \psi \rangle \quad \forall (r, y) \in \text{icr}(\Gamma)
\]
and by the first part of Lemma 2.5.7.
\[
\lambda \nu + \langle \theta_Y, \psi \rangle \leq \lambda r + \langle y, \psi \rangle \quad \forall (r, y) \in \Gamma
\]

**WTS 1:** Show \(\psi \in P_Y^+\). Assume not, i.e. \(\psi \notin P_Y^+\). If \(\psi \notin P_Y^+\), then there exists \(\hat{y} \in P_Y\) (i.e. \(\hat{y} \geq \theta_Y\)) such that \(\langle \hat{y}, \psi \rangle < 0\). Then \(\langle \alpha \hat{y}, \psi \rangle < 0\) for all \(\alpha > 0\). By assumption, the primal has a feasible solution \(\pi\). Then by picking \(\alpha > 0\), we have \(\alpha \hat{y} \geq \theta_Y \geq G(\pi)\). Then
\[
(f(\pi), \alpha \hat{y}) \in \Gamma
\]
By letting \(\alpha\) go to \(\infty\) we can make \(\lambda f(\pi) + \langle \alpha \hat{y}, \psi \rangle\) arbitrarily small and thus strictly less than \(\lambda \nu + \langle \theta_Y, \psi \rangle\), which violates (3.4.6). Thus, we conclude \(\psi \in P_Y^+\).

**WTS 2:** Show \(\lambda > 0\). Assume not, i.e. \(\lambda \leq 0\). Consider first \(\lambda < 0\). By hypothesis \((\pi, \pi)\) is an intrinsic core point with \(\pi \leq \theta_Y\), so \((\hat{r}, \hat{y})\) is in \(\Gamma\) for arbitrarily large \(\hat{r}\). Then by (3.4.6),
\[
\lambda \nu + \langle \theta_Y, \psi \rangle \leq \lambda \hat{r} + \langle \hat{y}, \psi \rangle, \quad \forall \hat{r} \geq \pi
\]
Since the left hand side of the inequality is fixed, a negative \(\lambda\) implies this inequality will be violated for sufficiently large \(\hat{r}\). Next consider \(\lambda = 0\). Then (3.4.5) becomes
\[
0 < \langle y, \psi \rangle \quad \forall (r, y) \in \text{icr}(\Gamma)
\]
By hypothesis, \((\pi, \pi)\) is in \(\Gamma\) and \(\pi \leq \theta_Y\). We know it is the case that \(\psi \in P_Y^+\), so \(\langle \pi, \psi \rangle \leq 0\) and this contradicts (3.4.7). Therefore, \(\lambda > 0\). Since \((f(x), G(x)) \in \Gamma\) for all \(x \in \Omega\) we have from (3.4.6)
\[
\lambda \nu + \langle \theta_Y, \psi \rangle \leq \lambda f(x) + \langle G(x), \psi \rangle, \quad \forall x \in \Omega
\]
\[
\lambda \nu \leq \lambda f(x) + \langle G(x), \psi \rangle, \quad \forall x \in \Omega
\]
\[
\nu \leq f(x) + \langle G(x), \psi / \lambda \rangle, \quad \forall x \in \Omega
\]
\[
\nu \leq \inf_{x \in \Omega} f(x) + \langle G(x), \psi / \lambda \rangle = L(\psi / \lambda)
\]
But $P^+_Y$ is a cone, so $\psi \in P^+_Y$ and $\lambda > 0$ implies $\psi/\lambda \in P^+_Y$ and hence dual feasible. Thus we have $L(\psi/\lambda) \geq \nu$. By 3.4.6, $v(D) = \nu$ so it follows that $L(\psi/\lambda) = \nu$.

\begin{proof}
We know by either Corollary 3.4.7 or Corollary 3.4.8, that $\mu = \nu$ so there is no duality gap. By Corollary 3.4.11 we know there is a $\psi \in P^+_Y$ such that $L(\psi) = \mu = \nu$. By Lemma 3.4.4, $(\nu, \theta_Y) \in \text{lin}(\Gamma)$. By hypothesis $\Gamma = \text{lin}(\Gamma)$, so $(\nu, \theta_Y) = (\mu, \theta_Y) \in \Gamma$. This implies there is an $\hat{x} \in \Omega$ such that $f(\hat{x}) = \mu = \nu$ and $G(\hat{x}) \preceq \theta_Y$, that is we have strong duality.
\end{proof}

\begin{corollary}
Let $\Gamma$, $\mu$ and $\nu$ be as defined in (3.4.1)–(3.4.3). If $\nu$ is finite, if $\Gamma$ has a nonempty intrinsic core $(\tau, \eta)$ with $\eta \preceq \theta_Y$, and $\Gamma = \text{lin}(\Gamma)$, then there is an $\hat{x} \in \Omega$ such that $f(\hat{x}) = \mu = \nu$ and $G(\hat{x}) \preceq \theta_Y$, that is we have strong duality.
\end{corollary}

\begin{proof}
We know by Corollary 3.4.12, that $\mu = \nu$ so there is no duality gap. By Corollary 3.4.11 we know there is a $\psi \in P^+_Y$ such that $L(\psi) = \mu = \nu$. By Lemma 3.4.4, $(\nu, \theta_Y) \in \text{lin}(\Gamma)$. By hypothesis $\Gamma = \text{lin}(\Gamma)$, so $(\nu, \theta_Y) = (\mu, \theta_Y) \in \Gamma$. This implies there is an $\hat{x} \in \Omega$ such that $f(\hat{x}) = \mu = \nu$ and $G(\hat{x}) \preceq \theta_Y$, that is we have strong duality.
\end{proof}

\begin{remark}
The hypothesis $\Gamma$ has a nonempty intrinsic core $(\tau, \eta)$ with $\eta \preceq \theta_Y$ is used to prove the existence of an optimal dual solution, and the hypothesis $\Gamma = \text{lin}(\Gamma)$, is used to prove the existence of an optimal primal solution.
\end{remark}

\begin{complementary slacks}
If there exists an $\bar{x} \in \Omega$ with $G(\bar{x}) \preceq \theta_Y$ and $\bar{\psi} \in P^+_Y$ such that $\langle G(\bar{x}), \bar{\psi} \rangle = 0$, then the primal dual pair $(\bar{x}, \bar{\psi})$ satisfy complementary slackness. We leave the following Corollary as Exercise 3.6.
\end{complementary slacks}

\begin{corollary}
If $(\bar{x}, \bar{\psi})$ are a primal-dual pair that satisfy complementary slackness then the primal-dual solution satisfies strong duality.
\end{corollary}

\begin{corollary}
If $\bar{x}$ is an optimal primal solution, $\bar{\psi}$ is an optimal dual solution, and $\mu = \nu$, then this primal-dual pair satisfies complementary slackness.
\end{corollary}

\begin{proof}
See Exercise 3.7.
\end{proof}

Figure 3.3 gives a summary of the sufficient conditions described in this section. One final note is that all of these are sufficient conditions, not necessary conditions for no duality gap and strong duality.

\section{Optimal value functional}

We have spent a great deal of effort establishing conditions for no duality gap and strong duality. This is only really interesting if it turns out that the Lagrangian dual is a useful object, particularly in terms of solving the problem of interest. In this section we demonstrate why the Langrangian dual is interesting from a computational standpoint.

Consider the primal problem $(P)$, where the right-hand side of the conic inequality $G(x) \preceq_P y$ is an arbitrary $y \in Y$. Not all such choices of $y$ lead to a feasible problem, those that do are captured in the set

\[ \Lambda := \{ y \in Y : \text{there exists an } x \in \Omega \text{ such that } G(x) \preceq y \}. \] (3.5.1)
CHAPTER 3. ALGEBRAIC LAGRANGIAN DUALITY

\[ \nu = v(D) \]

\[ \nu \text{ finite} \]
\[ \text{icr}(\Gamma) \neq \emptyset \]

no duality gap +
optimal dual solution

\[ \exists (\bar{r}, \bar{y}) \in \text{icr}(\Gamma) \]
\[ \text{s.t. } \bar{y} \preceq \theta_Y \]

optimal primal solution
+ strong duality

\[ \Gamma = \text{lin}(\Gamma) \]

\[ \exists \bar{x} \text{ feasible with} \]
\[ G(\bar{x}) \in \text{cor}(-P_Y) \]

Figure 3.3: A summary of conditions for no duality gap and strong duality.

Lemma 3.5.1. If \( \Omega \) is a convex subset of \( X \), and \( G \) is a convex mapping, then and is \( \Lambda \) is a convex subset in \( Y \).

Proof. See Exercise 3.8.

Define the optimal value functional on defined on \( \Lambda \) by

\[ \omega(y) = \inf_{x \in \Omega} \{f(x) : G(x) \preceq y\} \quad (3.5.2) \]

The function \( \omega \) tracks how the optimal value changes as the right-hand side changes. Note that \( \omega(y) = +\infty \) when \( y \notin \Lambda \), so we restrict the domain of \( \omega \) to be \( \Lambda \). The following are some properties of the optimal value functional:

Lemma 3.5.2. If \( \Omega \) is a convex subset of \( X \), and \( f \) and \( G \) are convex mappings, then the optimal value functional \( \omega \) defined in (3.5.2) is:

(i) monotonically decreasing; that is, \( y \preceq y' \) implies \( \omega(y) \geq \omega(y') \) and

(ii) convex.

Proof. See Exercise 3.9.

Recall the dual functional \( L(\psi) \) defined by

\[ L(\psi) = \inf_{x \in \Omega} \{f(x) + \langle G(x), \psi \rangle\} . \]

This functional has a particularly nice structure as given in the next result.
Proposition 3.5.3. The dual functional $L(\psi)$ is concave over $Y'$ and can be written as

$$L(\psi) = \inf_{y \in \Lambda} \{ \omega(y) + \langle y, \psi \rangle \} = \inf_{x \in \Omega} \{ f(x) + \langle G(x), \psi \rangle \}$$ (3.5.3)

where $\omega$ is the optimal value functional defined in (3.5.2).

*Proof.* See Exercise 3.11. $\square$

Okay, let’s step back and take an inventory of where we are.

- We have a primal problem we want to optimize, however this may be difficult due to the constraints $G(x) \preceq \theta_Y$
- We formed a problem relaxation based on the dual functional $L(\psi)$
- The Lagrangian dual, $\sup_{\psi \in Y'} L(\psi)$, has two great features: 1) there no complicating $G(x) \preceq \theta_Y$ constraints; and 2) we are trying to find the supremum of a concave function which makes the local optimality problem go away.
- Under mild assumptions, we may even have strong duality so we can get by solving the easier Lagrangian dual!

Wow, life is great! Really? You think so? What are you, one of those starry eyed people who goes through life thinking the glass is half full! Wake up and get real – it is half empty! Let’s say we solve our “relatively easy” Lagrangian problem and find an optimal $\bar{\psi} \in Y'$. What makes you think that when you solve the problem

$$\inf_{x \in \Omega} f(x) + \langle G(x), \bar{\psi} \rangle$$

that the resulting $\bar{x}$ will be primal optimal, or even primal feasible? See, I told you the glass was half empty. However, as much as I truly dislike taking an optimistic view about anything, I have to admit that we indeed have found an optimal primal solution to a closely related problem. This result is due to Everett [10] and is actually quite practical.

Proposition 3.5.4 (Everett’s Theorem). If $\bar{\psi} \in P_Y^+$ and

$$\bar{x} = \arg\min_{x \in \Omega} \{ f(x) + \langle G(x), \bar{\psi} \rangle \}$$

then $f(\bar{x}) = \omega(G(\bar{x}))$.

In many interesting cases, $G(\bar{x})$ will be sufficiently close to $\theta_Y$ for us to claim victory. Typically the input data is not all that accurate to begin with. Now, for those of you who make me ill and think that the glass is half full, guess what? I am not going to prove this result. It is homework! (See Exercise 3.12).
Example 3.5.5.

\[
\begin{align*}
& \text{min } 40x_1 + 60x_2 + 70x_3 + 160x_4 \\
& 16x_1 + 35x_2 + 45x_3 + 85x_4 \geq 81 \\
& x_1, x_2, x_3, x_4 \in \{0, 1\}
\end{align*}
\]

This is a binary knapsack problem. This problem structure arises from capital budgeting problems. See Lorie and Savage [19]. In this application setting variable \( x_i = 1 \) if project \( i \) is undertaken and \( x_i = 0 \) if project \( i \) is not undertaken. The coefficients of \( x_i \) in the objective function represent the cost of each project. The coefficients in the knapsack constraint represent the capital return of each of the projects. The right hand side of 81 on the knapsack constraint represents the capital return that must be achieved. Dualizing the budget constraint gives

\[
L(\psi) = \min \{81\psi + (40x_1 + 60x_2 + 70x_3 + 160x_4 - \\
\psi(16x_1 + 35x_2 + 45x_3 + 85x_4)) : x_i \in \{0, 1\}, i = 1, \ldots 4\}
= \min \{81\psi + (40 - 16\psi)x_1 + (60 - 35\psi)x_2 + (70 - 45\psi)x_3 \\
+ (160 - 85\psi)x_4 : x_i \in \{0, 1\}, i = 1, \ldots 4\}.
\]

For any \( \psi \geq 0 \), \( L(\psi) \) is very easy to optimize. Since \( \Omega = \{x : x_i \in \{0, 1\}\} \), for any \( \psi = \bar{\psi} \), we just “peg” \( x_i = 1 \) if \((c_i - \bar{\psi}a_i) < 0\) and \( x_i = 0 \) if \((c_i - \bar{\psi}a_i) \geq 0\). Then the optimal value of \( L(\psi) \) for any \( \psi = \bar{\psi} \) is

\[
L(\bar{\psi}) = \bar{\psi}b + \sum_{i=1}^{n} \min\{0, (c_i - \bar{\psi}a_i)\}.
\]

There are alternative optima when \((c_i - \bar{\psi}a_i) = 0\) since we can peg \( x_i \) to either 0 or 1. For example, if \( \bar{\psi} = 2 \) then

\[
L(2) = \min \{162 + 8x_1 - 10x_2 - 20x_3 - 10x_4 : x_i \in \{0, 1\}, i = 1, \ldots 4\} = 162 - 40 = 122.
\]

Letting \( \psi \) vary over the nonnegative orthant gives the following family of solutions (the alternative optimal solution for the right endpoint of the interval is also given).

\[
\begin{array}{c|cccc|cccc}
0 \leq \psi \leq 14/9 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
14/9 < \psi \leq 12/7 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
12/7 < \psi \leq 32/17 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
32/17 < \psi \leq 2.5 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2.5 < \psi & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]
This gives the following $L(\psi)$,

$$
L(\psi) = \begin{cases} 
81\psi, & 0 \leq \psi \leq 14/9 \\
70 + 36\psi, & 14/9 < \psi \leq 12/7 \\
130 + \psi, & 12/7 < \psi \leq 32/17 \\
290 - 84\psi, & 32/17 < \psi \leq 2.5 \\
330 - 100\psi, & 2.5 < \psi 
\end{cases}
$$

The optimal value of $\psi$ is $\bar{\psi} = 32/17$ and $L(32/17) = 131 + (15/17)$. The optimal primal solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 1$. The optimal primal solution value is 160. Thus, there is a substantial duality gap. The optimal primal dual solution fails to satisfy the complementary slackness condition of the optimality conditions because there is positive slack of 1 unit on the knapsack constraint and it has a positive optimal dual variable. However, if we required a capital return of only 80 units, instead of 81 units, then $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 0$ and $\bar{\psi} = 32/17$, constitute an optimal primal-dual solution. Why? By requiring one less unit of return we can save $(160 - 130) = 30$ cost units!

### 3.6 An Extension with Affine Mappings

An important condition used to show that there is no duality gap is that $\Gamma$ has an intrinsic point $(\bar{r}, \bar{y})$ with $\bar{y} \preceq \theta Y$. By Theorem 3.4.9, if there exists $\bar{x} \in \Omega$ such that $G(\bar{x})$ is in the core of $N_Y$, then $(f(\bar{x}) + \epsilon, \theta Y)$ is in the intrinsic core of $\Gamma$. We need $G(\bar{x})$ to be a core point of $N_Y$ in order to guarantee an intrinsic core point in $\Gamma$. Can we get by with $G(\bar{x})$ in the intrinsic core of $N_Y$? Unfortunately, no! The following example illustrating this is due to Angelo Mancini of the Booth School of Business at the University of Chicago.

**Example 3.6.1.** Consider the linear program

$$
\begin{align*}
\min & \quad -x_1 + x_2 \\
\text{s.t.} & \quad x_1 + x_2 \geq 2 \\
& \quad -x_1 - x_2 \geq -2 \\
& \quad x_1 - x_2 \geq 0
\end{align*}
$$

By the first two constraints, in any feasible solution $x_1 + x_2 = 2$. We cannot have a core point in the feasible region. We show this leads to no intrinsic core in $\Gamma$.

Consider the feasible point $x_1 = 3/2$ and $x_2 = 1/2$ with objective value -1. Then $(r, y) = (f(x), b - Ax)$ is the point $(-1, 0, 0, -1)$ and this implies $(0, 100, 100, 0) \in \Gamma$. Next consider the feasible point $x_1 = 1$ and $x_2 = 1$. Then $(r, y) = (f(x), b - Ax)$ is the point $(0, 0, 0, 0)$ and this implies $(0, 1, 1, 1) \in \Gamma$. Then taking an affine combination of $(0, 100, 100, 0)$ and $(0, 1, 1, 1)$ gives the point

$$
-1(0, 100, 100, 0) + 2(0, 1, 1, 1) = (0, -98, -98, 2) \in \text{aff}(\Gamma)
$$
Assume there is an intrinsic core point \((\overline{r}, \overline{y})\) with \(\overline{y} \preceq \theta Y\). In this specific example, \(\overline{y} \preceq \theta Y\) is \(\overline{y}_1 \leq 0, \overline{y}_2 \leq 0,\) and \(\overline{y}_3 \leq 0\). If \((\overline{r}, \overline{y})\) is a point in the intrinsic core, then there must be a \(\lambda \in (0, 1)\) such that
\[
(1 - \lambda)(\overline{r}, \overline{y}) + \lambda(0, -98, -98, 2) \in \Gamma
\]
Now herein lies the problem. If such a \(\lambda\) exists to give us a point in \(\Gamma\), it must be the case that there exists an \(\overline{x} \in \mathbb{R}^3\) such that
\[
b - Ax \preceq \begin{bmatrix}
(1 - \lambda)\overline{y}_1 - 98\lambda \\
(1 - \lambda)\overline{y}_2 - 98\lambda \\
(1 - \lambda)\overline{y}_3 + 2\lambda
\end{bmatrix}
\]
Consider the first two components
\[
2 - (\overline{x}_1 + \overline{x}_2) \leq (1 - \lambda)\overline{y}_1 - 98\lambda \\
2 - (\overline{x}_1 + \overline{x}_2) \leq (1 - \lambda)\overline{y}_1 - 98\lambda
\]
Adding the two inequalities gives
\[
0 \leq (1 - \lambda)\overline{y}_1 - 98\lambda + (1 - \lambda)\overline{y}_2 - 98\lambda
\]
and this is
\[
196\lambda \leq (1 - \lambda)\overline{y}_1 + (1 - \lambda)\overline{y}_2
\]
and this is impossible since \(\overline{y}_1 \leq 0\) and \(\overline{y}_2 \leq 0\).

Chris and Kipp would like to express their undying gratitude to Angelo for ruining a perfectly good Sunday afternoon with this example.

Consider the modified primal problem \((GLPM)\).

\[
\begin{align*}
\text{min } & f(x) \\
(GLPM) \quad & G(x) \preceq_P Y, \quad \theta_Y \\
& H(x) \preceq_P Z, \quad \theta_Z \\
& x \in \Omega
\end{align*}
\]
with \(H(x)\) an affine mapping from the vector space \(X\) into the vector space \(Z\).

\[
\Gamma := \{(r, y, z) \in \mathbb{R} \times Y \times Z : \exists x \in \Omega \text{ s.t. } f(x) \leq r, G(x) \preceq y, H(x) \preceq z\}. \tag{3.6.5}
\]

Writing \((P)\) as \((GLPM)\) with two sets of constraints has no effect on any of duality results in Section 3.4. However, by writing out two sets of constraints, we have greater
flexibility in achieving the sufficient conditions for zero duality gap and strong duality. If $H(x)$ is an affine mapping, and the cone $P_Z$ consists of the singleton $\theta_Z$, then obviously we cannot achieve our Slater core point assumption. We show in Lemma 3.6.4 that if there is a feasible primal point $\bar{x} \in \Omega$ and $G(\bar{x})$ is a core point of $N_Y$, then we can relax the condition that $H(\bar{x})$ is a core point of $N_Z$ to requiring that $\theta_Z$ is a core point of $\Upsilon$ defined below in (3.6.6).

$$\Upsilon := \{ z \in Z : \exists x \in \Omega \text{ s.t. } H(x) = z \} \quad (3.6.6)$$

Before proving the key Lemma 3.6.4, we need a brief digression. In our main Lemma 3.6.4 we are going to assume that $\theta_Z$ is a core point of $\Upsilon$. This means that if $q$ is an arbitrary point in $Z$, for a sufficiently small $\lambda > 0$, $\theta_Y + \lambda q$ is in $\Upsilon$. This in turn, implies by definition of $\Upsilon$, that there is $\hat{x} \in \Omega$ such that $H(\hat{x}) = \theta_Y + \lambda q$. We are going to show that by making $\lambda$ sufficiently small, it is legitimate to also assume $G(\hat{x})$ is core point of $N_Y$. This may seem like a great leap of faith, but not really.

Lemma 3.6.2. If $G : X \to Y$ is a convex mapping, and $G(\bar{x})$ is a core point of $N_Y$, and $\hat{x} \in X$, then there exists an $\tilde{\alpha} \in (0,1)$ such that $G(\hat{x})$ is a core point of $N_Y$ where $\hat{x}$ is defined by $\hat{x} = (1 - \tilde{\alpha})\bar{x} + \tilde{\alpha} \tilde{x}$.

Proof. By convexity of $G : X \to Y$, if $\alpha \in (0,1)$,

$$G((1 - \alpha)\bar{x} + \alpha \tilde{x}) \preceq (1 - \alpha)G(\bar{x}) + \alpha G(\tilde{x})$$

Since $G(\bar{x})$ is a core point of $N_Y$, for sufficiently small $\alpha > 0$,

$$(1 - \alpha)G(\bar{x}) + \alpha G(\tilde{x}) \in N_Y$$

Since $G(\bar{x})$ is a core point of $N_Y$, and for sufficiently small $\epsilon$, the points, $(1 - \alpha)G(\bar{x}) + \alpha G(\tilde{x})$ lie in $N_Y$, we can apply Theorem 2.4.20 and conclude that there is an $\tilde{\alpha} \in (0,1)$ such that $(1 - \tilde{\alpha})G(\bar{x}) + \tilde{\alpha} G(\tilde{x})$ is a core point of $N_Y$. Take $\hat{x} = (1 - \tilde{\alpha})\bar{x} + \tilde{\alpha} \tilde{x}$ and by convexity of the $G$ mapping

$$G(\hat{x}) \preceq (1 - \tilde{\alpha})G(\bar{x}) + \tilde{\alpha} G(\tilde{x})$$

Now show $G(\hat{x})$ is a core point in $N_Y$. Let $p$ be an arbitrary element of $Y$. Since $(1 - \tilde{\alpha})G(\bar{x}) + \tilde{\alpha} G(\tilde{x})$ is core point of $N_Y$, there is a $\lambda > 0$ such that

$$(1 - \tilde{\alpha})G(\bar{x}) + \tilde{\alpha} G(\tilde{x}) + \lambda p \preceq \theta_Y$$

and this implies

$$G(\hat{x}) + \lambda p \preceq \theta_Y.$$ 

Since $p$ was an arbitrary direction, $G(\hat{x})$ is a core point of $N_Y$. \qed
CHAPTER 3. ALGEBRAIC LAGRANGIAN DUALITY

Remark 3.6.3. Let \( q \) be an arbitrary point in \( Z \). If \( \theta_Z \) is a core point of \( \Upsilon \), for a sufficiently small \( \lambda > 0 \), \( \theta_Y + \lambda q \) is in \( \Upsilon \). By definition of \( \Upsilon \), there is a \( \hat{x} \in X \) such that \( H(\hat{x}) = \lambda q \). Since \( H \) is an affine mapping, for every \( \alpha \in (0, 1) \),

\[
H((1 - \alpha)\overline{x} + \alpha \hat{x}) = (1 - \alpha)H(\overline{x}) + \alpha H(\hat{x}) = (1 - \alpha)\theta_Z + \alpha \lambda q
\]

so the image of \((1 - \alpha)\overline{x} + \alpha \hat{x}\) under \( H \) is on the line connecting \( \theta_Z \) and \( \lambda q \). By Lemma 3.6.2, we can pick an \( \hat{\alpha} \) such that \( \hat{x} = (1 - \hat{\alpha})\overline{x} + \hat{\alpha} \hat{x} \) and \( G(\hat{x}) \) is a core point of \( N_Y \). But for this \( \hat{x} \), \( H(\hat{x}) \) is on the line segment connecting \( \theta_Z \) and \( \lambda q \). Therefore, we can assume, without loss, we have picked \( \lambda \) small enough so that not only does \( H(\hat{x}) = \lambda q \), but \( G(\hat{x}) \) is in the core of \( N_Y \).

Lemma 3.6.4. If

1. \( H : X \rightarrow Z \) is an affine mapping
2. \( P_Z = \theta_Z \)
3. there is an \( \overline{x} \in \Omega \) with \( H(\overline{x}) \preceq \theta_Z \) and \( G(\overline{x}) \) is a core point of \( N_Y \)
4. \( \theta_Z \) is a core point of \( \Upsilon \)

then there exists an \( \epsilon > 0 \), such that the point \((\epsilon + 1, \theta_Y, \theta_Z)\) is in the core of \( \Gamma \) defined in (3.6.6).

Proof. Let \((r, p, q)\) be an arbitrary element of \( \mathcal{R} \times Y \times Z \). By definition of core point, \((\epsilon + 1, \theta_Y, \theta_Z)\) is a core point of \( \Gamma \) if there exists \( X > 0 \) such that for all \( \lambda \in [0,X) \)

\[
(\epsilon + 1 + \lambda r, \theta_Y + \lambda p, \theta_Z + \lambda q) \in \Gamma
\]

By part 4. of the hypothesis, \( \theta_Z \) is a core point of \( \Upsilon \). This means that regardless of the choice of \( q \), we can move a small distance from \( \theta_Y \) in the direction of \( q \) and remain in \( \Upsilon \). Then there is a \( \lambda_1 > 0 \) and an \( \hat{x} \in \Omega \) such that

\[
H(\hat{x}) = \theta_Z + \lambda_1 q
\]

Since \( H : X \rightarrow Z \) is affine, if \( \alpha \in [0, 1] \) then

\[
H((1 - \alpha)\overline{x} + \alpha \hat{x}) = (1 - \alpha)H(\overline{x}) + \alpha H(\hat{x}) = (1 - \alpha)\theta_Z + \alpha(\theta_Z + \lambda_1 q) = \theta_Z + \alpha \lambda_1 q
\]

Since \( \Omega \) is convex, and \( \hat{x} \) and \( \overline{x} \) are in \( \Omega \), we know \((1 - \alpha)\overline{x} + \alpha \hat{x} \in \Omega \). Therefore, if we pick any \( \lambda \) in the interval \((0, \lambda_1)\), and set \( \alpha = \lambda/\lambda_1 \), the point \((1 - \alpha)\overline{x} + \alpha \hat{x} \in X \) maps under \( H \) to the point \( \theta_Z + \lambda q \) in \( Z \).
3.6. AN EXTENSION WITH AFFINE MAPPINGS

By hypothesis 3., \( \overline{x} \) is in the core of \( N_Y \), and by Remark 3.6.3 we assume \( \hat{x} \) is also in the core of \( N_Y \). Then by definition of core point, there exists \( \lambda_2 > 0 \) such that for all \( \lambda \in [0, \lambda_2) \),

\[
G(\overline{x}) \preceq_Y \lambda p, \quad G(\hat{x}) \preceq_Y \lambda p
\]

Then

\[
G((1 - \alpha)\overline{x} + \alpha \hat{x}) \preceq_Y (1 - \alpha)G(\overline{x}) + \alpha G(\hat{x}) \preceq_Y (1 - \alpha)\lambda p + \alpha \lambda p \preceq_Y \lambda p = \theta_Y + \lambda p
\]

Therefore, as \( \alpha \) varies from 0 to 1, the point \( (1 - \alpha)\overline{x} + \alpha \hat{x} \in X \) maps under \( G \) to a point \( G((1 - \alpha)\overline{x} + \alpha \hat{x}) \) in \( Y \) that precedes \( (\preceq) \lambda p \) for all \( \lambda \) from 0 to \( \lambda_2 \).

Now for the objective function. Let

\[
\epsilon = \max\{f(\overline{x}), f(\hat{x}), 0\}
\]

Then

\[
f((1 - \alpha)\overline{x} + \alpha \hat{x}) \leq (1 - \alpha)f(\overline{x}) + \alpha f(\hat{x}) \leq (1 - \alpha)\epsilon + \alpha \epsilon = \epsilon
\]

Pick \( \lambda_3 = 1/|r| \). Therefore, as \( \alpha \) varies from 0 to 1, the point \( (1 - \alpha)\overline{x} + \alpha \hat{x} \in X \) maps under \( f \) to a point \( f((1 - \alpha)\overline{x} + \alpha \hat{x}) \) in \( R \) that cannot exceed \( \epsilon + 1 + \lambda \) for all \( \lambda \) from 0 to \( \lambda_3 \).

Now pick \( \overline{\lambda} = \min\{\lambda_1, \lambda_2, \lambda_3\} \). Then given any \( \lambda \) in the interval \( [0, \overline{\lambda}] \) the point \( (1 - \alpha)\overline{x} + \alpha \hat{x} \) for \( \alpha = \lambda/\lambda_1 \) maps to a point in \( (R \times Y \times Z) \) under \( f, G, \) and \( H \) respectively such that

\[
f((1 - \alpha)\overline{x} + \alpha \hat{x}) \leq \epsilon + 1 + \lambda r
\]

\[
G((1 - \alpha)\overline{x} + \alpha \hat{x}) \preceq \theta_Y + \lambda p
\]

\[
H((1 - \alpha)\overline{x} + \alpha \hat{x}) = \theta_Y + \lambda q
\]

and this implies

\[
(\epsilon + 1 + \lambda r, \theta_Y + \lambda p, \theta_Z + \lambda q) \in \Gamma.
\]

Note that by definition, \( \overline{\lambda} \leq \lambda \) so \( \alpha = \lambda/\lambda_1 \) can never exceed 1 so we are taking a convex combination of \( \overline{x} \) and \( \hat{x} \).

\[\Box\]

**Remark 3.6.5.** From Lemma 3.6.4 we have \( (\epsilon + 1, \theta_Y, \theta_Z) \) a core point of \( \Gamma \) which is needed in the conditions for the results in Section 3.4.
3.7 Linear Programming

A very general class of problems that fit into the general framework of (P) are those where \( f \) and \( G \) are linear. These problems are known in the literature as infinite dimensional conic LP’s and have the general form:

\[
\inf \langle x, \phi \rangle_X \\
\text{s.t. } b - A(x) \preceq_{P_Y} \theta_Y \\
x \in \Omega
\]

where \( x \in \Omega \subseteq X, b \in Y, P \) is a pointed, convex cone in \( Y \) and \( A : X \to Y \) is a linear mapping of vector spaces \( X \) and \( Y \). That is, \( A(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 A(x^1) + \alpha_2 A(x^2) \) for all \( x^1, x^2 \in X \) and \( \alpha_1, \alpha_2 \in \mathbb{R} \). We assume \( \Omega = X \). The problem’s label of (GLP) stands for general LP as this is the most general LP formulation that we consider.

Classical (finite dimensional) linear programming has \( X = \mathbb{R}^n, Y = \mathbb{R}^m, \) and \( P = \mathbb{R}^{m+} \). In this setting, (GLP) becomes the familiar:

\[
\min c^\top x \\
\text{s.t. } Ax \geq b.
\]

as it is commonly seen in the literature, where \( A \) is an \( m \times n \) matrix, \( c \in \mathbb{R}^n, x \in \mathbb{R}^n, \) and \( b \in \mathbb{R}^m \).

Turning (GLP) into (FLP) under the assumption of finite dimensions is not entirely straightforward: it depends on an adept understanding of the content in Section 2.2 and Section 2.6. Every piece of the optimization problem when one compares (GLP) and (FLP) looks a bit different:

(i) The objective turns from \( \langle x, \phi \rangle \) to \( c^\top x \).

(ii) The constraint turns from \( b - A(x) \preceq_p \theta_Y \) to \( Ax \geq b \).

(iii) The inf turns into min.

We discuss these changes in detail now.

(i) We are able to rewrite the objective using the following reasoning: in view of Example 2.2.3, the linear functional \( \phi \) takes on a form \( \phi_c \) for some \( c \in \mathbb{R}^n \) (see (2.2.1) for a definition of \( \phi_c \)) and uses the identity:

\[
\langle x, \phi_c \rangle = c^\top x.
\]

(ii) The constraint has two major changes, we have a functional notation \( A(x) \) replaced with a vector-matrix product \( Ax \). The reason is that the linear map \( A \) usually is
represented by its associated matrix, also denoted by $A$. In this setting, $A$ is an $m \times n$ matrix and $A(x)$ is the vector-matrix product $Ax$.

The second change is in swapping the $\preceq_P$ ordering for the $\leq$ ordering and putting $b$ on the right hand side. In standard linear programming formulations $P$ is the positive orthant in $\mathbb{R}^m$:

$$\mathbb{R}_+^m = \{ y \in \mathbb{R}^m : y_i \geq 0, i = 1, \ldots, m \}.$$

where the notation $\geq$ is standard shorthand for $\succeq_{\mathbb{R}_+^m}$.

(iii) This is an important issue discussed later in the section. The fact that every feasible and bounded (we are referring to objective function value, not feasible region) finite dimensional linear program satisfies strong duality is something usually taken for granted. This is not always the case for infinite dimensional conic linear programs.

Returning to general problem, let’s apply the Lagrangian approach to (GLP). Choose a dual vector $\psi \in Y$ from $P_+^T$ (to assure weak duality) and form the Langrangian dual:

$$\sup_{\psi \in P_+^T} \left\{ \inf_{x \in \Omega} \langle x, \phi \rangle_X + \langle b - A(x), \psi \rangle_Y \right\}.$$ 

cleaning up a bit by taking the term $\langle b, \psi \rangle_Y$ out of the minimization in $x$ yields:

$$\sup_{\psi \in P_+^T} \left\{ \langle b, \psi \rangle + \inf_{x \in \Omega} \{ \langle x, \phi \rangle_X - \langle A(x), \psi \rangle_Y \} \right\}. \quad (*)$$

Now consider the inner minimization in $x$. Notice that the the terms in angle braces $\langle x, \phi \rangle_X$ and $\langle A(x), \psi \rangle_Y$ represent linear functionals over two different spaces $X$ and $Y$. How should we combine them? Moreover, $x$ appears in a different way in both terms: in the latter term, $x$ enters as an argument of a linear map. We would like to simplify further, say by collecting terms on $X$, but it seems we are stuck.

The dual for a finite dimensional linear program is much cleaner. The dual of the finite linear program (FLP) is:

$$\max \ b^\top y \quad \text{s.t.} \ A^\top y = c \quad \text{FLPD}$$

$$y \geq 0.$$ 

Even more puzzling than turning (GLP) into (FLP) is turning (*) into (FLPD). The key part we focus on here is how to drop the inf in (*) and end up with a maximization

---

3 Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping of vector spaces. Let $e^i \in \mathbb{R}^n$ be the $i$th standard basis vector for $\mathbb{R}^n$. The matrix $M_A$ (often simply called $A$ despite the potential for confusion) has as its $i$th column the image of $e^i$ under the mapping $A$. That is, $(M_A)_i = A(e^i)$.
problem. Consider (*) in the case of the finite linear program (FLP). In this case the dual vector is \( \psi \in (\mathbb{R}^m)' \). However, in view of the isomorphism described in Example 2.2.3 we know \( \psi = \psi_y \) for some \( y \in \mathbb{R}^m \) where \( \langle b, \psi_y \rangle = y(b) = y^\top b \).

The expression inside the inner minimization of (*) yields:

\[
c^\top x - (Ax)^\top y.
\]

We immediately trip our “transpose” reflex and spring free the \( x \) the term \( (Ax)^\top y \) by writing

\[
(Ax)^\top y = x^\top (A^\top y).
\]

We are now ready to combine terms on \( x \) to yield a new minimization problem:

\[
\inf_{x \in X} x^\top (c - A^\top y).
\]

We now apply standard logic and observe that this optimization problem has a value of \(-\infty\) unless \( c - A^\top y = 0 \). This yields (FLPD).

Can we say something similar for a general LP? The answer is a resounding YES! It is enabled by a powerful “dual” (arghh, another dual concept!) to the linear map \( A \) called its adjoint.

**Definition.** Given a linear mapping \( A \) there is an associated mapping \( A' \) called the adjoint that maps \( Y' \) to \( X' \) with the following property:

\[
\langle x, A'(\psi) \rangle_X = \langle A(x), \psi \rangle_Y \quad \forall x \in X, \psi \in Y'
\]  

(3.7.1)

The notation is a bit confusing, so let’s take a moment to unpack it. The left-hand side of (3.7.1) represents the value of the linear functional \( A'(\psi) \) when evaluated at the vector \( x \). The right-hand side can be written equivalently as \( \psi(A(x)) \) where \( \psi \) is a linear functional on \( Y' \) and \( A(x) \) is an element of \( Y \), so things work out. Thus, an equivalent way to express (3.7.1) is

\[
A'(\psi)(x) = \psi(A(x))
\]

but the “angle brace” statement is often preferred to avoid use of the two sets of parantheses on both sides of the equation needed here.\(^4\) For yet another way to look at the definition of the adjoint check out Figure 3.4, which shows that we can think of \( A' \) as the composition map \( \psi \circ A \). Think of \( A'(\psi) \) as an “equivalent” notation for \( \psi \circ A \). Both \( A'(\psi) \) and \( \psi \circ A \) are linear functionals that take an element of \( x \in X \) and map it to the real numbers.

\(^4\)To obscure things even further some authors drop the parantheses in the statement \( A(x) \) and write \( Ax \) when \( A \) is a linear map. We avoid this notation as much as possible, but we may sometimes fall prey to its convenience.
Example 3.7.1. In this example, we show that the matrix notion of transpose is really just a special case of an adjoint. Consider a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$. Again abusing notation we use $A$ to refer both to the mapping and its associated $m \times n$ matrix. We ask you to keep track of which is which. We use the notation $A'$ for the adjoint of the mapping $A$, and $A^\top$ for the transpose of the matrix $A$.

Let’s think about the adjoint of $A$. The adjoint of $A$ is a mapping from $(\mathbb{R}^m)'$ to $(\mathbb{R}^n)'$. We claim the adjoint map of $A$ is as follows:

$$A' : (\mathbb{R}^m)' \to (\mathbb{R}^n)'$$
$$\psi_b \mapsto \phi_{A^\top b} \quad \text{(3.7.2)}$$

where $A^\top$ is the usual transpose of the matrix $A$. In other words, it takes an arbitrary linear functional of $\mathbb{R}^m$, $\psi_b$, into the linear functional of $\mathbb{R}^n$, $\phi_{A^\top b}$. A sanity check reveals that $A^\top b$ is in fact a vector in $\mathbb{R}^n$.

Let’s verify that $A'$ is indeed the adjoint of $A$. To do this we check (3.7.1), that is we verify

$$\langle x, A'(\psi_b) \rangle_{\mathbb{R}^n} = \langle A(x), \psi_b \rangle_{\mathbb{R}^m} \quad \text{(3.7.4)}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$. We start by developing the left-hand
This verifies that $A'$ is indeed the adjoint of $A$. Since the adjoint $A'$ is determined entirely by the matrix $A^\top$, we often confound notation and say the adjoint of $A$ is the matrix $A^\top$.

As long as you can keep track, then no worries!

We have gone on a detour through understanding adjoints, but now we are in a position to return to developing the dual of our general LP from where we left it in (*), reproduced here for convenience:

$\sup_{\psi \in P^+_Y} \left\{ \langle b, \psi \rangle + \inf_{x \in X} \left\{ \langle x, \phi \rangle - \langle A(x), \psi \rangle \right\} \right\}.$

We immediately recognize the second term in the inner minimization ($\langle A(x), \psi \rangle_Y$) is the definition of the adjoint in (3.7.1) and rewrite it as $\langle x, A'(\psi) \rangle_X$. Collecting terms in $x$ we rewrite (*) as:

$\sup_{\psi \in P^+_Y} \left\{ \langle b, \psi \rangle + \inf_{x \in X} \left\{ \langle x, \phi - A'(\psi) \rangle_X \right\} \right\}.$

Finally, we employ our usual trick! The inner minimization shoots off to $-\infty$ unless $\phi - A'(\psi) = \theta_X'$, where $\theta_X'$ is the linear functional that takes every element in $X$ to 0. We write the algebraic dual as a generalization of (FLPD):

$\sup \langle b, \psi \rangle$

s.t. $\phi - A'(\psi) = \theta_X'$

$\psi \in P^+_Y.$

Now that we have our primal-dual pair in (GLP) and (GLPD) we turn to the questions of duality gap and strong duality. Let’s start by examining our familiar case: LP in finite dimensions with primal dual pair (FLP) and (FLPD). There are numerous proofs of the basic fact that strong duality always holds for finite dimensional LP’s. Our proof is based

5if some of the notation that follows confuses you, check out Appendix A.2
on the very general principle encapsulated in Corollary 3.4.12, which gives hints to results in more abstract settings.

First a general statement about the algebraic closure of generalized polyhedra in arbitrary vector spaces.

**Lemma 3.7.2.** Let $Q = \{ x \in \mathbb{R}^n : Ax \geq b \}$ be a polyhedron in $\mathbb{R}^n$. Then $\text{lin}(Q) = Q$.

*Proof.* See Exercise 3.15

Assume the rows of $A$ that define $Q = \{ x \in \mathbb{R}^n : Ax \geq b \}$ are $a^i, i = 1, \ldots, m$. Define

$$I^\leq := \{ i : (a^i)^\top x = b_i, \forall x \in Q \}$$

$$I^\geq := \{ 1, 2, \ldots, m \} \setminus I^\leq$$

Denote by $A^\leq$, the submatrix of $A$ consisting of the rows of $A$ indexed by $I^\leq$. Likewise, $A^\geq$ is the submatrix of $A$ consisting of the rows of $A$ indexed by $I^\geq$. Similarly for $A^\leq$, $b^\leq$, and $A^\geq$.

**Lemma 3.7.3.** If the polyhedron $Q = \{ x \in \mathbb{R}^n : Ax \geq b \}$ is not empty, then $Q$ can be expressed as $Q = \{ x \in \mathbb{R}^n : A^\leq x = b^\leq, A^\geq x \geq b^\geq \}$ and there is an $\pi$ such that $A^\leq \pi = b^\leq$ and $A^\geq \pi > b^\geq$.

*Proof.* Kipp is very tired and stressed so this left as Exercise 3.17.

**Lemma 3.7.4.** If the polyhedron $Q = \{ x \in \mathbb{R}^n : Ax \geq b \}$ is not empty, then 0 is a core point of

$$\Upsilon = \{ z : \exists x \in \mathbb{R}^n \text{ s.t. } b^\leq - A^\leq x = z \}$$

*Proof.* Yes, you guessed it, still very tired and stressed so this left as Exercise 3.18.

**Theorem 3.7.5** (Strong duality for finite dimensional LP’s). Consider the primal dual pair (FLP) and (FLPD). One of the following three conditions must hold:

(i) the primal is infeasible;

(ii) the primal is unbounded;

(iii) strong duality always holds; there is a primal feasible solution $\pi$, a dual feasible solution $\pi$, and $c^\top x = b^\top y$.

*Proof.* Assume that the primal is both feasible and bounded. There are three things to check: (i) $\Gamma$ has a nonempty intrinsic core $(\pi, \pi)$ with $\pi \leq \theta_\Upsilon$, (ii) $\Gamma = \text{lin}(\Gamma)$ and (iii) $\nu$ finite.
(i) If the primal is feasible, by Lemma 3.7.3 we can break the constraints $Ax \geq b$ into the set $A^=x = b^= \text{ which corresponds to } H(x) \preceq_{P_Z} \theta_Z$ and $b^\geq -A^\geq x \leq 0$ which corresponds to $G(x) \preceq_{P_Y} \theta_Y$. Furthermore, by Lemma 3.7.3 there is a feasible $\bar{x}$ with $b - A\bar{x} < 0$. Then $\bar{x}$ is a core point of $N_Y$. By Lemma 3.7.4, $0 = \theta_Z$ is a core point of $Y = \{z : A^=x = z\}$. Then by Lemma 3.6.4, $\Gamma$ has a nonempty intrinsic core point $(\epsilon + 1, \theta_Y, \theta_Z)$.

(ii) Our approach is to show that $\Gamma$ is a polyhedron and thus leverage Lemma 3.7.2 to establish $\Gamma$ is algebraically closed. Clearly, $\Gamma$ is the projection of the polyhedron
\[
\{(r, y, x) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n : c^T x \leq r, Ax \geq b - y\}
\]
on its $r$ and $y$ coordinates. It is well-known (using Fourier-Motzkin) the the projection of a polyhedron is a polyhedron. This implies that $\Gamma$ is a polyhedron, establishing the result.

(iii) The last step is to check $\nu$ is finite. By (ii), $\Gamma = \text{lin}(\Gamma)$ so $\nu = \mu$. By Theorem 3.4.2, we know $\mu = v(P)$. Since the primal is both feasible and not unbounded, $\mu$ is finite. This implies that $\nu = \mu$ is finite.

We have strong duality for finite dimensional linear programming problems. What can we say about general linear programs? It is possible for there to be a duality gap? Indeed, there can be as the following example demonstrates.

**Example 3.7.6** (Due to Gale, as cited in [3]). Consider the following program (PG): 

\[
\text{PG} : \begin{bmatrix} \mathbb{R}^\infty \\ \mathbb{R}^\infty \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{R}^2 \\ (\mathbb{R}^2)' \end{bmatrix} : \text{PGD}
\]

with primal-dual pair:

\[
\text{min } x_0 \\
\text{s.t. } -x_0 - \sum_{i=1}^\infty ix_i + 1 \leq 0 \\
\sum_{i=1}^\infty x_i \leq 0 \\
x_i \geq 0, \quad i = 0, 1, 2, \ldots \\
x \in \mathbb{R}^\infty
\]

(PG)
and

$$\begin{align*}
\text{max} & \quad w_1 \\
\text{s.t.} & \quad w_1 \leq 1 \\
& \quad iw_1 - w_2 \leq 0 \quad i = 1, 2, \ldots \\
& \quad w_1, w_2 \geq 0.
\end{align*}$$

(PGD)

Here the cone $P_Y = \Re^2$ and $\Omega = \Re^\infty_+ = \{x \in \Re^\infty : x_i \geq 0 \forall i = 1, 2, \ldots\}$. Exercise 3.20 asks you to verify that (PGD) is indeed the dual of (PG). We are a bit sloppy, since we are assuming here the dual variables are in $\Re^2$ when in fact they are in $\Re^2'$. This valid because by Proposition 2.2.5, $\Re^2$ is self-dual.

We now argue that there is a duality gap. We claim that $v(PG) = 1$. The second constraint in (PG) combined with the nonnegativity constraints on the $x_i$’s imply $x_i = 0$ for $i = 1, 2, \ldots$. Thus, the first constraint effectively becomes

$$x_0 \geq 1.$$ 

The objective is to minimize $x_0$ so we take $x_0 = 1$. The feasible region is $\{(x_0, 0, 0, \ldots) : x_0 \geq 1\}$ and the optimal solution is $(1, 0, \ldots)$ with an optimal value of 1.

Now show $v(PGD) = 0$. The non negativity of $w_1$ and $w_2$, along with the second set of constraints of (PGD), implies that $w_1 = 0$. This implies that $v(PGD) = 0$.

Clearly this problem must fail to satisfy the sufficient conditions for “no duality gap” described in Section 3.4. In particular, we show that the conditions of Corollary 3.4.8 do not hold. That is, there is no intrinsic core point $(\bar{r}, \bar{y})$ of $\Gamma$ with $\bar{y} \preceq \theta_Y$. To establish this we first of all show that the affine hull of $\Gamma$ is $\Re^3$ and hence the $\text{cor}(\Gamma) = \text{icr}(\Gamma)$. Then we show that no point in $\Gamma$ satisfying $\bar{y} \preceq \theta_Y$ can be a core point. The set $\Gamma \in \Re^3$ is:

$$\Gamma = \left\{ (r, y_1, y_2) : \exists x \geq 0 \text{ s.t. } \begin{array}{ll}
x_0 \leq r, \\
-x_0 - \sum_{i=1}^{\infty} ix_i + 1 \leq y_1, \\
\sum_{i=1}^{\infty} x_i \leq y_2
\end{array} \right\}. \quad (3.7.5)$$

We first claim that $\text{aff}(\Gamma) = \Re^3$ and hence $\text{icr}(\Gamma) = \text{cor}(\Gamma)$. This follows from the fact that the following four affinely independent vectors lie in $\Gamma$:

$$(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1).$$

It is straightforward to check that this points are in $\Gamma$. For example, $(0, 1, 0) \in \Gamma$ since the point $x = (0, 0, \ldots)$ satisfies the constraints in the definition of $\Gamma$ in (3.7.5). The first three points are linearly independent and therefore are affinely independent. It is also clear that $(1, 1, 1) \notin \text{aff}(\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\})$. If $(1, 1, 1)$ is written as a linear combination of $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ the multipliers must sum to 3 and not 1. This implies that the affine hull of $\Gamma$ has dimensions greater than or equal to 3 (see Exercise 2.10). Clearly, every affine subspace in $\Re^3$ has dimension at most three, and $\text{aff}(\Gamma) = \Re^3$. 

Now, consider a point \((\vec{r}, \vec{y})\) in \(\Gamma\) with \(\vec{y} \preceq \theta_Y\). This means \(\vec{y} \in -\mathbb{R}^2\) and hence \(\vec{y}_1 \leq 0\) and \(\vec{y}_2 \leq 0\). However, if \((\vec{r}, \vec{y}) \in \Gamma\) the nonnegativity of the \(x_i\) imply \(\vec{y}_2 \geq 0\). Taken together this implies that \(\vec{y}_2 = 0\). However, no such point can be a core point of \(\Gamma\). From any point of the form \((r, y_1, 0)\) we cannot move towards the point \((0, 0, -1)\) and remain in \(\Gamma\). Thus, the conditions of Corollary 3.4.8 are violated. This allows for the possibility of a duality gap.

We provide more applications of infinite dimensional linear programs in Chapter ??.

### 3.8 Notes

To the author’s knowledge, the main results in of this chapter do not appear elsewhere in the literature. Many references [3, 4, 16, 22, 28] discuss an algebraic dual problem, but do not give duality results for the generic primal problem \((P)\). An exception to this is a recent paper by Pintér [23] who explores strong duality in the linear case. His proof of strong duality, however, is not amenable to the more general convex case considered here.

The statement of Theorem 3.4.6 is partly inspired by duality results found in Chapter 3 of Anderson and Nash [3].

### 3.9 Exercises

**Exercise 3.1.** Prove Lemma 3.4.1: Show the set \(\Gamma\) defined in (3.4.1) is convex.

**Exercise 3.2.** Prove Lemma 3.4.4. If \(\nu\) is finite, and the intrinsic core of \(\Gamma\) is not empty, then \((\nu, \theta_Y) \in \text{lin}(\Gamma)\).

**Exercise 3.3.** Assume \(A\) is a convex set in the vector space \(X\) with a nonempty intrinsic core. Show that if \(x\) can be strictly separated from \(A\) it can be strictly separated from \(\text{lin}(A)\).

**Exercise 3.4.** (Minimum not achieved, positive duality gap) Nice example from page 280 of Boyd and Vandenberghe text book. Consider the primal problem:

\[
\begin{align*}
\text{inf} & \quad e^{-x_1} \\
\text{s.t.} & \quad x_1^2/x_2 \leq 0 \\
& \quad (x_1, x_2) \in \Omega = \{(x_1, x_2) : x_2 > 0\}
\end{align*}
\]

**a.** Formulate the dual problem.

**b.** Find the optimal primal value.

**c.** Find the optimal dual value.
d. Draw the set $\Gamma$ for this problem, where $\Gamma$ is defined in (3.4.1).

e. Describe how this problem fails to meet the hypothesis of Corollary 3.4.8 that guarantees no duality gap.

**Exercise 3.5** (Minimum not achieved, no duality gap (*Boyd and Vandenberghe survey page 64*)).

$$\inf t \begin{bmatrix} x & 1 \\ 1 & t \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \succeq 0$$

a. Formulate the dual problem.

b. What is the optimal primal value?

c. What is the optimal dual value?

d. Do the hypotheses of Corollary 3.4.7 hold?

e. Do the hypotheses of Corollary 3.4.8 hold?

f. Do the hypotheses of Corollary 3.4.12 hold?

**Exercise 3.6.** Prove Corollary 3.4.14: If $(\bar{x}, \bar{\psi})$ are a primal-dual pair that satisfy complementary slackness then the primal-dual solution satisfies strong duality.

**Exercise 3.7.** Prove Corollary 3.4.15: If $\bar{x}$ is an optimal primal solution, $\bar{\psi}$ is an optimal dual solution, and $\mu = \nu$, then this primal-dual pair satisfies complementary slackness.

**Exercise 3.8.** Prove Lemma 3.5.1: If $\Omega$ and $G(\Omega)$ are convex sets, then so is $\Lambda$.

**Exercise 3.9.** Prove Lemma 3.5.2: Show that if $f$ and $G$ are convex mappings and $\Omega$ is a convex set then the optimal value functional $\omega$ defined in (3.5.2) is

(i) monotonically decreasing; that is, for $y_1, y_2 \in \Lambda$ with $y_1 \preceq y_2$ implies $\omega(y_1) \geq \omega(y_2)$

and

(ii) convex.

**Exercise 3.10** (A. Mancini). Assume $f$ and $G$ are convex and $\Gamma$ is defined by (3.4.1)). Show that the epigraph of $\omega$ is a subset of $\text{lin}(\Gamma)$. In other words, show

$$\{(r, y) \in \mathbb{R} \times \Lambda : \omega(y) \leq r\} \subseteq \text{lin}(\Gamma). \quad (3.9.1)$$
Exercise 3.11. Prove Proposition 3.5.3: Show that the dual functional \( L(\psi) \) is concave over \( P_Y^+ \) and can be written as:

\[
L(\psi) = \inf_{y \in \Lambda} \{\omega(y) + \langle y, \psi \rangle\} \quad (3.9.2)
\]

Exercise 3.12 (Everett’s Theorem). If \( \bar{\psi} \in P_Y^+ \) and

\[
\bar{x} = \arg\min_{x \in \Omega} \{f(x) + \langle G(x), \bar{\psi} \rangle\}
\]

then \( f(\bar{x}) = \omega(G(\bar{x})) \).

Exercise 3.13. Prove Lemma 3.7.2. Assume \( H : X \to Z \) is an affine mapping and \( P_Z = \theta_Z \). If \( \bar{\pi} \in \Omega = X \), \( -G(\bar{\pi}) \) is a core point of \( P_Y \), and \( H(\bar{\pi}) \preceq P_Z \), then \( (f(\bar{\pi}), \theta_Y, \theta_Z) \) is in the intrinsic core of \( \Gamma \) defined in (3.4).

Exercise 3.14. If \( (f(\bar{\pi}), G(\bar{\pi})) \) in the intrinsic core of \( \Gamma \) (defined in (3.4.1)) and \( G(\bar{\pi}) \preceq \theta_Y \), then \( (f(\bar{\pi}), \theta_Y) \) is in the intrinsic core of \( \Gamma \).

Exercise 3.15. Prove Lemma 3.7.2: Let \( Q = \{x \in \mathbb{R}^n : Ax \geq b\} \) be a polyhedron in \( \mathbb{R}^n \). Then \( \text{lin}(Q) = Q \).

Exercise 3.16. Assume that the linear program (FLP) is feasible and bounded. If the set \( I^\perp \) is empty, then every feasible solution is an optimal primal solution, there is an optimal dual solution, and the optimal primal and dual solution values are equal. Do not appeal to any of the results in this Chapter in the proof. Prove by “first principles”.

Exercise 3.17. Prove Lemma 3.7.3 If the polyhedron \( Q = \{x \in \mathbb{R}^n : Ax \geq b\} \) is not empty, then \( Q \) can be expressed as \( Q = \{x \in \mathbb{R}^n : A^\perp x = b^\perp, A^{\ge} x \geq b^{\ge}\} \) and there is an \( \bar{x} \) such that \( A^\perp \bar{x} = b^\perp \) and \( A^{\ge} \bar{x} > b^{\ge} \).

Exercise 3.18. Prove Lemma 3.7.4: If the polyhedron \( Q = \{x \in \mathbb{R}^n : Ax \geq b\} \) is not empty, then \( 0 \) is a core point of

\[
\mathcal{Y} = \{z : \exists x \in \mathbb{R}^n \text{ s.t. } b^\perp - A^\perp x = z\}
\]

Exercise 3.19. Consider a semi-infinite program where there are possibly an infinite number of variables, but a finite number of constraints:

\[
\text{SIPV} : \begin{bmatrix} X \\ X' \end{bmatrix} \to \begin{bmatrix} \mathbb{R}^m \\ \mathbb{R}^m \end{bmatrix} : \text{DSIPV}
\]

The primal problem (SIPV) is

\[
\min f(x) \\
A(x) \geq b \\
x \in X
\]

where \( X \) is an arbitrary vector space (possibly infinite dimensional), \( A : X \to Y \) is a linear mapping, \( f : X \to \mathbb{R} \) is a convex functional, and \( b \in \mathbb{R}^m \).
3.9. EXERCISES

- If the primal (SPIV) is feasible and \( \nu \) finite, then there is no duality gap, i.e. \( v(SIPV) = v(DSIPV) \).
- If \( \Gamma = \text{lin}(\Gamma) \), then strong duality holds in addition to a zero duality gap.

**Exercise 3.20.** Verify that (PGD) is indeed the Lagrangian dual of (PG).

**Exercise 3.21.** (M. Stern) Assume \( X \) and \( Y \) are vector spaces. Define the vector space \( W = X \times Y \). Let \( \phi_W \) be an arbitrary linear functional in \( W' \). Show that there exists \( \phi_X \in X' \) and \( \phi_Y \in Y' \) such that

\[
\langle (x, y), \phi_W \rangle = \langle x, \phi_X \rangle + \langle y, \phi_Y \rangle
\]

for all \( (x, y) \in W \).

**Exercise 3.22.** In Exercise 3.22 you were asked to show

\[
\{ (r, y) \in \mathbb{R} \times \Lambda : \omega(y) \leq r \} \subseteq \text{lin}(\Gamma).
\]

Show by example that

\[
\{ (r, y) \in \mathbb{R} \times \Lambda : \omega(y) \leq r \} \subseteq \Gamma
\]

is false.

**Exercise 3.23.** In Exercise 3.22 you were asked to show

\[
\{ (r, y) \in \mathbb{R} \times \Lambda : \omega(y) \leq r \} \subseteq \text{lin}(\Gamma).
\]

Show by example that

\[
\{ (r, y) \in \mathbb{R} \times \Lambda : \omega(y) \leq r \} = \text{lin}(\Gamma)
\]

is false.
Chapter 4

Duality in Topological Vector Spaces

In the previous chapter we developed one of our main results – strong duality – in the pristine world of abstract vector spaces. Up until this point, all of our optimization is safely in the realm of linear algebra! This chapter marks a transition into a slightly less innocent world, a world with topology. We do so to address some outstanding issues with algebraic duality results. Despite their beauty, we are motivated to sharpen the theory to meet more specific needs.

There are basically two motivations for moving to topological vector spaces. First, the algebraic dual of vector spaces can be quite nasty in general and something we might like to avoid. This is discussed in the following section. Second, when we have a well-behaved topology (namely, a locally convex topology) we can streamline our separation theory. Indeed, we can drop some of the intrinsic core conditions which showed up frequently in Chapters 2 and 3. We explore this in Section 4.7

4.1 Motivation #1: Algebraic duals are nasty!

Through Chapters 2 and 3 we developed a geometric theory of optimization and duality. The Lagrangian dual was based on the algebraic dual of a vector space, and conditions involved the algebraic closure and intrinsic core. We believe this approach gives a very intuitive picture of the essence of optimization theory. Everything essentially boils down to line segments! Nonetheless, there are limitations to a purely algebraic approach. The first important limitation is that the algebraic dual can be an unwieldy monster! Recall our canonical representation of elements of $X'$ (the algebraic dual of vector space $X$):

$$X' \cong \mathbb{R}^\mathbb{N}$$
where \( \mathcal{H} \) is a Hamel basis of \( X \) and \( \mathbb{R}^\mathcal{H} \) is the space of all functions from \( \mathcal{H} \) to \( \mathbb{R} \). This representation can be a problem, in particular, when \( \mathcal{H} \) is nasty in some way. The following example demonstrates this possibility.

**Example 4.1.1.** Recall the vector space \( X = \mathbb{R}^\mathbb{N} = \{(x_1, x_2, \ldots) : x_i \in \mathbb{R}\} \). This is the space of all real sequences, which we encountered at several places in the previous chapters. This vector space is quite familiar to us from our introductory real analysis days. Its algebraic dual, however, is much less familiar. In fact, we can say very little about it. To see this, let’s try to understand the Hamel basis \( H \) of \( \mathbb{R}^\mathbb{N} \) in order to get a sense of our canonical representation \( (\mathbb{R}^\mathbb{N})' \cong \mathbb{R}^\mathcal{H} \).

Recall that the Hamel basis of \( \mathbb{R}^\mathbb{N} \) (see Example 2.2.7) is \( \{e_1, e_2, \ldots\} \). This Hamel basis is clearly countable under the simple bijection \( n \mapsto e_n \) for \( n \in \mathbb{N} \). Although the vectors \( e_n \) lie in \( \mathbb{R}^\mathbb{N} \), and are linearly independent, but they do not produce a basis of \( \mathbb{R}^\mathbb{N} \). Consider the element \( (1, 1, \ldots) \in \mathbb{R}^\mathbb{N} \). It is clear that \( (1, 1, \ldots) \) cannot be written as a finite combination of the vectors in the set \( \{e_1, e_2, \ldots\} \). Therefore the Hamel basis of \( \mathbb{R}^\mathbb{N} \) has strictly more elements than a Hamel basis of \( \mathbb{R}^\mathbb{N} \).

How large is a Hamel basis of \( \mathbb{R}^\mathbb{N} \)? Is there an easy way to describe it? We do not have easy answers to these questions. This is particularly troubling for the canonical representation of the algebraic dual of \( \mathbb{R}^\mathbb{N} \). If we do not even know what a Hamel basis looks like, how can we talk about functions over it?

One answer to this conundrum is to restrict attention to well-behaved subsets of \( \mathbb{R}^\mathbb{N} \). These include the spaces \( \ell^p \) which consists of those real sequences \( x = (x_1, x_2, \ldots) \) where

\[
\sum_{i=1}^{\infty} |x_i|^p < \infty.
\]

We discuss such spaces and their duals in Chapter 6. These spaces are commonly used in applications.

Of course, not all algebraic duals are nasty. We saw in the case of finite dimensions every vector space is self-dual: if we understand the space itself then we understand its dual. It is not only finite dimensional vector spaces which have this property, we show in Chapter 6 Hilbert spaces are also self-dual.

The following example illustrates how algebraic duals easily become unwieldy. Recall the finite dimensional linear programming\(^1\) example from Section 3.7:

\[
\begin{align*}
\min & \quad c^\top x \\
\text{s.t.} & \quad Ax \geq b.
\end{align*}
\]

(FLP)

where \( x \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \) and \( A \) is a an \( m \) by \( n \) real matrix.

\(^1\)what most sane people simply call linear programming: ignorance is bliss!
In Section 3.7 we showed that the Lagrangian dual is:

\[
\begin{align*}
\text{max} & \quad b^\top y \\
\text{s.t.} & \quad A^\top y = c \\
& \quad y \geq 0.
\end{align*}
\]  

(FLPD)

where \( y \in \mathbb{R}^m \). Some gymnastics were needed to write \( y \in \mathbb{R}^m \) instead of considering linear functionals \( \psi_y \in (\mathbb{R}^m)' \). We use more of this reasoning below.

We have done nothing more than find the dual of a linear program. This is certainly something you have done in any linear programming course you have taken. Now, if you had a particularly mean and unreasonable professor, perhaps like the one who teaches 36900 at the University of Chicago, you were probably asked to take the dual of the dual and show you get the primal back again. Under duress in 36900, your first question was likely how to reintroduce the primal variables \( x \). One idea is to reintroduce \( x \) as the Lagrangian dual variable. Can we justify this?

Schematically, the original primal-dual pair is:

\[
\text{FLP : } \left[ \begin{array}{c} \mathbb{R}^n \\ (\mathbb{R}^n)' \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathbb{R}^m \\ (\mathbb{R}^m)' \end{array} \right] : \text{FLPD}
\]

and the new primal-dual pair (with the old dual taken as our new primal) looks like:

\[
\text{FLPD : } \left[ \begin{array}{c} \mathbb{R}^m \\ (\mathbb{R}^m)' \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathbb{R}^n \\ (\mathbb{R}^n)' \end{array} \right] : \text{FLPDD}
\]

By Proposition 2.2.5, \( \mathbb{R}^m \) and \( \mathbb{R}^n \) are self dual which implies \( \mathbb{R}^m \cong (\mathbb{R}^m)' \) and \( \mathbb{R}^n \cong (\mathbb{R}^n)' \). Thus, we are justified in thinking of the dual variables to (FLPD) as being elements of the original space. That is, it is legitimate to reintroduce \( x \) as a dual variable. This was the fact you probably took for granted in an introductory linear programming class. We know better now that there is something going on in the background that makes this work.

To apply our Lagrangian approach to (FLPD) we need to interpret it as a special case of our generic problem (1.2.1) with the roles of \( x \) and \( y \) flipped to fit the decision variables as they appear in (FLPD):

\[
\begin{align*}
\text{inf} & \quad f(y) \\
\text{s.t.} & \quad G(y) \preceq P_X \theta_X \\
& \quad y \in \Omega.
\end{align*}
\]  

(P)

Comparing (FLPD) and (P) we see two important differences:

(i) The objective in (FLPD) is a “max” problem, whereas in (P) it is an “inf” problem. Changing a “max” into a “min” is straightforward by changing the objective from \( b^\top y \)
CHAPTER 4. DUALITY IN TOPOLOGICAL VECTOR SPACES

Maximizing the objective \( b^\top y \) is the same as minimizing the objective \(-b^\top y\). The fact that the “\( \min \)” can be seen as an “\( \inf \)” is a consequence of strong duality established in Theorem 3.7.5. It turns out we can apply the Lagrangian approach in a straightforward manner without switching “\( \max \)” and “\( \min \)”. This is the approach we employ here. In this case weak duality is reversed: the dual feasible solutions yield upper bounds on the primal and the dual problem is to find the least upper bound.

(ii) The constraint “\( G(y) \preceq_{P_X} \theta_X \)” in (P) looks different than the constraint “\( A^\top y = c \)” in (FLPD). How do we turn our generic constraint into an equality? Some thought reveals that this arises when the cone is the trivial one: \( \{0\} \). Thus, if we take \( P_{\mathbb{R}^n} = \{0\} \) we may interpret “\( A^\top y = c \)” in terms of the general cone constraint

\[
 c - A^\top x \preceq_{P_{\mathbb{R}^n}} 0.
\]

Then \( P_{\mathbb{R}^n}^+ = (\mathbb{R}^n)' \) and any vector in \( (\mathbb{R}^n)' \) is a valid vector of Lagrange multipliers for \( c - A^\top x \preceq_{P_{\mathbb{R}^n}} 0 \). In plain English, if we have equality constraints, the Lagrange multipliers are unrestricted.

Now apply the Lagrangian approach and take the dual of the dual. To establish weak duality we require \( x \in P_{\mathbb{R}^n}^+ \) (by Theorem 3.2.1). As noted in point (ii) above, \( P_{\mathbb{R}^n}^+ = \{0\}^+ = (\mathbb{R}^n)' \). Now, we are ready to form the Lagrangian dual of (FLPD):

\[
\min_{x \in \mathbb{R}^n} \left\{ \max_{y \geq 0} \left\{ b^\top y + x^\top (c - A^\top y) \right\} \right\} \\
= \min_{x \in \mathbb{R}^n} \left\{ x^\top c + \max_{y \geq 0} \left\{ b^\top y - x^\top A^\top y \right\} \right\} \\
= \min_{x \in \mathbb{R}^n} \left\{ x^\top c + \max_{y \geq 0} \left\{ (b^\top - x^\top A^\top) y \right\} \right\} \\
= \min_{x \in \mathbb{R}^n} \left\{ c^\top x + \max_{y \geq 0} \left\{ y^\top (b - Ax) \right\} \right\}
\]

where the last step uses the adjoint property \( x^\top A^\top y = y^\top Ax \).

For the inner maximization to remain bounded we require \( b - Ax \leq 0 \) and the Lagrangian dual becomes:

\[
\min c^\top x \\
\text{s.t. } Ax \geq b
\]

(FLPDD)

which is precisely the same as (FLP)!

You have probably seen this before, maybe not using the same argument as above, but something equivalent. We shall see in time that several things need to conspire to make
4.1. MOTIVATION #1: ALGEBRAIC DUALS ARE NASTY!

this work out. To appreciate this, let’s try the same thing on our general linear program from Section 3.7:

\[
\begin{align*}
\text{inf} & \quad \langle x, \phi \rangle_X \\
\text{s.t.} & \quad b - A(x) \leq \theta_Y.
\end{align*}
\]

which has Lagrangian dual:

\[
\begin{align*}
\text{sup} & \quad \langle b, \psi \rangle \\
\text{s.t.} & \quad \phi - A'(\psi) = \theta_{X'} \\
& \quad \psi \in P_Y^+.
\end{align*}
\]

where \( A' : Y' \to X' \) is adjoint of linear mapping \( A : X \to Y \). In problem (GLPD) the variable space is \( Y' \) and the constraint space is \( X' \).

In order to form the Lagrangian function by linearly penalizing the constraint “\( \phi - A'(\psi) = \theta_{X'} \)” we need a linear functional over \( X' \). We call this set of functionals the second algebraic dual of \( X \) and denote it \( X'' \) (see Exercise 2.5). Luckily, in the case of \( \mathbb{R}^n \) the second algebraic dual is again isomorphic to \( \mathbb{R}^n \). This was one reason we could reconstruct the primal in the finite dimensional LP case. In the general case, things aren’t so happy. In fact things can get really unhappy! Perhaps the first algebraic dual was somewhat manageable, but once you go the second algebraic dual, then all bets are off.

**Example 4.1.2.** Consider again Example 3.7.6. The problem schematic looked like:

\[
\begin{align*}
\text{PG} : \quad & \begin{bmatrix} \mathbb{R}^\infty \end{bmatrix} \to \begin{bmatrix} \mathbb{R}^2 \end{bmatrix} : \text{PGD} \\
& \begin{bmatrix} \mathbb{R}^N \end{bmatrix} \leftarrow \begin{bmatrix} \mathbb{R}^2 \end{bmatrix}
\end{align*}
\]

This was a relatively easy pill to swallow. We can make sense of all the spaces involved, and we were able to analyze the problem to show the existence of a duality gap.

Now consider forming the dual of (PGD). We would need to consider a primal-dual pair fitting the schematic:

\[
\begin{align*}
\text{PGD} : \quad & \begin{bmatrix} \mathbb{R}^2 \end{bmatrix} \to \begin{bmatrix} \mathbb{R}^N \end{bmatrix} : \text{PGDD} \\
& \begin{bmatrix} \mathbb{R}^2 \end{bmatrix} \leftarrow \begin{bmatrix} (\mathbb{R}^N)' \end{bmatrix}
\end{align*}
\]

In Example 4.1.1 we detailed the hazards of conceptualizing the algebraic dual of \( \mathbb{R}^N \). So although the first dual was still within reach conceptually for this problem, taking the dual of the dual got us into confusing waters.

Turning again to the general case, our problem schematic for taking the dual of (GLPD) is illustrated in Figure 4.1 where again \( X'' \) and \( Y'' \) are the second algebraic duals of \( X \) and \( Y \).

We use Hebrew letters, typically \( \mathcal{A} \), to denote elements of \( X'' \). This emphasizes the fact that that there are “vectors” different than those found in both \( X \) and \( X' \). New levels of
functionals call for new levels of alphabets! We might have to invent an alphabet if this goes much further. Can anyone write Klingon? The Lagrangian dual of (GLPD) is:

$$\inf_{\mathcal{X}'} \left\{ \sup_{\psi \in P_+} \{ (b, \psi)_Y + \langle \phi - A'(\psi), \mathcal{D} \rangle_{X'} \} \right\}$$

and isolating the inner sup gives:

$$\inf_{\mathcal{X}''} \left\{ (\phi, \mathcal{D})_{X'} + \sup_{\psi \in P_+} \{ (b, \psi)_Y - \langle A'(\psi), \mathcal{D} \rangle_{X'} \} \right\}$$

But now here is the key problem! How do we isolate for $\psi$ in the inner maximization? We would ideally like to write:

$$\langle A'\psi, \mathcal{D} \rangle = \langle \psi, A(\mathcal{D}) \rangle = \langle \psi, A(x) \rangle$$
But we cannot do this since \( A(\mathcal{M}) \) has no meaning! The homomorphism \( A \) is a map from \( X \) to \( Y \) but \( \mathcal{M} \) is in \( X'' \) and \( X'' \) is not isomorphic to \( X \) in general. To analyze this further we really need to add some additional structure!

**Key Idea:** the algebraic dual \( X' \) (or the second algebraic dual \( X'' \)) is too big. We would like to restrict it to a subspace so that the dual problem is more tractable. We need to characterize dual vectors in a way that leads to an interesting dual optimization problem. Several examples are explored in Chapter 6. In the following sections we show that introducing a topology on the vector space \( X \) can make \( X' \) more manageable and the Lagrangian dual easier to solve.

### 4.2 Topological spaces

So how can we use topology to tame an unwieldy algebraic dual? The answer is to restrict the set of linear functionals. The approach is to add additional structure to \( X \) which allows us to rule out certain linear functionals from consideration.

The linear functionals we remove are those that are not continuous. *Wait!* A linear functional which is not continuous! How dare you! To make such a bold statement we need to be clear about what we mean by continuous. We are not thinking about continuity in terms being able to draw it without lifting our pencil! We consider an abstract theory of “continuity”, which is a topic in topology. It turns out that by restricting ourselves to *topological vector spaces* our duality theory becomes more manicured and potentially more useful. We start by defining topological spaces:

**Definition.** Let \( X \) be a set and let \( \tau \) be a family of subsets of \( X \). We call \( \tau \) a topology of \( X \) and \((X, \tau)\) a topological space if:

1. \( \emptyset, X \in \tau \)
2. \( \tau \) is closed under finite intersections; that is, if \( U_1, \ldots, U_n \in \tau \) then \( \bigcap_{i=1}^n U_i \in \tau \).
3. \( \tau \) is closed under arbitrary unions; that is, for any set \( A \) let \( \{U_\alpha\}_{\alpha \in A} \) be a family of open sets, then \( \bigcup_{\alpha \in A} U_\alpha \in \tau \). It is acceptable for \( A \) to be an uncountable set.

The elements of \( \tau \) are commonly called open sets. An open set that contains the vector \( x \) is called a *neighborhood* of \( x \). The complements of open sets are called closed sets. In particular, the family of closed sets and open sets are in one-to-one correspondence. Working with a topology \( \tau \) directly can be somewhat tricky (as we shall see in the examples below), since there is a large number and variety of open sets. However, in many cases there are nice sub-families of \( \tau \) that give rise to the entire topology.

**Definition.** A basis \( \mathcal{B} \) of a topology \( \tau \) is a sub-family of \( \tau \) such that every element of \( \tau \) can be expressed as a union of elements in \( \mathcal{B} \).
Here is a useful characterization of a basis of a topology:

**Proposition 4.2.1.** A subset $B$ of a topology is a basis if and only if for every $x \in X$ and every neighborhood $U$ of $x$, there is an element $B$ of $B$ such that $x \in B \subseteq U$.

**Proof.** ($\Rightarrow$) Let $B$ and $x$ be given. Suppose $U$ is a neighborhood of $x$. Now, since $B$ is a basis we can express $U$ as a union of elements of $B$, say $\{B_i\}_{i \in I}$ where $I$ is some index set. This implies $x \in B_j$ for some $j \in I$. Thus, $x \in B_j \subseteq U$.

($\Leftarrow$) Let $U$ be an arbitrary open set. Our goal is to show that we can express $U$ as a union of elements from $B$. Now, for all $x \in U$ there exists a $B_x \in B$ such that $x \in B_x \subseteq U$. Thus

$$U = \bigcup_{x \in U} B_x.$$ (4.2.1)

This gives the desired result.  

Alternatively, one can define a topology by simply giving a basis $B$ which *generates* it. Let $B$ be an arbitrary family of subsets of $X$ and let $\tau(B)$ is the set of all unions of elements of $B$. In order for $\tau(B)$ to be a topology (and hence $B$ to be a basis), $B$ needs to satisfy the following two criteria:

(B1) For all $x \in X$ there exists a basis element $B \in B$ so that $x \in B$.

(B2) Let $B_1$ and $B_2$ are two basis elements and denote their intersection $I = B_1 \cap B_2$. Then for each $x \in I$ there exists another basis element $B_3$ such that $x \in B_3 \subseteq I$.

Property (B2) ensures that no intersections are needed to derive elements of the topology.

To wrap our heads around these definitions, we consider a few examples (Exercise 4.1 asks you to show that these are topological spaces).

**Example 4.2.2.**

(i) (Indiscrete topology). The family $\tau = \{\emptyset, X\}$ is a topology, called the *indiscrete topology*.

(ii) (Discrete topology). The power set of $X$ – that is, the family of all subsets of $X$ – is a topology on $X$ called the *discrete topology*.

(iii) The reals $\mathbb{R}$ have a topology called the *usual* or *standard* topology. It is defined by a basis consisting of all open intervals $(a, b)$ where $a < b$.

(iv) (Metric topology) Suppose $d$ is a metric on $X$; that is, $d : X \times X \to \mathbb{R}$ has the following properties: for all $x, y, z \in X$

(M1) $d(x, y) \geq 0$ (nonnegativity)

(M2) $d(x, y) = 0$ if and only if $x = y$

(M3) $d(x, y) = d(y, x)$ (symmetry)
4.2. **TOPOLOGICAL SPACES**

(M4) \(d(x, z) \leq d(x, y) + d(y, z)\) (triangle inequality)

Let \(B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}\) be the *open ball of radius \(\epsilon\) about \(x\). The set of open balls \(\{B_\epsilon(x) : x \in X, \epsilon > 0\}\) form a basis of the *metric topology* of \(X\).

(v) (Product topology) Let \(X\) and \(Y\) be topological spaces. The *product topology* on \(X \times Y\) is the topology having as basis the collection \(\mathcal{B}\) of all sets of the form \(U \times V\) where \(U\) is an open subset of \(X\) and \(V\) is an open subset of \(Y\).

(vi) (Relative topology). Let \((X, \tau)\) be a topological space and let \(Y\) be a subset of \(X\).

The *relative topology* \(\tau_Y\) of \(Y\) consists of the open sets \(U \cap Y\) where \(U\) is an open set of \(X\).

Our main interest in introducing notions of topology is to refine our algebraic model. These refinements come in two flavors: the first is the notion of continuity. The second are analogues to the ideas of core and algebraic closure introduced in Chapter 2 in the topological setting.

**Definition.** Let \(X\) and \(Y\) be topological spaces with topologies \(\tau\) and \(\sigma\) respectively. A function \(f : X \to Y\) is *continuous* if for every open set \(V \in \sigma\) we have \(f^{-1}(V) \in \tau\). That, the inverse image of an open set is open.

**Remark 4.2.3.** Suppose \(\sigma\) has a basis \(\mathcal{B}\). In Exercise 4.2 you are asked to show it suffices to show that \(f^{-1}(V)\) for basis elements \(V \subseteq \mathcal{B}\).

An immediate alternate statement of continuity is the following:

**Lemma 4.2.4.** Let \(X\) and \(Y\) be topological spaces with topologies \(\tau\) and \(\sigma\) respectively. A function \(f : X \to Y\) is continuous if for every \(\sigma\)-closed set \(V \in \sigma\), \(f^{-1}(V)\) is \(\tau\)-closed. That, the inverse image of a closed set is closed.

**Proof.** See Exercise 4.3.

If a continuous function \(f : X \to Y\) is a bijection and its inverse is also continuous then we call \(f\) a *homeomorphism* of \(X\) and \(Y\). When there is a homeomorphism between \(X\) and \(Y\) they are called *homeomorphic*. When \(X\) and \(Y\) are homeomorphic, open sets in \(X\) and open sets in \(Y\) perfectly correspond: \(U\) is an open set in \(X\) if and only if \(f(U)\) is in \(Y\).

There are numerous equivalent conditions that guarantee a function is continuous (see for instance Chapter 2 of Aliprantis and Border [2]). The following is useful in our development.

**Lemma 4.2.5.** Let \(X\) and \(Y\) be topological spaces with topologies \(\tau\) and \(\sigma\), respectively. Let \(f : X \to Y\). Then \(f\) is continuous if and only if for each \(x \in X\), and each neighborhood \(V \in \sigma\) of \(f(x)\), there is a neighborhood \(U \in \tau\) of \(x\), such that \(f(U) \subseteq V\).

**Proof.** See Exercise 4.4.
Remark 4.2.6. In the previous lemma, when \( \sigma \) is given by a basis \( B \), it suffices to consider neighborhoods \( V \) of \( f(x) \) which are basis elements.

We say \( f \) is continuous at the point \( x \) if the condition of Lemma 4.2.5 holds: for each neighborhood \( V \) of \( f(x) \), there is a neighborhood \( U \) of \( x \) such that \( f(U) \subseteq V \).

It is now easier to imagine linear functionals that are not continuous. Indeed, the only continuous linear functional over a vector space \( X \) endowed with the indiscrete topology is the “zero” functional \( \phi \) which maps everything to 0. Quite a departure from drawing functions without lifting your pencil! Nothing is continuous! By contrast, every functional (including nonlinear functionals) is continuous when the vector space is endowed with the discrete topology (see Exercise 4.5).

4.3 Topological vector spaces

As you might guess, the discrete and indiscrete topologies are of limited interest to us. We want the topology to somehow “make sense” with the underlying vector space structures of the sets we are using. We are interested in topological vector spaces. The basic requirement beyond (T1)-(T3) for a topological space to be a topological vector space is “compatibility” with the underlying operations of the vector space: vector addition and scalar multiplication. More precisely, \((X, \tau)\) is a topological vector space if it is a topological space and vector addition

\[
a : \ X \times X \to X \quad \quad (x, y) \mapsto x + y
\]

and scalar multiplication

\[
m : \ \mathbb{R} \times X \to X \quad \quad (\alpha, x) \mapsto \alpha x
\]

are continuous functions. The topology \( \tau \) is linear if \((X, \tau)\) is a topological vector space.

Let’s examine more carefully what it means for the vector space operations to be continuous. The topologies on \( X \times X \) and \( \mathbb{R} \times X \) are the associated product topologies inherited from the usual topology on \( \mathbb{R} \) and the topology \( \tau \) on \( X \).

Continuity of vector addition means (using Lemma 4.2.5) that if \( x^i \in X \) for \( i = 1, 2 \), and if \( V \) is a neighborhood of \( x^1 + x^2 \), then there exists a neighborhood \( V_1 \times V_2 \) of \( (x^1, x^2) \) in the product topology where \( a(V_1 \times V_2) = V_1 + V_2 \subseteq V \). From the definition of the product topology, \( V_1 \) is a neighborhood of \( x^i \) for \( i = 1, 2 \). Similarly, the continuity of scalar multiplication is: if \( x \in X \) and \( \alpha \in \mathbb{R} \) and \( V \) is a neighborhood of \( \alpha x \), then there exists a neighborhood \( B_\tau(\alpha) \times W \), of \( (\alpha, x) \) such that \( m(B_\tau(\alpha) \times W) = \{ \beta y : \beta \in B_\tau(\alpha), y \in W \} \subseteq V \). More compactly, \( \beta W \subseteq V \) whenever \( |\beta - \alpha| < r \).
It takes some time to fully understand the significance of the continuity of vector space operations. Some reflection reveals that it provides a lot of structure. The following two lemmas illustrate structure provided by topological vector spaces.

**Lemma 4.3.1 (Invariance Lemma).** Consider the mappings \( T_a(x) = a + x \) and \( M_\lambda(x) = \lambda x \) where \( a, x \in X \) and \( \lambda \in \mathbb{R} \) is nonzero. Then \( T_a \) and \( M_\lambda \) are homeomorphisms of \( X \) onto \( X \).

*Proof.* See Exercise 4.6. \( \square \)

An implication is that a topological vector space is entirely described by neighborhoods of the origin \( \theta \). Here is the reasoning: since \( T_a \) is a homeomorphism, \( U \) is open if and only if \( T_a(U) \) is open. This is true for all \( a \in X \). Then given a set \( U \) which contains \( x \), \( U \) is is a neighborhood of \( x \) if and only if \( U - x \) is a neighborhood of the origin. In other words, all open sets are translates of neighborhoods of the origin. This motivates the following definition.

**Definition.** A collection \( B \) of neighborhoods of \( \theta \) is called a *neighborhood base at \( \theta \)* if all neighborhoods of \( \theta \) contain an element of \( B \).

The significance of finding a neighborhood base at \( \theta \) for a topological vector space \((X, \tau)\), is that any \( V \in \tau \) can be written as the union of translates of elements of \( B \).

**Lemma 4.3.2.** If \( B \) is a neighborhood base at \( \theta \) in a topological vector space, \((X, \tau)\), and \( V \in \tau \), then \( V \) is the union of translates of elements of \( B \).

*Proof.* Combine the Invariance Lemma 4.3.1 with the proof technique of Proposition 4.2.1. Assume \( V \) is an arbitrary element of \( \tau \). For all \( x \in V \), by the Invariance Lemma 4.3.1, \( V - x \) is a neighborhood of \( \theta \). Since \( B \) is a neighborhood base at \( \theta \), there exists \( B_x \in B \) such that \( B_x \subseteq V - x \). Then

\[
V = \bigcup_{x \in V} (B_x + x).
\]

where again by the Invariance Lemma 4.3.1, \( x + B_x \in \tau \) for all \( x \in V \) and \( B_x + x \subseteq V \). \( \square \)

We now describe some of the properties of neighborhood bases. First some definitions:

**Definition.** Let \( A \) be a subset of a vector space \( X \). The set \( A \) is:

- *absorbing* if for any \( x \in X \), some multiple of \( A \) includes the line segment joining \( x \) and the origin. That is, there exists a \( \alpha_0 \) such that \( \alpha x \in A \) for every \( 0 \leq \alpha \leq \alpha_0 \). In other words, \( \theta \) is a core point of \( A \).

- *circled* if for each \( x \in A \) the line segment joining \( x \) and \( -x \) lies in \( A \); that is, \([-x, x] \subseteq A \).
Lemma 4.3.3 (Structure Lemma). If \((X, \tau)\) is a topological vector space, then there exists a neighborhood base of the origin \(B\) with the following properties:

(NB1) all elements of \(B\) contain the origin;

(NB2) let \(B_1\) and \(B_2\) be elements in \(B\), then there exists \(B_3 \in B\) such that \(B_3 \subseteq B_1 \cap B_2\);

(NB3) all elements of \(B\) are absorbing;

(NB4) all elements of \(B\) are circled; and

(NB5) for each \(B \in B\) there exists a \(W \in B\) with \(W + W \subseteq B\).

Proof. Since \((X, \tau)\) is a topological space, then \(\tau\) has a basis (it could trivially be all of \(\tau\)). Now, consider the collection \(N\) of all basis elements which are neighborhoods of the origin. Note that \(N\) trivially satisfies (B2). Indeed, if we intersect two elements \(B_1\) and \(B_2\) of \(N\) then clearly \(\theta \in B_1 \cap B_2\). Hence by the definition of basis there must exist another basis element \(B_3 \in N\) such that \(\theta \in B_3 \subseteq B_1 \cap B_2\). Thus we have \(B_3 \in N\) and so (NB2) holds.

Our goal is remove elements from \(N\) that violate (NB3) to (NB5). After removing all such elements we call the resulting set \(B\). It remains to argue that \(B\) satisfies (NB2). Note that all of these sets contain the origin are thus (NB1) is trivially satisfied.

We first show that all elements of \(N\) satisfy (NB3), which follows from the continuity of scalar multiplication. Consider any \(B \in N\). For all \(x \in X\), \(\theta = 0 \cdot x \in B\). Applying continuity of scalar multiplication implies that there exists a neighborhood \(W\) of \(x\), and a scalar \(r > 0\), such that \(\beta W \subseteq B\) whenever \(|\beta - 0| = |\beta| < r\). Note that \(\beta x \in \beta W \subseteq B\) and so \(\beta x \in B\) for \(|\beta| < r\). This implies that \(B\) is absorbing. Thus we need not remove any elements from \(N\) which violate (NB3), since none of them do.

We next argue that if \(V \in N\) is not circled, then we are then safe to remove \(V\) from our collection and still satisfy (NB2). This will follow from the following fact: every \(V \in N\) contains a circled neighborhood \(B\) of \(\theta\). Note that the continuity of scalar multiplication by \(0\) implies that there exists a neighborhood \(W\) of \(\theta = 0 \cdot x\), and an \(r \in \mathbb{R}\) such that for all \(x \in W\) and all \(|\beta| < r\), we have \(\beta x \in V\). Define \(B = \bigcup_{|\beta| < r} \beta W\). Since each \(\beta W \subseteq V\), \(B\) is contained in \(V\).

Take \(y \in B\). We need to show \([-y, y] \subseteq B\). Note that, \(y = \beta w\) for some \(\beta \in [-r, r]\) and \(w \in W\). Note that when \(\beta \in [-r, r]\) this means \([-\beta, \beta] \subseteq [-r, r]\). Thus, \([-\beta w, \beta w] \subseteq B\). That is, \([-y, y] \subseteq B\), as required.

We have thus shown that every non-circled neighborhood \(V\) always contains a circled neighborhood. Can we still satisfy (B2) if we throw out \(V\)? If it so happened that \(V \subseteq B_1 \cap B_2\) for some \(B_1 \cap B_2 \in N\) then we also have

\[B \subseteq V \subseteq B_1 \cap B_2.\]
This implies that $V$ is not needed in the collection $\mathcal{N}$ in order to satisfy condition (B2) and so it can be safely removed.

Lastly, all elements of $\mathcal{N}$ satisfy (NB5) due to the continuity of vector addition. Let $B \in \mathcal{N}$. Now $\theta \in B$ and $\theta = x + (-x)$. So by continuity of vector addition we know there exists neighborhoods of the origin $U_x, U_{-x}$ such that $(x + U_x) + (-x + U_{-x}) \subseteq V$. Since the elements of $\mathcal{N}$ satisfy (NB2) we have a $W \in \mathcal{N}$ such that $W \subseteq U_x \cap U_{-x}$. This implies

$$W + W = (x + W) + (-x + W) \subseteq (x + U_x) + (-x + U_{-x}) \subseteq V,$$

as required.

Thus, the set $B = \mathcal{N} \setminus \{V \in \mathcal{N} : V \text{ not circled}\}$ satisfies (NB1)-(NB5).

We would like to able to define a topological vector space specifying a neighborhood base of the origin. Suppose we are given a collection $\mathcal{B}$ of subsets, each of which contains the origin. We define a proposed topology $\tau(\mathcal{B})$ as follows: a set $U$ is in $\tau(\mathcal{B})$ provided

$$U \in \tau(\mathcal{B}) \text{ if and only if } \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x + B \subseteq U. \tag{4.3.1}$$

If it so happens that $\tau(\mathcal{B})$ is a linear topology, then $\mathcal{B}$ is a neighborhood base of the origin in the topology. The following result characterizes when $\tau(\mathcal{B})$ is indeed a linear topology.

The result can be seen as a kind of converse of the Structure Lemma:

**Lemma 4.3.4.** Let $\mathcal{B}$ be a collection of sets that satisfy properties (NB1)-(NB5) and $\tau(\mathcal{B})$ the collection of subsets of $X$ that satisfy (4.3.1). Then $(X, \tau(\mathcal{B}))$ is a topological vector space with $\mathcal{B}$ as a neighborhood base of the origin.

**Proof.** To show that $\tau(\mathcal{B})$ is a linear topology with $\mathcal{B}$ as a neighborhood base of the origin, we need to establish the following:

(i) $\tau(\mathcal{B})$ satisfies (T1)-(T3),

(ii) vector addition is continuous, and

(iii) scalar multiplication is continuous.

We treat each of these in turn.

(i) To establish (T1), note that the empty set trivially satisfies (4.3.1) and thus $\emptyset \in \tau$. Also, since for all $x \in X$ and $B \in \mathcal{B}$ we have $x + B \subseteq X$ it follows that $X$ satisfies (4.3.1) and is thus in $\tau(\mathcal{B})$.

To establish (T2), consider a finite collection of open sets in $\tau(\mathcal{B})$: $U_1, \ldots, U_n$. We will show $\cap_{i=1}^n U_i \in \tau(\mathcal{B})$. That is, for all $x \in \cap_{i=1}^n U_i$ we find a $B \in \mathcal{B}$ such that $x + B \subseteq \cap_{i=1}^n U_i$.

Take $x \in \cap_{i=1}^n U_i$. This means $x \in U_i$ for all $i = 1, \ldots, n$. Thus since the $U_i$ are open in $\tau(\mathcal{B})$, there exists $B_i \in \mathcal{B}$ such that $x + B_i \subseteq U_i$. Now, by property (NB2) there exists $B \in \mathcal{B}$ with $B \subseteq \cap_{i=1}^n B_i$. Thus for all $i$:

$$x + B \subseteq x + \cap_{i=1}^n B_i \subseteq x + B_i \subseteq U_i.$$
The we can conclude $x + B \subseteq \bigcap_{i=1}^n U_i$ and thus by (4.3.1), $\bigcap_{i=1}^n U_i$ is closed.

To establish (T3), consider an arbitrary collection of open sets $\{U_{\alpha}\}_{\alpha \in A}$ in $\tau(\mathcal{B})$ where $A$ is an arbitrary index set. Take $x \in \bigcup_{\alpha \in A} U_{\alpha}$. This means there exists an $\bar{\alpha}$ such that $x \in U_{\bar{\alpha}}$. Since $U_{\bar{\alpha}}$ is open we have a $B \in \mathcal{B}$ such that $x + B \subseteq U_{\bar{\alpha}} \subseteq \bigcup_{\alpha \in A} U_{\alpha}$.

This implies $\bigcup_{\alpha \in A} U_{\alpha}$ is open, establishing (T3). Thus we have shown $\tau(\mathcal{B})$ is a topology.

(ii) Next we turn to showing continuity of vector addition. Take $x, y \in X$ and let $U$ be an neighborhood of $x + y$. Note that by (4.3.1) there exists a $B \in \mathcal{B}$ such that $(x + y) + B \subseteq U$.

Now, by (NB5) there exists a $W \in \mathcal{B}$ such that $W + W \subseteq B$. Now consider the sets $x + W$ and $y + W$. They are clearly neighborhoods of $x$ and $y$ respectively, and

$$(x + W) + (y + W) = (x + y) + W + W \subseteq (x + y) + B \subseteq U.$$  

This implies that vector addition is continuous.

(ii) Finally, we show the continuity of scalar multiplication. Let $\lambda \in \mathbb{R}$ and $x \in X$ and take $V$ to be an arbitrary neighborhood of $\lambda x$. Since $V$ is open there exists $B \in \mathcal{B}$ such that $\lambda x + B \subseteq V$. By (NB5) there exists $W \in \mathcal{B}$ with $W + W \subseteq B$. That is, we have

$$\lambda x \in \lambda x + W + W \subseteq \lambda x + B \subseteq V$$

Our goal is to find an $r > 0$ and a neighborhood $U$ of $x$ such that $\alpha U \subseteq V$ whenever $\alpha \in [\lambda - r, \lambda + r]$. The main idea is as follows, we will express $U = x + B'$ and take $\alpha y \in \alpha U$ and express it as follows:

$$\alpha y = \alpha y + (\lambda x - \lambda x) + \alpha x - \alpha x$$

$$= \lambda x + (\alpha - \lambda)x + \alpha(y - x).$$

The idea is to stick both $(\alpha - \lambda)x$ and $\alpha(y - x)$ into $W$ for all $\alpha \in [\lambda - r, \lambda + r]$. If we can achieve this then:

$$\alpha y \in \lambda x + W + W \subseteq V$$

for all $\alpha$, as required.

So our goal is clear: we must define $r$ (and thus the range for $\alpha$) and $U$ via $B'$ so that

$$(\alpha - \lambda)x \in W$$  \hspace{1cm} (4.3.2)

and

$$\alpha(y - x) \in W.$$  \hspace{1cm} (4.3.3)
for all $\alpha \in [\lambda - r, \lambda + r]$ and $y - x \in B'$.

Now is where we finally get to use properties (NB3) and (NB4). In particular we use the fact that since $W \in B$ it is absorbing and circled to find our radius $r$ and in turn establish (4.3.2). Since $W$ is absorbing we can move from $\theta$ towards $x$ and stay in $W$. Thus we can find an $r > 0$ so that $[\theta, rx] \subseteq W$ for some $r > 0$. Since $W$ is circled this means $[-rx, rx] \subseteq W$. In particular, for $\alpha \in [\lambda - r, \lambda + r]$ we have

$$(\alpha - \lambda)x \in [-rx, rx] \subseteq W$$

which establishes (4.3.2).

To establish (4.3.3) note that $\alpha(y - x) \in \alpha B'$. Thus we would like to choose $B'$ so that $\alpha B' \in W$. Now $\alpha \leq \alpha + r$ so if we take $n \geq \alpha + r$ and set $B' = \frac{1}{n}W$ to ensure $\alpha B' = \frac{1}{n}W \subseteq W$. This establishes (4.3.3). We conclude that scalar multiplication is continuous.

**Remark 4.3.5.** In the proofs that follow we will often use Invariance Lemma (Lemma 4.3.1) and the Structure Lemma (Lemma 4.3.3). We describe in some detail now a typical usage. In the course of a proof we will need to show some property for an arbitrary neighborhood $U$ on some point $x$ in a topological vector space. It often suffices to exhibit that property for the translate $x + B$ of some absorbing and circled neighborhood $B$ of the origin.

The reasoning is as follows. Let $(X, \tau)$ be a topological vector space. By the Structure Lemma, $X$ has a neighborhood base $B$ of the origin consisting of absorbing and circled sets. Now, by the Invariance Lemma every open set in $\tau$ can be seen as a translate of an open set of the origin. Open sets of the origin are precisely unions of elements of the neighborhood base of the origin. Again leveraging the Invariance Lemma, this allows to conclude that every neighborhood of $x \in X$ is a union of absorbing and circled neighborhoods of the origin which are translated to $x$.

To see why this is useful, suppose in the course of a proof you would like to argue every neighborhood $U$ of a point $x$ intersects the set $A$ (possibly in attempts to show $x$ is in the closure of $A$). It suffices to show $x + B$ intersects $A$ all absorbing and circled neighborhoods $B$ of the origin. This is due to the fact that an arbitrary neighborhood of $x$ is a union sets of the form $x + B$. This idea is used extensively in proofs in the remainder of the chapter.

### 4.4 Continuity of linear functionals

With all this structure, you might begin to wonder if we can still have linear functionals which are not continuous. Indeed, continuous functions on a topological vector space are also highly structured. The next two results illustrate this by showing continuity of a linear functional boils down to its properties at the origin:

**Proposition 4.4.1.** Let $(X, \tau)$ be a topological vector space, then a linear functional $\phi$ is continuous if and only if it is continuous at the origin.
We thus conclude that φ is defined as above. The Invariance Lemma assures that x so that δ for all y exists an open set U−a,b of the origin in X with φ(U) ∈ (−a, b).

Now, consider an arbitrary element x ∈ X. We argue that φ is continuous at x by appealing to Lemma 4.2.5 and the remark that followed. Every open interval containing φ(x) in ℜ can be seen as a translate of an open interval from the origin (ℜ with the standard topology is a topological vector space so this can be seen as an application of the by the Invariance Lemma). Thus we may think of an arbitrary open interval of φ(x) as having the form φ(x) + (−a, b) where a, b > 0. Now consider the open set x + U−a,b where U−a,b is defined as above. The Invariance Lemma assures that x + U−a,b is indeed open. For any x + a ∈ x + U−a,b we have φ(x + u) = φ(x) + φ(u) ∈ φ(x) + (−a, b). That is, φ is continuous at x.

Remark 4.4.2. Note that the proof could equivalently be stated in terms of an arbitrary point p in the space. That is, φ is continuous if and only if it is continuous at a point p ∈ X.

The following relates continuity of a linear functional to its boundedness at 0. Note, that we have no notion of boundedness on the space X. The idea of boundedness requires a norm, which we do not have on every topological vector space. Here we refer to boundedness in ℜ in terms of its standard norm, the absolute value |·|.

Proposition 4.4.3. Let (X, τ) be a topological vector space. A linear functional φ is continuous if and only if it bounded on a neighborhood of θ.

Proof. (⇐) From Proposition 4.4.1 it suffices to show continuity at the origin θ in X. Since f is bounded above on a neighborhood U of θ we have that |φ(y)| ≤ M for each y ∈ U. Clearly, M ≥ 0. To show continuity at θ we need to show that for a given ε > 0 there exists an open set U(ε) of θ so that

|φ(y) − φ(θ)| = |φ(y)| < ε

for all y ∈ U(ε). This is easily achieved by setting U(ε) = δ(ε)U where δ(ε) > 0 is chosen so that δ(ε)M < ε. Indeed, if y ∈ U(ε) = δ(ε)U then y = δ(ε)z for some z ∈ U. Thus,

|φ(y)| = |φ(δ(ε)z)| = δ(ε)|φ(z)| ≤ δ(ε)M < ε.

Note δ(ε)U is an open set by the Invariance Lemma (Lemma 4.3.1).

(⇒) We show the contrapositive. Let ε > 0 be given. Since φ is unbounded on all neighborhoods U of θ there exists a u ∈ U such that |φ(u)| > ε. This means φ(U) ⊈ (−ε, ε). We thus conclude that φ is not continuous at θ (and thus not continuous).
These results demonstrate that in a topological vector space the bar is high for finding linear functionals which are not continuous. However, even in this case we can have the “unthinkable”:

**Example 4.4.4** (A linear functional which is not continuous in a topological vector space). Once again consider the vector space $X = \mathbb{R}^\infty$ of finitely nonzero real sequences. For each $x \in X$, we let $L(x)$ be the last index of $x$ that is not zero. Define the linear functional $\phi \in X'$ as follows. Given an $x = (a_1, a_2, \ldots)$ let

$$\phi(x) = \sum_{i=1}^{L(x)} ia_i. \quad (4.4.1)$$

We leave it as an exercise (Exercise 4.11) to prove that $\phi$ in (4.4.1) is a linear functional.

Next we define a topology on $X$ via Lemma 4.3.4. Define

$$B = \{B_{1/n} : n = 1, 2, \ldots\} \quad (4.4.2)$$

where

$$B_{1/n} = \left\{ x : \max\{|a_i| : i = 1, 2, \ldots\} < \frac{1}{n} \right\}.$$

We leave it as an exercise to show $B$ satisfies (NB1)-(NB5) (Exercise 4.12) and thus defines a linear topology $\tau$. That is, $(\mathbb{R}^\infty, \tau)$ is a topological vector space.

The WTS is that the linear functional is not continuous at $\theta$. We apply Lemma 4.2.5. Since $\phi(\theta) = 0$ we need to find a neighborhood $V$ of $0$ such that for every basis element $B_{1/n}$ we have $\phi(B_{1/n}) \nsubseteq V$. This will allow us to conclude that $\phi$ is not continuous at $0$ since all neighborhoods of $\theta$ can be expressed as unions of the $B_{1/n}$ (indeed, $B$ is a neighborhood basis of the origin).

Choose $V = (-1, 1)$. This is obviously an open neighborhood of $0$ in $\mathbb{R}$. We now show $\phi(B_{1/n}) \nsubseteq (-1, 1)$ for all $n = 1, 2, \ldots$. Define a set of points

$$x^n = (a_{n1}, a_{n2}, \ldots, a_{ni}, \ldots) \in X, \; n = 1, 2, \ldots$$

by

$$a_{ni} = \begin{cases} \frac{1}{n+1}, & i = n + 2 \\ 0, & i \neq n + 2 \end{cases}$$

Then for $n = 1, 2, \ldots$ we have

$$\max\{|a_{ni}| : i = 1, 2, \ldots\} = \frac{1}{n+1} < \frac{1}{n}.$$
so $x^n \in B^n_1$ for $n = 1, 2, \ldots$ but

$$\phi(x^n) = \frac{n+2}{n+1} > 1.$$  

This implies $\phi(B^n_1) \notin (-1, 1)$ for all $n = 1, 2, \ldots$. We conclude that $\phi$ is not continuous.

Linear functionals which are not continuous turn out to be the ugly cousins in the linear functional extended family (later we will insult them further by calling them dense). However, like most oddities in the family tree they have an interesting back story, which we explore in the next two sections. First we need to introduce a few additional notions from topology are tightly connected with ideas of continuity of linear functionals.

## 4.5 Interior and closure

We introduce the following topological notions, which have close analogues to the algebraic notions of core and algebraic closure:

**Definition.** Let $A$ be a set in a topological vector space $(X, \tau)$.

(i) A point $x \in A$ is an an **interior point** of $A$ if it is contained in an open subset of $A$; that is, there exists a neighborhood $U$ of $x$ such that $U \subseteq A$. The set of all interior points of $A$ is called the **interior** of $A$ and is denoted $\text{int}(A)$. When we want to emphasize the topology $\tau$ we write $\text{int}_{\tau}(A)$.

(ii) A point $x \in A$ is a **relative interior point** if there exists an open set $U$ such that $U \cap \text{aff}(A) \subseteq A$. In other words, there $x$ is an interior point of $A$ in the topological space $(\text{aff}(A), \tau_{\text{aff}(A)})$ where $\text{aff}(A)$ is the affine hull of $A$ and $\tau_{\text{aff}(A)}$ is the relative topology. The set of all relative interior points is called the **relative interior** of $A$ and is denoted relint$(A)$.

(iii) A point $x$ is a **closure point** of $A$ if every neighborhood of $x$ intersects $A$; that is, if $U$ is an open set which contains $x$ then $A \cap U \neq \emptyset$. The set of all closure points of $A$ is called is the (topological) **closure** of $A$, denoted $\text{cl}(A)$. When we want to emphasize the topology $\tau$ we write $\text{cl}_{\tau}(A)$.

(iv) A set $A$ is **dense** if its closure is the entire space; that is, $\text{cl}(A) = X$.

All of the above concepts, besides the relative interior of $A$, can be defined in arbitrary topological spaces, although they will not be our focus here. The relative interior requires a notion of affine hull, which only makes sense when the underlying set has a vector space structure. A summary of the analogies to our algebraic notions can be found in Table 4.1.

There are some alternate characterizations of the interior and closure which turn out to be useful.
4.5. INTERIOR AND CLOSURE

Table 4.1: Related algebraic and topological notions.

<table>
<thead>
<tr>
<th>Algebraic notion</th>
<th>Topological notion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core, cor((A))</td>
<td>Interior, int((A))</td>
</tr>
<tr>
<td>Algebraic closure, lin((A))</td>
<td>Topological closure, cl((A))</td>
</tr>
<tr>
<td>Intrinsic core, icr((A))</td>
<td>Relative interior, relint((A))</td>
</tr>
<tr>
<td>Ubiquitous</td>
<td>Dense</td>
</tr>
</tbody>
</table>

**Lemma 4.5.1.** Let \((X, \tau)\) be a topological space and \(A\) a subset of \(X\). Then,

(i) \(\text{int}(A)\) is the largest (with respect to containment) open set which is contained in \(A\);

(ii) \(\text{cl}(A)\) is the smallest (with respect to containment) closed set which contains \(A\).

In particular, this implies that \(\text{int}(A)\) is an open set and \(\text{cl}(A)\) is a closed set.

*Proof.* See Exercise 4.13. \(\square\)

An implication of this result is the following:

**Lemma 4.5.2.** Let \((X, \tau)\) be a topological space and \(A\) a subset of \(X\). Then

(i) \(A\) is open if and only if \(A = \text{int}(A)\) and

(ii) \(A\) is closed if and only if \(A = \text{cl}(A)\).

*Proof.*

(i) If \(A\) is open, then from Lemma 4.5.1(i) we have \(A \subseteq \text{int}(A)\). We always have \(\text{int}(A) \subseteq A\) and so \(A = \text{int}(A)\). Conversely, if \(A = \text{int}(A)\) then since \(\text{int}(A)\) is open (Lemma 4.5.1) \(A\) is also open.

(ii) Since \(\text{cl}(A)\) is closed (by Lemma 4.5.1) if \(A = \text{cl}(A)\) then \(A\) is closed. Conversely, if \(A\) is closed then \(A \supseteq \text{cl}(A)\) since \(\text{cl}(A)\) is the smallest closed set which contains \(A\). For the opposite containment we employ the fact that for all sets \(B\) (not necessarily closed)

\[(\text{cl}(B))^c = \text{int}(B^c).\] \hspace{1cm} (4.5.1)

This is useful since \(A^c \supseteq \text{int}(A^c)\) from Lemma 4.5.1(i) and so by taking complements we have \(A \subseteq \text{cl}(A)\), as required.

It remains to show (4.5.1). Note that \(B \subseteq \text{cl}(B)\) and so by taking complements \(B^c \supseteq (\text{cl}(B))^c\). This means \((\text{cl}(B))^c\) is an open set which lies in \(B^c\) and so \(\text{int}(B^c)\) contains \((\text{cl}(B))^c\). For the opposite containment, note that \(B^c \supseteq \text{int}(B^c)\) and so \(B \subseteq (\text{int}(B^c))^c\). Since \(\text{int}(B^c)\) is open, we have \((\text{int}(B^c))^c\) closed and so \(\text{cl}(B) \subseteq (\text{int}(B^c))^c\). Taking complements we have that \((\text{cl}(B))^c\) contains \(\text{int}(B^c)\). We have thus shown (4.5.1). \(\square\)
The following are a few useful facts about open and closed sets in a topological vector space:

**Lemma 4.5.3.** Let $(X, \tau)$ be a topological vector space and $A$ a subset of $X$.

(i) The Minkowski sum of an open set and an arbitrary set is open.

(ii) If $B$ is open, then for any set $A$ we have $\text{cl}(A) + B = A + B$

(iii) For any subset $A$ of a topological space, $\text{int}(A) = (\text{cl}(A^c))^c$.

(iv) $\text{int}(x + A) = x + \text{int}(A)$

(v) $\text{cl}(x + A) = x + \text{cl}(A)$

**Proof.** See Exercise 4.14.

**Lemma 4.5.4.** Let $M$ be a subspace of a topological vector space $X$. Then $\text{cl}(M)$ is also a subspace of $X$.

**Proof.** Here we establish closure under vector addition, the idea for scalar multiplication is similar. Let $x, y \in \text{cl}(M)$, we show $x + y \in \text{cl}(M)$. Let $V$ be any neighborhood of $x + y$. The goal is to show $V \cap M \neq \emptyset$.

Since the vector addition operation is continuous in a topological vector space, there exists neighborhoods $V_x$ and $V_y$ of $x$ and $y$ respectively, with $V_x + V_y \subset V$. Since $x$ and $y$ are closure points of $M$ it follows that $V_x$ intersects $M$, say at $w_x$, and $V_y$ intersects $M$, say at $w_y$. Since $w_x \in V_x$ and $w_y \in V_y$ we know $w_x + w_y \in V_x + V_y \subseteq V$. Note that $w_x + w_y \in M$ since $w_x, w_y \in M$ and $M$ is a subspace. Therefore, $w_x + w_y \in V \cap M$ and thus $V \cap M \neq \emptyset$.

As in Chapter 2 where we saw interesting connections between the core and the algebraic closure (see for instance Corollary 2.4.21) we have the following useful result on the topological side:

**Lemma 4.5.5.** Let $A$ be a convex subset of a topological vector space and suppose $\text{int}(A) \neq \emptyset$. Then for all $t \in [0, 1)$:

$$t \text{cl}(A) + (1 - t) \text{int}(A) \subseteq \text{int}(A).$$

**Proof.** First, note that the left hand side is open. By definition $\text{int}(A)$ is an open set, and since a topological vector space is homeomorphic under scalar multiplication (Lemma 4.3.1) this implies that $(1 - t) \text{int}(A)$ is open. Note that the left hand side is the Minkowski sum of a set with an open set, which is open (by Lemma 4.5.3)(i). Thus if we can show the left hand side is contained in $A$ we conclude it is a subset of $\text{int}(A)$, the largest set which is both open and contains $A$. 

4.5. INTERIOR AND CLOSURE

Now, let \( x \in \text{cl}(A) \) and \( y \in \text{int}(A) \) and suppose, by way of contradiction, that there exists a \( w = tx + (1 - t)y \in [y, x] \notin A \) for \( 0 \leq t < 1 \). Since \( A \) is convex this implies that \([w, x] \cap A = \emptyset\). We will contradict this conclusion by constructing an element of \( A \) which lies in \([w, x]\).

Since \( y \in \text{int}(A) \) there exists a neighborhood of the origin \( B \) such that \( y + B \) is a neighborhood of \( y \) where \( y + B \subseteq A \). By the Invariance Lemma 1 \( \lambda B \) open for all \( \lambda \in \mathbb{R} \).

Since \( x \in \text{cl}(A) \) then for every \( \lambda \) there exists a point \( a_\lambda \in A \) which lies in \( x + \lambda B \). Let \( p_\lambda = a_\lambda - x \) be the direction from \( x \) to the point \( a_\lambda \) in \( A \) and notice that \( \lambda p_\lambda \in B \). Since \( B \) is circled this implies that \( y - \lambda p_\lambda \in y + B \) and hence in \( A \). See Figure 4.2 for a visual representation.

Now consider the line segment \([y - \lambda p_\lambda, a_\lambda]\), we argue that it intersects \((y, x)\) and give a formula for the point of intersection. An arbitrary point in the line segment \([y - \lambda p_\lambda, a_\lambda]\) is of the form:

\[
\mu(y - \lambda p_\lambda) + (1 - \mu)a_\lambda = \mu(y - \lambda p_\lambda) + (1 - \mu)(x + p_\lambda) = \mu y + ((1 - \mu) - \mu \lambda)p_\lambda + (1 - \mu)x
\]

(4.5.2)

where \( 0 \leq \mu \leq 1 \). A point on this line segment lies in \((y, x)\) if there exists a \( 0 < \mu < 1 \) such that \((1 - \mu) - \mu \lambda = 0 \) (thus, we can drop the “\( p_\lambda \)” term in (4.5.2)). Working out the algebra this happens when \( \mu = \frac{1}{1 + \lambda} \) which is permissible (that is, \( 0 \leq \mu \leq 1 \)) whenever \( \lambda > 0 \). By plugging in for \( \mu \), the point of intersection \( z_\lambda \) can then expressed as (by plugging in for \( \mu \)):

\[
z_\lambda = \frac{1}{1 + \lambda} y + \frac{\lambda}{1 + \lambda} x.
\]

(4.5.3)

If we show that there exists a \( \lambda \) such that \( z_\lambda \in [w, x] \) then we are done, this is accomplished if \( \frac{\lambda}{1 + \lambda} \geq t \) or \( \lambda \geq \frac{t}{1 - t} \). Since (4.5.3) is valid for all \( \lambda > 0 \) this is easy to achieve. \( \square \)

Of course, the question that is dying to be asked is what is the precise connection between our algebraic and topological notions. The following two results elucidate some of these relationships. See also Section 4.9.

**Theorem 4.5.6.** Let \( A \) be a (not necessarily convex) subset of a topological vector space. Then:

(i) \( \text{int}(A) \subseteq \text{cor}(A) \) and

(ii) \( \text{lin}(A) \subseteq \text{cl}(A) \).

**Proof.** (i) If \( \text{int}(A) = \emptyset \) the result is trivial, so assume otherwise. Let \( x \in \text{int}(A) \). We show \( x \) is a core point. Let \( y \) be an arbitrary direction in \( X \). Since \( x \in \text{int}(A) \) there exists a neighborhood \( U \) of \( x \) such that \( U \subseteq A \). By the Structure Lemma there exists an absorbing and circled neighborhood of the origin \( B \) where

\[ x \in x + B \subseteq U \subseteq A. \]
Figure 4.2: Proof of Lemma 4.5.5

Since $B$ is absorbing there exists an $\alpha_0 > 0$ such that $\alpha y \in B$ for $0 \leq \alpha \leq \alpha_0$. This implies in turn that $x + \alpha y \in x + B \subseteq A$ for $0 \leq \alpha \leq \alpha_0$. Hence $[x, x + \alpha_0 y] \subseteq A$. We conclude $x \in \text{cor}(A)$.

(ii) Since $A$ is not necessarily convex, we cannot assume $\text{lin}(A) = \text{lina}(A)$. Take $x \in \text{lin}(A)$. If $x \in A$ then immediately $x \in \text{cl}(A)$. So we may assume that $x \in \text{lina}(A)$. Then, there exists a $y \in A$ such that $[y, x] \subseteq A$. The WTS is that for every open set $U$ of $x$ we have $U \cap A \neq \emptyset$. By the remark following the Structure Lemma (Lemma 4.3.3 we may assume that $U$ is some translate of an absorbing and circled neighborhood $B$ of the origin; that is, $U = x + B$. We now show that $[y, x] \cap U \neq \emptyset$. Since $B$ is absorbing there exists some $\alpha_0$ such that $\alpha(y - x) \in B$ for $0 \leq \alpha \leq \alpha_0$ and thus $x + \alpha(y - x) \in U$ for similarly constrained $\alpha$. Hence, for $0 \leq \alpha \leq \min\{\alpha_0, 1\}$ small enough (i.e. strictly less than 1), $x + \alpha(y - x) \in [y, x] \subseteq A$ and hence $x + \alpha(y - x) \in A \cap U$. Thus we can conclude $x \in \text{cl}(A)$. \hfill \qed

These results can be tightened for convex subsets with nonempty interior:

**Theorem 4.5.7.** Let $A$ be a convex subset of a topological vector space $\text{int}(A) \neq \emptyset$. Then we have:

(i) $\text{int}(A) = \text{cor}(A)$ and

(ii) $\text{cl}(A) = \text{lin}(A)$.

**Proof.** (i) The $(\subseteq)$ inclusion follows immediately from Theorem 4.5.6(i). For the opposite inclusion, let $x \in \text{cor}(A)$. We want to show $x \in \text{int}(A)$. By hypothesis, the interior of $A$ is
4.6. HYPERPLANES IN TOPOLOGICAL VECTOR SPACES

nonempty, so let \( p \in \text{int}(A) \). Since \( x \in \text{cor}(A) \) for some \( \alpha > 0 \) we have \( y = x + \alpha(x - p) \in A. \) From Lemma 4.5.5 we have \( (y, p) \subseteq \text{int}(A) \) and since \( x \in (y, p) \) it follows that \( x \in \text{int}(A) \).

(ii) The \((\supseteq)\) inclusion follows immediately from Theorem 4.5.6(ii). For the opposite inclusion, suppose \( x \in \text{cl}(A) \). We want to show that \( x \in \text{lin}(A) \). Take \( p \in \text{int}(A) \).

By Lemma 4.5.5 we have \( tx + (1 - t)p \in \text{int}(A) \) for all \( t \in [0, 1) \). This implies that \( [p, x) \subseteq \text{int}(A) \subseteq A \) and thus \( x \in \text{lin}(A) \).

Note that this theorem does not apply when the interior of \( A \) is empty. In other words, just because a set does not have an interior point does not imply that it does not have a core point. We may also find cases where \( \text{cl}(A) \) strictly contains \( \text{lin}(A) \) when \( A \) is convex but has an empty interior (see Example ??). Another implication is that for a convex set, in every topology where the interior of the set is nonempty, the interiors are equal.

In terms of the connection between the relative interior and the intrinsic core we have the following result:

**Corollary 4.5.8.** Let \( A \) be a convex subset of a topological vector space \( \text{relint}(A) \neq \emptyset \). Then \( \text{relint}(A) = \text{icr}(A) \).

**Proof.** The result is immediate by restricting to \((\text{aff}(A), \tau_{\text{aff}(A)})\) and noting the fact that \( \text{icr}(A) = \text{cor}_{\text{aff}(A)}(A) \) and then applying Theorem 4.5.7. \( \square \)

These last results prove useful in leveraging what we learned in Chapter 2 to the current setting. Indeed, the algebraic theory in Chapter 2 in light of the connections to topological theory highlighted in this section drives most of our separating hyperplane and Lagrangian duality theory for topological vector spaces.

### 4.6 Hyperplanes in topological vector spaces

Let’s return to our narrative about (dis)continuous linear functionals, now with more ideas in tow. Recall from Chapter 2 the intimate connection between hyperplanes and linear functionals; that is, every hyperplane in a vector space \( X \) arises in the form \( H = [\phi, \alpha] \) where \( \phi \in X' \) and \( \alpha \in \mathbb{R} \). In this section we characterize which linear functionals are continuous via topological properties of hyperplanes.

First, we observe the following:

**Theorem 4.6.1.** Let \( X \) be a topological vector space with topology \( \tau. \) Let \( H \) be a hyperplane in \( X \), then \( H \) is either closed or dense in \( X \).

**Proof.** See Exercise 4.15. \( \square \)

---

\(^2\)We implicitly use the fact \( \text{lina}(A) = \text{lin}(A) \) for convex sets.
Wow, hyperplane are dense! Didn’t see that one coming. Think about how far we have come since starting with the picture of a line in $\mathbb{R}^2$. Now, hyperplanes are very useful things. However, much like people, they are always useful when they are dense. The reason for this, as shown below in Theorem 4.6.2, is that dense hyperplanes have restricted usefulness when it comes to separation.

**Theorem 4.6.2.** Let $A$ be a convex set with nonempty interior in a topological vector space $X$. Then there does not exist a dense hyperplane $H = [\phi, \alpha]$ where $A$ lies entirely on one side of $H$. That is, there are points $x, y \in A$ where $\phi(x) > \alpha$ and $\phi(y) < \alpha$. In particular, if $x \notin \text{int}(A)$ then $x$ cannot be separated from $A$ using a dense hyperplane.

**Proof.** Let $\phi$ be an arbitrary linear functional on $X$. Then, $H' = [\phi, 0]$ is a hyperplane. Since every hyperplane is proper, there exists an $r \in X$ that is not in $H'$. That is, $\phi(r) \neq 0$. Without loss of generality suppose $\phi(r) > 0$. By hypothesis, $\overline{x} \in \text{int}(A)$. Thus, there exists an open set $U$ such that $a \in U \subseteq A$. If $H = [\phi, \alpha]$ is a dense hyperplane parallel to $H'$, then there exists an $h \in H \cap U$. Thus, $h$ is an interior point of $A$ (since $U$ is open set within $A$). By the Structure Lemma there exists an absorbing and circled neighborhood of the origin $B$ such that $h + B \subseteq A$. Since $B$ is circled and absorbing there exists an $\alpha_0 > 0$ such that $\alpha r \in B$ for all $-\alpha_0 \leq \alpha \leq \alpha_0$. This implies $h \pm \alpha_0 r \in h + B \subseteq A$ and $h - \alpha_0 r \in h + B \subseteq A$. By construction, $\phi(r) > 0$,

$$\begin{align*}
\phi(h + \alpha_0 r) &= \alpha + \alpha_0 \phi(r) > \alpha \\
\phi(h - \alpha_0 r) &= \alpha - \alpha_0 \phi(r) < \alpha
\end{align*}$$

where $\phi(h) = \alpha$ since $h \in H$. Thus we have elements $h + \alpha_0 r, h - \alpha_0 r \in A$ on “both sides” of $H$. \hfill \square

Finally, the central connection between hyperplanes and linear functionals:

**Theorem 4.6.3.** Let $X$ be a topological vector space. The hyperplane $H = [\phi, \alpha]$ is closed if and only if $\phi$ is a continuous linear functional.

**Proof.** ($\Leftarrow$) Note that $\{\alpha\}$ is a closed set in $\mathbb{R}$ and so $\phi^{-1}(\alpha) = H$ is closed when $\phi$ is continuous, via Lemma 4.2.4.

($\Rightarrow$) First we introduce some notation. Recall that $H = [\phi, \alpha] = \{x \in X : \phi(x) = \alpha\}$. In the course of this proof, we will fix $\phi$ and allow the right-hand side to change. Thus, to emphasize the role of the right-hand side we will use the notation $H(\alpha) = [\phi, \alpha]$. We will also describe halfspaces associated with a hyperplane by denoting:

$$\begin{align*}
H^+(\alpha) &= \{x \in X : \phi(x) > \alpha\} \\
H^-(\alpha) &= \{x \in X : \phi(x) < \alpha\}
\end{align*}$$

---

*This proof is based on the development in Chapter III of [4] (see Theorem 2.7, page 114)*
Now, take an arbitrary \( x \in X \) and suppose \( \phi(x) = \gamma \). To show \( \phi \) is continuous at \( x \), for every \( \epsilon > 0 \) we must find an open set \( U \) where for every \( y \in U \) we have \( \phi(y) \in (\gamma - \epsilon, \gamma + \epsilon) \). We construct a \( U \) as follows:

\[
U = \phi^{-1}(\gamma - \epsilon, \gamma + \epsilon) = \{ x \in X : \gamma - \epsilon < \phi(x) < \gamma + \epsilon \} = H^+(\gamma - \epsilon) \cap H^-(\gamma + \epsilon).
\]

Now if we can show \( H^+(\gamma) \) and \( H^-(\gamma) \) are open then \( U \) is open. This implies \( \phi(U) \in (\gamma - \epsilon, \gamma + \epsilon) \), as required.

Thus, it remains to show that the halfspaces \( H^+(\gamma) \) and \( H^-(\gamma) \) are open. Suppose \( H^+(\gamma) \) is not open (the argument for \( H^-(\gamma) \) is symmetric). Then, there exists a \( u \in H^+(\gamma) \) such that every neighborhood of \( u \) intersects \( (H^+(\gamma))^c \) where

\[
(H^+(\gamma))^c = \{ x \in X : \phi(x) \leq \gamma \}.
\]

Our goal is to show that \( u \in \text{cl}(H(\gamma)) \). This yields a contradiction. Why? We know that \( H(\alpha) \) is closed by hypothesis. So by Lemma 4.5.3(iv) this implies that \( H(\gamma) \) is closed. However, if \( u \in \text{cl}(H(\gamma)) \) then it is a closure point which is not in \( H(\gamma) \) (recall \( u \in H^+(\gamma) \)), a contradiction of the fact \( H(\gamma) \) is closed.

Thus, it remains to show that \( u \in \text{cl}(H(\gamma)) \). We have the property that neighborhood set \( V \) of \( u \) intersects \( (H^+(\gamma))^c \). Thus, for every absorbing and circled neighborhood of the origin \( B \) we have \( (u + B) \cap (H^+(\gamma))^c \neq \emptyset \). We claim \( (u + B) \cap H(\gamma) \neq \emptyset \). This in turn implies that \( u \in \text{cl}(H(\gamma)) \). The reason for this is that every open neighborhood of \( u \) can be expressed as a union of sets of the form \( u + B \) where \( B \) is an absorbing and circled neighborhood of the origin. So, if \( (u + B) \cap H(\gamma) \neq \emptyset \) for every such set, then this implies that every neighborhood of \( u \) intersects \( H(\gamma) \). For further discussion on this point see the remark following the Structure Lemma (Lemma 4.3.3).

So, it finally remains to show that \( (u + B) \cap H(\gamma) \neq \emptyset \) for every absorbing and circled neighborhood of the origin \( B \). We know there exists an \( x \in (u + B) \cap (H^+(\gamma))^c \). If \( \phi(x) = \gamma \) then we’re done. If \( \phi(x) < \gamma \) then we can use linearity to find a \( z \) such that \( \phi(z) = \gamma \). Indeed, we know \( \phi(u) > \gamma \) and \( \phi(x) < \gamma \). Suppose

\[
\phi(u) = \gamma + \delta_u,
\phi(x) = \gamma - \delta_x
\]

where \( \delta_u, \delta_x > 0 \). We leave the reader to check that for

\[
\lambda = \frac{\delta_x}{\delta_u + \delta_x}
\]

we have

\[
\phi(\lambda u + (1 - \lambda)x) = \gamma.
\]
Since \( 0 \leq \lambda \leq 1 \) we conclude that
\[
(u, x) \cap H(\gamma) \neq \emptyset.
\] (4.6.1)

Now, \( x \in u + B \) since \( B \) is circled this means that \( (\theta, x - u) \subseteq B \). Thus, by translating to \( u \) we have:
\[
[u, x] \subseteq u + B.
\] (4.6.2)

Combining (4.6.1) and (4.6.2) implies \( (u + B) \cap H(\gamma) \neq \emptyset \), as required.

To unfurl to the logic going back, this allows to conclude that \( u \in \text{cl}(H(\gamma)) \). This contradicts the fact \( H(\gamma) \) is closed. So we may assume that \( H^+(\gamma) \) is open. A similar argument allows us to conclude that \( H^-(\gamma) \) is open. When both these sets are open we can show that \( \phi \) is continuous. \( \square \)

**Remark 4.6.4.** Combining Theorems 4.6.1 and 4.6.3 implies that a hyperplane \( H = [\phi, \alpha] \) comes in two flavors: where \( H \) is closed and \( \phi \) is continuous, or \( H \) is dense and \( \phi \) is not continuous.

In view of Theorem 4.6.2 and Theorem 4.6.3 looking at linear functionals which are not continuous have limited use for separation. Indeed, if your convex set \( A \) has a non-empty interior in our topology, then no dense hyperplane can be used to separate points from \( A \). This is one motivation for restricting to continuous linear functionals (the other, explored in Chapter 6 is the fact that the continuous linear functionals may have appealing structure). One way to tame algebraic dual \( X' \) is to throw out linear functionals which are not continuous in your topology! This motivates the following definition:

**Definition.** Let \( (X, \tau) \) be a topological vector space. The subset of \( X' \) that consists of all continuous linear functionals on \( X \) is called the topological dual of \( X \), and is denoted by \( X^* \). There are many topological duals and they depend on the topology in question. When it is necessary to be explicit about which topology is used, the topological dual is denoted by \( X^*_\tau \).

If \( X \) is ordered by a pointed cone \( P \) we may similarly define its topological dual cone \( P^* \) as the set of those continuous linear functionals on \( X \) that are nonnegative on \( P \). In other words,
\[
P^* = \{ \phi \in X^* : \langle x, \phi \rangle \geq 0, \forall x \in P \}.
\] (4.6.3)

When we want to emphasize the topological dual cone for a particular topology \( \tau \) we write \( P^*_\tau \).

Note that different topologies may have different sets of continuous linear functionals and so their topological duals may differ. Thus, the choice of topology on your vector space is an important modeling consideration. Never thought you would think of something as abstract as a topology as an issue to model? Think again! We take this topic up in further detail in Chapter 6.
4.7 Motivation #2: Improved separation

With a developed theory of hyperplanes in topological vector spaces in hand we turn to our theory of separation. The essence of “topological separation” is given in the following theorem, our basic separation theorem in a topological vector space. It derives directly from our Basic Separation Theorem in Chapter 2.

**Theorem 4.7.1.** Assume that $A$ and $B$ are convex sets in the linear topological space $X$. If $A$ has a nonempty interior, then $A$ and $B$ can be separated by a closed hyperplane if and only if $\text{int}(A) \cap B = \emptyset$.

**Proof.** Since $A$ has a nonempty interior, it follows from Theorem 4.5.7 that $\text{cor}(A) = \text{int}(A) \neq \emptyset$. By Theorem 2.5.9, $A$ and $B$ can be separated by a hyperplane if

$$\text{int}(A) \cap B = \text{cor}(A) \cap B = \emptyset.$$ 

Let $H = [\phi, \alpha]$ be such a hyperplane. Let $\bar{x} \in B$. We know $H$ separates $\bar{x}$ from $A$ so in particular $\bar{x} \notin \text{int}(A)$. Thus by Theorem 4.6.2, $H$ cannot be dense in $X$. If $H$ is not dense in $X$, then by Theorem 4.6.3 it is closed. \qed

There are, of course, a panorama of similar looking statements to the above. A brief review of the literature will acquaint you with dozens. The basic program is to take results from the algebraic side and draw analogies based on Table 4.1 – for instance, swapping “core” and “interior” in a result – taking some care to state the separating hyperplanes must be closed (i.e. not dense). At this point, one might ask themselves: what have we really bought besides a simple “topology-to-algebra” dictionary. As we discussed above, we have shrunk the family of linear functionals we consider for separation and tamed, to some extent, the algebraic dual. There is something more. In some topological vector spaces we get tighter results for strong separation.

Recall Theorem 2.5.16 which states that for a convex set $A$ with a non-empty intrinsic core, if $x \notin \text{lin}(A)$ then $x$ can be strictly separated from $A$. Wouldn’t it be great if we could drop the requirement for a non-empty intrinsic core? It turns out we can do this in some topological vector spaces.

We start with the following simple idea. Suppose we have a closed subset $A$ with an empty interior in which we hope to separate via a closed hyperplane from a point $x \notin A$. Since neither the set $A$ nor the singleton $\{x\}$ have an interior point, we cannot apply Theorem 4.7.1 directly. However, there is still hope. Note that $x \in A^c$ which is an open set and hence, in a topological vector space, there exists an absorbing, circled neighborhood of the origin $B$ such that $x + B \subseteq A^c$. Notice that the set $x + B$ has an interior point (it is open!). In particular, $x$ is in the interior. If it so happens that the set $x + B$ was convex then we could apply Theorem 4.7.1 switching the roles of $A$ and $x + B$! This motivates the following definition:
Definition. A topological vector space \((X, \tau)\) is locally convex if it has a neighborhood base of \(\theta\) consisting of convex, circled and absorbing sets.

Note that every topological vector space has a neighborhood base of \(\theta\) consisting of circled and absorbing sets (via the Structure Lemma), but the property of “convexity” is an added bonus which we can use to our advantage.

Theorem 4.7.2. Assume \(A\) is a convex set in a locally convex topological vector space. If \(x \notin \text{cl}(A)\) then \(x\) can be strongly separated from \(\text{cl}(A)\) by a closed hyperplane.

Proof. (See Figure 4.3 for a visualization) Since \(x \notin \text{cl}(A)\) we have an open convex neighborhood \(B\) of the origin such that \((x + B) \cap A = \emptyset\). It is clear that \(x\) is in the interior of \(x + B\) and both \(\text{cl}(A)\) and \(x + B\) are convex. By Theorem 4.7.1 there is a closed hyperplane \(H = [\phi, \gamma]\) which separates \(x + B\) and \(A\); that is, \(\phi(x + u) \leq \gamma \leq \phi(a)\) for all \(a \in \text{cl}(A)\) and \(b \in B\). Now, since \(B\) is absorbing then for any \(y \in X\) we have \(\alpha y \in B\) for some \(\alpha > 0\). Since \(B\) is also circled, without loss of generality we may assume that \(\phi(y) > 0\). Now, \(x + \alpha y \in B\) and so
\[
\gamma \geq \phi(x + \alpha y) = \phi(x) + \alpha \phi(y)
\]
which implies that \(\phi(x) + \delta < \gamma \leq \phi(a)\) for some \(\delta = \alpha \phi(y) > 0\) and for all \(a \in \text{cl}(A)\). Hence \(x\) can be strongly separated from \(\text{cl}(A)\).

This yields the immediate corollary:

Corollary 4.7.3. Let \(A\) be a closed set in a locally convex topological vector space and let \(x \notin A\). Then \(x\) can be strongly separated from \(A\).

Theorem 4.7.2 is a key result in developing Lagrangian duality theory in locally convex topological spaces, which is the subject of the next chapter. Chapter 6 features examples of locally convex spaces, which feature in most applications of the theory we have developed so far. Many authors start with assuming locally convex spaces in view of this proximity to applications. The development up to this point gives a clear understanding of what features of the space are being leveraged at what point in the theory. There are no mysteries as to why locally convex spaces are convenient to consider. Strong separation from points boils down being closed, and we can drop our worries about interiors and core points in many applications.

4.8 A brief summary of separation

4.8.1 Separation is algebraic

Without any knowledge of topology we can understand what it means to separate a point \(x\) from a convex set \(A\). All we need is a hyperplane \(H = [\phi, \alpha]\) where \(x\) lies on one side of the hyperplane (that is, \(\phi(x) \leq \alpha\)) and the entire set \(A\) lies on the other side (that is, \(\phi(a) \geq \alpha\))
for all $a \in A$). In other words, separation is algebraic, it involves only the properties of the underlying vector space (in defining convexity and hyperplanes) and the reals (which yields the ordering $\leq$ on which we compare values). For instance, if we apply Theorem 4.7.2 for one linear topology $\tau$ and find a separating hyperplane $H$, then this hyperplane works irrespective of topology. So which topology might be most useful in identifying separating hyperplanes?

There are many possible topologies on a vector space $X$. To give some notion of ordering or comparing topologies we make some definition. Let $X$ be a set and suppose $\tau$ and $\tau'$ are two topologies on $X$. We say $\tau$ is finer or (stronger) than $\tau'$ if $\tau' \subseteq \tau$. In other words, $\tau$ contains more open sets than $\tau'$. An equivalent way to say this is that $\tau'$ is coarser (or weaker) than $\tau$. Note that it may be that two topologies are neither stronger or weaker than each other, in which case we can say these topologies are incomparable. A useful way to characterize when one topology is finer than another is captured in the following result:

**Lemma 4.8.1.** Let $\mathcal{B}$, $\mathcal{B}'$ be bases for topologies $\tau$ and $\tau'$ respectively. The following are equivalent:

(a) $\tau'$ is finer than $\tau$

(b) for all $x \in X$ and all basis elements $B \in \mathcal{B}$ containing $x$ there exists a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. 

Figure 4.3: Proof of Theorem 4.7.2
Proof. \((b \Rightarrow a)\) Let \(U\) be an open set in \(\tau\). We show \(U \in \tau'\). Let \(x \in U\). By Proposition 4.2.1 there exists a basis element \(B_x \in \mathcal{B}\) such that \(x \in B_x \subseteq U\). By hypothesis there exists a \(B'_x \in \mathcal{B}'\) so that \(x \in B'_x \subseteq B \subseteq U\). Thus \(U = \bigcup_{x \in U} B'_x\) and so \(U\) is an open set in \(\tau'\).

\((a \Rightarrow b)\) Let \(B\) be an arbitrary set in \(\mathcal{B}\) and let \(x \in B\). The goal is to show that there exists a \(B'_x \in \mathcal{B}'\) such that \(x \in B'_x \subseteq B\). Now, since \(\tau'\) is finer than \(\tau\) every set in \(\tau\) also lies in \(\tau'\). In particular, all elements of \(\mathcal{B}\) lie in \(\tau'\). Thus \(B\) can be expressed as a union

\[
B = \bigcup_{i \in I} B'_i
\]

where \(B'_i\) are elements of \(\mathcal{B}'\) and \(I\) is an index set. Now, since \(x \in B\) we know \(x \in B'_j\) for some \(j \in I\). Thus, \(x \in B'_j \subseteq B\). This yields the desired result.

The following proposition collects some implications of one topology being weaker than another:

**Lemma 4.8.2.** Let \(\tau\) and \(\tau'\) be two topologies on \(X\). Suppose \(\tau\) is finer than \(\tau'\). Then the following hold:

(i) if a set is closed in \(\tau'\) then it is closed in \(\tau\),

(ii) if a linear functional \(\phi\) is continuous in \(\tau'\) then it is continuous in \(\tau\),

(iii) if a set has a nonempty interior in \(\tau'\) then it has a nonempty interior in \(\tau\),

(iv) for any set \(A\), \(\text{int}_{\tau'}(A) \subseteq \text{int}_\tau(A)\), and

(v) for any set \(A\), \(\text{cl}_\tau(A) \subseteq \text{cl}_{\tau'}(A)\).

**Proof.** See Exercise 4.18.

This result has significant implications for our separation theory. Suppose we have a convex set \(A\) and a point \(x\) and we are interesting in finding a separating hyperplane which separates \(A\) from \(x\). Since separation is an algebraic condition we can either separate \(x\) from \(A\) or we cannot, irrespective of topology. Let’s see how a finer topology might help us as compared to a coarser topology in assessing which is the case.

The finer the topology, the more sets (possibly including \(A\)) have non-empty interiors (Lemma 4.8.2(iv)). If the topology is sufficiently fine so that \(\text{int}(A)\) is nonempty then Theorem 4.6.2 allows to narrow our search to hyperplanes which are not dense. This is clearly a positive when trying to decide on separation: there are fewer candidate hyperplanes to consider.

A similar story emerges when we consider our basic separation theorem Theorem 4.7.1 of two convex sets \(A\) and \(B\). Here \(B\) is a single point \(\{x\}\), and the necessary and sufficient condition for separation is that \(x \notin \text{int}(A)\). The finer the topology, the more chance we
4.8. A BRIEF SUMMARY OF SEPARATION

have for $A$ to have a non-empty interior and hence for the theorem to apply. The theorem yields a necessary and sufficient condition to decide separation, which is clearly to our advantage. Notice, however, that once $A$ has a nonempty interior in a topology $\tau'$ then Theorem 4.5.7(i) implies $\text{int}_\tau(A) = \text{int}_{\tau'}(A)$ for every finer topology $\tau$ (both are equal to the $\text{cor}(A)$ which does not depend on topology). Thus, the necessary and sufficient condition for separation does not always become stronger as the topology strengthens.

Finally in view of Theorem 4.7.2 a finer locally convex topology is also an advantage for deciding strong separation. Indeed, by Lemma 4.8.2(v) the finer the topology the smaller the closure of $A$ (in a containment sense). The smaller the closure, the more definitive we are being able to say whether $x$ can be strongly separated from $A$. Indeed, Theorem 4.7.2 says that if $x \notin \text{cl}(A)$ in some topology then we can strongly separate $x$ from $A$ (in fact, $\text{cl}(A)$). However, it does not speak to $x \in \text{cl}(A) \setminus A$. The smaller the closure, the smaller the set $\text{cl}(A) \setminus A$ becomes. Thus, the set of vectors where our theorem cannot ascertain separation shrinks. Notice, however, that once $A$ has a nonempty interior in a topology $\tau'$ then Theorem 4.5.7(ii) implies $\text{cl}_\tau(A) = \text{cl}_{\tau'}(A)$ for every finer topology $\tau$. Indeed, both are equal to the $\text{lin}(A)$, which does not depend on topology. Thus, the sufficient condition for claiming strong separation from $x$ ($x \notin \text{cl}(A)$) eventually amounts to asking whether $x \notin \text{lin}(A)$, which is the tightest condition we have at our disposal. The following section explores this idea further.

4.8.2 Hierarchy of sufficient conditions

It is useful to think about all of the separation results found in Section 2.5 and Section 4.7 in terms of a hierarchy sufficient conditions for separation. Again, suppose we are interested in whether we can strongly separate a point $x$ from a convex set $A$. Where $x$ is in relation to the set, its algebraic closure, and its topological closure are significant in assessing which sufficient conditions guarantee whether $x$ can be strongly separated.

We have learned that

$$\text{int}(A) \subseteq \text{cor}(A) \subseteq \text{icr}(A) \subseteq A \subseteq \text{lin}(A) \subseteq \text{cl}(A) \tag{4.8.1}$$

where the interior and closure are defined for any linear topology (see Theorem 4.5.6). If $A$ is convex and has a nonempty interior in that topology then $\text{int}(A) = \text{cor}(A)$, $\text{lin}(A) = \text{cl}(A)$ and $\text{icr}(A) = \text{cor}(A)$ (see Theorem 4.5.7 and Lemma 2.4.7). The possibility still exists for all except the first of the containments in (4.8.1) to be strict (in this case $\text{int}(A)$ and $\text{cor}(A)$ must be empty). This situation is illustrated in Figure 4.4.

Here we isolate our discussion to a given topology $\tau$ which is locally convex. If $x \notin \text{cl}(A)$ then no further conditions on $A$ are needed to strongly separate, we immediately have $x$ strong separated from $A$ by Theorem 4.7.2. If, however, $x \in \text{cl}(A) \setminus \text{lin}(A)$ we still have a sufficient condition for separation but it is stricter: we now require $A$ to have a nonempty intrinsic core by Theorem 2.5.16. Finally, if $x \in \text{lin}(A) \setminus A$ then all bets are off in terms of strong separation.
This leads us to consider (weak) separation. Here the algebraic result essentially drives everything. If \( x \in A \) but not in the core then we can separate via Theorem 2.5.14 (in fact we just need it outside of the intrinsic core, but we will not make such a fine distinction in this discussion). Thus being outside the core is sufficient for separation. From Theorem 4.7.1 we know that when the interior of the set is empty and \( x \) lies outside of the interior then we have a sufficient condition for separation. This is not too surprising, however, since in this case int(\( A \)) = cor(\( A \)) and things essentially follow from Theorem 2.5.14. What is gained there is the fact it is sufficient to look at closed hyperplanes in our topology \( \tau \), which may be useful for characterizing the hyperplane in applications.

**Example 4.8.3.** Recall Example 2.5.1 which illustrated a convex set in \( \mathbb{R}^\infty \) which could not be even weakly separated from the origin, a point which the set does not contain. This set fails all of the sufficient conditions found in Theorems 2.5.14, 2.5.16 and 4.7.2. In Example 2.4.11 we discovered the set has an empty intrinsic core and hence both Theo-
rem 2.5.14 and Theorem 2.5.16 do not apply. We also learned that \( \text{lin}(A) = X \) and hence by Theorem 4.5.6 we have \( \text{cl}(A) = X \) and so there are no sets outside of the closure of \( A \) in any topology. Thus, Theorem 4.7.2 is also not applicable.

The proliferation of separating hyperplane results underscores two things: (i) how important these results are in convex analysis and optimization, and (ii) how nuanced the topic is, particularly when adding the layer of differing topologies.

### 4.8.3 The Kozbur convex set

We define a convex set \( K \) whose linear closure is a strict subset of all topological closures arising from locally convex topologies. The example is due to Damian Kozbur of the Booth School of Business, at the University of Chicago.

Let \( X \) be a vector space with Hamel basis \( \mathcal{H} \). For each \( x \in X \) there exists a \textit{finite} and \textit{unique} subset of \( \mathcal{H} \), which we denote by \( \mathcal{H}_x \), whereby

\[
x = \sum_{h \in \mathcal{H}} x_h h
\]

with \( x_h \neq 0 \). Define the \textit{Kozbur convex set}:

\[
K = \left\{ x \in X : \sum_{h \in \mathcal{H}} x_h \geq \frac{1}{|\mathcal{H}_x|}, x \geq 0 \right\}.
\]

An element \( x \) of \( K \) has \( x_h > 0 \) for all \( h \in \mathcal{H}_x \) and \( x_h = 0 \) otherwise.

**Proposition 4.8.4.** The set \( K \) is convex.

**Proof.** Let \( x, y \in K \) and \( \lambda \in (0, 1) \) and show \( w = \lambda x + (1 - \lambda)y \in K \). Observe:

\[
\mathcal{H}_w = \mathcal{H}_x \cup \mathcal{H}_y
\]

and by the inclusion-exclusion principle

\[
|\mathcal{H}_w| = |\mathcal{H}_x \cup \mathcal{H}_y| = |\mathcal{H}_x| + |\mathcal{H}_y| - |\mathcal{H}_x \cap \mathcal{H}_y| \geq \max\{|\mathcal{H}_x|, |\mathcal{H}_y|\}. \tag{4.8.2}
\]

Equation (4.8.2) implies

\[
\sum_{h \in \mathcal{H}_w} w_h = \lambda \sum_{h \in \mathcal{H}_w} x_h + (1 - \lambda) \sum_{h \in \mathcal{H}_w} y_h
\]

\[
= \lambda \sum_{h \in \mathcal{H}_x} x_h + (1 - \lambda) \sum_{h \in \mathcal{H}_y} y_h
\]

\[
\geq \min \left\{ \frac{1}{|\mathcal{H}_x|}, \frac{1}{|\mathcal{H}_y|} \right\}. \tag{4.8.3}
\]
By (4.8.2), $|H_w| \geq \max\{|H_x|, |H_y|\}$ and

$$\frac{1}{|H_w|} \leq \min \left\{ \frac{1}{|H_x|}, \frac{1}{|H_y|} \right\}.$$  

From (4.8.3),

$$\sum_{h \in H_w} w_h \geq \frac{1}{|H_w|}$$

and this implies $w \in K$. \hfill \Box

**Proposition 4.8.5.** The origin $\theta_X$ is not linearly accessible from $K$. That is, $\theta_X \notin \text{lin}(K)$.

**Proof.** Since $\theta_X$ is not in $K$, it suffices to show $\theta_X$ is not linearly accessible from any point $x$ in $K$. Let $x \in K$. To establish $\theta$ is not linearly accessible from $x$, show $(x, \theta) \notin K$. Apply the definition linear accessible and find a $\lambda \in (0, 1)$ such that $\lambda x + (1 - \lambda) \theta = \lambda x \notin K$.

Showing $\lambda x \notin K$ requires finding a $\lambda$ such that

$$\sum_{h \in H_x} (\lambda x)_h < \frac{1}{|H_x|}. \quad (4.8.4)$$

However, $H_{\lambda x} = H_x$ and (4.8.4) reduces to

$$\lambda \sum_{h \in H_x} x_h < \frac{1}{|H_x|}.$$  

Taking $M = \sum_{h \in H_x} x_h$ and $n = |H_x|$ it suffices to have

$$\lambda < \frac{1}{Mn}.$$  

Thus $\lambda x \notin K$ for such $\lambda$ and it follows that $\theta \notin \text{lin}(K)$. \hfill \Box

**Theorem 4.8.6.** If $\mathcal{H}$ is uncountable, then the zero vector cannot be strongly separated from the set $K$.

**Proof.** Strong separation requires a linear functional $\phi$, and a $\delta > 0$, so that either $\phi(x) > \phi(\theta) + \delta = \delta$ for all $x \in K$, or $\phi(x) < -\delta$ for all $x \in K$. To show no such linear functional exists, show that for all $\phi \in X'$ and all $\delta > 0$, there exists a $y \in K$ such that $\phi(y) \leq \delta$ and $\phi(y) \geq -\delta$.

The idea is to construct $y \in K$ with support $\mathcal{H}_y$ that becomes arbitrarily large, making $1/|\mathcal{H}_y|$ arbitrarily small. Show how this drives $\phi(y)$ arbitrarily close to 0 regardless of which linear functional $\phi$ is used.
This requires the assumption of $\mathcal{H}$ uncountable. Given arbitrary $\delta > 0$, show there is an integer $N$ such that

$$F_N = \{ h \in \mathcal{H} : |\phi(h)| \leq \delta N \} \quad (4.8.5)$$

is uncountable. Suppose otherwise, that each set $F_N$ is countable. Then $\mathcal{H} = \bigcup_{N \in \mathbb{N}} F_N$ is countable, since the countable union of sets with countable cardinality is countable. This contradicts the fact that the Hamel basis $\mathcal{H}$ is uncountable. Therefore, $F_N$ is uncountable for some $N$.

Construct $y$ by choosing a support $S \subseteq F_N$ with $|S| = N$ (this is possible since $F_N$ is uncountable) and define

$$y = \sum_{h \in S} \frac{1}{N^2} h.$$

By construction, $\mathcal{H}_y = S$ and $y \in K$ because $\sum_{h \in S} (1/N^2) = 1/N = 1/|S|$. Putting this together gives

$$\phi(y) = \sum_{h \in \mathcal{H}_y} \frac{1}{N^2} \phi(h) \leq \sum_{h \in \mathcal{H}_y} \frac{1}{N^2} |\phi(h)| \leq N \left( \frac{1}{N^2} \right) \delta N = \delta.$$

Similarly, we have

$$\phi(y) = \sum_{h \in \mathcal{H}_y} \frac{1}{N^2} \phi(h) \geq -\sum_{h \in \mathcal{H}_y} \frac{1}{N^2} |\phi(h)| \geq N \left( \frac{1}{N^2} \right) (-\delta N) = -\delta.$$

For any linear functional $\phi$, $\phi(\theta) = 0$. We have shown for any $\delta > 0$, there are points in $K$ within $\delta$ of zero. This implies that $\phi$ cannot be used to strongly separate $\theta$ from $K$. This argument holds for all linear functionals $\phi$ and thus $\theta$ cannot be strongly separated from $K$.

**Example 4.8.7.** Show the above construction fails when the dimension of the vector space is countable (that is $\mathcal{H}$ is at most countable in size). The key step that requires uncountability is the existence of the set $S$ with cardinality $|N|$. Suppose the Hamel basis $\mathcal{H}$ is countable. Then the elements of the Hamel basis can be enumerated as $\mathcal{H} = \{ h_1, h_2, \ldots \}$. Define the following linear functional: $\phi(h_i) = i$, for $i = 1, 2, \ldots$. For this linear functional $\phi$, the set $F_N$ has cardinality equal to the set $\{ i : i \leq \delta N \}$, and this set has cardinality $\delta N$. If $\delta < 1$, then the cardinality is less than $|N|$ and the desired $S$ cannot be constructed. Thus, the construction in the proof does not hold in the countable case.

**Remark 4.8.8.** The origin cannot be strongly separated from the Kozbur convex set. However, as Damian Kozbur has pointed out, the origin can be strictly separated from $K$. To see this, define a linear functional $\phi(h) = 1.0$ for all $h \in \mathcal{H}$. For this linear functional, if $x \in K$, $\phi(x) \geq \frac{1}{|\mathcal{H}_x|} > 0$, but $\phi(\theta) = 0$. 
4.8.4 Some implications

This existence of this set $K$ demonstrates that the nonempty intrinsic core condition in Theorem 2.5.16 is needed in order to ensure strong separation. We also have the following powerful implication (alluded to above):

**Corollary 4.8.9.** Let $X$ be a vector space with uncountable dimension. There exists convex sets where the linear closure is a strict subset of the topological closure is any locally convex linear topology on $X$.

**Proof.** Let $(X, \tau)$ be a locally convex topological vector space on uncountable dimension and consider the set Kozbur convex set $K$ constructed above. Note that $\theta$ cannot be strongly separated from $K$ and thus by the contrapositive of Theorem 4.7.2 $\theta$ must lie in $\text{cl}_\tau(K)$. Since $\theta \notin \text{lin}(K)$ this implies $\text{lin}(K) \subset \text{cl}_\tau(K)$. 

**Open Question 1:** Consider a convex set $K$ and a point $x \notin K$. Is it always possible to strictly separate $x$ from $\text{lin}(K)$?

**Open Question 2:** Can we construct an optimization problem that has a zero duality gap, but a positive duality gap if we restrict the linear functionals to the continuous linear functionals in any topology.

4.9 The Barvinok topology

In Theorem 4.5.6 we saw that for convex sets $A$, $\text{int}_\tau(A) \subseteq \text{cor}(A)$ and $\text{lin}(A) \subseteq \text{cl}_\tau(A)$. From the discussion in the previous section we learned that the finer the topology, the more potentially useful it is in terms of separation. The downside, of course, is that the finer the topology, the more linear functionals are continuous, leading a blowup in size and complexity of the topological dual. This is a clear tradeoff.

A very natural question is whether there is a topology $\tau$ where $\text{int}_\tau(A)$ always equals $\text{cor}(A)$ and $\text{cl}_\tau(A)$ always equals $\text{lin}(A)$ for arbitrary sets $A$. This section describes some things we know about the topic and some things we conjecture. These are evolving areas of contemplation and research, below is essentially a snapshot of current thinking.

A first idea in defining a topology where the cor $-$ lin theory maps to the int $-$ cl theory is one where the open sets correspond to the core concept. Recall the following definition:

**Definition.** A set $A$ is *algebraically open* if $\text{cor}(A) = A$.

There is an alternate characterization of algebraically open sets which turns out to be useful. The intuitive idea is that an algebraically open sets intersects lines in unions of “open” line segments (if we allow for unbounded line segments of the form $[a, \infty)$ or even
4.9. THE BARVINOK TOPOLOGY

[−∞, ∞). More precisely, let x and y be two points in a vector space \( X \). The line from x in direction y has the form

\[
L_{xy} = \{ x + \lambda y : \lambda \in \mathbb{R} \}.
\]

Consider the intersection of the line \( L \) with a set \( A \). We can parameterize the intersection \( L_{xy} \cap A \) as follows:

\[
I_{xy}(A) = \{ \lambda \in \mathbb{R} : x + \lambda y \in A \}.
\]

These notions yield our alternate characterization:

**Proposition 4.9.1.** Let \( X \) be a vector space. A subset \( A \) of \( X \) is algebraically open if and only if for all \( x, y \in X \) the intersection of the line \( L_{xy} \) with \( A \) corresponds to an open subset of \( \mathbb{R} \). More precisely, \( I_{xy}(A) \) is open in the usual topology of \( \mathbb{R} \).

**Proof.** See Exercise 4.19.

If \( A \) is convex then there is an even more streamlined characterization of being algebraically open: the intersection of \( A \) with an arbitrary line is an open line segment. This uses the above proposition and the well-known fact that the intersection of a line and a convex set is always a line segment.

**Example 4.9.2.** An important example of an algebraically open set for our purposes are open halfspaces. Let \( H = [\phi, \alpha] \) be a hyperplane. Then the open halfspaces

\[
H^+ = [\phi, \alpha]^+ = \{ x \in X : \phi(x) > \alpha \}
\]

\[
H^- = [\phi, \alpha]^- = \{ x \in X : \phi(x) < \alpha \}
\]

are both algebraically open.

All open sets in a topological vector space are algebraically open:

**Lemma 4.9.3.** Let \( X \) be a topological vector space and \( A \subseteq X \) be an open set. Then \( A \) is an algebraically open set.

**Proof.** This is an immediate consequence of Theorem 4.5.7. If \( A = \emptyset \) then the result is trivial. Since \( A \) is open \( \text{int}(A) = A \) and so when \( A \) is nonempty \( \text{int}(A) \) is nonempty. So by Theorem 4.5.7(i) \( \text{int}(A) = \text{cor}(A) \). This implies \( A = \text{cor}(A) \); that is, \( A \) is algebraically open.

Now, consider the topology where we say a set \( A \) is (topologically) open precisely when it is algebraically open. It is relatively straightforward exercise to show that this collection of open sets forms a topology on \( X \). We call this the algebraic topology (See Exercise 4.20). Lemma 4.9.3 reveals that the algebraic topology is extremely fine: it is as fine as any linear open intervals.
topology. Note that every vector space has the algebraic topology, no additional structure beyond the “algebraic” properties of scalar multiplication and vector addition are needed to define it.

This seems a natural enough topology, but you might be surprised to learn than in general that the algebraic topology is not linear! This can be demonstrated by a simple example:

**Example 4.9.4.** Consider topologies in \( \mathbb{R}^2 \). It can be shown that there exists a unique Hausdorff\(^6\) linear topology on \( \mathbb{R}^2 \) (see for instance [2], Theorem 5.21), which is exactly the usual (Euclidean) topology in \( \mathbb{R}^2 \). It is immediate from Lemma 4.9.3 that algebraic topology is itself Hausdorff.

We construct a set which is open in the algebraic topology but not open in the usual topology on \( \mathbb{R}^2 \). This shows that these topologies are not equal, and due to the uniqueness of the usual topology, we can conclude that the algebraic topology is not linear.

Here is the set in question:

\[
A = \mathbb{R}^2 \setminus \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 = x_1^2\}
\]

which is illustrated in Figure 4.5. Let \( B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 = x_1^2\} \) and thus \( A = \mathbb{R}^2 \setminus B \). Consider the point \((0, 0)\). This point is not in the interior of \( A \) in the usual topology, every open ball around \((0, 0)\) contains points which satisfy \( x_1 > 0 \) and \( x_2 = x_1^2 \). Notice, however that \((0, 0)\) is a core point: given any point \( y \) in the space the line \( \{ty : t \in \mathbb{R}\} \) intersects \( B \) in at most one place. It is does intersect, there exists a unique \( t_y \) that yields the intersection. Then simply take the line segment \( \{ty : 0 \leq t \leq t_y\} \). This clearly lies in \( A \) and hence \((0, 0)\) is a core point.

In search of a linear topology which works well with our cor – lin machinery, we define the following:

**Definition.** The Barvinok topology\(^7\) is the topology whose basis consists of the convex algebraically open sets. In other words, the open sets in the Barvinok topology are unions of convex algebraically open sets.

To ensure the Barvinok topology is well-defined we argue that the set of convex algebraically open sets indeed form a basis of a topology: in other words, they satisfy properties (B1) and (B2) described after Proposition 4.2.1.

To establish (B1) note that every \( x \in X \) lies in some open halfspace. Indeed, take any linear functional \( \phi \). The open halfspace \( [\phi, \phi(x) - 1]^+ \) contains \( x \) is algebraically open.

---

\(^6\)A topological space \((X, \tau)\) is Hausdorff if for every two points \(x, y \in X\) there exists open sets \(U_x\) and \(U_y\) such that \(x \in U_x\), \(y \in U_y\) and \(U_x \cap U_y = \emptyset\).

\(^7\)This topology is named after Alexander Barvinok, from whose book [4] we learned about this topology.
Figure 4.5: The set (4.9.1) (shaded in blue) which is algebraically open in $\mathbb{R}^2$ but not open in the usual topology.

To establish (B2) let $B_1$ and $B_2$ be two algebraically open subsets. Recall that the intersection of two convex sets is convex. So it remains to show that $B_1 \cap B_2$ is algebraically open. This follows from the following fact (Exercise 4.21):

$$\text{cor}(B_1 \cap \text{cor}(B_2)) = \text{cor}(B_1) \cap \text{cor}(B_2).$$

Since $B_1$ and $B_2$ are algebraically open, the left-hand side is $B_1 \cap B_2$. Now, since $B_1 \cap B_2$ is convex and algebraically open, it lies in the Barvinok topology. Property (B2) is then immediate.

The next theorem shows that the Barvinok topology is a well-behaved topology on $X$:

**Theorem 4.9.5.** Let $X$ be an arbitrary vector space. Then $(X, \tau_B)$ is a locally convex, Hausdorff topological vector space.

**Proof.** There are several things to prove:

(i) Vector addition is continuous,

(ii) Scalar multiplication is continuous,

(iii) $\tau_B$ is Hausdorff: that is, for every $x, y \in X$ there exists open sets $U_x$ and $U_y$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.

(iv) $\tau_B$ is locally convex.
Let’s prove each of these in turn:

(i) By definition we must show that if \( x_1, x_2 \in X \) and \( V \) is a neighborhood of \( x_1 + x_2 \) there must exist neighborhoods \( V_1 \) and \( V_2 \) of \( x_1 \) and \( x_2 \) respectively such that \( V_1 + V_2 \subseteq V \).

It suffices to show this for \( V \) which are algebraically open and convex, since this is the base of the topology. Define \( V = x_1 + x_2 + B \) where \( B \) is a convex algebraically open subset of the origin. By Exercise 4.22 we know that \( \frac{1}{2}B \) is algebraically open. Define \( V_i = x_i + \frac{1}{2}B \) for \( i = 1, 2 \). Then observe:

\[
V_1 + V_2 = (x_1 + \frac{1}{2}B) + (x_2 + \frac{1}{2}B) = (x_1 + x_2) + (\frac{1}{2}B + \frac{1}{2}B) \subseteq x_1 + x_2 + B
\]

where the last step follows by Exercise 4.23. This establishes continuity of vector addition.

(ii) Details are omitted.

(iii) Suppose we are given two points \( x \neq y \) in the space. We construct a hyperplane \( H = [\phi, \alpha] \) such that \( \phi(x) < \alpha \) and \( \phi(y) > \alpha \). Since halfspaces are algebraically open we will have exhibited disjoint open sets which contain \( x \) and \( y \) respectively. This establishes the Hausdorff property.

The construction works as follows. Consider the midpoint \( \frac{x + y}{2} \) and consider the maximal affine subspace containing this midpoint and not containing \( x \). The existence of such a space is proved using Zorn’s lemma (see Exercise 4.24). Denote this affine subspace by \( F \).

If we can show that \( F \) is a hyperplane, say \( F = [\phi, \alpha] \) we are done: it will provide a separating hyperplane. The reasoning is as follows. We have \( x \notin F \) we must also have \( y \notin F \). The reason is that \( F \) contains the midpoint and so if it contains \( y \) then it will contain \( x \) since \( x \) is an affine combination of \( y \) and the midpoint \( \frac{x + y}{2} \). We want to show \( x \) and \( y \) are on either side of the hyperplane. Now

\[
\phi \left( \frac{x + y}{2} \right) = \alpha
\]

so if \( \phi(x) > \alpha \) and \( \phi(y) > \alpha \) then by linearity \( \phi \left( \frac{x + y}{2} \right) > \alpha \), a contradiction. A similar argument derives a contradiction of \( \phi(x) < \alpha \) and \( \phi(y) < \alpha \). Thus without loss of generality we may assume \( \phi(x) > 0 \) and \( \phi(y) < 0 \). That is \( [\phi, \alpha] \) separates \( x \) and \( y \).

So it only remains to show that \( F \) is a hyperplane. It is sufficient to show \( F \) has codimension 1. Suppose otherwise. Then there exists a vector \( z \) that is outside the
affine hull of $x$ and $F$. This violates the maximality of the Zorn construction of the maximal set of affinely independent points: you could have added $z$. The result then follows.

(iv) This is by definition of the topology, it has a basis of convex sets.

**Theorem 4.9.6.** The Barvinok topology is the finest locally convex topological vector space.

*Proof.* Let $(X, \tau)$ be a locally convex topological vector space and let $B$ be a basis of convex sets for $\tau$. By Lemma 4.8.1 it is enough to show that for every $x \in X$ and every basis element $B \in B$ containing $x$, there exists an algebraically open convex set containing $x$ which lies inside of $B$.

Since $B$ is a basis element in a locally convex topological vector space, $B$ is convex and so by Lemma 2.4.3 and Corollary 2.4.23 we know $\text{cor}(B)$ is convex and algebraically open. Thus, $\text{cor}(B)$ is a basis element of the Barvinok topology, and of course lies inside of $B$. This establishes the result.

**4.9.1 Connections to core and algebraic closure**

We present two interesting streams of results regarding the Barvinok topology. The first stream relates to its close connection to the cor $-$ lin concepts of Chapter 2. The second stream relates to its close association with separating hyperplanes.

Here is the key result of this subsection:

**Theorem 4.9.7.** Let $A$ be a convex set. Then (i) $\text{cor}(A) = \text{int}_B(A)$ and (ii) if in addition $\text{cor}(A)$ is nonempty, then $\text{lin}(A) = \text{cl}_B(A)$.

*Proof.* (i) The fact that $\text{int}(A)$ is contained in the core is immediate from Theorem 4.5.6(i). For the reverse inclusion, we argue that $\text{cor}(A)$ is convex and algebraically open and thus $\tau_B$-open. Then since $\text{cor}(A)$ is contained in $A$ this will imply $\text{cor}(A) \subseteq \text{int}_B(A)$ (which is the largest $\tau_B$-open set contained in $A$ via Lemma 4.5.1).

Thus, it remains to argue that $\text{cor}(A)$ is convex and algebraically open when $A$ is convex. These facts are established in Lemma 2.4.3 and Corollary 2.4.23 respectively.

(ii) The fact that $\text{lin}(A)$ is contained in $\text{cl}_B(A)$ is immediate from Theorem 4.5.6(ii). For the reverse inclusion start with $y \in \text{cl}(A)$. We argue that $y \in \text{lin}(A)$. Since $\text{cor}(A)$ we can take an $x \in \text{cor}(A)$. By Part (i), $\int(A) = \text{cor}(A)$ and thus $x \in \int(A)$. Hence, by Lemma 4.5.5 we know $\lambda \text{cl}(A) + (1 - \lambda) \text{int}(A) \subseteq \text{int}(A)$ and thus $(x, y) \in \int(A) \subseteq A$. This implies that $y \in \text{lin}(A)$.

$\square$
This result tells us that the Barvinok topology creates a topological definition \( \text{cor}(A) \) and \( \text{lin}(A) \) for convex sets \( A \) with nonempty core. What about in other circumstances? Example 4.9.4 shows that \( \text{int}_{\tau_B}(A) \) need not equal \( \text{cor}(A) \) when \( A \) is nonconvex. Note that the interior in the Barvinok topology is the usual interior since there is a unique linear Hausdorff topology in \( \mathbb{R}^2 \) ([2], Theorem 5.21).

Characterizing precisely when \( \text{lin}(A) = \text{cl}_{\tau_B}(A) \) is an open question. We know of non-convex sets \( A \) where \( \text{cor}(A) \supseteq \text{int}_{\tau_B}(A) \) and convex sets with empty core where \( \text{lin}(A) \subseteq \text{cl}_{\tau_B}(A) \):

**Example 4.9.8 (Basu).** Consider the set \( \mathbb{Q} \) (the set of rational numbers) in \( \mathbb{R} \). By Theorem 4.9.5 the Barvinok topology is a linear Hausdorff topology. Since there is a unique Hausdorff linear topology in \( \mathbb{R} \) ([2], Theorem 5.21), the Barvinok topology is simply the usual topology. Hence \( \text{cl}(\mathbb{Q}) = \mathbb{R} \) since the rationals are dense in the reals in the usual topology. However, \( \mathbb{Q} \) is algebraically closed. Indeed, we can argue \( \text{lina}(\mathbb{Q}) = \emptyset \). Let \( r \in \mathbb{R} \) and consider any interval \([q,r)\) where \( q \in \mathbb{Q} \). It is well known that every nonempty interval in \( \mathbb{R} \) contains irrational numbers. Thus \([q,r) \cap \mathbb{R} \setminus \mathbb{Q} \) is nonempty. Thus \( r \notin \text{lina}(\mathbb{Q}) \).

**Example 4.9.9 (Kozbur).** Damian Kozbur (from our class) gives a general construction of a convex set \( A \) where \( \text{lin}(A) \) is a strict subset of \( \text{cl}_{\tau_B}(A) \). The construction works for every vector space with an uncountable dimension. The proof method is as follows, it can be shown \( \theta \notin \text{lin}(A) \) whereas we cannot strongly separate \( \theta \) from \( A \). However, by Theorem 4.7.2 if \( \theta \notin \text{cl}_{\tau_B}(A) \) then it can be strongly separated. This implies \( \theta \in \text{cl}(A) \). We are still verifying the details of this result and will provide more details.

### 4.9.2 Equivalence with separation

We next explore the Barvinok topology as a natural setting for geometric separation.

**Proposition 4.9.10.** All linear functionals over \( X \) are \( \tau_B \)-continuous.

**Proof.** Let \( \phi \) be a linear functional over \( X \). To show that \( \phi \) is continuous it suffices to show \( \phi^{-1}(a,b) \) is open in the Barvinok topology. Here \( (a,b) \) is an arbitrary open interval in \( \mathbb{R} \). This suffices because the open intervals form a basis of the usual topology on \( \mathbb{R} \).

Note that

\[
\phi^{-1}(a,b) = \{ x \in X : a < \phi(x) < b \}.
\]

This set is the intersection of two halfspaces, which are open in the Barvinok topology. Since the intersection of two open sets is open we conclude that \( \phi^{-1}(a,b) \) is open, as required.

---

*We learned this fact from Amitabh Basu*
Proposition 4.9.10 implies that the Barvinok topological dual of $X$ is precisely the algebraic dual. Thus, considering the Barvinok topology to solve duals of optimization problems falls prey to all of the challenges discussed in Section 4.1.

Now, by Theorem 4.7.2 we know in locally convex topological vector spaces a sufficient condition for strongly separation a point $x$ from the closure of a convex set is that $x$ is not contained in the closure. In the Barvinok topology, this is both necessary and sufficient.

**Theorem 4.9.11.** Let $A$ be a convex set. The point $x \in X$ can be strongly separated from $A$ if and only if $x \notin \text{cl}_{rB}(A)$.

**Proof.** ($\Rightarrow$) This is immediate by Theorem 4.7.2 since the Barvinok topology is locally convex.

($\Leftarrow$) Suppose $x$ can be strongly separated from $A$. Thus, there exists an open halfspace $[\phi, \alpha]^+$ which contains $x$ and does not intersect $A$. Since halfspaces are open sets in the Barvinok topology, this immediately implies that $x \notin \text{cl}_{rB}(A)$.

Another way to understand this connection is encapsulated in the following result:

**Theorem 4.9.12.** We can (weakly/strongly) separate point $x$ from $A$ using a hyperplane if and only if we can (weakly/strongly) separate $x$ from $\text{cl}_{rB}(A)$ in the Barvinok topology with the same hyperplane.

**Proof.** Let’s consider weak separation. Suppose $[\phi, \alpha]$ weakly separates $A$ from $x$. Then $A$ is contained in the half-space $\{y : \phi(y) \leq \alpha\}$ and $\phi(x) \geq \alpha$. Observe that $\{y \in X : \phi(y) \leq \alpha\}$ contains $A$ and is a closed set in the Barvinok topology (its complement is a open halfspace). Since the closure is the smallest closed set which contains $A$, this closed half-space must $\text{cl}_{rB}(A)$. Thus the hyperplane $[\phi, \alpha]$ weakly separates $x$ from $\text{cl}_{rB}(A)$.

Let’s look at strong separation. Suppose $[\phi, \alpha]$ strongly separates $A$ from $x$. Then we can assume (using the definition of strong separation) that $A$ is contained in the half-space $\{y : \phi(y) \leq \alpha\}$ and $\phi(x) > \alpha + \epsilon$ for some $\epsilon > 0$. Again, $\{y : \phi(y) \leq \alpha\}$ is a closed set and so it contains $\text{cl}_{rB}(A)$. We conclude $[\phi, \alpha]$ strongly separates $x$ from $\text{cl}_{rB}(A)$.

The results in the section are quite powerful, and provide a different perspective, and indeed alternate proofs for many earlier “algebraic” results. Theorem 4.9.7 and Theorem 4.9.12 allow us to rephrase most results involving a convex set with a nonempty intrinsic core in terms of the Barvinok topology. These include our weak and strong separation results of Chapter 2, as well as the “no duality gap” and strong duality results of Chapter 3.

### 4.10 Notes

Sections 4.2 and 4.3 are standard topics in any book on topology (for instance, see [21]) and topological vector spaces (for instance see Chapter 5 of [2]). The connection between the
concepts of core and interior in Theorem 4.5.7 is due to Holmes [16]. The “algebraic topology” discussed in Section 4.9 is partly inspired by definitions of slightly different topologies described in [4] and [29]. The authors are indebted to Amitabh Basu for clarifying discussions which inspired our writing of Section 4.9.

4.11 Exercises

Exercise 4.1. Show that all the examples in Example 4.2.2 are indeed topologies.

Exercise 4.2. Let $X$ and $Y$ be topological spaces with topologies $\tau$ and $\sigma$ respectively and suppose $\sigma$ has a basis $B$. Show that a function $f : X \to Y$ is continuous if and only for every basis element $B \in B$ we have $f^{-1}(B) \in \tau$.

Exercise 4.3. Prove Lemma 4.2.4: Let $X$ and $Y$ be topological spaces with topologies $\tau$ and $\sigma$ respectively. A function $f : X \to Y$ is continuous if for every $\sigma$-closed set $V \in \sigma$, $f^{-1}(V)$ is $\tau$-closed. That, the inverse image of a closed set is closed.

Exercise 4.4. Prove Lemma 4.2.5: Let $X$ and $Y$ be topological vectors spaces and $f : X \to Y$. Show that $f$ is continuous if and only if for each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood of $x$ such that $f(U) \subset V$.

Exercise 4.5. (i) Show that the only continuous linear functional over the topological vector space $X$ endowed with the indiscrete topology is the zero functional.

(ii) Show that all functionals (including nonlinear functionals) are continuous on a set endowed with the indiscrete topology.

Exercise 4.6. Prove Lemma 4.3.1: Consider the mappings $T_a(x) = a + x$ and $M_\lambda(x) = \lambda x$ for all $x \in X$. Then $T_a$ and $M_\lambda$ are homeomorphism of $X$ onto $X$.

Exercise 4.7 (Norm topology). Let $X$ be a vector space. We may define a functional $|| \cdot ||$ on $X$ called a norm if it satisfies the following axioms:

1. (N1) $||x|| \geq 0$
2. (N2) $||x|| = 0 \Rightarrow x = \theta$
3. (N3) $||x + y|| \leq ||x|| + ||y||$
4. (N4) $||\alpha x|| = |\alpha||x||$

where $x, y \in X$, $\alpha \in \mathbb{R}$ and $| \cdot |$ is the absolute value functional of $\mathbb{R}$ (which is itself a norm). Define an open ball of radius $\epsilon > 0$ centered at $x \in X$ by:

$$B_\epsilon(x) = \{ y \in X : ||x - y|| < \epsilon \}.$$
4.11. EXERCISES

(i) Show that the topology \( \tau \) with basis consisting of all open balls (i.e. the claimed basis is the set \( \{ B_\epsilon(x) : \epsilon > 0, x \in X \} \) is linear. We call \( \tau \) the norm topology.

(ii) Show the norm functional \( x \mapsto ||x|| \) is continuous with respect to the norm topology.

**Exercise 4.8.** Let \( C[0,1] \) denote the set of all real-valued continuous functions defined on domain \([0,1]\). That is, if \( x \in C[0,1] \) then \( x \) maps closed interval \([0,1] \subseteq \mathbb{R} \) into \( \mathbb{R} \). Define the following functional \( || \cdot ||_\infty \) on \( C[0,1] \)

\[
||x||_\infty = \sup_{0 \leq t \leq 1} |x(t)|.
\]

Show that \( C[0,1] \) is a topological vector space with a basis neighborhood basis of \( \theta \) consisting of the sets:

\[
B_\epsilon(\theta) = \{ y \in C[0,1] : ||y||_\infty < \epsilon \}
\]

where \( \epsilon \) ranges over all positive real numbers.

**Exercise 4.9 (Paired vector spaces and the weak topology).** Let \( X \) and \( Z \) be vector spaces. Suppose there exists a mapping \( (\cdot|\cdot) : X \to Z \)

with the following two properties:

(P1) The mapping \( (\cdot|\cdot) \) is bilinear; that is linear in both of its arguments. In other words,

\[
(x^1 + x^2|z) = (x^1|z) + (x^2|z) \text{ for all } x^1, x^2 \in X, z \in Z, \text{ and}
\]

\[
(x|z^1 + z^2) = (x|z^1) + (x|z^2) \text{ for all } x \in X, z^1, z^2 \in Z.
\]

(P2) The mapping is non-degenerate; that is,

\[
(x|z) = 0 \text{ for all } z \in Z \text{ implies } x = \theta_X, \text{ and}
\]

\[
(x|z) = 0 \text{ for all } x \in Z \text{ implies } z = \theta_Z.
\]

Such a mapping \( (\cdot|\cdot) \) is called a pairing between \( X \) and \( Z \) and we called \( X \) and \( Z \) paired vector spaces.\(^9\)

The goal of this question is to explore how we can use the pairing \( X \) of \( Z \) to define a linear topology on \( X \) and the nature of the linear functionals which are continuous in that topology.

\(^9\)Confusingly, such a mapping is also called a duality of vector spaces, which puts the size of the list of things called “dual” or “duality” a bit too long for Kipp and Chris to stomach.

In a related note, a more standard notation for a pairing is the “angle brace” notation \( \langle \cdot, \cdot \rangle \). Since we already use that elsewhere we stick to the \( (\cdot|\cdot) \) notation found above.
Define the following collection of sets. Let $A$ be a finite subset of $Z$ and define:

$$B_A = \{ x \in X : -1 < (x|z) < 1 \text{ for all } z \in A \}. \quad (4.11.1)$$

Note that $B_A$ is a subset of $X$. The collection is

$$\mathcal{B} = \{ B_A : A \subseteq Z, A \text{ finite} \}.$$ 

Show the following:

(i) There exists a linear topology on $X$ with $\mathcal{B}$ as a neighborhood base of $\theta_X$. This linear topology is called the \textit{weak topology} on $X$ and is denoted $\sigma(X, Z)$.

(ii) For every $z \in Z$ define a function $\phi_z$ on $X$ as follows: $\phi_z(x) = (x|z)$ for all $x \in X$. Show that for all $z \in Z$, $\phi_z$ is linear functional which is continuous the weak topology $\sigma(X, Z)$.

\textbf{Exercise 4.10.} Consider the following two subsets of $\mathbb{R}^N$:

$$\ell^\infty = \{ x = (x_1, x_2, \ldots) \in \mathbb{R}^N : \sup \{|x_i| : i \in \mathbb{N}\} < \infty \}$$

and

$$\ell^1 = \{ x = (x_1, x_2, \ldots) \in \mathbb{R}^N : \sum_{i=1}^{\infty} |x_i| < \infty \}$$

Show that the mapping $(\cdot|\cdot) : \ell^\infty \times \ell^1 \to \mathbb{R}$ defined by:

$$(x|y) = \sum_{i=1}^{\infty} x_i y_i$$

is a pairing of $\ell^\infty$ and $\ell^1$.

\textbf{Exercise 4.11.} Prove that the functional defined in (4.4.1) is a linear functional.

\textbf{Exercise 4.12.} Show that $\mathcal{B}$ defined in (4.4.2) is a neighborhood basis of $\theta$, that is we need to establish that each there exists a basis element than contains $\theta$ and property (B2).

\textbf{Exercise 4.13.} Prove Lemma 4.5.1: Let $A$ be a subset of a topological space. Show

(i) $\text{int}(A)$ is the largest (with respect to containment) open set which is contained in $A$ and

(ii) $\text{cl}(A)$ is the smallest (with respect to containment) closed set which contains $A$.

\textbf{Exercise 4.14.} Prove Lemma 4.5.3: Let $(X, \tau)$ be a topological vector space and $A$ a subset of $X$. Show
(i) The Minkowski sum of an open set and an arbitrary set is open.

(ii) If $B$ is open, then for any set $A$ we have $\text{cl}(A) + B = A + B$

(iii) For any subset $A$ of a topological space, $\text{int}(A) = (\text{cl}(A^c))^c$.

(iv) $\text{int}(x + A) = x + \text{int}(A)$

(v) $\text{cl}(x + A) = x + \text{cl}(A)$

**Exercise 4.15.** Prove Theorem 4.6.1: Let $H$ be a hyperplane in $X$, then $H$ is either closed or dense in $X$.

**Exercise 4.16.** Let $A$ be a convex set. Suppose $x$ cannot be strongly separated from $\text{lin}(A)$. Show that $x \in \text{cl}(A)$ in every locally convex topology.

**Exercise 4.17.** Is a dense set necessarily ubiquitous? Vice versa?

**Exercise 4.18.** Prove Lemma 4.8.2: Let $\tau$ and $\tau'$ be two linear topologies on $X$. Suppose $\tau$ is finer than $\tau'$. Then show the following hold:

(i) if a set is closed in $\tau'$ then it is closed in $\tau$,

(ii) if a linear functional $\psi$ is continuous in $\tau'$ then it is continuous in $\tau$,

(iii) if a set has a nonempty interior in $\tau'$ then it has a nonempty interior in $\tau$,

(iv) for any set $A$, $\text{int}_{\tau'}(A) \subseteq \text{int}_{\tau}(A)$, and

(v) for any set $A$, $\text{cl}_{\tau}(A) \subseteq \text{cl}_{\tau'}(A)$.

**Exercise 4.19.** Prove Proposition 4.19: A set $A$ is algebraically open if and only if for all $x, y \in X$, $I_{xy}(A)$ is an open set in $\mathbb{R}$ in its usual topology.

**Exercise 4.20.** Show that the algebraic topology is indeed a topology.

**Exercise 4.21.** Show the following holds for any two sets $B_1$ and $B_2$:

$$\text{cor}(B_1 \cap \text{cor}(B_2)) = \text{cor}(B_1) \cap \text{cor}(B_2).$$

**Exercise 4.22.** Let $B$ be a convex algebraically open set which contains the origin. Then $\lambda B$ is a convex algebraically open set for all $\lambda > 0$.

**Exercise 4.23.** If $B$ is a convex set, then

$$\lambda B + (1 - \lambda)B \subseteq B$$

for $0 \leq \lambda \leq 1$.

**Exercise 4.24.** Let $a$ and $b$ be two points in a vector space $X$. Prove there exists a maximal set of affinely independent points which contain $a$ and do not contain $b$. 

Chapter 5

Topological Langrangian Duality

5.1 Motivation

In this Chapter give topological equivalents for Theorem 3.4.6 and Corollaries 3.4.7 through 3.4.14. The assumption of an intrinsic core was critical in proving duality results in vector spaces with no topology assumed. The key ingredient in the duality proofs was a separating hyperplane. In particular, we needed to strictly separate a point from the algebraic closure of a set. In order to do that the convex set had to have a nonempty intrinsic core. No such requirement is needed in the results below for locally convex topological vector spaces. When we have a topology, we use Theorem 4.7.2 to strongly separate a point from a closed set. In particular, assume $x$ is not in the closure of a convex set $A$ in a locally convex topological vector space $(X, \sigma)$. Then there is a convex, absorbing and circled neighborhood $B$ of the origin so that $x + B$ does not intersect $A$. This implies the existence of a hyperplane that strongly separates $x$ from the closure of $A$. This idea is important in optimization theory. Indeed, most optimization problems are stated in locally convex topological vector spaces. Recall our generic optimization problem:

$$
\mu = \inf_{x \in \Omega} f(x)
\text{ s.t. } G(x) \preceq_P \theta_Y \quad (P)
$$

where $\Omega$ is a convex set contained in vector space $X$, $f : X \to \mathbb{R}$ is a convex functional, $G : X \to Y$ is a convex mapping from $X$ into a vector space $Y$. The vector space $Y$ comes equipped with a partial order $\preceq_P$ defined by a convex pointed cone $P$ in $Y$. In this chapter we restrict this problem by making the following assumption:

**Assumption.** There exists a linear topology $\sigma$ so that $(Y, \sigma)$ is a *locally convex* topological vector space.
Our goal is to understand how having the locally convex linear topology $\sigma$ impacts duality theory. We pay special attention to how different topologies give rise to different Lagrangian dual problems and how this effects the duality gap.

5.2 No duality gap and strong duality in topological vector spaces

Recall the algebraic Lagrangian dual (3.3.1) of (P) (here given the label (AD)):

$$\sup_{\psi} \inf_x f(x) + \langle G(x), \psi \rangle$$

s.t. $\psi \in P^+$

$$x \in \Omega.$$  

(AD)

Now assume $(Y, \sigma)$ is a locally convex topological vector space ordered by the pointed cone $P$. We define a topological Lagrangian dual $(TD_\sigma)$, that depends on the choice of topology $\sigma$:

$$\sup_{\psi} \inf_x f(x) + \langle G(x), \psi \rangle$$

s.t. $\psi \in P^*_\sigma$

$$x \in \Omega.$$  

(TD$_\sigma$)

where $P^*_\sigma$ is the topological dual of the cone $P$ in topology $\sigma$.

At first glance the algebraic dual and the topological dual are pretty much the same, and indeed the only substantive difference is the the swapping of $P^*_\sigma$ for $P^+$. In other words, in the topological dual we restrict the choice of $\psi$ to be linear functionals that are not only nonnegative on $P$ as in the algebraic case (if you recall, this is what guaranteed us weak duality, see Theorem 3.2.1) but also continuous in the locally convex topology $\sigma$. With this observation, and using the notation $v(\cdot)$ to denote the value of a problem, we can easily show the following simple bounds:

$$v(TD_\sigma) \leq v(AD) \leq v(P) = \mu.$$  

(5.2.1)

The first inequality follows from the fact that $P^*_\sigma \subseteq P^+$ and the second inequality follows by weak duality. In this chapter we explore topological conditions whereby $v(TD_\sigma) = v(P)$. Later in Section 5.3, we examine the possibility that both inequalities in (5.2.1) are strict and how the value $v(TD_\sigma)$ depends on the choice of $\sigma$. For now, we take the locally convex topology $\sigma$ as given.

We mirror the development in Section 3.4. There is only one crucial difference: $Y$ is a locally convex topological vector space. Many of the results are very similar and follow immediately, or with minor adjustments, from analogous results in Section 3.4. We will
be careful to alert the reader to analogous results from Chapter 3 and highlight where the development in the proofs diverge.

Recall the key players in our development in the algebraic case:

\[
\begin{align*}
\Gamma &:= \{(r,y) \in \mathbb{R} \times Y : \exists x \in \Omega \text{ s.t. } f(x) \leq r, G(x) \preceq y\} \quad (5.2.2) \\
\mu &:= \inf \{r : \exists y \preceq \theta Y \text{ where } (r,y) \in \Gamma\} \quad (5.2.3) \\
\nu &:= \inf \{r : \exists y \preceq \theta Y \text{ where } (r,y) \in \text{lin}(\Gamma)\}. \quad (5.2.4)
\end{align*}
\]

In our topological theory we redefine \( \nu \) replacing the linear closure in its definition for an appropriate topological closure. Note that \( \Gamma \) lies in \( \mathbb{R} \times Y \) and not in \( Y \) so we cannot apply the topology \( \sigma \) directly to \( \Gamma \). We must work with the products of topological vector spaces. Several salient Lemmas are given below:

**Lemma 5.2.1.** Let \((X,\tau)\) and \((Y,\sigma)\) be locally convex topological vector spaces. Let \( \rho \) denote the product topology on \( X \times Y \) derived from \( \tau \) and \( \sigma \). Show that \((X \times Y, \rho)\) is a locally convex topological vector space.

*Proof.* See Exercise 5.1. \( \square \)

**Lemma 5.2.2.** Let \((X,\tau)\) and \((Y,\sigma)\) be topological vector spaces. Let \( W = X \times Y \) and let \( \rho \) be the product topology on \( W \). Consider a linear functional \( \phi_W \in W' \),

\[
\phi_W(x,y) = \phi_X(x) + \phi_Y(y)
\]

where \( \phi_X \in X' \) and \( \phi_Y \in Y' \) (In Exercise 3.21 we show all linear functions in \( W' \) have this form.) Then, if any two of \( \phi_W \), \( \phi_X \) and \( \phi_Y \) are continuous then so if the third.

*Proof.* See Exercise 5.2. \( \square \)

**Remark 5.2.3.** Note that in the setting we explore in this chapter \( X = \mathbb{R} \), which is a finite dimensional vector space. In Exercise 5.3 you are asked to show that every linear functional in \( \mathbb{R}^n \) is continuous in the usual (norm) topology. Thus, \( \phi_X \) is continuous. Then the linear functional \( \phi_W \) is continuous in \( \mathbb{R} \times Y \) if and only if \( \phi_Y \) is continuous in \( Y \). We use this fact in the results below.

Since the usual topology \( \tau_{\text{usual}} \) on \( \mathbb{R} \) is locally convex, it follows that if we define \( \rho \) to be the product topology of \( \tau_{\text{usual}} \) and \( \sigma \), which is also locally convex, then \( \rho \) is a locally convex topological vector space. In this chapter we consider different topologies \( \sigma \) on \( Y \), that give rise to different product topologies \( \rho \). When we want to emphasize the connection between \( \sigma \) and \( \rho \) we write \( \rho = \rho(\sigma) \). Define

\[
\nu_\rho = \inf\{r : \exists y \preceq \theta Y \text{ where } (r,y) \in \text{cl}_\rho(\Gamma)\}. \quad (5.2.5)
\]

From Theorem 4.5.6, \( \text{lin}(\Gamma) \subseteq \text{cl}_\rho(\Gamma) \) and it follows that \( \nu_\rho \leq \nu \). See Figure 5.2 for a visual representation.
CHAPTER 5. TOPOLOGICAL LANGRANGIAN DUALITY

As before (Theorem 3.4.2) \( \mu \) is equal to the value of the primal \( v(P) \). Indeed, the primal optimization is unchanged under different choices of topology for \( Y \). In the algebraic case we could equate \( \nu \) with the value of the algebraic dual \( (AD) \) under certain conditions, including an intrinsic core condition. To derive a topological analogue we need the following preliminary:

**Lemma 5.2.4** (cf. Theorem 3.4.4). If \( \nu_\rho \) is finite then \( (\nu, \theta_Y) \in \text{cl}_\rho(\Gamma) \).

*Proof.* See Exercise 5.5. \( \square \)

**Theorem 5.2.5** (cf. Theorem 3.4.6). Let \((Y, \sigma)\) be a locally convex topological vector space and \( \rho \) the associated product topology on \( \mathbb{R} \times Y \). If \( \nu_\rho \) is finite then \( \nu_\rho = v(TD_\sigma) \). In more detail,

\[
\nu_\rho = \sup_{\psi \in P_\sigma^+} \left\{ \inf_{x \in \Omega} f(x) + \langle G(x), \psi \rangle \right\} = v(TD_\sigma).
\]

*Proof.* The proof is essentially identical to the proof of Theorem 3.4.6. In a key point of that proof we need to separate the point \((\nu_\rho - \epsilon, \theta_Y)\) from \( \text{lin}(\Gamma) \). Here we need to separate from \( \text{cl}_\rho(\Gamma) \) instead. Thus, we invoke Theorem 4.7.2 instead of Theorem 2.5.16. This allows us to drop the intrinsic core assumption in Theorem 3.4.6.

Theorem 4.7.2 produces a *continuous* linear functional that allows us to strictly separate. From the proof of Theorem 3.4.6 we know we can take that continuous linear functional to be of the form \((1, \psi)\) where \( \psi \in P^+ \). Combining this with continuity we have \((1, \psi) \in (\mathbb{R} \times Y)^*_\rho \). However, we require is \( \psi \in P_\sigma^* \). That is, we need to argue \( \psi \) is continuous in \( \sigma \). This follows from Lemma 5.2 and the fact \((1, \psi)\) is continuous in the product topology of \( \mathbb{R} \times Y \). \( \square \)

With this result in hand we state a series of corollaries on “no duality gap” and strong duality. They are nearly identical to their corresponding results in the algebraic case. The difference is that duality gap refers to the *topological* duality gap

\[
gap(P)_\sigma = v(P) - v(TD_\sigma)
\]
as opposed to the algebraic duality gap

\[
gap(P) = v(P) - v(AD).
\]

To derive results on the topological duality gap we proceed as follows. We use Theorem 4.5.7 and Corollary 4.5.8, that give \( \text{int}_\rho(\Gamma) = \text{cor}(\Gamma) \) and \( \text{relint}_\rho(\Gamma) \subseteq \text{icr}(\Gamma) \) when \( \text{int}_\rho(\Gamma) \neq \emptyset \) and \( \text{relint}_\rho(\Gamma) \neq \emptyset \) respectively. With this, the arguments of the proofs of algebraic results apply essentially without change. The only substantive differences are that we reason about \( \nu_\rho \) (instead of \( \nu \)) and appeal to Theorem 5.2.5 that equates \( \nu_\sigma \) with \( v(TD_\sigma) \) (instead of Theorem 3.4.6 that equates \( \nu \) with \( v(AD) \)). Throughout, we take \( \Gamma, \mu \) and \( \nu_\rho \) as defined in (5.2.2), (5.2.3) and (5.2.5).
Corollary 5.2.6 (cf. Corollary 3.4.7). If $\nu_\rho$ is finite and $\Gamma = \text{cl}_\rho(\Gamma)$, then $\text{gap}_\sigma(P) = 0$.

Corollary 5.2.7 (cf. Corollary 3.4.8). If $\nu_\rho$ is finite and there exists a point $(\bar{r}, \bar{y})$ in the relative interior of $\Gamma$ with $\bar{y} \preceq \theta_Y$, then $\text{gap}_\sigma(P) = 0$.

Theorem 5.2.8 (cf. Theorem 3.4.9). If there exists $\bar{x} \in \Omega$ such that $G(\bar{x})$ in the interior of $N$ (the negative cone in $Y$) then $\text{gap}_\sigma(P) = 0$.

We also have adjusted results for strong duality:

Corollary 5.2.9 (cf. Corollary 3.4.11). If $\nu_\rho$ is finite, and there exists a point $(\bar{r}, \bar{y})$ in the relative interior of $\Gamma$ with $\bar{y} \preceq \theta_Y$, then there is a $\bar{\psi} \in P^*_Y$, which is a feasible solution to $(TD_\sigma)$, such that

$$\nu_\rho = \inf_{x \in \Omega} f(x) + \langle G(x), \bar{\psi} \rangle$$

Corollary 5.2.10 (cf. Corollary 3.4.12). If $\nu_\rho$ is finite, if $\Gamma$ has a relative interior point $(\bar{r}, \bar{y})$ with $\bar{y} \preceq \theta_Y$, and $\Gamma = \text{lin}(\Gamma)$, then there is an $\hat{x} \in \Omega$ such that $f(\hat{x}) = \mu = \nu$ and $G(\hat{x}) \preceq \theta_Y$. That is we have strong duality.

We re-define the complementary slackness condition in the topological case:

If there exists an $\bar{x} \in \Omega$ with $G(\bar{x}) \preceq \theta_Y$ and $\bar{\psi} \in P^*_Y$ such that $\langle G(\bar{x}), \bar{\psi} \rangle = 0$, then the primal dual pair $(\bar{x}, \bar{\psi})$ satisfy complementary slackness.

Corollary 5.2.11 (cf. Corollary 3.4.14). If $(\bar{x}, \bar{\psi})$ are a primal-dual pair relative to $(P)$ and $(TD_\sigma)$ that satisfy complementary slackness, then strong duality holds.

Figure 5.1 gives a summary of the sufficient conditions for the topological dual.

Remark 5.2.12. In presence of topological vector spaces one can also apply the results of Section 3.6 and break the constraint set $G(x) \preceq_{P_Y} \theta_Y$ into two constraint sets $G(x) \preceq_{P_Y} \theta_Y$ and $H(x) \preceq_{P_Z} \theta_Z$ where $H(x)$ is an affine mapping and $P_Z$ is the singleton $\theta_Z$. The appropriate modification of the hypotheses for Lemma 3.6.4 is to require $G(\bar{x})$ to be an interior point of $N_Y$ and for $\theta_Z$ to be an interior point of $\Upsilon$.

5.3 A brief summary of strong duality

In this chapter and Chapter 3 we gave sufficient conditions for strong duality. It is of course possible for strong duality to occur despite the failure of these sufficient conditions, we really say nothing here about necessity. In this section, as in Section 4.8, we hope to bring into focus precisely the difference between the results of this chapter and Chapter 3. For the purposes of this discussion assume $\mu, \nu$ and $\nu_\sigma$ are finite for all topologies $\sigma$ we
Figure 5.1: A summary of conditions for no duality gap and strong duality in the topological case.

Consider. This allows us to apply Theorems 3.4.6 and 5.2.5 and their corollaries assuming the hypothesis of finiteness is satisfied.

Figure 5.2 gives an intuitive visualization of what may arise. We explore its implications in detail. A first observation is that different topologies yield different Lagrangian duals. This is clear from the definition of \( \text{TD}_\sigma \); different choices of \( \sigma \) lead to different topological duals of the cone \( P \) and hence differently constrained problems. How does the choice of \( \sigma \) affect \( v(\text{TD}_\sigma) \) and the topological duality gap? Note that the primal is “algebraic” in the sense that its value \( \mu \) is unchanging with the topology. Similarly, for the value of the algebraic dual \( v(\text{AD}) \).

Suppose we now consider two topologies \( \sigma \) and \( \sigma' \) where \( \sigma' \) is coarser than \( \sigma \). In view of Lemma 4.8.2 we know that in the coarser topology fewer linear functionals are continuous and thus \( P_{\sigma'}^* \subseteq P_{\sigma}^* \). Again, this implies \( v(\text{TD}_{\sigma'}) \leq v(\text{TD}_\sigma) \). Similarly, again in view of Lemma 4.8.2 we know that \( \text{cl}_{\sigma'}(A) \subseteq \text{cl}_{\sigma}(A) \).

It is also straightforward to see that \( \nu \) (as defined in (3.4.3)) is greater than \( \nu_{\sigma} \) for all linear topologies \( \sigma \), because the algebraic closure \( \text{lin}(A) \) is a subset of every closure (via Theorem 4.5.6). This too is illustrated in Figure 5.2.

Comparing Theorem 3.4.6 and Theorem 5.2.5 we see in the topological case we always have \( \nu_{\sigma} \) equal to the value of the topological Lagrangian dual, whereas in the algebraic case, the result that \( \nu \) equals the value of the algebraic Lagrangian dual requires \( \Gamma \) to have a nonempty intrinsic core. For the remainder of the discussion in this section we assume that \( \Gamma \) does have a nonempty intrinsic core.

From Figure 5.2 we also see visually the possibility of different duality gaps depending
on the topology. If $\Gamma$ is strictly contained in $\text{lin}(\Gamma)$ and $\text{lin}(\Gamma)$ is strictly contained in $\text{cl}_\sigma(\Gamma)$ then we have the situation where the inequalities in (5.2.1) are strict. In particular, in such a setting, phrasing the problem in terms of the topology $\sigma$ *worsens* the duality gap. We would need to trade this off against the structure that this topology adds to the Lagrangian dual $\text{TD}_\sigma$.

Two of our results that yield no duality gap – Corollary 3.4.7 and Corollary 5.2.6 – can also be seen in the intuitive light of Figure 5.2. To ask that $\Gamma = \text{lin}(\Gamma)$ is potentially less stringent then asking for $\Gamma = \text{cl}_\sigma(\Gamma)$, and indeed when $\Gamma = \text{cl}_\sigma(\Gamma)$ the algebraic closure is also squeezed to equal $\Gamma$.

5.4 Notes

To the authors’s knowledge, the main results of Section 5.2 do not appear elsewhere in the literature. The case of normed spaces is treated in Chapter 8 of Luenberger [20]. For the linear case where $f$ and $G$ are both linear, several roughly equivalent statements can be found in Chapter 3 of Anderson and Nash [3]. See also Shapiro [28].
5.5 Exercises

Exercise 5.1. Prove Lemma 5.2.1: Let \((X, \tau)\) and \((Y, \sigma)\) be locally convex topological vector spaces. Let \(\rho\) denote the product topology on \(X \times Y\) derived from \(\tau\) and \(\sigma\). Show that \((X \times Y, \rho)\) is a locally convex topological vector space.

Exercise 5.2. Prove Lemma: Let \((X, \tau)\) and \((Y, \sigma)\) be topological vector spaces. Let \(W = X \times Y\) and let \(\rho\) be the product topology on \(W\). Consider a linear functional \(\phi_W \in W'\),

\[
\phi_W(x, y) = \phi_X(x) + \phi_Y(y)
\]

where \(\phi_X \in X'\) and \(\phi_Y \in Y'\) (In Exercise 3.21 we show all linear functions in \(W'\) have this form.) Then, if any two of \(\phi_W\), \(\phi_X\) and \(\phi_Y\) are continuous then so if the third.

Exercise 5.3. Suppose \(\phi\) is a linear functional in \(\mathbb{R}^n\) where \(\mathbb{R}^n\) is endowed with the standard norm topology. Show that \(\phi\) is continuous.

Exercise 5.4. In Corollary 5.2.10 is it necessary to require that \((f(\bar{x}), G(\bar{x}))\) in the relative interior of \(A\) or can we get by with only the hypothesis that \(A = \text{cl}_\sigma(A)\). Please answer yes or no and justify. m

Exercise 5.5. Prove Lemma 5.2.4. If \(\nu_\sigma\) is finite, then \((\nu_\sigma, \theta_Y) \in \text{cl}_\rho(\Gamma)\).

Exercise 5.6. We have used Theorem 4.7.2 repeatedly in this chapter. Theorem 4.7.2 states that if \(A\) is a closed set and \(x \notin A\), then we can strongly (which implies strictly) separate \(x\) from \(A\). However, what we have really done in this chapter is to say that if \(x \notin \text{cl}_\sigma(A)\) then we can strictly separate \(x\) from \(A\). In other words, we have made the tacit assumption that \(\text{cl}_\sigma(A)\) is closed. Prove that this is true.
Chapter 6

Locally Convex Spaces

*In theory, what remains in these notes is “applied”. – Guess and WTS*

In the vast majority of downstream applications of our theory we restrict ourselves two locally convex spaces. The essential idea, which was evident in Chapter 4, is that one needs to *shrink* the size of the dual space and thus limit the set of linear functionals under consideration. For the purposes of optimization, there is no loss in doing this if we still recover a strong duality result for our optimization problem. This indicates that we have looked at all the linear functionals we need in order to define a meaningful dual problem.

Our approach is to through defining *paired* vector spaces $X$ and $Z$ where $X$ is endowed with the *weak topology* (concepts defined below). In this approach we can essentially specify (under certain conditions) the set of continuous linear functions which we use in the topological dual. We then provide several examples of paired vector spaces and their associated optimizations problems.

### 6.1 Paired vector spaces

Let $X$ and $Z$ be vector spaces. A mapping

$$(\cdot | \cdot) : X \times Z \to \mathbb{R}$$

that satisfies:

(P1) $(\cdot | z)$ is a linear functional on $X$ for all $z \in Z$ and $(x | \cdot)$ is a linear functional on $Z$ for all $x \in X$.

(P2) $(x | z) = 0$ for all $z \in Z$ implies $x = \theta_X$ and $(x | z) = 0$ for all $x \in X$ implies $z = \theta_Z$.

is called a *pairing* of $X$ and $Z$. When such a pairing exists $X$ and $Z$ are called paired vector spaces.
A mapping that satisfies (P1) is called \textit{bilinear}. Property (P2) is termed \textit{non-degeneracy}. Thus, a pairing is a bilinear non-degenerate functional defined on the product of vector spaces $X \times Z$. One important thing that is easy to overlook is that $(x|z) \in \mathbb{R}$ for all $x \in X$ and $z \in Z$ when $(x|z)$ is a pairing. Thus, for instance taking $X = Z = \mathbb{R}^\mathbb{N}$ and $(x|z) = \sum_{i=1}^{\infty} x_i z_i$ is not a pairing since the infinite sum may diverge and thus not lie in $\mathbb{R}$.

**Example 6.1.1.** Here are some examples of paired vector spaces. It is an exercise (Exercise 6.1) to show that they are indeed paired.

(i) $X = Z = \mathbb{R}^n$. We may define the pairing as

$$(x|z) = x^\top z = \sum_{i=1}^{n} x_i z_i$$

where $x = (x_1, \ldots, x_n)$ and $z = (z_1, \ldots, z_n)$.

(ii) $X = \mathbb{R}^\infty$ and $Z = \mathbb{R}^\mathbb{N}$. We define the pairing as

$$(x|z) = \sum_{i=1}^{\infty} x_i z_i$$

where $x = (x_1, x_2, \ldots)$ and $z = (z_1, z_2, \ldots)$.

(iii) $X = \ell^\infty$ and $Z = \ell^1$ where for $x = (x_1, x_2, \ldots)$ and $z = (z_1, z_2, \ldots)$:

$$\ell^\infty = \left\{ x \in \mathbb{R}^\mathbb{N} : \sup_{i \in \mathbb{N}} |x_i| < \infty \right\} \quad (6.1.1)$$

and

$$\ell^1 = \left\{ x \in \mathbb{R}^\mathbb{N} : \sum_{i \in \mathbb{N}} |x_i| < \infty \right\} \quad (6.1.2)$$

with

$$(x|z) = \sum_{i=1}^{\infty} x_i z_i.$$ 

For notational convenience let $|||x||| = \sup_{i \in \mathbb{N}} |x_i|$ and $|||x|||_1 = \sum_{i \in \mathbb{N}} |x_i|$ for $x \in \mathbb{R}^\mathbb{N}$. It can be shown that $|||\cdot|||_\infty$ and $|||\cdot|||_1$ are norms on $\mathbb{R}^\mathbb{N}$ (in fact, Exercise ?? asks you to establish a more general result).

A pairing of $X$ and $Z$ induce a topology on $X$ termed the weak topology:
**Definition.** Suppose $X$ and $Z$ are paired vector spaces with pairing $(\cdot|\cdot)$. Let

$$B = \{B_A : A \subseteq Z \text{ finite} \}$$

where

$$B_A = \{x \in X : -1 < (x|z) < 1, z \in A \}$$

The topology $\sigma(X, Z)$ whose neighborhood base of the origin is $B$ is called the *weak topology* on $X$.

It remains to verify that $\sigma(X, Z)$ is indeed a topology, in fact it is a locally convex linear topology:

**Proposition 6.1.2.** The weak topology $\sigma(X, Z)$ defined above is a locally convex linear topology on $X$.

**Proof.** It suffices to show that $B$ satisfies (NB1)–(NB5) and each of element of $B$ is convex. Let $B_A \in B$. It is straightforward to see that $B_A$ is convex. Take $x, y \in B_A$ and $\lambda \in (0, 1)$ then

$$(\lambda x + (1 - \lambda)y|z) = \lambda(x|z) + (1 - \lambda)(y|z).$$

Since $(x|z) \in (-1, 1)$ and $(y|z) \in (-1, 1)$ we have $\lambda(x|z) + (1 - \lambda)(y|z) \in (-1, 1)$.

To establish (NB1) note that $(\theta_X, z) = 0$ for all $z \in A$ for $\theta_X \in B_A$ for all finite $A$.

Property (NB2) follows from the fact for all finite $A_1, A_2 \subseteq Z$ we have

$$B_{A_1} \cap B_{A_2} = \left\{ x \in X : -1 < (x|z) < 1, z \in A_1 \right\}$$

which itself precisely equal to $B_{A_1 \cup A_2}$.

To establish (NB3) we show that $B_A$ has $\theta_X$ as a core point. Take $x \in X$. We will show there exists a $\lambda > 0$ such that $[\theta, \lambda x] \in B_A$. Since $B_A$ is convex it suffices to show there that there exists a $\lambda$ such that $\lambda x \in B_A$. Now, $x \in B_A$ if $-1 < (x|z) < 1$ for all $z \in A$. Since the pairing is linear, we know $(\lambda x|z) = \lambda(x|z)$ and so for sufficiently small $\lambda$ we have $-1 < \lambda(x|z) < 1$. This choice of $\lambda$ depends on $z$ so we denote it $\lambda_z$. Since there are only finitely many $z \in A$ we may define $\underline{\lambda} = \min \{\lambda_z : z \in A\}$. Then $-1 < \underline{\lambda}(x|z) < 1$ for all $z \in A$. Again, since $\lambda(x|z) = (\lambda x|z)$ we have $\underline{\lambda} x \in B_A$.

(NB4) is immediate, since if $x \in B_A$ then

$$-1 < (x|z) < 1 \text{ for all } z \in A$$

and since $-(x|z) = (-x|z)$ we multiply (6.1.5) through by $-1$ to get

$$-1 < (-x|z) < 1 \text{ for all } z \in A$$
and thus $-x \in B_A$. That is, $B_A$ is circled.

Finally, to establish (NB5) take $B_A \in B$ and define

$$W = \{ x \in X : -1 < (x|2z) < 1, z \in A \}.$$ 

We claim $W + W \subseteq B_A$. Let $x \in W + W$, that is $x = w^1 + w^2$ where $w^1, w^2 \in W$. Thus,

$$(x|z) = (w^1 + w^2|z) = (w^1|z) + (w^2|z) = \frac{1}{2}(w^1|2z) + \frac{1}{2}(w^2|2z)$$

for all $z \in A$. Thus $(x|z) < \frac{1}{2}1 + \frac{1}{2}1 = 1$ and similarly $-1 < (x|z)$. We conclude $x \in B_A$. □

We now present the key result of the section (possibly the chapter) and the main reason we study paired vector spaces. First we need a lemma which is used in the course of the proof of our main result.

**Lemma 6.1.3.** Let $\phi, \phi_1, \phi_2, \ldots, \phi_n$ be linear functionals on $X$. Then

$$\phi \in \text{span}(\phi_1, \ldots, \phi_n) \iff \bigcap_{i=1}^{n} \ker \phi_i \subseteq \ker \phi$$

where $\ker \phi$ denotes the *kernel* of the functional $\phi$ which consists of those elements $x \in X$ which $\phi$ sends to 0. That is,

$$\ker \phi = \{ x \in X : \phi(x) = 0 \}$$

**Proof.** $(\Rightarrow)$ If $\phi \in \text{span}(\phi_1, \ldots, \phi_n)$ then $\phi = \sum_{i=1}^{n} \alpha_i \phi_i$ where $\alpha_i \in \mathbb{R}$. Suppose $x \in \bigcap_{i=1}^{n} \ker \phi_i$, i.e. $\phi_i(x) = 0$ for $i = 1, 2, \ldots$. This implies:

$$\phi(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x) = \sum_{i=1}^{n} \alpha_i 0 = 0$$

which implies $x \in \ker \phi$.

$(\Leftarrow)$ Conversely, suppose $\bigcap_{i=1}^{n} \ker \phi_i \subseteq \ker \phi$. We define the useful function $T : X \to \mathbb{R}^n$ where $T(x) = (\phi_1(x), \ldots, \phi_n(x))$. Since the $\phi_i$ are linear it is immediate that $T$ is a linear map.

We further define a mapping $\varphi : T(X) \to \mathbb{R}$ where $\varphi(\phi_1(x), \ldots, \phi_n(x)) = \phi(x)$. We show that this map is well-defined, i.e. if $T(x) = T(y)$ then $\varphi(T(x)) = \varphi(T(y))$. This requires that $\phi(x) = \phi(y)$ whenever $(\phi_1(x), \ldots, \phi_n(x)) = (\phi_1(y), \ldots, \phi_n(x))$. This follows from the fact $\bigcap_{i=1}^{n} \ker \phi_i \subseteq \ker \phi$. Indeed, $\phi_i(x) = \phi_i(y)$ for all $i = 1, \ldots, n$ implies $x - y \in \bigcap_{i=1}^{n} \ker \phi_i$. Thus, $x - y \in \ker \phi$ and so $\phi(x) = \phi(y)$, as required.
6.1. PAIRED VECTOR SPACES

It is straightforward to observe that $\varphi$ can be extended to be defined on all of $\mathbb{R}^n$ instead of $T(X)$ and remain a linear functional. Note that $\varphi$ is linear on $T(X)$ due to the linearity of the $\phi_i$ and $\varphi$. Note also that $T(X)$ is a subspace of $\mathbb{R}^n$, it the linear projection of the vector space $X$ into $\mathbb{R}^n$. Any basis of $T(X)$ can thus be extended to a basis of $\mathbb{R}^n$ and we may define $\varphi$ arbitrarily on the basis elements outside of the basis of $T(X)$ and extend linearly. This defines a linear functional.

Why is all of this useful? Since $\varphi$ is a linear functional we may write

$$\varphi(\phi_1(x), \ldots, \phi_n(x)) = \varphi \left( \sum_{i=1}^{n} \phi_i(x) e^i \right)$$

$$= \sum_{i=1}^{n} \phi_i(x) \varphi(e^i)$$

$$= \sum_{i=1}^{n} \alpha_i \phi_i(x)$$

where $\alpha_i = \phi_i(e^i)$ and $e^i$ is the $i$th standard basis element of $\mathbb{R}^n$. Since $\varphi(\phi_1(x), \ldots, \phi_n(x)) = \phi(x)$ by definition this implies $\phi = \sum_{i=1}^{n} \alpha_i \phi_i$. That is, $\phi \in \text{span}(\phi_1, \ldots, \phi_n)$.

**Theorem 6.1.4.** Suppose $X$ and $Z$ are paired vector spaces and let $\sigma(X, Z)$ denote the weak topology on $X$. Then, $X^*_{\sigma(X,Z)} \cong Z$. In other words, every linear functional of the form $\phi_z = \langle \cdot | z \rangle$ is continuous in $\sigma(X, Z)$ and every continuous linear functional $\phi$ is of the form $\langle \cdot | z \rangle$ for some $z \in Z$.

**Proof.** Define the mapping $J : Z \to X^*_{\sigma(X,Z)}$ where $J(z) = \phi_z$ where $\phi_z(x) = \langle x | z \rangle$. We want to show that $J$ is an isomorphism of vector spaces.

The fact that $J$ is a homomorphism of vector spaces follows directly from the bilinearity of $\langle \cdot | \cdot \rangle$. It remains to show that $J$ is a well-defined bijection. There are three steps involved:

(i) Show $J$ is well-defined. That is, $\phi_z$ is indeed $\sigma(X, Z)$ continuous (and thus in $X^*_{\sigma(X,Z)}$) for all $z \in Z$.

(ii) Show that $J$ is one-to-one, i.e. $J(z) = J(w)$ implies $z = w$.

(iii) Show that $J$ is onto, i.e. for all $\phi \in X^*_{\sigma(X,Z)}$ there exists a $z \in Z$ such that $\phi_z = \phi$.

We establish each step in turn.

(i) Let $z \in Z$ and we claim that $J(z) = \phi_z = \langle \cdot | z \rangle$ is continuous. From Proposition 4.4.1 if suffices to show that $\phi_z$ is continuous at the origin. Let $\epsilon > 0$. Our goal is to show that there exists a neighborhood $B$ of $0$ such that $\phi_z(B) \subseteq (-\epsilon, \epsilon)$. Note that

$$B' = \{ x \in X : -1 < \langle x | z \rangle < 1 \}$$
is a neighborhood of \( \theta \). Note that if \( x \in B' \) then \( \phi_z(x) \in (-1, 1) \). Thus we simply need to scale \( B \) so that \( \phi_z(x) \in (-\epsilon, \epsilon) \). By the Invariance Lemma, \( B = \epsilon B' \) is also a neighborhood of \( \theta \). Let \( x \in B \), then \( x = \epsilon x' \) for some \( x' \in B \). Thus

\[
\phi_z(x) = \phi_z(\epsilon x') = \epsilon \phi_z(x') \subseteq \epsilon(-1, 1) = (-\epsilon, \epsilon).
\]

This implies \( \phi_z \) is continuous and thus \( J \) is well-defined.

(ii) The fact \( J \) is one-to-one follows from the fact \( (\cdot|\cdot) \) is non-degenerate. Indeed \( J(z) = J(w) \) precisely if \( (x|z) = (x|w) \) for all \( x \in W \). Rearranging and using linearity this amounts to \( (x|z - y) = 0 \) for all \( x \in X \). Since the pairing is non-degenerate this immediately implies that \( z - y = 0 \). In other words, \( x = y \) and hence \( J \) is one-to-one.

(iii) We will use Lemma 6.1.3 in the course of this part of the proof. The task is to show that for every element \( \phi \in X_{\sigma(X,Z)}^* \) there exists a \( z \in Z \) such that \( \phi(x) = (x|z) \).

The proof idea is to show that there exists \( \phi_i = (\cdot|z_i) \) for \( z_i \in Z \) \( i = 1, 2, \ldots \) so that \( \phi = \sum_{i=1}^n \lambda_i z_i \). In this case \( \phi = (\cdot|\sum_{i=1}^n \lambda_i z_i) \) and since \( \sum_{i=1}^n \lambda_i z_i \in Z \) this implied \( \phi \in J(Z) \), as needed. We use the equivalence in Lemma 6.1.3 to establish the fact.

Here is the approach, since \( \phi \) is continuous (since its in \( X_{\sigma(X,Z)}^* \)) we know it bounded on some neighborhood of the origin. That is, there exists an \( M \) such that \( |\phi(x)| < M \) for all \( x \in B \). We are free to choose \( B \) to be a basis element of the weak topology so

\[
B = \{ x \in X : -1 < (x|z^i) < 1 \text{ for } i = 1, \ldots, n \}.
\]

Let \( \phi_i = (\cdot|z^i) \) for \( i = 1, \ldots, m \). Now choose \( \epsilon > 0 \). Consider the set \( \frac{\epsilon}{M} B \). Note that for \( y \in \frac{\epsilon}{M} B \) we have

\[
|\phi(y)| = |\phi(\frac{\epsilon}{M} x)|
\]

for some \( x \in B \). This allows us to conclude

\[
|\phi(y)| = \frac{\epsilon}{M} |\phi(x)| < \frac{\epsilon}{M} M = \epsilon.
\]

Denote by \( L = \cap_{i=1}^n \ker \phi_i \). Observe that \( L \subseteq \frac{\epsilon}{M} B \) for all \( \epsilon > 0 \) and so \( |\phi(x)| < \epsilon \) for all \( x \in L \) and for all \( \epsilon > 0 \). Thus we may conclude that \( \phi(x) = 0 \) for all \( x \in L \). That is, \( \cap_{i=1}^n \ker \phi_i \subseteq \ker \phi \). So by Lemma 6.1.3 we have \( \phi = \sum_{i=1}^n \alpha_i \phi_i \), as required. \( \Box \)

### 6.2 Linear programs with paired spaces

In this section we turn to inserting paired spaces into our general linear programming framework. Recall the general primal and dual LPs:

\[
\begin{align*}
\text{inf} \quad & \langle x, \phi \rangle_X \\
\text{s.t.} \quad & b - A(x) \preceq_Y \theta_Y
\end{align*}
\]

(GLP)
and

$$\sup \langle b, \psi \rangle$$
\[\text{s.t. } \phi - A'(\psi) = \theta_{X'}, \quad \psi \in P_Y^+. \tag{GLPD} \]

The general problem schematic is

$$\text{GLP : } \begin{bmatrix} X \\ X' \end{bmatrix} \rightarrow \begin{bmatrix} Y \\ Y' \end{bmatrix} : \text{GLPD}$$

where $X'$ and $Y'$ are the algebraic duals of $X$ and $Y$ respectively. Now assume $X$ and $Y$ are paired with vector spaces $Z$ and $W$ respectively. That is, there exists pairings $(\cdot|\cdot)_X : X \times Z \to \mathbb{R}$ and $(\cdot|\cdot)_Y : Y \times W \to \mathbb{R}$.

Theorem 6.1.4 established that $Z$ can be embedded into a subspace of $X'$. Similarly, $W$ is embedded as a subspace of $Y'$. We would like to replace $X'$ and $Y'$ in the problem schematic by $Z$ and $W$ respectively, thus re-phrasing the problem away from the language of linear functionals. This involves a little work.

We make the following assumptions:

(i) The primal objection linear functional $\phi$ is continuous in the weak topology $\sigma(X, Z)$.

Thus by Theorem 6.1.4, $\phi(x) = (x|z)$ for some $z \in Z$.

(ii) The dual decision variable $\psi$ is continuous in the weak topology $\sigma(Y, W)$. Again by Theorem 6.1.4 this means $\psi(y) = (y|w)$ for some $w \in W$.

(iii) The linear mapping $A$ is $\sigma(X, Z)$-$\sigma(Y, W)$-continuous.

The significance of this last assumption is apparent from the following result.

**Theorem 6.2.1.** Let $X$ and $Y$ be vector spaces paired with vector spaces $Z$ and $W$ respectively. Let $A : X \to Y$ be a linear mapping. Let $A' : Y' \to X'$ denote its (algebraic) adjoint. Then $A$ is $\sigma(X, Z)$-$\sigma(Y, W)$-continuous if and only if $A'$ maps $\sigma(Y, W)$-continuous linear functionals to $\sigma(X, Z)$-continuous linear functionals.

**Proof.** See Exercise 6.2. \qed

Another way to state this result is that when $A$ is continuous then $A'(Y^*_{\sigma(Y, W)}) \subseteq X^*_{\sigma(X, Z)}$.

**Definition.** Let $X$ and $Y$ be vector spaces paired with vector spaces $Z$ and $W$ respectively. When $A$ is $\sigma(X, Z)$-$\sigma(Y, W)$-continuous we may define the *paired adjoint* of $A$ as follows:

$$A^* : W \to Z$$

which is defined as follows: $A^*(w)$ maps to that $z \in Z$ whereby $A'(\psi_w) = \phi_z$. In other words, $A^*(w) = z \in Z$ such that $A'(\psi_w)(x) = (x|z)$. 
Note that if $A$ is not continuous then $A^\ast$ is not guaranteed to be well-defined. The reason is that $A'(\psi_w)$ could in fact be discontinuous and hence not equal to $\phi_z$ for some $z$, which is necessarily a continuous linear functional. Note also that this definition implicitly uses Theorem 6.1.4 since we need the fact an isomorphism exists between continuous linear functionals and elements of $W$ and $Z$. See Figure 6.1 for a visualization and further explanation.

Figure 6.1: The paired adjoint $A^\ast$. The mapping $J_W : W \to Y'$ and $J_Z : Z \to X'$ are as defined in Theorem 6.1.4. That is, $J_W(w) = \psi_w = (\cdot | w)$. We require that $A$ is continuous so that $A'(\psi_w)$ is a continuous linear functional and thus maps under $J_Z^{-1}$ to an element of $Z$.

Using the above results and definitions, we can specialize our general linear programs to the case of paired spaces, which we call the “paired” linear program:

$$\begin{align*}
\inf & \ (x|c)_X \\
\text{s.t.} & \ b - A(x) \preceq_P \theta_Y \quad \text{(PLP)}
\end{align*}$$

and

$$\begin{align*}
\sup & \ (b|w)_Y \\
\text{s.t.} & \ c - A^\ast(w) = \theta_Z \quad \text{(PLPD)}
\end{align*}$$

where $x \in X$, $c \in Z$, $b \in Y$ and $w \in W$. We may define $P^\ast$ as follows:

$$P^\ast = \{ w \in W : (y|w) \geq 0 \ \text{for all} \ y \in P \} \quad \text{(6.2.1)}$$

where again $P$ is the positive cone in $Y$. Problem (PLP) has the following schematic:

$$\begin{align*}
\text{PLP} : \begin{bmatrix} \text{X} \\ \text{Z} \end{bmatrix} & \to \begin{bmatrix} \text{Y} \\ \text{W} \end{bmatrix} : \text{PLPD}
\end{align*}$$
In the next section we describe a specific instance of (PLP) and derive its dual.

6.3 Countably Infinite Linear Programming

Let \( X = \ell^\infty \) and \( Z = \ell^1 \) be paired according the pairing defined in Example 6.1.1(iii). We similarly the spaces \( Y = \ell^\infty \) and \( W = \ell^1 \). Our goal is to describe a linear programming problem set in these spaces. The pairings were described earlier in Example 6.1.1(iii). It remains to specify the continuous linear mapping \( A \) and the cones \( P \) and \( P^* \).

We start by defining the linear mapping \( A : \ell^\infty \to \ell^\infty \):

\[
A(x) = \left( \sum_{j=1}^{\infty} a_{ij} x_j : i = 1, 2, \ldots \right). \tag{6.3.1}
\]

The linear map \( A \) is determined by a doubly infinite matrix \((a_{ij} : i, j = 1, 2, \ldots)\), which we also denote by \( A \). Note that \( A(x) \) is a vector in \( \mathbb{R}^\infty \) (that is, a real sequence). In fact, it is not guaranteed that \( A(x) \in \ell^\infty \). This is assured by assuming

\[
\sup \left\{ \sum_{j=1}^{\infty} |a_{ij}| : i = 1, 2, \ldots \right\} = M_1 < \infty. \tag{6.3.2}
\]

This implies, in particular, that each row \( a_i = (a_{ij} : j = 1, 2, \ldots) \) of \( A \) is in \( \ell^1 \) (that is, for all \( i \), \( \sum_{j=1}^{\infty} |a_{ij}| \in \ell^1 \)). Additionally, the sequence of row sums lies in \( \ell^\infty \) with the bound \( M_1 \).

**Proposition 6.3.1.** Let \( A : \ell^\infty \to \ell^\infty \) be a linear mapping where \( A(x) \) is defined by (6.3.1) and satisfies the condition (6.3.2). Then \( A \) is well-defined and \( \sigma(\ell^\infty, \ell^1) \)-\( \sigma(\ell^\infty, \ell^1) \)-continuous.

**Proof.** First we show \( A(x) \in \ell^\infty \) for all \( x \in \ell^\infty \), thus establishing \( A(x) \) is well-defined. Fix \( x \in \ell^\infty \). There exists a real number \( M(x) > 0 \) such that \( \sup \{|x_j| : j = 1, 2, \ldots\} < M(x) \). Let \( y = A(x) \) with \( y_i = \sum_{j=1}^{\infty} a_{ij} x_j \). To show \( A(x) \in \ell^\infty \) we require

\[
\sup \{|y_i| : i = 1, 2, \ldots\} < \infty.
\]

Note that we may write

\[
y_i = \sum_{j=1}^{\infty} a_{ij} x_j = (a_i|x) < \infty
\]

where \((\cdot|\cdot)\) is the pairing of \( \ell^\infty \) and \( \ell^1 \). This is possible because \( a_i \in \ell^1 \).
Now,

\[ |y_i| = |(a_i|x)| = \left| \sum_{j=1}^{\infty} a_{ij} x_j \right| \leq \sum_{j=1}^{\infty} |a_{ij} x_j| = \sum_{j=1}^{\infty} |a_{ij}| |x|. \]

and since \( \sup \{|x_j| : j = 1, 2, \ldots \} < M(x) \) we may write

\[ |y_i| < M(x) \sum_{j=1}^{\infty} |a_{ij}|. \]

By (6.3.2) \( \sum_{j=1}^{\infty} |a_{ij}| < M_1 \) and

\[ |y_i| < M(x) M_1. \]

The bound \( M(x) M_1 \) is uniform (works for all \( i = 1, 2, \ldots \)) and so \( y \in \ell^\infty \).

Next we establish \( A \) is continuous by showing the adjoint \( A' \) maps continuous linear functionals to continuous linear functionals (and then applying Theorem 6.2.1). Recall, the set of continuous linear functionals on \( \ell^\infty \) is isomorphic to \( \ell^1 \) (Theorem 6.1.4). Thus, \( \psi \) continuous implies \( \psi = (|\cdot| w) \) for some \( w \in \ell^1 \). Thus,

\[ A'(\psi)(x) = \psi \circ A(x) = \psi(A(x)) = (A(x)|w). \]

Our goal is to show that \( A'(\psi) \) is a continuous linear functionals on \( X = \ell^\infty \). Again, since the set of continuous linear functionals on \( \ell^\infty \) is isomorphic to \( \ell^1 \), it suffices to show that \( A'(\psi) \) arises as a linear functional of the form \( (|\cdot| z) \) for some \( z \in \ell^1 \). Letting \( x \in \ell^\infty \) and \( A(x) = y \) with \( y_i = \sum_{j=1}^{\infty} a_{ij} x_j \) we may express

\[ (A(x)|w) = \sum_{i=1}^{\infty} y_i w_i \]

\[ = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} x_j \right) w_i \]

\[ = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} w_i x_j. \quad (6.3.3) \]

Note that the quantity \( (A(x)|w) \) is finite since \( A(x) \in \ell^\infty \) (established above) and \( w \in \ell^1 \) by assumption. So we can apply Fubini’s Theorem to change the order of summation in
(6.3.3) which yields

\[(A(x)|w) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} w_i x_j = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij} w_i \right) x_j = \sum_{j=1}^{\infty} z_j x_j \]

where \(z = (z_j : j = 1, 2, \ldots)\) with \(z_j = \sum_{i=1}^{\infty} a_{ij} w_i\).

Finally we show \(z \in \ell^1\):

\[
\sum_{j=1}^{\infty} |z_j| = \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} a_{ij} w_i \right| \\
\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{ij} w_i| \\
= \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} |a_{ij}| \right) |w_i| \\
\leq M_1 \sum_{i=1}^{\infty} |w_i| < \infty
\]

since \(\sum_{j=1}^{\infty} |a_{ij}| \leq M_1\) by (6.3.2) and \(w \in \ell^1\). We again used Fubini’s theorem to interchange the order of summation (which is justified since the sum is finite).

Thus we conclude \((A(x)|w) = (x|z)\) where \(z \in \ell^1\), and so \(A'(\psi)\) is a continuous linear functional on \(X\).

In the course of the proof we established the existence of the paired adjoint \(A^* : \ell^1 \to \ell^1\) where

\[
A^*(w) = \left( \sum_{j=1}^{\infty} a_{ij} w_i : i = 1, 2, \ldots \right).
\]

(6.3.4)

Thus, \(A^*\) can be seen as the doubly infinite “transpose” of \(A\).

We now turn to defining the cones \(P\) and \(P^*\). We simplest cone to work with is

\[
P = \left\{ y = (y_1, y_2, \ldots) \in \ell^\infty : y_i \geq 0 \text{ for all } i = 1, 2, \ldots \right\}.
\]
The expression $y \geq P \theta_Y$ is routinely shortened to $y \geq 0$.

Applying the definition of $P^*$ in (6.2.1):

$$P^* = \left\{ w \in \ell^1 : \sum_{i=1}^{\infty} y_i w_i \geq 0 \text{ for all } y \in P \right\}.$$ 

Noting $e^i \in P$ for all $i \in \{1, 2, \ldots\}$, if $w_j < 0$ for some $j$ then $\sum_{i=1}^{\infty} e_i w_i = w_j < 0$. Thus,

$$P^* \subseteq \{ w \in \ell_1 : w_i \geq 0 \text{ for all } i = 1, 2, \ldots \}.$$ 

The opposite containment is trivial, since in this case every term in the sum $\sum_{i=1}^{\infty} y_i w_i$ is nonnegative and so the sum is nonnegative. Thus,

$$P^* = \{ w \in \ell_1 : w_i \geq 0 \text{ for all } i = 1, 2, \ldots \}.$$ 

The expression $w \in P^*$ is routinely shortened to $w \geq 0$.

We can now specialize (PLP) and (PLPD) to the countably infinite setting:

$$\inf \sum_{j=1}^{\infty} c_j x_j$$

s.t. \begin{align*}
\sum_{j=1}^{\infty} a_{ij} x_j &\geq b_i & \text{for } i = 1, 2, \ldots \\
x &\in \ell^\infty
\end{align*}

and

$$\sup \sum_{i=1}^{\infty} b_i w_i$$

s.t. \begin{align*}
\sum_{i=1}^{\infty} a_{ij} w_i & = c_j & \text{for } j = 1, 2, \ldots \\
w &\in \ell^1 \\
w &\geq 0
\end{align*}

where $c \in \ell^1$ and $b \in \ell^\infty$. Problem (CILP) has the following schematic:

$$\text{CILP : } \begin{bmatrix} \ell^\infty \\ \ell^1 \end{bmatrix} \rightarrow \begin{bmatrix} \ell^\infty \\ \ell^1 \end{bmatrix} : \text{CILPD}$$

We may apply all results of Chapter 5 to this setting to ascertain whether there is a duality gap or if strong duality holds.
We consider a Markov decision process (MDP) where the state space is countable and time is discrete and infinite.\footnote{For an introduction to MDP’s see Puterman [24].} Let the index $t$ denote the time with $t \in \{1, 2, \ldots\}$. Let $S = \{1, 2, \ldots\}$ denote the state space and $A$ a finite set of actions available in each state. Rewards are assumed to be stationary and uniformly bounded: $r_t(s, a) = r(s, a)$ and $|r(s, a)| < M$ for all $t \in \{1, 2, \ldots\}$, $s \in S$ and $a \in A$. The transition probabilities are also stationary: $p_t(j|s, a) = p_j(s, a)$ for all $t \in \{1, 2, \ldots\}$, $j, s \in S$ and $a \in A$. We assume future rewards are discounted at rate $\lambda$ per time period.

The decision-maker must choose an action for each state. Let $d : S \rightarrow A$ denote a mapping of states to actions. The value $d(s)$ is the action undertaken in state $s$. We restrict attention to deterministic and stationary decision rules (see Puterman [24] for a definitions and discussion).

Let $v(s, d)$ denote the expected value of following decision rule $d$ starting in state $s$. The goal of the decision maker who faces an MDP is to choose an optimal decision rule $d^*$ to maximize $v(s, d)$. A key result in this area is that the optimal value $v(s)$ for state $s \in S$ satisfies the following “optimality equation”:

\[
    v(s) = \max_{a \in A} \left\{ r(s, a) + \lambda \sum_{j \in S} p(j|s, a)v(j) \right\}.
\]

(6.4.1)

The optimal decision rule can be determined after solving for $v(s)$. It can also be shown (see for instance Puterman [24]) that finding a solution to the optimality equation is equivalent to solving for $v = (v(s) : s \in S)$ in the LP:

\[
    \begin{align*}
    \inf \sum_{s \in S} \alpha(s)v(s) \\
    \text{s.t. } v(s) - \lambda \sum_{j \in S} p(j|s, a)v(s) & \geq r(s, a) \quad \text{for } s \in S, a \in A \\
    v & \in \mathbb{R}^S
    \end{align*}
\]

(6.4.2)

where $\sum_{s \in S} \alpha(s) = 1$ and $\alpha(s) > 0$.\footnote{One possible choice for $\alpha$ is $\alpha(s) = \frac{\delta^{s-1}}{1 - \delta}$ with $0 < \delta < 1$.}

A couple things to note about this formulation. We have assumed that the rewards are uniformly bounded by $M$ and thus $r \in \ell^\infty$. How can we consider $r$ be a real-valued sequence? The set $S \times A$ is countable, so we may simply use the bijection of $S \times A$ into $\mathbb{N}$ to index the elements of $r$ accordingly. It is also clear that $\alpha \in \ell^1$. It is natural then, to take $v \in \ell^\infty$. This guarantees that the objective function is finite valued.
Next we will show that our linear mapping \( E : \mathbb{R}^S \to \mathbb{R}^{S \times A} \), which is implicitly defined (6.4.2), satisfies (6.3.2). The entries of the matrix \( E \) associated with our linear mapping are \( E(s,a),j = 1 - \lambda p(j|s,a) \) if \( s = j \) and \( E(s,a),j = -\lambda p(j|s,a) \) if \( s \neq j \). Thus, for \( (s,a) \in S \times A \)

\[
\sum_{j \in S} E(s,a),j = 1 - \lambda \sum_{j \in S} p(j|s,a) = 1 - \lambda
\]

since \( \sum_{j \in S} p(j|s,a) = 1 \). Thus

\[
\sup \left\{ \sum_{j \in S} |E(s,a),j| : i = 1, 2, \ldots \right\} = 1 - \lambda.
\]

Our problem fits the framework of the previous section and we may express the primal as:

\[
\inf \sum_{s \in S} \alpha(s)v(s)
\]

s.t. \( v(s) - \lambda \sum_{j \in S} p(j|s,a)v(s) \geq r(s,a) \) for \( s \in S, a \in A \) (MDP-LP)

\[
v \in \ell^\infty.
\]

To state the paired dual of (MDP-LP) we need to determine the form of \( E^* \). Let \( w \in \ell^1 \), a generate dual variable to the paired dual which is indexed as \( w(s,a) \) for \( s \in S \) and \( a \in A \). Using (6.3.4) we can express the paired adjoint as:

\[
E^*(w)_j = \sum_{s \in S} \sum_{a \in A} E(s,a),j w(s,a)
\]

\[
= \sum_{a \in A} w(j,a) - \lambda \sum_{s \in S} \sum_{a \in A} p(j|s,a)w(s,a).
\]

The paired dual of (MDP-LP) is thus:

\[
\sup \sum_{i=1}^\infty r(s,a)w(s,a)
\]

s.t. \( \sum_{a \in A} w(j,a) - \lambda \sum_{s \in S} \sum_{a \in A} p(j|s,a)w(s,a) = \alpha(j) \) for \( j \in S \) (MDP-LPD)

\[
w \in \ell^1
\]

\[
w(s,a) \geq 0.
\]

The dual variables of (MDP-LPD) have a natural interpretation in terms of an optimal decision rule for the MDP. See Puterman [24] for further discussion.

Our problem fits about primal-dual framework and so we can leverage all the results of Chapter 5 in this setting: weak duality, complementary slackness, no duality results and strong duality. It turns out special conditions can be put on this problem to guarantee no duality gap analytically. See Chapter 4 of Ghate [13] for further details.
6.5 Notes

Our treatment in Sections 6.1 and 6.2 follows the treatments in Aliprantis and Border [2] and Barvinok [4]. Sections 6.3 and 6.4 closely follow the treatment found in Ghate [13].

6.6 Exercises

Exercise 6.1. Show that the examples in Example 6.1.1 are indeed paired vector spaces.

Exercise 6.2. Prove Theorem 6.2.1: Let $X$ and $Y$ be vector spaces paired with vector spaces $Z$ and $W$ respectively. Let $A : X \to Y$ be a linear mapping. Show that $A$ is $\sigma(X,Z) - \sigma(Y,W)$-continuous if and only if its adjoint $A'$ maps $\sigma(Y,W)$-continuous linear functionals to $\sigma(X,Z)$-continuous linear functionals.
Chapter 7

Semidefinite Programming

7.1 Motivation

The traveling salesperson problem (TSP) is an extremely important and well-known problem. The TSP was described in Chapter 1 and the following formulation was given where \( x_{ij} = 1 \) if city \( i \) is the \( j \)th city in the tour, and 0 if not.

\[
\begin{align*}
\min & \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n-1} c_{ik} x_{ij} x_{k,j+1} \\
\sum_{i=1}^{n} x_{ij} &= 1, \quad j = 1, \ldots, n \quad (7.1.1) \\
\sum_{j=1}^{n} x_{ij} &= 1, \quad i = 1, \ldots, n \quad (7.1.2) \\
x_{ij} &\in \{0,1\}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n \quad (7.1.3)
\end{align*}
\]

This is a difficult model to solve. Not only is it an integer program, but the objective function is quadratic and nonconvex. Many important applications, including numerous combinatorial optimization problems, can be formulated as quadratic integer programs. Several applications are discussed later in the chapter. Quadratic integer programs are an important problem class. Branch and bound is the most common approach for solving problems with integer variables. At each node in the tree a relaxation of the original problem is solved. Ideally the problem should be easy to solve, while at the same time provide good lower. One approach to generating problem relaxation is to reformulate the problem by introducing a new variable \( y_{ik} \) for each integer product term \( x_{ij} x_{k,j+1} \). Then in the objective function, each \( x_{ij} x_{k,j+1} \) is replaced by \( y_{ik} \) and the following constraint is added for each \( i, j, \) and \( k \).

\[
x_{ij} + x_{k,j+1} \leq 1 - y_{ik}, \quad \forall i, j, k
\]

181
The resulting model is a linear integer program. However, it generates very poor relaxation (lower bounds). Another alternative is to generate lower bounds using semidefinite programming reformulations. Semidefinite programming (SDP) has been used extensively to generate good lower bounds for combinatorial optimization and quadratic integer programs. There are also numerous other applications of SDP. See the excellent survey paper by Vandenberghe and Boyd [30].

In Section 7.2 the primal semidefinite programming problem (SDP) is defined. It is an example of the general linear program defined in Chapter 3. The adjoint map is defined and the dual problem is derived. The vector spaces used in this chapter are isomorphic to their dual spaces and hence the dual of the dual is the primal back again. This is shown rigorously in Section 7.3. The main application, the 0/1 quadratic programming problem, is the topic of Section 7.4. Relaxations to this problem based on the semidefinite primal (SDP) are given. In Section 7.5 the Lagrangian dual for the 0/1 quadratic problem is given. The bound generated by the Lagrangian dual is shown to equal the optimal value of the (SDP) dual problem. A summary of the lower bound relationships is given in Section 7.6. In Section 7.7 semidefinite programming is used to reformulate non convex quadratic programs into convex quadratic programs. Several important combinatorial optimization models are introduced in Section 7.8 and then semidefinite programming relaxations are given.

7.2 The Semidefinite Programming Primal and Dual

Semidefinite programming is based on the vector space $X = \mathcal{S}^n$, where $\mathcal{S}^n$ is the set of $n \times n$ symmetric matrices. It is straightforward to show that under matrix addition and scalar multiplication, $\mathcal{S}^n$ satisfies the vector space axioms given in Section 2.1.

Define $E_{ij}$ to be the $n \times n$ symmetric where elements $(i, j)$ and $(j, i)$ are both one, and all other elements are zero. By definition, $E_{ij} = E_{ji}$. If $M \in \mathcal{S}^n$ with $M = (a_{ij})$, then

$$M = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} E_{ij}$$  \hspace{1cm} (7.2.1)

Since $M$ is an arbitrary symmetric matrix, (7.2.1) implies $\mathcal{S}^n$ is a finite dimensional vector space. In this vector space, the $E_{ij}$ for $i = 1, \ldots, n$ and $j = i, \ldots, n$ are linearly independent. It follows that the dimension of $\mathcal{S}^n$ is $n(n+1)/2$ and the matrices $E_{ij}$ for $i = 1, \ldots, n$, $j = i, \ldots, n$ define a Hamel basis for $\mathcal{S}^n$. Because the dimension of $\mathcal{S}^n$ is $n(n+1)/2$, and therefore finite, $\mathcal{S}^n$ is isomorphic to $\mathbb{R}^{n(n+1)/2}$ and is self dual:

$$\mathcal{S}^n \cong \mathbb{R}^{n(n+1)/2}$$  \hspace{1cm} (7.2.2)

$$\mathcal{S}^n \cong (\mathcal{S}^n)'$$  \hspace{1cm} (7.2.3)
Given that $S^n$ is self-dual, and there is a one-to-one correspondence between symmetric matrices and the linear functionals in $(S^n)'$, it is of interest to see exactly how each symmetric matrix maps to a linear functional. Given the representation in (7.2.1) of an arbitrary symmetric matrix $M$ in terms of its Hamel basis, if $\phi \in (S^n)'$, then

$$\langle M, \phi \rangle_{S^n} = \phi(M) = \phi\left(\sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} E_{ij}\right)$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} \phi(a_{ij} E_{ij})$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} \phi(E_{ij})$$

Taking $z_{ji} = z_{ij} = \frac{1}{2} \phi(E_{ij})$ when $i \neq j$ and $z_{ii} = \phi(E_{ij})$ defines an element $Z = (z_{ij})$ in $S^n$. Then

$$\phi(M) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \phi(E_{ij})$$

$$= \sum_{i=1}^{n} a_{ii} \phi(E_{ii}) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{2} a_{ij} \phi(E_{ij}) + \sum_{j=1}^{n-1} \sum_{i=1}^{j} \frac{1}{2} a_{ji} \phi(E_{ji})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_{ij}$$

Thus for each $Z \in S^n$, the map $Z \mapsto \phi_Z$ is defined by

$$\phi_Z(M) = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} z_{ij} \quad (7.2.4)$$

The summation given in (7.2.4) is the sumproduct of the matrices $Z$ and $M$. Users of Excel may well be familiar with function. In matrix algebra this is also the trace function. See Appendix A.2.4 for more information on the trace function. The trace function is denoted by $\text{Tr}$ and it is common to write

$$\phi_Z(M) = \text{Tr}(ZM) = \text{Tr}(MZ) = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} z_{ij} \quad (7.2.5)$$

In equation (7.2.5) it is important to understand that $Z$ is not a linear functional, it is a symmetric matrix. However, since $(S^n)' \cong S^n$, given a linear functional $\phi \in (S^n)'$, there is a corresponding symmetric matrix $Z \in S^n$, such that given any symmetric matrix $M$, $\text{Tr}(MZ)$ is equal to the real number $\langle M, \phi \rangle$. 


We have a vector space $S^n$ and a nice characterization of the linear functionals that define the dual vector space. The next step required to define an optimization problem over the vector space $S^n$ is to make $S^n$ an ordered vector space. Based on Theorem 2.6.3, this requires finding a pointed, convex cone in $S^n$ to be the positive cone that defines $\preceq$. A natural candidate for the positive cone is set of positive semidefinite matrices $S^n_+$. The set of positive semidefinite matrices give us exactly what we need and define a pointed, convex cone.

**Lemma 7.2.1.** The set of positive semidefinite matrices $S^n_+$ is a pointed, convex cone.

**Proof.** First show $S^n_+$ is a cone. Let $M$ be an arbitrary element of $S^n_+$. Show $\lambda M \in S^n_+$ for all $\lambda \geq 0$. If $M \in S$, then for all $x \in \mathbb{R}^n$, $x^\top M x \geq 0$ and this implies for nonnegative $\lambda$

$$x^\top \lambda M x = \lambda x^\top M x \geq 0$$

and this implies $\lambda M \in S^n_+$ and that $S^n_+$ is a cone. If $M_1$ and $M_2$ are elements of $S^n_+$ a similar argument shows $\lambda M_1 + (1 - \lambda)M_2$ is in $S^n_+$ for $\lambda \in (0, 1)$ so $S^n_+$ is a convex cone.

Next show that $S^n_+$ is pointed. It suffices to show that if $M \in S^n_+$ and $M$ nonzero, then $-M \notin S^n_+$. But $M$ positive semidefinite implies $-M$ negative semidefinite. A matrix can be both positive and negative semidefinite if and only if it is the zero matrix. Therefore is $S^n_+$ is also pointed. $\square$

Not only is $S^n_+$ a convex, pointed cone, but it is self dual.

**Lemma 7.2.2.** The positive semidefinite cone $S^n_+$ is self dual.

**Proof.** The dual cone $(S^n_+)', \text{ consists of the linear functionals } \phi \in (S^n)\text{ s}uch that } \langle M, \phi \rangle \geq 0 \text{ for all } M \in S^n_+. \text{ Since } (S^n)' \cong (S^n) \text{ proving } (S^n_+)' \cong S^n_+ \text{ requires showing } \phi \in (S^n_+)' \text{ if and only if there is } Z \in S^n_+ \text{ such that } \phi = \phi_Z. \text{ Alternatively, } \phi_Z \in (S^n_+)' \text{ if and only if } Z \in S^n_+.

(Show $\Leftarrow$) Assume $Z \in S^n_+$ and show that this implies $\phi_Z \in (S^n_+)'$. If $Z \in S^n_+$, for all $M \in S^n_+$

$$\langle M, \phi_Z \rangle = \text{Tr}(MZ)$$

and by Lemma A.2.4, $\text{Tr}(MZ) \geq 0$. Since $M$ is arbitrary, $\phi_Z \in (S^n_+)'$.

(Show $\Rightarrow$) Assume $\phi \in (S^n_+)'$. Then there is a corresponding $Z \in S^n$. Show $Z$ is positive semidefinite. By definition of $\phi_Z \in (S^n_+)'$

$$\langle M, \phi_Z \rangle = \text{Tr}(MZ) \geq 0$$

for all $M \in S^n_+$. Let $x$ be any vector in $\mathbb{R}^n$. Then by part i) of Proposition A.2.3, $xx^\top$ is in $S^n_+$ so $\text{Tr}((xx^\top)Z) \geq 0$, and by part iii) of Proposition A.2.3,

$$x^\top Z x = \text{Tr}((xx^\top)Z) \geq 0$$

Therefore $Z$ is positive semidefinite this completes the proof. $\square$
Define a primal problem which is a general linear program in the form \((GLP)\) defined in Section 3.7. The primal problem is based on (7.2.6).

\[
SDP : \begin{bmatrix} S^n \\ (S^n)' \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{R}^m \\ (\mathbb{R}^m)' \end{bmatrix} : SDPD
\]  

(7.2.6)

The semidefinite programming primal problem is

\[
\begin{align*}
\inf & \quad \text{Tr}(CZ) \\
\text{(SDP)} & \quad b - A(Z) = 0 \\
\end{align*}
\]

(7.2.7)

(7.2.8)

(7.2.9)

where the mapping \(A : S^n \rightarrow \mathbb{R}^m\) is defined by

\[
A(Z) := \begin{bmatrix} \text{Tr}(A_1 Z) \\ \vdots \\ \text{Tr}(A_m Z) \end{bmatrix}.
\]

In the primal objective function (7.2.7), we write the linear functional as \(\phi_C\) to emphasize the fact the \((S^n)'\) is isomorphic to \(S^n\). In the primal constraint set (7.2.8), the right hand side 0 is the zero vector in \(\mathbb{R}^m\), i.e. it is \(\theta_{\mathbb{R}^m}\). In terms of modified general linear program \((GLPM)\), these constraints could be written more formally as

\[
b - A(Z) \preceq_C \theta_{\mathbb{R}^m}
\]

where the cone \(C\) is the singleton \(\theta_{\mathbb{R}^m}\). Since \(A(Z)\) is a linear mapping these are affine constraints and correspond to constraint set (3.6.3) of \((GLPM)\). Constraint set (7.2.9) correspond to the \(Z \in \Omega\) constraint set and this constraint set does not get dualized. The constraint that \(Z \in S^n_+\) is often expressed as \(Z \succeq 0\).

**Example 7.2.3. Linear Programming as a Semidefinite Program** Consider a linear program in standard form.

\[
\begin{align*}
\min & \quad c^\top x \\
Ax & = b \\
x & \geq 0
\end{align*}
\]

Let \(a^i\) for \(i = 1, \ldots, m\), denote the rows of matrix \(A\). Define the following matrices,

\[
Z = \text{Diag}(x), \quad C = \text{Diag}(c), \quad A_1 = \text{Diag}(a^1), \quad A_2 = \text{Diag}(a^2), \quad \cdots, \quad A_m = \text{Diag}(a^m)
\]
Then the good old fashioned linear program in standard form is equivalent to

\[
\min \operatorname{Tr}(CZ) \\
b_i - \operatorname{Tr}(A_i Z) = 0, \quad i = 1, \ldots, m \\
Z \succeq 0
\]

which is a semidefinite program in standard form. Note the constraint \( x \geq 0 \) is equivalent to \( Z \succeq 0 \).

Before taking the dual of \((\text{SDP})\) it is necessary to understand the adjoint mapping that takes linear functionals in the space \((\mathbb{R}^m)'\) and maps them into \((\mathcal{S}^n)'\).

\[ A' : (\mathbb{R}^m)' \rightarrow (\mathcal{S}^n)' \quad \psi_x \mapsto \phi_M \]

where \( x \in \mathbb{R}^m \) and \( M \in \mathcal{S}^n \). The **WTS** is:

**WTS:** \( \langle A(Z), \psi_x \rangle_{\mathbb{R}^m} = \langle Z, A'(\psi_x) \rangle_{\mathcal{S}^n} \)

Here is the derivation,

\[
\langle A(Z), \psi_x \rangle_{\mathbb{R}^m} = \psi_x(A(Z)) \\
= \psi_x \left( \begin{bmatrix} \operatorname{Tr}(A_1 Z) \\ \vdots \\ \operatorname{Tr}(A_m Z) \end{bmatrix} \right), \quad A_i \in \mathcal{S}^n \\
= \sum_{i=1}^m x_i \operatorname{Tr}(ZA_i) \\
= \operatorname{Tr} \left( Z \left( \sum_{i=1}^m x_i A_i \right) \right) \quad \text{see Appendix A.2.4} \\
= \operatorname{Tr}(ZM) \quad \text{where } M = \sum_{i=1}^m x_i A_i \\
= \phi_M(Z) \\
= \langle Z, \phi_M \rangle_{\mathcal{S}^n} = \langle Z, A'(\psi_x) \rangle_{\mathcal{S}^n}
\]

But \((\mathcal{S}^n)' \cong \mathcal{S}^n\) and \((\mathbb{R}^m)' \cong \mathbb{R}^m\) so

\[ A' : \mathbb{R}^m \rightarrow \mathcal{S}^n \\
x \mapsto \sum_{i=1}^m x_i A_i = M \]
At long last, take the dual of (SDP). The Lagrangian function is

\[
L(\psi_x) = \inf \{ \langle Z, \phi_C \rangle_S^n + \langle b - A(Z), \psi_x \rangle_{\mathbb{R}^m} : Z \in S^n_+ \}
\]

\[
= \inf \{ \langle Z, \phi_C \rangle_S^n + \langle b, \psi_x \rangle_{\mathbb{R}^m} - \langle A(Z), \psi_x \rangle_{\mathbb{R}^m} : Z \in S^n_+ \}
\]

\[
= \langle b, \psi_x \rangle_{\mathbb{R}^m} + \inf \{ \langle Z, \phi_C \rangle_S^n - \langle A(Z), \psi_x \rangle_{\mathbb{R}^m} : Z \in S^n_+ \} \quad (7.2.10)
\]

From the adjoint derivation, \(\langle Z, A'(\psi_x)\rangle_S^n = \langle A(Z), \psi_x \rangle_{\mathbb{R}^m}\), and it is valid to substitute \(\langle Z, A'(\psi_x)\rangle_S^n\) for \(\langle A(Z), \psi_x \rangle_{\mathbb{R}^m}\) in (7.2.10) and \(L(\psi_x)\) becomes

\[
L(\psi_x) = \langle b, \psi_x \rangle_{\mathbb{R}^m} + \inf \{ \langle Z, \phi_C \rangle_S^n - \langle Z, A'(\psi_x)\rangle_S^n : Z \in S^n_+ \}
\]

\[
= \langle b, \psi_x \rangle_{\mathbb{R}^m} + \inf \{ \langle Z, \phi_C - A'(\psi_x)\rangle_S^n : Z \in S^n_+ \}
\]

If a \(\psi_x\) is selected such that there is a \(Z \in S^n_+\) with \(\langle Z, \phi_C - A'(\psi_x)\rangle_S^n < 0\), then \(L(\psi_x)\) is unbounded since \(S^n_+\) is a cone and \(Z \in S^n_+\) implies \(\lambda Z \in S^n_+\) for positive \(\lambda\). Therefore, it is necessary to select \(\psi_x\) such that \(\langle Z, \phi_C - A'(\psi_x)\rangle_S^n \geq 0\) for all \(Z \in S^n_+\). This means that the \(\psi_x\) should be selected the linear functional, \(\phi_C - A'(\psi_x)\) is the dual cone, i.e. and element of \((S^n_+)'\). Therefore, the dual linear program is

\[
\sup \{ \langle b, \psi_x \rangle_{\mathbb{R}^m} \} \quad (7.2.11)
\]

\[
(SDPD) \quad \text{s.t.} \quad \phi_C - A'(\psi_x) \in (S^n_+)', \quad \psi_x \in (\mathbb{R}^m)' \quad (7.2.12)
\]

By Lemma 7.2.2, the positive semidefinite cone \(S^n_+\) is self dual, that is \((S^n_+)' \cong S^n_+\). Since \(\phi_C - A'(\psi_x)\) maps to \((C - \sum_{i=1}^{m} x_i A_i)\) this implies \((C - \sum_{i=1}^{m} x_i A_i)\) must be positive semidefinite. The dual problem is then

This is equivalent to

\[
\sup b^\top x \quad (7.2.14)
\]

\[
(SDPD) \quad \text{s.t.} \quad C - \sum_{i=1}^{m} x_i A_i \in S^n_+ \quad (7.2.15)
\]

\[
x \in \mathbb{R}^m \quad (7.2.16)
\]

In constraint set (7.2.12), \(\phi_C - A'(\psi_x) \in (S^n_+)'\) is used instead of the more cumbersome,

\[
\phi_C - A'(\psi_x) \succeq_{(S^n_+)' \times (S^n_+)} \theta_{S^n_+},
\]

and constraint set (7.2.15), \(C - \sum_{i=1}^{m} x_i A_i \in S^n_+\) is used instead of the more cumbersome,

\[
C - \sum_{i=1}^{m} x_i A_i \succeq_{S^n_+} \theta_{S^n_+}.
\]
CHAPTER 7. SEMIDEFINITE PROGRAMMING

Problem (SDP) is an example of the modified general linear program (GLPM) introduced in Chapter 3. The conditions for no duality gap and strong duality that were developed for (GLPM) apply here. A key condition is the Slater condition, which in the context of this problem is for \( \theta_{\mathbb{R}^m} \) to be a core point of

\[
\Upsilon := \{ z \in \mathbb{R}^m : \exists Z \in \mathcal{S}^n \text{ s.t. } b - A(Z) = z \}. \tag{7.2.17}
\]

Finally note that since we have a finite dimensional vector space and are self dual, we do not need to consider a topology.

7.3 Dual of the Dual

Just for fun, let’s take the dual of the dual. Since everything is finite dimensional we should get the primal back. The primal dual mappings for (SDP) and (SDPD) are

\[
\begin{align*}
\text{SDP} & : \left[ \begin{array}{c} \mathcal{S}^n \\ (\mathcal{S}^n)' \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathbb{R}^m \\ (\mathbb{R}^m)' \end{array} \right] : \text{SDPD} \\
\text{SDPD} & : \left[ \begin{array}{c} (\mathbb{R}^m)' \\ (\mathbb{R}^m)'' \end{array} \right] \rightarrow \left[ \begin{array}{c} (\mathcal{S}^n)' \\ (\mathcal{S}^n)' \end{array} \right] : \text{SDPDD}
\end{align*}
\]

Treating (SDPD) as expressed in (7.2.11)-(7.2.13) as the primal, and taking its dual, yields the mappings

\[
\begin{align*}
\text{SDPD} & : \left[ \begin{array}{c} (\mathbb{R}^m)' \\ (\mathbb{R}^m)'' \end{array} \right] \rightarrow \left[ \begin{array}{c} (\mathcal{S}^n)' \\ (\mathcal{S}^n)' \end{array} \right] : \text{SDPDD} \\
\text{SDPD} & : \left[ \begin{array}{c} (\mathbb{R}^m) \\ (\mathbb{R}^m)' \end{array} \right] \rightarrow \left[ \begin{array}{c} (\mathcal{S}^n) \\ (\mathcal{S}^n)' \end{array} \right] : \text{SDPDD}
\end{align*}
\]

However, to make life easier, we work with the (SDPD) expressed by (7.2.14)-(7.2.16) and the primal dual mappings are

\[
\begin{align*}
\text{SDPD} & : \left[ \begin{array}{c} (\mathbb{R}^m) \\ (\mathbb{R}^m)' \end{array} \right] \rightarrow \left[ \begin{array}{c} (\mathcal{S}^n) \\ (\mathcal{S}^n)' \end{array} \right] : \text{SDPDD}
\end{align*}
\]

First consider the adjoint mapping. In deriving (SDPD) from (SDP), the adjoint mapping \( A' \), was from \( (\mathbb{R}^m)' \) to \( (\mathcal{S}^n)' \) and \( \psi \) was used to represent a linear functional in the dual space \( (\mathbb{R}^m)' \) and \( \phi \) was used to represent a linear functional in the dual space \( (\mathcal{S}^n)' \). In deriving (SDPDD) from (SDPD), we denote the adjoint mapping by \( A'' \) and keep the convention that \( \psi \) represents a linear functional in the dual space \( (\mathbb{R}^m)' \) and \( \phi \) represents a linear functional in the dual space \( (\mathcal{S}^n)' \).

\[
A'' : (\mathcal{S}^n)' \rightarrow (\mathbb{R}^m)'
\]

\[
\phi_Z \mapsto \psi_d
\]

where \( Z \in \mathcal{S}^n \) and \( b \in \mathbb{R}^m \). We now abuse the adjoint notation and using \( x \) instead of \( \psi_x \) write the constraints \( C - \sum_{i=1}^m x_i A_i \in \mathcal{S}_+^n \) in (7.2.15) as \( C - A'(x) \in \mathcal{S}_+^n \), where \( A'(x) = \sum_{i=1}^m x_i A_i \). Now show

\[
\langle A'(x), \phi_Z \rangle_{\mathcal{S}^n} = \langle x, A''(\phi_Z) \rangle_{\mathbb{R}^m}
\]
Here is the derivation,

\[
\langle A'(x), \phi Z \rangle_{S^n} = \phi Z(A'(x)) = \phi Z\left(\sum_{i=1}^{m} x_i A_i \right) \quad A_i \in S^n
\]

\[
= \sum_{i=1}^{m} x_i \phi Z(A_i)
\]

\[
= \sum_{i=1}^{m} x_i \text{Tr}(A_i Z)
\]

\[
= \left[ \begin{array}{c}
\text{Tr}(A_1 Z) \\
\vdots \\
\text{Tr}(A_m Z)
\end{array} \right] \top x
\]

\[
= d \top x
\]

\[
= \psi_d(x)
\]

\[
= \langle x, A''(\phi Z) \rangle_{\mathbb{R}^m}
\]

where

\[
d = \left[ \begin{array}{c}
\text{Tr}(A_1 Z) \\
\vdots \\
\text{Tr}(A_m Z)
\end{array} \right]
\]

(7.3.1)

But \((S^n)' \cong S^n\) and \((\mathbb{R}^n)' \cong \mathbb{R}^n\) so

\[
A'' : S^n \to \mathbb{R}^m
\]

\[
Z \mapsto \left[ \begin{array}{c}
\text{Tr}(A_1 Z) \\
\vdots \\
\text{Tr}(A_m Z)
\end{array} \right]
\]

Using the result, \(\langle x, A''(\phi Z) \rangle_{\mathbb{R}^m} = \langle A'(x), \phi Z \rangle_{S^n}\) take the Lagrangian dual of (7.2.11)-(7.2.13). The Lagrangian function is

\[
L(\phi Z) = \sup_{x \in \mathbb{R}^m} \left\{ b \top x + \langle C - A'(x), \phi Z \rangle_{S^n} \right\}
\]

\[
= \langle C, \phi Z \rangle_{S^n} + \sup_{x \in \mathbb{R}^m} \left\{ b \top x - \langle A'(x), \phi Z \rangle_{S^n} \right\}
\]

\[
= \langle C, \phi Z \rangle_{S^n} + \sup_{x \in \mathbb{R}^m} \left\{ b \top x - \langle x, A''(\phi Z) \rangle_{\mathbb{R}^m} \right\}
\]

\[
= \langle C, \phi Z \rangle_{S^n} + \sup_{x \in \mathbb{R}^m} \left\{ b \top x - d \top x \right\}
\]

\[
= \text{Tr}(CZ) + \sup_{x \in \mathbb{R}^m} \left\{ b \top x - d \top x \right\}
\]
where \( d \) is defined in (7.3.1). Since the supremum is taken over all of \( \mathbb{R}^m \), this supremum will be positive infinity if there is value of \( d_i \neq b_i \). Therefore the dual problem is

\[
\inf \text{Tr}(CZ)
\]

s.t. \[
\begin{bmatrix}
    b - & a_1 & \cdots & a_m \\
    \frac{1}{2} & Z \
    \vdots & \vdots & \ddots & \vdots \\
    0 & a_m & \cdots & 1
\end{bmatrix} = 0 \\
Z \succeq 0
\]

and this is exactly the original primal (SDP) given by (7.2.7)-(7.2.9).

### 7.4 Semidefinite Relaxations of 0/1 Quadratic Programs

Many important applications are formulated as quadratic 0/1 integer programming problems. See, for example, the combinatorial optimization problems given in Section 7.8 of this chapter. In this section we generate lower bounds on the optimal solution value of non-convex quadratic 0/1 programs by solving a convex semidefinite programming problem.

#### 7.4.1 A Simple Unconstrained Quadratic Program

The general procedure for formulating a semidefinite relaxation of a 0/1 quadratic program is first motivated by illustrating the procedure on the unconstrained 0/1 quadratic programming. More general quadratic programs are considered in Section 7.4.2. The 0/1 quadratic program with no constraints is

\[
(QP) \quad \min \{ x^\top Q x \mid x_i \in \{0, 1\}, i = 1, \ldots, n \}
\]

where \( Q \in \mathcal{S}^n \) (but not necessarily \( \mathcal{S}^n_+ \)). From part (iii) of Proposition A.2.3,

\[
x^\top Q x = \text{Tr}(Q(xx^\top))
\]

Rewrite (QP) as

\[
(QP) \quad \min \text{Tr}(Q(xx^\top)) \\
x_i^2 - x_i = 0, \quad i = 1, \ldots, n
\]

Next create an equivalent problem to (QP) by introducing an \( X \in \mathcal{S}^n \) where \( X = xx^\top \). Problem (QP) is now

\[
(QP) \quad \min \text{Tr}(QX) \\
x_i^2 - x_i = 0, \quad i = 1, \ldots, n \\
X = xx^\top
\]
Next replace the constraints $x_i^2 - x_i = 0$, $i = 1, \ldots, n$ with the equivalent $\text{diag}(X) = x$. Problem $(QP)$ is now

$$
(QP) \quad \min \text{Tr}(QX) \\
\text{diag}(X) = x \\
X = xx^\top
$$

The constraints $X = xx^\top$ are nonconvex. Replace them with $X \succeq xx^\top$ and replace problem $(QP)$ with a relaxation $(QPR)$.

$$(QPR) \quad \min \text{Tr}(QX) \\
\text{diag}(X) = x \\
X - xx^\top \succeq 0$$

Why is problem $(QPR)$ a relaxation of problem $(QP)$? From Corollary A.2.10, $X - xx^\top \succeq 0$ is a convex constraint. Again, by Corollary A.2.10, rewrite $(QPR)$ as

$$(QPR) \quad \min \text{Tr}(QX) \\
\text{diag}(X) = x \\
\begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0$$

We can also express $\text{diag}(X) = x$ in terms of matrix “trace constraints.” In terms of matrix trace, $\text{diag}(X) = x$ is expressed as

$$\text{Tr}((E_{j+1,j+1} - \frac{1}{2}(E_{1j} + E_{j1}))Z) = 0, \quad j = 1, \ldots, n \quad (7.4.1)$$

where

$$Z = \begin{bmatrix} z_{11} & x^\top \\ x & X \end{bmatrix}$$

and $E_{ij}$ is an $n + 1 \times n + 1$ matrix of all 0 elements, except for a 1 in component $(i, j)$ and $(j, i)$. Since $Z$ must be symmetric, each $j$, in constraint $(7.4.1)$ forces diagonal element $z_{jj}$ to equal elements $z_{1j}$ and $z_{j1}$. The constraint $\text{Tr}(E_{11}Z) = 1$ forces $z_{11} = 1$. Problem $(QPR)$ can then be written as

$$(QPR) \quad \min \text{Tr}(\overline{Q}Z) \\
\text{Tr}(E_{11}Z) = 1 \\
\text{Tr} \left( (E_{j+1,j+1} - \frac{1}{2}(E_{1j} + E_{j1}))Z \right) = 0, \quad j = 1, \ldots, n$$

$$Z \succeq 0$$
where

\[
\mathcal{Q} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & Q
\end{bmatrix}
\]

Problem (QPR) now has the format of the primal semidefinite program (SDP) and provides a continuous, convex relaxation of the original quadratic 0/1 program.

### 7.4.2 Semidefinite Relaxations for Quadratic Programs

The methodology used in Section 7.4.1 to give a semidefinite programming relaxation of an unconstrained 0/1 quadratic program is extended in this section to include linear equality and inequality constants, and quadratic equality constraints. Much of the material in this section is from Galli and Letchford [12]. Consider the quadratically constrained quadratic program (QCQP). Notice constraints (7.4.6). These constraints are equivalent to requiring \(x_i \in \{0, 1\}\) for \(i = 1, \ldots, n\).

\[
\begin{align*}
\text{(QCQP)} & \quad \min x^\top Q_0 x + c_0^\top x \\
& \quad \text{s.t.} \quad Ax = b \in \mathbb{R}^t \\
& \quad \quad\quad\quad\quad Dx \leq f \in \mathbb{R}^p \\
& \quad \quad\quad\quad\quad x^\top Q_i x + c_i^\top x = h_i \quad i = 1, \ldots, m \\
& \quad \quad\quad\quad\quad x_i^2 - x_i = 0 \quad i = 1, \ldots, n
\end{align*}
\]

Problem (QCQP) can be viewed a binary quadratic program with the binary conditions replaced with constraint set (7.4.6). Using the standard “trick” from part (iii) of Proposition A.2.3, \(x^\top Qx = \text{Tr}(Qxx^\top)\) rewrite (QCQP) as

\[
\begin{align*}
\text{(QCQP)} & \quad \min \text{Tr}(Q_0(xx^\top)) + c_0^\top x \\
& \quad \quad\quad\quad\quad\quad\quad\quad Ax = b \in \mathbb{R}^t \\
& \quad \quad\quad\quad\quad\quad\quad\quad Dx \leq f \in \mathbb{R}^p \\
& \quad \quad\quad\quad\quad\quad\quad\quad \text{Tr}(Q_i(xx^\top)) + c_i^\top x = h_i \quad i = 1, \ldots, m \\
& \quad \quad\quad\quad\quad\quad\quad\quad x_i^2 - x_i = 0 \quad i = 1, \ldots, n
\end{align*}
\]
Next, create an equivalent problem to (QP) by introducing an $X \in S^n$ where $X = xx^\top$. Problem (QCQP) is then

$$\begin{align*}
\min & \quad \text{Tr}(Q_0X) + c_0^\top x \\
\text{subject to} & \quad Ax = b \in \mathbb{R}^t \\
& \quad Dx \leq f \in \mathbb{R}^p \\
& \quad \text{Tr}(Q_iX) + c_i^\top x = h_i, \quad i = 1, \ldots, m \\
& \quad x_i^2 - x_i = 0, \quad i = 1, \ldots, n \\
& \quad X = xx^\top 
\end{align*}$$

Now replace the constraints $x_i - x_i^2 = 0, i = 1, \ldots, n$ with the equivalent $\text{diag}(X) = x$. Problem (QCQP) is

$$\begin{align*}
\min & \quad \text{Tr}(Q_0X) + c_0^\top x \\
\text{subject to} & \quad Ax = b \in \mathbb{R}^t \\
& \quad Dx \leq f \in \mathbb{R}^p \\
& \quad \text{Tr}(Q_iX) + c_i^\top x = h_i, \quad i = 1, \ldots, m \\
& \quad \text{diag}(X) = x \\
& \quad X = xx^\top 
\end{align*}$$

The constraints $X = xx^\top$ are nonconvex. Form a relaxation and replace them with the constraints $X \succeq xx^\top$. Problem (QCQPR) below is a relaxation of (QCQP).

$$\begin{align*}
\min & \quad \text{Tr}(Q_0X) + c_0^\top x \\
\text{subject to} & \quad Ax = b \in \mathbb{R}^t \\
& \quad Dx \leq f \in \mathbb{R}^p \\
& \quad \text{Tr}(Q_iX) + c_i^\top x = h_i, \quad i = 1, \ldots, m \\
& \quad \text{diag}(X) = x \\
& \quad X - xx^\top \succeq 0 
\end{align*}$$

**Key Question:** Why is problem (QCQPR) a relaxation of problem (QCQP)?

By Corollary A.2.10, $X - xx^\top \succeq 0$ is a closed convex constraint set. Also, by A.2.10 we can rewrite (QCQPR) as the equivalent
\[
\begin{align*}
\text{(QCQPR)} & \quad \min Tr(QX) + c^\top x \\
& \quad (a^i)^\top x = b_i, \quad i = 1, \ldots, t \\
& \quad (d^i)^\top x \leq f_i, \quad i = 1, \ldots, p \\
& \quad \text{Tr}(Q_iX) + c_i^\top x = h_i, \quad i = 1, \ldots, m \\
& \quad \text{diag}(X) = x, \\
& \quad \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0 \\
\end{align*}
\]

Now put the relaxation in standard \((SDP)\) primal form (in the definition of \(Z\), \(S\) is an \(p \times p\) symmetric matrix).

\[
Z = \begin{bmatrix}
z_{11} & x^\top & * \\
x & X & * \\
* & * & S \\
\end{bmatrix}
\]

\[
F_0 = \begin{bmatrix}
0 & \frac{1}{2}c_0^\top & 0 \\
\frac{1}{2}c_0 & Q_0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[
F'_i = \begin{bmatrix}
0 & \frac{1}{2}(a^i)^\top & 0 \\
\frac{1}{2}a^i & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad i = 1, \ldots, t
\]

\[
H'_i = \begin{bmatrix}
0 & \frac{1}{2}(d^i)^\top & 0 \\
\frac{1}{2}d^i & 0 & 0 \\
0 & 0 & E_{i,i} \\
\end{bmatrix}, \quad i = 1, \ldots, p
\]

where \(D_i\) is row \(i\) of \(D\) used to define the \(Dx \leq f\) constraints.

\[
G'_i = \begin{bmatrix}
0 & \frac{1}{2}c_i^\top & 0 \\
\frac{1}{2}c_i & Q_i & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad i = 1, \ldots, m
\]

The \((SDP)\) standard form is:
7.4. SEMIDEFINITE RELAXATIONS OF 0/1 QUADRATIC PROGRAMS

\[
\begin{align*}
\inf & \quad \text{Tr}(F_0Z) \\
\text{Tr}(F_i'Z) & = b_i, \quad i = 1, \ldots, t \quad (7.4.13) \\
\text{Tr}(H_i'Z) & = f_i, \quad i = 1, \ldots, p \quad (7.4.14) \\
\text{Tr}(G_i'Z) & = h_i, \quad i = 1, \ldots, m \quad (7.4.15) \\
\text{Tr}(E_{11}Z) & = 1 \quad (7.4.16) \\
\text{Tr}((E_{j+1,j+1} - \frac{1}{2} (E_{1j} + E_{j1}))Z) & = 0, \quad j = 1, \ldots, n \quad (7.4.17) \\
\end{align*}
\]

\[
Z \succeq 0 \quad (7.4.18)
\]

\[
\begin{align*}
\text{Note:} & \quad \text{we are a bit lose with our notation. In (7.4.17), } E_{11} \text{ is an } (n + 1 + p) \times (n + 1 + p) \\
& \quad \text{matrix where element (1,1) is equal 1 and all others zero. However, in the definition of } H_i', \quad E_{ii} \text{ is a } p \times p \text{ matrix with element (i,i) equal to 1 and all others zero.}
\end{align*}
\]

**Lemma 7.4.1.** If \( Z \) is a feasible solution to (7.4.13)-(7.4.19) then \( x, X \) are feasible to (7.4.7)-(7.4.12).

**Proof.** If

\[
\begin{bmatrix} A & * \\ * & * \end{bmatrix} \succeq 0,
\]

then by Lemma A.2.9, \( A \succeq 0 \). Then \( Z \succeq 0 \) implies

\[
\begin{bmatrix} z_{11} & x^\top \\ x & X \end{bmatrix} \succeq 0
\]

and by \( \text{Tr}(E_{11}Z) = 1 \) in (7.4.17), \( z_{11} = 1 \) so

\[
\begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0
\]

Also, \( Z \succeq 0 \) implies that the diagonal elements are nonnegative, in particular, \( Z_{ii} \geq 0, \) \( i = n + 2, \ldots, n + 1 + p. \) But

\[
\langle H_i', Z \rangle = D_i^\top x + Z_{n+1+i,n+1+i} = f_i, \quad i = 1, \ldots, p
\]

and this implies

\[
D_i^\top x \leq f_i, \quad i = 1, \ldots, t.
\]

\( \square \)
CHAPTER 7. SEMIDEFINITE PROGRAMMING

Next observe that because of the unity constraint (7.4.17), it is possible to incorporate the right hand side constants into the matrices below.

\[
F_i = \begin{bmatrix} -b_i & \frac{1}{2}(a_i)^\top & 0 \\ \frac{1}{2}a_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, t
\]

where \( A_i \) is row \( i \) of \( A \).

\[
H_i = \begin{bmatrix} -f_i & \frac{1}{2}(d_i)^\top & 0 \\ \frac{1}{2}d_i & 0 & 0 \\ 0 & 0 & E_{ii} \end{bmatrix}, \quad i = 1, \ldots, p
\]

where \( D_i \) is row \( i \) of \( D \).

\[
G_i = \begin{bmatrix} -h_i & \frac{1}{2}c_i^\top & 0 \\ \frac{1}{2}c_i & Q_i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, m
\]

The slightly revised (SDP) is:

\[
\begin{align*}
\inf & \quad \text{Tr}(F_0 Z) \\
\text{Tr}(F_i Z) & = 0, \quad i = 1, \ldots, t \quad \lambda \\
\text{Tr}(H_i Z) & = 0, \quad i = 1, \ldots, p \quad \mu \\
\text{Tr}(G_i Z) & = 0, \quad i = 1, \ldots, m \quad \nu \\
-1 + \text{Tr}(E_{11} Z) & = 0 \quad \theta \\
\text{Tr}((E_{j+1,j+1} - \frac{1}{2}(E_{1j} + E_{j1})) Z) & = 0, \quad j = 1, \ldots, n \quad \phi \\
Z & \succeq 0
\end{align*}
\]

Taking the dual (see Section 7.3) of (7.4.20)-(7.4.26) where the dual variables are respectively, \( \lambda, \mu, \nu, \theta, \) and \( \phi \) gives:

\[
\sup -\theta
\]

\[
F_0 + \sum_{i=1}^{t} \lambda_i F_i + \sum_{i=1}^{p} \mu_i H_i + \sum_{i=1}^{m} \nu_i G_i
\]

\[
+ \theta E_{11} + \sum_{j=1}^{n} \phi_j (E_{j+1,j+1} - \frac{1}{2}E_{1j+1} - \frac{1}{2}E_{j+1,1}) \succeq 0
\]
7.5. THE LAGRANGIAN DUAL AND THE SDP RELAXATION

In (7.4.28) every matrix is symmetric and has dimension \((n + 1 + p) \times (n + 1 + p)\).

Following Galli and Letchford [12], define

\[
\begin{align*}
\bar{h} &= -\sum_{i=1}^{t} \lambda_i b_i - \sum_{i=1}^{p} \mu_i f_i - \sum_{i=1}^{m} \nu_i h_i \\
\bar{e} &= c_0 + \sum_{i=1}^{t} \lambda_i a_i + \sum_{i=1}^{p} \mu_i d_i + \sum_{i=1}^{m} \nu_i c_i - \phi \\
\bar{Q} &= Q_0 + \sum_{i=1}^{m} \nu_i Q_i + \text{Diag}(\phi)
\end{align*}
\]

Using this notation, the SDP dual problem is rewritten as

\[
\begin{align*}
\sup_{\theta} -\theta
\begin{bmatrix}
\theta + \bar{h} & \frac{1}{2} \bar{e}^T & 0 \\
\frac{1}{2} \bar{e} & \bar{Q} & 0 \\
0 & 0 & \text{Diag}(\mu)
\end{bmatrix}
\succeq 0
\end{align*}
\]

This simplifies to the following SDP dual (since \(\theta\) it is valid to switch the sign). This is the dual of \((QCQPR)\) and is denoted by \((QCQPRD)\).

\[
\begin{align*}
\max \theta \\
\begin{bmatrix}
\bar{h} - \theta & \frac{1}{2} \bar{e}^T \\
\frac{1}{2} \bar{e} & \bar{Q}
\end{bmatrix}
\succeq 0
\end{align*}
\]

The positive semidefinite constraint in (7.4.33) implies that \(\text{Diag}(\mu)\) is positive semidefinite, that is, \(\mu\) is nonnegative which is constraint (7.4.37).

7.5 The Lagrangian Dual and the SDP Relaxation

In Section 7.4 a problem relaxation, based on a semidefinite program, was given for 0/1 quadratic programming problems. In this Section, we examine an alternative approach for
generating lower bounds using Lagrangian relaxation. Before doing so, certain properties 
of quadratic functions are required in the analysis. These properties are stated in Section 7.5.1. These results are used in Section 7.5.2 to compare the Lagrangian bounds with the semidefinite programming bounds.

7.5.1 Quadratic programming properties

This material is from Lemaréchal and Oustry [18]. Lemma 7.5.1 given below appears as Lemma 3.6 from Lemaréchal and Oustry. Define

\[ q(y) = y^\top Q y + b^\top y + c \]

where \( Q \in S^n \).

**Lemma 7.5.1.** \( \inf q(y) > -\infty \) if and only if \( Q \succeq 0 \) and \( b \in \mathcal{R}(Q) \) where \( \mathcal{R}(Q) = \{ b \mid b = Qy \ \text{for some} \ y \in \mathbb{R}^n \} \).

**Proof of Lemma 7.5.1:** *Necessity:* If \( Q \) is not positive semidefinite then there exists \( \overline{y} \in \mathbb{R}^n \) such that \( \overline{y}^\top Q \overline{y} < 0 \). We can also assume without loss that \( \overline{y}^\top b \leq 0 \). Why? Then \( q(\alpha \overline{y}) \to -\infty \) as \( \alpha \to \infty \).

If \( Q \succeq 0 \), then it is still necessary to have \( b \in \mathcal{R}(Q) \). Assume no such \( b \) exists. Then by the Fredholm alternative, if \( Qx = b \) has no solution, then there exists \( \overline{y} \in \mathbb{R}^n \) such that \( \overline{y}^\top Q = 0 \) and \( \overline{y}^\top b \neq 0 \). Without loss, assume \( \overline{y}^\top b < 0 \). Then for \( \alpha > 0 \),

\[ q(\alpha \overline{y}) = \alpha^2 \overline{y}^\top Q \overline{y} + \alpha \overline{y}^\top b + c = \alpha \overline{y}^\top b + c \]

which goes to \(-\infty\) as \( \alpha \to \infty \) since \( \alpha \overline{y}^\top b < 0 \).

*Sufficiency:* If \( Q \succeq 0 \) then \( q(y) \) a convex function. Take the partials of \( q(y) \) and set equal to zero. The resulting system is \( 2Qy = -b \). Since \( b \in \mathcal{R}(Q) \) this system has a solution and hence a finite optimal value. \( \square \)

We now state and prove Lemma 3.7 from Lemaréchal and Oustry.

**Lemma 7.5.2.** If \( Q \succeq 0 \) and \( b \in \mathcal{R}(Q) \), then \( \overline{y} = -\frac{1}{2}Q^\dagger b \) is the optimal solution to the problem \( \inf_{y \in \mathbb{R}^n} q(y) \) and the optimal objective function value is \( q(\overline{y}) = c - \frac{1}{4} b^\top Q^\dagger b \).

**Proof of Lemma 7.5.2** First show that \( \overline{y} = -\frac{1}{2}Q^\dagger b \) is an optimal solution to to \( \inf_{y \in \mathbb{R}^n} q(y) \). Given the hypothesis \( Q \succeq 0 \) and \( b \in \mathcal{R}(Q) \), the optimal solution to \( \inf_{y \in \mathbb{R}^n} q(y) \) is given by the solution to the first order conditions \( \nabla q(y) = 0 \) which is

\[ \nabla q(y) = 2Qy + b = 0 \]
This implies that the optimal solution is the solution to the system \( Qy = -b/2 \). We show \( y = -\frac{1}{2}Q^\dagger b \) satisfies this system. By hypothesis \( b \in \mathcal{R}(Q) \) so from Lemma ?? (Kipp – put in correct reference) \( QQ^\dagger b = b \). Then if we set \( y = -\frac{1}{2}Q^\dagger b \),

\[
Qy = Q(-\frac{1}{2}Q^\dagger b) \\
= -\frac{1}{2}QQ^\dagger b \\
= -\frac{1}{2}b
\]

Thus, \( y = -\frac{1}{2}Q^\dagger b \) is a solution to \( Qy = -b/2 \). Given that \( y = -\frac{1}{2}Q^\dagger b \) is optimal,

\[
q(y) = (-\frac{1}{2}Q^\dagger b)^\top Q(-\frac{1}{2}Q^\dagger b) + b^\top (-\frac{1}{2}Q^\dagger b) + c \\
= \frac{1}{4}(Q^\dagger b)^\top QQ^\dagger b - \frac{1}{2}b^\top Q^\dagger b + c \\
= \frac{1}{4}(Q^\dagger b)^\top b - \frac{1}{2}b^\top Q^\dagger b + c \\
= \frac{1}{4}b^\top (Q^\dagger)^\top b - \frac{1}{2}b^\top Q^\dagger b + c \\
= \frac{1}{4}b^\top Q^\dagger b - \frac{1}{2}b^\top Q^\dagger b + c \\
= -\frac{1}{4}b^\top Q^\dagger b + c \quad \square
\]

### 7.5.2 Lagrangian dual and SDP dual equivalence

We show that a strengthening of the classic Lagrangian dual gives a lower bound equal to the optimal value of the semidefinite programming relaxation. The quadratically constrained quadratic program \((QCQP)\) is

\[
\begin{align*}
\text{(QCQP)} \quad & \min x^\top Q_0 x + c_0^\top x \\
\text{s.t.} \quad & Ax = b \in \mathbb{R}^d \quad (\lambda) \\
& Dx \leq f \in \mathbb{R}^p \quad (\mu) \\
& x^\top Q_i x + c_i^\top x = h_i \quad i = 1, \ldots, m \quad (\nu) \\
& x_i^2 - x_i = 0 \quad i = 1, \ldots, n \quad (\phi)
\end{align*}
\]

Taking the Lagrangian dual of (7.4.2)-(7.4.6) using \( \lambda, \mu, \nu, \) and \( \phi \) gives:
\[
x^\top Q_0 x + c_0^\top x + \sum_{i=1}^{t} \lambda_i((a_i)^\top - b_i) + \sum_{i=1}^{t} \mu_i((d_i)^\top - f_i)
+ \sum_{i=1}^{m} \nu_i(x^\top Q_ix + c_i^\top - h_i) + \sum_{i=1}^{m} \phi_i(x_i^2 - x_i)
\]

\[
= x^\top \left( Q_0 + \sum_{i=1}^{m} \nu_i Q_i + \text{Diag}(\phi) \right) x + (c_0 + \sum_{i=1}^{t} \lambda_i a_i + \sum_{i=1}^{m} \nu_i c_i - \phi)^\top x
- \sum_{i=1}^{t} \lambda_i b_i - \sum_{i=1}^{m} \nu_i h_i
\]

\[
= x^\top \overline{Q} x + \overline{c}^\top x + \overline{h}
\]

where \( \overline{Q} \) is defined by

\[
\overline{Q} := Q_0 + \sum_{i=1}^{m} \nu_i Q_i + \text{Diag}(\phi)
\]

which is exactly the \( \overline{Q} \) defined in (7.4.31) used to form in problem \((QCQPRD)\). Likewise for \( \overline{h} \) (see (7.4.29)) and \( \overline{c} \) (see (7.4.30)). Then the Lagrangian function is

\[
L(\lambda, \nu, \mu, \phi) = \min\{x^\top \overline{Q} x + \overline{c}^\top x + \overline{h} \mid x \in \mathbb{R}^n\}
\]

(7.5.6)

The Lagrangian dual problem is then

\[
(LD) \quad \max\{L(\lambda, \nu, \mu, \phi) \mid \mu \geq 0, \lambda, \nu, \phi, \text{free}\}
\]

(7.5.7)

Recall from Lemma 7.5.1, that a necessary and sufficient condition for \( L(\lambda, \nu, \mu, \phi) \) to be finite is for \( \overline{Q} \succeq 0 \) and \( \overline{c} \in \mathcal{R}(\overline{Q}) \). Thus, the Lagrangian dual can be rewritten as

\[
(LD) \quad \sup_{\overline{Q} \succeq 0, \overline{c} \in \mathcal{R}(\overline{Q})} L(\lambda, \nu, \mu, \phi)
\]

Recall of course that \( \overline{Q} \) is function of \( \nu \) and \( \phi \) and \( \overline{c} \) is a function of \( \lambda \), \( \mu \), \( \nu \), and \( \phi \). Refer back to (7.4.30) and (7.4.31) for the definitions of \( \overline{c} \) and \( \overline{Q} \), respectively. Given these constraints on the dual variables to make the Lagrangian function finite, we can rewrite
the Lagrangian objective function using Lemma 7.5.2:

\[
(LD) \quad \sup \bar{h} - \frac{1}{4} c^\top Q^\dagger c
\]

\[
\bar{Q} \succeq 0 \\
\bar{c} \in \mathcal{R}(\bar{Q}) \\
\mu \geq 0, \lambda, \nu, \phi, \text{free}
\]

We further rewrite the Lagrangian dual problem by introducing a new variable \(\theta\) as

\[
(LD) \quad \sup \theta
\]

\[
\theta \leq \bar{h} - \frac{1}{4} c^\top Q^\dagger c
\]

\[
\bar{Q} \succeq 0 \\
\bar{c} \in \mathcal{R}(\bar{Q}) \\
\mu \geq 0, \lambda, \nu, \phi, \text{free}
\]

rewriting the first equation gives

\[
(LD) \quad \sup \theta
\]

\[
0 \leq \bar{h} - \theta - \frac{1}{4} c^\top Q^\dagger c
\]

\[
\bar{Q} \succeq 0 \\
\bar{c} \in \mathcal{R}(\bar{Q}) \\
\mu \geq 0, \lambda, \nu, \phi, \text{free}
\]

Now refer back to Schur’s Lemma (see Lemma A.2.7). Let \(\bar{h} - \theta\) play the role of \(P\) and \((1/2) \bar{c}\) the role of \(S\). Then part (ii) of Schur’s Lemma corresponds to the constraints constraints (7.5.9)-(7.5.11). Then the Lagrangian dual (7.5.8)-(7.5.12) is equivalent to

\[
\sup \theta
\]

\[
(QCQPRD) \quad \begin{bmatrix} \bar{h} - \theta & \frac{1}{2} c^\top \\ \frac{1}{2} c & \bar{Q} \end{bmatrix} \succeq 0 \\
\mu \geq 0
\]

which is exactly the SDP dual. So assuming our Slater condition is satisfied, the SDP relaxation of (7.4.2)-(7.4.6) is equal to the SDP dual optimal value which is equal to the
Lagrangian dual value. If there is a duality gap, then the primal semidefinite relaxation
will provide a better lower bound on the optimal value of \((QCQP)\).

Research Question: We have examined the SDP relaxation of \((7.4.2)-(7.4.6)\). Assume
for the moment that without constraint \((7.4.6)\) the problem was convex. Then another
relaxation of the primal is to replace \((7.4.6)\) with \(x_i \leq 1\) and \(x_i \geq 0\). How would this
convex relaxation compare with SDP relaxation?

7.6 Lower bound value comparisons

In general, problem \((QCQP)\) is a difficult nonconvex problem and it is typical to solve a
relaxation of this problem. Obviously we would like to find convex relaxations, and these
relaxations to be as tight as possible. Here are the relationships so far. Denote the optimal
value of problem \((\ast)\) by \(v(\ast)\).

Proposition 7.6.1. (Quadratic Programming Bounds)

1. \(v(QCQP) \geq v(QCQPR)\)
2. \(v(QCQPR) \geq v(QCQPRD)\)
3. \(v(QCQPRD) = v(LD)\)

Proof of Proposition 7.6.1: Part 1 follows from the fact that we defined \((QCQPR)\) as
a relaxation of \((QCQP)\). Part 2 is a consequence of weak duality. Problem \((QCQPRD)\) is
the dual of \((QCQPR)\). Part 3, the equality between the Lagrangian dual of \((QCQP)\) and
the dual problem \((QCQPRD)\) was shown in Section 7.5.2. □

A consequence of Proposition 7.6.1 is that if there is a duality gap between problems
\((QCQP)\) and \((QCQPRD)\), then solving \((QCQP)\) will yield better (larger) lower bounds
than the Lagrangian dual.

Next we compare the bounds derived from semi-definite relaxation with that of a simple
continuous relaxation of the original problem. Consider the equality constrained quadratic
0–1 program:

\[
\min_x x^\top Qx + c^\top x \\
\text{s.t. } Ax = b \\
x \in \{0,1\}^n
\]  

(QP)

Without loss, assume that \(\text{diag}(Q) = 0\). Indeed, suppose \(Q_{ii} \neq 0\) then the form \(x^\top Qx^\top\)
involves the term \(Q_{ii}x_i^2\). However, since \(x_i^2 = x_i\) we can simply adjust \(i\)th entry of \(c\) to
be $c'_i = c_i + Q_{ii}$. It is therefore valid to assume that $\text{diag}(Q) = 0$ through an appropriate change to $c$.

We consider two relaxations of (QP): the continuous relaxation

$$\min_x x^\top Qx + c^\top x$$

s.t. $Ax = b$

$$0 \leq x \leq 1$$

and the semidefinite relaxation

$$\min_{X,x} \langle Q, X \rangle + c^\top x$$

s.t. $Ax = b$

$$\text{diag}(X) = x$$

$$\begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0.$$ 

**Proposition 7.6.2.** $v(QPCR) \geq v(QPSDR)$

**Proof.** See Exercise 7.1. \qed

### 7.7 A quadratic problem reformulation

This is from Billionnett, Elloumi, and Plateau [5]. The original problem is a quadratic programming problem with arbitrary objective function and linear equality constraints with all variables restricted to be 0 or 1. The binary restriction is treated by adding the constraint $x_i^2 = x_i$ for all $i$. Problem (QP) is

$$\begin{align*}
(QP) \quad & \min x^\top Q_0 x + c_0^\top x \\
& Ax = b \\
& x_i^2 = x_i, \quad i = 1, \ldots, n
\end{align*}$$

We create an equivalent problem by adding redundant constraints. In particular, we multiply each equality constraint $\sum_{j=1}^n a_{kj}x_j - b_k = 0, k = 1, \ldots, m$ by each $x_i, i = 1, \ldots, n$.

$$\begin{align*}
(QP) \quad & \min x^\top Q_0 x + c_0^\top x \\
& Ax = b \\
& x_i^2 = x_i, \quad i = 1, \ldots, n \\
& x_i \left( \sum_{j=1}^n a_{kj} x_j - b_k \right) = 0, \quad k = 1, \ldots, m, \quad i = 1, \ldots, n
\end{align*}$$
Next define:

\[ Q_{ki} = \frac{1}{2} A_{ki} + \frac{1}{2} A_{ki}^\top \]
\[ c_{ki}^\top = [0, 0, \ldots, -b_k, 0, \ldots, 0] \]

In the definition of the \( Q_{ki} \), \( A_{ki} \) is a matrix with all zero elements, except for column \( i \) which row \( k \) of matrix \( A \). In the definition of \( c_{ki} \), the \(-b_k\) is in component \( i \). For example, if \( i = 2 \) and \( n = 3 \), then

\[
Q_{k2} = \begin{bmatrix}
0 & \frac{1}{2} a_{k1} & 0 \\
\frac{1}{2} a_{k1} & a_{k2} & \frac{1}{2} a_{k3} \\
0 & \frac{1}{2} a_{k3} & 0
\end{bmatrix}
\]

Then

\[
x^\top Q_{k2} x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix}
0 & \frac{1}{2} a_{k1} & 0 \\
\frac{1}{2} a_{k1} & a_{k2} & \frac{1}{2} a_{k3} \\
0 & \frac{1}{2} a_{k3} & 0
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \\
x_3
\end{bmatrix} = a_{k1} x_1 x_2 + a_{k2} x_2^2 + a_{k3} x_2 x_3 = x_2 (a_{k1} x_1 + a_{k2} x_2 + a_{k3} x_3)
\]

Using the definition of \( Q_{ki} \) gives \((QP)\) expressed as

\[
(QP) \quad \min x^\top Q_0 x + c_0^\top x \\
A x = b \\
x_i^2 = x_i, \quad i = 1, \ldots, n \\
x^\top Q_{ki} x + c_{ki}^\top x = 0, \quad k = 1, \ldots, m, \quad i = 1, \ldots, n
\]

The SDP reformulation using the usual \( X = xx^\top \) and for the millionth time using the fact that \( x^\top Q_{ki} x = \langle Q_{ki}, x \rangle \) gives the equivalent

\[
(QP) \quad \min \langle Q_0, X \rangle + c_0^\top x \\
A x = b \\
\text{diag}(X) = x \\
\langle Q_{ki}, X \rangle + c_{ki}^\top x = 0, \quad k = 1, \ldots, m, \quad i = 1, \ldots, n \\
X = xx^\top
\]

The SDP relaxation of \((QP)\) is
7.7. A QUADRATIC PROBLEM REFORMULATION

\[(QPR)\quad \min(Q_0, X) + c_0^\top x \]
\[
Ax = b \\
\text{diag}(X) = x \\
\langle Q_{ki}, X \rangle + c_{ki}^\top x = 0, \quad k = 1, \ldots, m, \ i = 1, \ldots, n \\
\begin{bmatrix} 1 & x^\top & X \end{bmatrix} \succeq 0
\]

Problem \((QPR)\) is the same as problem \((SDQP)\) in Billionnet, Elloumi, and Plateau [5]. Taking the dual of \((QPR)\) exactly as in Section ?? gives

\[
(QPRD) \quad \begin{bmatrix} \bar{h} - \theta & \frac{\max \theta}{2} c^\top \\ \frac{1}{2} \bar{c} & \bar{Q} \end{bmatrix} \succeq 0 \quad \mu \geq 0 \quad \lambda, \nu, \theta, \phi \text{ free}
\]

\[
\bar{h} = -\sum_{k=1}^{m} \lambda_k b_i \\
\bar{c} = c_0 + \sum_{k=1}^{m} \lambda_k a_k + \sum_{i=1}^{n} \sum_{k=1}^{m} \nu_{ki} c_{ki} - \phi \\
\bar{Q} = Q_0 + \sum_{i=1}^{n} \sum_{k=1}^{m} \nu_{ki} Q_{ki} + \text{Diag}(\phi)
\]

**Notation Note:** In our development our \(\nu_{ki}\) dual variables correspond to the \(\alpha_{ki}\), the \(\phi_i\) to the \(u_i\), and the \(\lambda_k\) to the \(\beta_k\) in Billionnet, Elloumi, and Plateau [5]. Based on Section ??, which shows the equivalence of the Lagrangian dual with \((QPRD)\), problem \((QPRD)\) has the same objective function value as

\[
(LD) \quad \max L(\lambda, \nu, \phi) \quad \bar{Q} \succeq 0 \\
\bar{c} \in \mathcal{R}(\bar{Q}) \\
\nu, \phi \in \mathbb{R}^n \\
\lambda \in \mathbb{R}^m
\]
where

\[ L(\lambda, \nu, \phi) = \min \{ x^\top Q x + c^\top x + h \} \]

**Lemma 7.7.1.** Let \((\lambda, \nu, \phi)\) be an optimal solution to problem \((LD)\) with

\[ L(\lambda, \nu, \phi) = x^\top Q x + c^\top x + h \]

then

\[ x_i^2 \leq \bar{x}_i, \quad i = 1, \ldots, n \]

**Proof of Lemma 7.7.1:** Prove the contrapositive. Let \((\lambda, \nu, \phi)\) be an arbitrary feasible dual solution. That is, for \((\lambda, \nu, \phi)\) we have \(Q \succeq 0\) and \(c \in \mathbb{R}^n\). Let

\[ x = \text{argmin} \{ x^\top Q x + c^\top x + h \} \]

We know \(x\) exists by Lemma 7.5.1. Show if there is a \(j\) with \(x_j^2 > \bar{x}_j\) then \((\lambda, \nu, \phi)\) is not an optimal dual solution. We do this by showing that we can strictly improve the objective function value.

Define a new \(\hat{\phi} = \phi + \epsilon e_j\). Redefine \(\hat{c}\) and \(\hat{Q}\) accordingly. Since we are adding a positive number to diagonal element \(i\) it is true that \(\hat{Q} \succeq 0\). Ignore the \(\hat{c} \in \mathbb{R}(\hat{Q})\) issue for now.

Since \(\hat{Q} \succeq 0\) it follows that \(L(\lambda, \nu, \hat{\phi})\) is not negative infinity and has a solution in \(\hat{x} \in \mathbb{R}^n\). Show

\[ \hat{x}^\top \hat{Q} \hat{x} + \hat{c}^\top \hat{x} + \hat{h} > \bar{x}^\top \bar{Q} \bar{x} + \bar{c}^\top \bar{x} + \bar{h} \]

\[ \hat{x}^\top \hat{Q} \hat{x} + \hat{c}^\top \hat{x} = (\hat{x}^\top \bar{Q} \hat{x} + \hat{c}^\top \hat{x} + \epsilon \hat{x}^\top \text{Diag}(e_i) \hat{x} - \epsilon \hat{x}_i) \]

But

\[ \hat{x}^\top \bar{Q} \hat{x} + \hat{c}^\top \hat{x} \geq \bar{x}^\top \bar{Q} \bar{x} + \bar{c}^\top \bar{x} \]

\[ \epsilon \hat{x}^\top \text{Diag}(e_i) \hat{x} - \epsilon \hat{x}_i = \epsilon (\hat{x}^2 - \hat{x}) > 0 \]

and

\[ L(\lambda, \nu, \hat{\phi}) = \hat{x}^\top \hat{Q} \hat{x} + \hat{c}^\top \hat{x} > L(\lambda, \nu, \phi) \]

Now show \(\hat{c} \in \mathcal{R}(\hat{Q})\). Assume that this is not the case. Then by the Fredholm alternative there exists a \(u\) such that \(u^\top \hat{Q} = 0\) and \(u^\top \hat{c} \neq 0\). By definition, \(q_i = \hat{q}\) for \(i \neq j\) and \(\hat{q}_j = q + \epsilon e_j\) when \(i = j\). Then \(u^\top \hat{Q} = 0\) implies \(u^\top q_i = 0\) for all \(i \neq j\) and

\[ u^\top \hat{q}_j = u^\top (q_j + \epsilon e_j) = 0 \]
which implies
\[ u^\top \eta_j = -\epsilon u^\top e_j = -\epsilon u_j \]

Next observe
\[ u^\top Q u = (u^\top Q)u = (-\epsilon \eta_j e_j)u = -\epsilon u_j^2 \]

Also, since \( \bar{c} \in \mathcal{R}(\bar{Q}) \) there exist \( \alpha_i, i = 1, \ldots, n \) such that
\[ \bar{c} = \sum_{i=1}^{n} \alpha_i \bar{q}_i \]

Then
\[ \bar{c}^\top u = \sum_{i=1}^{n} \alpha_i \bar{q}_i^\top u = \alpha_j \bar{q}_j^\top u = -\alpha_j (\epsilon u^\top e_j) = -\epsilon \alpha_j u_j \]

and
\[ u^\top Q u + \bar{c}^\top u + \bar{h} = -\epsilon (u_j^2 + \alpha_j u_j) + \bar{h} \]

Now argue that \( u_j^2 + \alpha_j u_j \neq 0 \). First show \( u_j \neq 0 \). If \( u_j = 0 \), then \( u^\top e_j = 0 \) which implies
\[ 0 = u^\top (\eta_j + \epsilon e_j) = u^\top \eta_j = 0 \]

so \( u^\top \eta = 0 \). But, \( u^\top \hat{c} \neq 0 \) so
\[ u^\top \hat{c} = u^\top (\alpha_j \eta_j + \epsilon e_j) = u^\top \alpha_j \eta_j = u^\top \bar{c} \]

so we have \( u^\top \hat{c} = u^\top \bar{c} \neq 0 \) and \( u^\top \eta = 0 \) which contradicts \( \bar{c} \in \mathcal{R}(\bar{Q}) \). Therefore \( u_j \neq 0 \).

So now the worry is having \( u_j^2 + \alpha_j u_j = 0 \) for a nonzero \( u_j \). If \( u_j \neq 0 \) but \( u_j^2 + \alpha_j u_j = 0 \) then we can scale \( u \) by \( \lambda \neq 0 \) and \( \lambda \neq 1 \) and have
\[ \lambda u^\top \eta \lambda u + \bar{c}^\top \lambda u + \bar{h} = -\epsilon (u_j^2 \lambda^2 + \alpha_j u_j \lambda) + \bar{h} \]

and \( u_j^2 + \alpha_j u_j = 0 \) implies \( u_j^2 \lambda^2 + \alpha_j u_j \lambda \neq 0 \) for \( \lambda \neq 1 \). So we can assume without loss that \( u_j^2 + \alpha_j u_j \neq 0 \). We can also assume without loss that \( u_j \) has the same sign as \( \alpha_j \) since we could replace \( u \) with \(-u\). Then, without loss, \( u_j^2 + \alpha_j u_j > 0 \). Then as \( \lambda \to \infty \), \( u_j^2 \lambda^2 + \alpha_j u_j \lambda \to \infty \). Then as \( \lambda \to \infty \)
\[ \lambda u^\top \eta \lambda u + \bar{c}^\top \lambda u + \bar{h} = -\epsilon (u_j^2 \lambda^2 + \alpha_j \lambda u_j) + \bar{h} \]

goes to \(-\infty\). Then assuming \( \hat{c} \notin \mathcal{R}(\hat{Q}) \) leads to the contradiction that \( L(\bar{\lambda}, \bar{v}, \hat{\phi}) = -\infty \). Therefore \( \hat{c} \notin \mathcal{R}(\hat{Q}) \) □
Consider the following problem that is equivalent to \((QP)\) problem and denoted by 
\((C(QP))\) in Billionnett, Elloumi, and Plateau.

\[
(C(QP)) \max L'(\nu, \phi) \quad \bar{Q} \succeq 0 \\
\nu, \phi \in \mathbb{R}^n
\]

where

\[
L'(\nu, \phi) = \min \{ x^\top \bar{Q} x + \hat{c}^\top x \mid Ax = b \}
\]

and

\[
\hat{c} = c_0 + \sum_{i=1}^{n} \sum_{k=1}^{m} \nu_{ki} c_{ki} - \phi
\]

**Observations:**

1. Problem \(L'(\nu, \phi)\) is a relaxation of problem \((QP)\) since we have dropped the binary constraints on the \(x\) vector.
2. When \(x\) is binary and \(A\bar{x} = b\),
\[
\bar{x}^\top \bar{Q} \bar{x} + \hat{c}^\top \bar{x} = \bar{x}^\top \bar{Q} \bar{x} + c_0^\top \bar{x}
\]
3. Solving problem \((C(QP))\) gives the best convex relaxation of problem \((QP)\) is the sense of the largest lower bound.

**Want to Show (WTS):** Show that if \(\bar{\nu}, \bar{\phi}\), and \(\bar{\lambda}\) are optimal dual values for \((LD)\), then \(\bar{\nu}, \bar{\phi}\) are optimal dual values for \((C(QP))\). Note that if we can show problem \((QPRD)\) satisfies a Slater condition, then \(v(QPR) = v(QPRD)\) and by part 3. of Proposition 7.6.1 \(v(QPR) = v(QPRD) = v(LD)\).

Since \(\bar{Q} \succeq 0\), \(x^\top \bar{Q} x + \hat{c}^\top x\) is convex. It follows that

\[
v(L'(\nu, \phi)) = \max_{\lambda \in \mathbb{R}^n} \min \{ x^\top \bar{Q} x + \hat{c}^\top x + \lambda (Ax - b) \}
\]

Then we can rewrite \((C(QP))\) as
\[ (C(QP)) \quad \max_{\nu, \phi, \lambda} \min \{ x^\top Q x + \tau^\top x + h \} \]
\[ \overline{Q} \succeq 0 \]
\[ \nu, \phi \in \mathbb{R}^n \]
\[ \lambda \in \mathbb{R}^m \]

To prevent the inner minimization from being $-\infty$ we add the redundant constraint $\tau \in \mathcal{R}(\overline{Q})$.

\[ (C(QP)) \quad \max_{\nu, \phi, \lambda} \min \{ x^\top Q x + \tau^\top x + h \} \]
\[ \overline{Q} \succeq 0 \]
\[ \tau \in \mathcal{R}(\overline{Q}) \]
\[ \nu, \phi \in \mathbb{R}^n \]
\[ \lambda \in \mathbb{R}^m \]

By 7.7.1 problem \((C(QP))\) is equivalent to

\[ (C(QP)) \quad \max_{\nu, \phi, \lambda} \min \{ x^\top Q x + \tau^\top x + h \mid 0 \leq x_i \leq 1, \; i = 1, \ldots, n \} \]
\[ \overline{Q} \succeq 0 \]
\[ \tau \in \mathcal{R}(\overline{Q}) \]
\[ \nu, \phi \in \mathbb{R}^n \]
\[ \lambda \in \mathbb{R}^m \]

By construction, \(v(LD) = v(C(QP))\) and we can use the optimal dual values from problem \((LD)\) to find the tightest reformulation relaxation. If we can verify the Slater condition then this value is also equal to \(v(QPR)\). Verifying Slater is left for another time. □

### 7.8 Combinatorial Optimization Applications

#### 7.8.1 Maximum cut

Consider an undirected, edge-weighted graph \(G\) with no self-loops (that is, no edges which are adjacent to only a single node). Let \(V\) denote the vertices of the graph \(G\) and \(E\) the set edges or ordered pairs of nodes; that is, \(E = \{ij : i, j \in V, i < j\}\). Since the edges are undirected, \(ij\) and \(ji\) point to the same edge in the graph, and hence in \(E\) we only take the representation where \(ij\) is such that \(i < j\). Thus, we assume that \(G\) is the complete graph on \(n\) vertices, but allow some of its edge-weights to be zero. Let \(n = |V|\). The edge-weights are captured in the weighted \(n\) by \(n\) adjacency matrix \(A = (a_{ij})_{i,j=1}^n\). Again, since \(G\) is
undirected, the entries \( a_{ij} \) and \( a_{ji} \) in \( E \) refer to the same edge and hence have the same weight; that is, \( a_{ij} = a_{ji} \). This implies that \( A \) is symmetric.

A classic problem in combinatorial optimization is to separate the vertex set \( V \) into two subsets – \( S \) and \( \bar{S} = V \setminus S \) – such that the weight of edges which lie in between \( S \) and \( \bar{S} \) is maximized. The problem of finding such a cut is called MAX-CUT. More formally, let \( \delta(S) = \{ij \in E : i \in S, j \in \bar{S}\} \) denote those edges with one end in \( S \) and the other in \( \bar{S} \). To solve the MAX-CUT problem on graph \( G \) is to find an optimal solution to the problem:

\[
z_{mc} = \max_{S \subseteq V} \sum_{ij \in \delta(S)} a_{ij} \tag{7.8.1}
\]

It is important to note that if \( ij \) is considered as part of \( \delta(S) \) then \( ji \) is not. Indeed from the discussion above, for \( kl \in E \) we have \( k < l \) and hence at most one of \( ij \) and \( ji \) lie in \( E \) (and thus \( \delta(S) \)). This avoids double counting in the summation over multiple representations of the same edge.

Let’s formulate this as a quadratic program. We introduce the vector \( x \in \{-1, 1\}^n \) to be characteristic of a choice of the \( S \). That is, we let \( x_i = 1 \) if \( i \in S \) and \( x_i = -1 \) if \( i \in \bar{S} \). Note that the condition \( x \in \{-1, 1\}^n \) is easily expressible as a quadratic constraint: \( x_i^2 = 1 \) for all \( i = 1, \ldots, n \).

The product \( x_i x_j \) now inherits the following useful interpretation:

\[
x_i x_j = \begin{cases} 1 & \text{if } i, j \in S \text{ or } i, j \in \bar{S} \\ -1 & \text{if } (i \in S \text{ and } j \in \bar{S}) \text{ or } (i \in \bar{S} \text{ and } j \in S) \end{cases} \tag{7.8.2}
\]

In other words, the value of the product \( x_i x_j \) reveals whether or not \( ij \in \delta(S) \).

The question remains as to how we can formulate the objective of (7.8.1) in terms of the variables \( x_i \). Observe that

\[
\sum_{ij \in \delta(S)} a_{ij} = \sum_{ij \in E} \alpha_{ij} a_{ij}
\]

where \( \alpha_{ij} = 1 \) if \( ij \in \delta(S) \) and 0 otherwise. Notice that \( \alpha_{ij} \) is a close cousin of the product term \( x_i x_j \) as described in (7.8.2). Indeed, one can verify that:

\[
\alpha_{ij} = \frac{1 - x_i x_j}{2}
\]

and hence we can express the objective function in (7.8.1) equivalently as:

\[
\sum_{ij \in \delta(S)} a_{ij} = \sum_{ij \in E} \frac{1 - x_i x_j}{2} a_{ij}.
\]

This is a quadratic function in the \( x_i \) and we are happy. However, one final step is to express the objective function compactly as a quadratic form. To do so, consider the
matrix \( L = \text{Diag}(Ae) - A \) called the Laplacian of \( A \). The \((i,j)\)th entry of \( L \) is:

\[
\ell_{ij} = \begin{cases} 
-a_{ij} & \text{for } i \neq j \\
\sum_{k \neq j} a_{ik} & \text{for } i = j
\end{cases}
\]  

(7.8.3)

We claim that we can express our objective function as a quadratic form in \( L \). That is,

\[
\sum_{ij \in E} \frac{1 - x_i x_j}{2} a_{ij} = \frac{1}{4} x^T L x.
\]  

(7.8.4)

We start by developing the right-hand side:

\[
\frac{1}{4} x^T L x = \frac{1}{4} \sum_{i,j=1}^{\infty} \ell_{ij} x_i x_j
\]

\[
= \frac{1}{4} \left( \sum_{i \neq j} (-a_{ij}) x_i x_j + \sum_{i} \sum_{k \neq i} a_{ik} x_i^2 \right)
\]

\[
= \frac{1}{4} \left( - \sum_{i,j=1}^{\infty} a_{ij} x_i x_j + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \right).
\]  

(7.8.5)

In the second equality we simply break up the sum over pairs \((i,j)\) into those terms where \(i \neq j\) and those where \(i = j\) and use the alternate values for \(\ell_{ij}\) under those conditions. The last equality just cleans things up a bit and uses the facts that \(a_{ii} = 0\) (there are no self loops in the graph) and \(x_i^2 = 1\) for all \(i\).

To get things looking more comparable, let’s process the left-hand side for a moment:

\[
\sum_{ij \in E} \frac{1 - x_i x_j}{2} a_{ij} = \frac{1}{2} \left( \sum_{ij \in E} a_{ij} - \sum_{ij \in E} (-a_{ij} x_i x_j) \right).
\]  

(7.8.6)

Note that

\[
\sum_{ij \in E} a_{ij} = \frac{1}{2} \sum_{i,j=1}^{\infty} a_{ij}
\]

(recall our discussion about double counting) and similarly that

\[
\sum_{ij \in E} (-a_{ij}) x_i x_j = \frac{1}{2} \sum_{i,j=1}^{\infty} (-a_{ij}) x_i x_j.
\]

Plugging these into (7.8.6) yields (7.8.5) (after a bit of algebra). This establishes our claim in (7.8.4).

To Do:
1. Put in an example graph and the corresponding quadratic program. Show what $L$ is for the graph.

2. Show the SDP relaxation of the quadratic program.

### 7.8.2 Quadratic assignment problem

The basic idea of the quadratic assignment problem is to assign a set of “facilities” to a set set of “locations.” In this development we assume that there are $n$ facilities and $n$ locations. There are there input parameter matrices. The matrix $F = (f_{ij})$ of required flow between facilities $i$ and $j$. The matrix $D = (d_{kl})$ of distances between locations $k$ and $l$. The matrix $B = (b_{ik})$ of costs of assigning facility $i$ to location $k$. Let $x_{ik} = 1$ if we assign facility $i$ to location $k$, 0 otherwise. An integer programming formulation is:

$$
\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} f_{ij} d_{kl} x_{ik} x_{jl} + \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} x_{ik} \quad (7.8.7)
$$

$$
\sum_{i=1}^{n} x_{ik} = 1, \quad k = 1, \ldots, n \quad (7.8.8)
$$

$$
\sum_{k=1}^{n} x_{ik} = 1, \quad i = 1, \ldots, n \quad (7.8.9)
$$

$$
x_{ik} \in \{0, 1\}, \quad i = 1, \ldots, n, \quad k = 1, \ldots, n \quad (7.8.10)
$$

In the objective function (7.8.7) if facility $i$ is assigned to location $k$ and facility $j$ is assigned location $l$ we have $x_{ik} = 1$ and $x_{jl} = 1$ so $x_{ik} x_{jl} = 1$ and we incur cost $f_{ij} d_{kl}$. Constraint (7.8.8) enforces the condition that each location must be assigned a facility. Constraint (7.8.9) enforces the condition that each facility must be assigned a location.

**Applications:**

- Consider an application where the facilities are ships and the locations are loading docks. In this case $x_{ik}$ is 1 if ship $i$ is assigned to loading dock $k$. The parameter $f_{ij}$ represents the tonnage that must be transferred from ship $i$ to ship $j$ and the parameter $d_{kl}$ is the distance between loading dock $k$ and loading dock $l$.

- Consider an application where the facilities are airplanes and the locations are terminal gates. In this case $x_{ik}$ is 1 if plane $i$ is assigned to terminal gate $k$. The parameter $f_{ij}$ represents the number of passengers on plane $i$ that have a connecting flight on plane $j$ and the parameter $d_{kl}$ is the distance between terminal gate $k$ and terminal gate $l$. 
Consider the traveling salesperson problem (TSP). If there are \( n \) cities, the TSP can be viewed as finding a permutation of cities other than the home city. That is \( x_{ik} = 1 \) if city \( i \) is the \( k \)th city in the tour. In terms of cost, let \( f_{ij} \) be the distance between cities \( i \) and \( j \) for all \( i \) and \( j \). Then set \( d_{kl} = 1 \) if \( l = k + 1 \), otherwise \( d_{kl} = 0 \). Thus if city \( i \) is city number \( k \) in the path and city \( j \) is city \( k + 1 \) in the path we have \( x_{ik} = 1 \) and \( x_{j,k+1} = 1 \) and we incur cost \( c_{ij} \). Finally, let \( b_{ik} = 0 \) for all \( i \) and \( k \).

Now let’s formulate in terms of semidefinite programming. Let \( X \) be a permutation matrix and \( X_n \) denote the set of \( n \times n \) permutation matrices. The quadratic assignment problem is

\[
\min \langle F, XDX^\top \rangle + \langle B, X \rangle
\]

\[
X \in X_n
\]

Of course we need to better characterize the condition \( X \in X_n \) so we now write the problem as

\[
\min \langle F, XDX^\top \rangle + \langle B, X \rangle
\]

\[
XX^\top = I
\]

\[
X^2_{ij} - X_{ij} = 0
\]

Much research has gone into various relaxations for the quadratic assignment problem, especially semidefinite programming relaxations. See for example, [31].

### 7.8.3 Stable set problem

The primal formulation for the stable set problem is

\[
\max \ w^\top x
\]

\[
x_i x_j = 0, \quad \forall (i, j) \in E \subseteq S^n
\]

\[
x_i^2 = x_i, \quad i = 1, \ldots, n
\]

In Figure 7.8.3, the nodes \{1, 2, 3\} are a stable set. An example constraint is \( x_2x_7 = 0 \). The stable set problem can also be written as

\[
\max \ w^\top x
\]

\[
x^\top (\frac{1}{2}E_{ij} + \frac{1}{2}E_{ji})x = 0, \quad \forall (i, j) \in E
\]

\[
x_i^2 = x_i, \quad i = 1, \ldots, n
\]
Figure 7.1: Illustrating a Stable Set

Note that the matrix \( \frac{1}{2}E_{ij} + \frac{1}{2}E_{ji} \) is symmetric. Let’s apply our standard SDP relaxation and get

\[
\begin{align*}
\max w^\top x \\
\langle \frac{1}{2}(E_{ij} + E_{ji}), X \rangle &= 0, \quad \forall (i, j) \in E \\
\Diag(X) &= x \\
X - xx^\top &\succeq 0 \\
X &\in \mathcal{S}^n
\end{align*}
\]

This can be written as

\[
\begin{align*}
\max w^\top \Diag(X) \\
\langle \frac{1}{2}(E_{ij} + E_{ji}), X \rangle &= 0, \quad \forall (i, j) \in E \\
X - \Diag(X)\Diag(X)^\top &\succeq 0
\end{align*}
\]

Observe that

\[
\langle \frac{1}{2}(E_{ij} + E_{ji}), X \rangle = \frac{1}{2}(X_{ij} + X_{ji})
\]

But \( X_{ji} = X_{ij} \) since \( X \in \mathcal{S}^n \) so this becomes \( X_{ij} = 0 \) Then we have

\[
\begin{align*}
\max w^\top \Diag(X) \\
X_{ij} &= 0, \quad \forall (i, j) \in E \\
X - \Diag(X)\Diag(X)^\top &\succeq 0
\end{align*}
\]
This is Theorem 4.8 in Lemaréchal and Oustry.

**Question:** What does this have to do with the László number mentioned in Lemaréchal and Oustry? See also their reference [8].

### 7.9 The tres quadratics

The problem is

\[
(P) \quad \min x^\top Q_0 x + c_0^\top x \\
Ax = b \\
x^\top Q_i x + c_i^\top x = d_i, \quad i = 1, \ldots, p
\]

Assume \( A \) is an \( m \times n \) matrix of rank \( m \). We make no positive semidefinite assumptions about \( Q_i, i = 1, \ldots, p \). Here are the tres quadratics:

#### 7.9.1 Quadratic uno

The original objective function quadratic \( x^\top Q_0 x \) which may be nonconvex. We define **quadratic uno**, \( Q_1(x) \) to be

\[
Q_1(x) = x^\top Q_0 x \tag{7.9.1}
\]

#### 7.9.2 Quadratic dos

First, form a partial Lagrangian that does not include the equality constraints.

\[
L(x, \mu) = \min \{ x^\top Q_0 x + c_0^\top x + \sum_{i=1}^p (\mu_i(x^\top Q_i x + c_i^\top x - d_i)) \mid Ax = b \}
\]

Define **quadratic dos**, \( Q_2(x, \mu) \) by:

\[
Q_2(x, \mu) = x^\top Q_2(\mu)x \tag{7.9.2}
\]

\[
Q_2(\mu) = (Q_0 + \sum_{i=1}^p \mu_i Q_i) \tag{7.9.3}
\]

so the Lagrangian problem is

\[
L(\mu) = \min \{ Q_2(x, \mu) + c_0^\top x + \sum_{i=1}^p (\mu_i(c_i^\top x - d_i)) \mid Ax = b \}
\]
and the dual problem \((D)\) is

\[
\max\{L(\mu) \mid \mu \in \mathbb{R}^p\}
\]

We make the assumption that there is at least one \(\mu\) such that \(L(\mu) > -\infty\). Then for \(\mu\)
\(Q_2(\mu)\) is positive semidefinite over the kernel of \(A\). Of course, it may be the case for some
problems \(L(\mu) = -\infty\) for all \(\mu\). Consider the following example.

Example:

\[
\min x^\top \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ \end{bmatrix} x \\
\text{s.t.} \quad x_2^2 + x_4^2 = 1 \\
\quad x_2 = 4 \\
\quad x_1 - x_3 = 0 
\]

\[
L(\mu) = \min x^\top \begin{bmatrix} -1 & -1 + \mu & -1 + \mu \\ -1 + \mu & -1 & -1 + \mu \\ \end{bmatrix} x - \mu \\
\text{s.t.} \quad x_2 = 4 \\
\quad x_1 - x_3 = 0 
\]

In this case, \(Ax = b\) corresponds to

\[
\{ (x_1, x_2, x_3, x_4) \mid x_2 = 4, x_1 - x_3 = 0 \} 
\]

Solutions to \(Ax = b\) are characterized by

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \end{bmatrix} 
\]

Regardless of the value of \(\mu\), we can make \(L(\mu)\) go to negative infinity by setting \(x_2 = 4\),
\(x_4 = 0\), and letting \(x_1\) and \(x_3\) go to \(\infty\).
7.9. THE TRES QUADRATICS

7.9.3 Quadratic tres

Now the really neat result of Faye and Roupin [11] is their extension of Debreu’s Lemma. In what follows we drop the overline on the $\mu$ and assume we are talking about $\mu$ such that $Q_2(\mu)$ is positive semidefinite over the kernel of $A$. Then by Faye and Roupin there is a quadratic function $F(x) = x^\top Fx$ (I am getting loose here and dropping the linear and constant term, put back in later) that is constant over the set $\{x \mid Ax = b\}$ that we can use to convexify quadratic dos over $\mathbb{R}^n$. This defines a new quadratic.

\begin{align*}
Q_3(x, \mu, F) &= x^\top Q_3(\mu, F)x \\
Q_3(\mu, F) &= Q_2(\mu) + F
\end{align*}

(7.9.4) \hspace{1cm} (7.9.5)

Now here are some really neat observations/conjectures that need to be proved rigorously (I have been very cavalier with the linear and constant terms.).

**Key Observation 1 (Partial Lagrangian Results):** First define:

$$
\Gamma(\mu, \lambda) = \min_{x \in \mathbb{R}^n} \{Q_3(x, \mu) + c_0^\top x + \sum_{i=1}^{p} \left( \mu_i (c_i^\top x - d_i) \right) + \lambda(Ax - b) \}
$$

Notice in the definition of $\Gamma(\mu, \lambda)$ we are working with quadratic tres. Then, due to the fact that $Q_2(\mu)$ is positive semidefinite over the kernel of $A$ and $F(x)$ is constant over $\{x \mid Ax = b\}$ we have

$$
L(\mu) = \max_{\lambda} \Gamma(\mu, \lambda)
$$

Then we have

$$
\max_{\mu} L(\mu) = \max_{\mu, \lambda} \Gamma(\mu, \lambda)
$$

**Key Observation 2 (Problem Reformulation Equivalence):** The original problem ($P$) is equivalent to the following reformulation with convex objective function:

$$
\min_{x} x^\top Q_3(\mu, F)x + c_0^\top x \\
\quad Ax = b \\
\quad x^\top Q_i x + c_i^\top x = d_i, \quad i = 1, \ldots, p
$$

and even better is the result is the the best reformulation given $F$ is the $\mu$ that optimizes $L(\mu)$.

**Key Observation 3 (Semidefinite Result):** We can add the redundant constraints to the original problem ($P$) and get:
CHAPTER 7. SEMIDEFINITE PROGRAMMING

\[
\min x^\top Q_0 x + c_0^\top x \\
Ax = b \\
x^\top Q_i x + c_i^\top x = d_i, \quad i = 1, \ldots, p \\
F(x) = 0
\]

Note we have added the \( F(x) = 0 \) redundant constraints. Now take the standard lifting semidefinite relaxation of this problem. This is the bidual of the semidefinite problem that is equivalent to the Lagrangian dual

\[
\max_\mu L(\mu) = \max_{\mu, \lambda} \Gamma(\mu, \lambda)
\]

KEY RESEARCH QUESTIONS:

- How do we find the best \( F(x) \)?
- What kinds of results do we get, if instead of Lagrangian approaches, we try similar approaches based on barrier functions?
- What about special structures? There are a ton of important applications where all of the quadratic nonlinear terms are of the form \( x_i y_j \) where the \( x \) variables are indexed over a set \( I \) and the \( y \) variables are indexed over a set \( J \). Is there a good way to define \( F \) for this broad class of problems? I also think what Fatma was doing fit into this category.

7.10 The tres relaxations

Consider the problem \((P)\):

\[
(P) \quad \min x^\top Q_0 x + c_0^\top x \\
Ax = b \\
x_i \in \{0, 1\}, \quad i = 1, \ldots, n
\]

Relaxation One (the continuous relaxation):

\[
(PR1) \quad \min x^\top Q_0 x + c_0^\top x \\
Ax = b \\
x_i \leq 1, \quad i = 1, \ldots, n \\
x_i \leq 0, \quad i = 1, \ldots, n
\]
Relaxation Two (the semidefinite relaxation):

\[(PR2) \quad \min \langle Q_0, X \rangle + c_0^\top x \]

\[Ax = b \]

\[\text{diag}(X) = x \]

\[
\begin{bmatrix}
1 & X^T \\
x & X
\end{bmatrix} \succeq 0
\]

Relaxation Three (the semidefinite reformulation): Construct a the problem reformulation based on convexifying the objective function. Then take the linear relaxation.

**KEY RESEARCH QUESTION:** Compare and contrast the three relaxations.

### 7.11 Exercises

**Exercise 7.1.** Prove Proposition 7.6.2, \( v(QPCR) \geq v(QPSDR) \)

**Exercise 7.2.** In Section 7.4.2, a semidefinite programming relaxation is given for 0/1 quadratic programs with quadratic equality constraints. Why were quadratic inequality constraints not included in the formulation?
Appendix A

Appendices

A.1 Partially ordered sets and Zorn’s Lemma

Let $S$ be a set. A binary relation (or simply a relation) $\leq$ is a subset of $S \times S$ where the ordered pair $(x, y) \in \leq$ implies there is some relation of interest between $x$ and $y$. We use the notation $x \leq y$ when $(x, y) \in \leq$. The notation evokes a specific example of a binary relation, the “inequality” relation $\leq$ on the reals, where we have $\leq$ consists of those ordered pairs $(x, y)$ where $x$ is to the left of $y$ on the real line. Of course, the relation $\leq$ on the real line is so familiar to us that to think about it in these terms seems a bit funny, but nonetheless it is useful formalization of the concept of “relation”.

Let $\leq$ be a binary relation on the set $S$. We call $\leq$ a partial order on $S$ if for all $x, y, z \in S$:

(V1) $x \leq x$,
(V2) $x \leq y$ and $y \leq z$ implies $x \leq z$,
(V3) $x \leq y$ and $y \leq x$ implies $x = y$.

Property (V1) is called reflexivity, (V2) transitivity and (V3) antisymmetry.

The pair $(S, \leq)$ where $\leq$ is a partial order on $S$ is called a partially ordered set or simply a poset. When the relation $\leq$ is implicit we may simply call $S$ a poset to lighten the notation.

Example A.1.1.

Let $X$ be a set. Let $S$ denote the set of all subsets of $X$. There is a natural relation on these subsets: we say $A \leq B$ if $A \subseteq B$. It is straightforward to verify that $\leq$ defined in this way is a partial order. We sometimes call this relation “containment” and say things like “$S$ is partially ordered under containment”.

If a binary relation $\leq$ on a set $S$ has, in addition to (V1)-(V3), the following property:
For every \( x, y \in S \) either \( x \preceq y \) or \( y \preceq x \).

then the poset \((S, \preceq)\) is called a totally ordered set. Not every set is totally ordered:

**Example A.1.2.** Consider the partially ordered set \((\mathbb{R}^2, \leq)\) where \( \leq \) is the “usual” order on \( \mathbb{R}^2 \); that is, \( x \leq y \) when \( x_1 \leq y_1 \) and \( x_2 \leq y_2 \). Consider the two points \((2,1)\) and \((1,2)\) then clearly \((2,1) \not\preceq (1,2)\) and \((1,2) \not\preceq (2,1)\).

If \( x \) are \( y \) are two elements in \( S \) which violate condition (V4) we say \( x \) and \( y \) are incomparable. Even if a partially ordered set \((S, \preceq)\) is not totally ordered, there may exists subsets which are totally ordered under \( \preceq \). We call these subsets chains in \( S \).

**Example A.1.3.** Let \( S \) be all subsets of the finite set \( X = \{a, b, c\} \) partially ordered under containment. Clearly there are incomparable elements in \( S \); for instance, \( \{a, b\} \) and \( \{a, c\} \). However, we can easily find chains in \( S \), one such example being \( C = \{\{a\}, \{a, b\}\} \).

Let \((S, \preceq)\) be a partially ordered set. Consider a subset \( A \) of \( S \). An upper bound on \( A \) in \( S \) is any element \( u \) in \( S \) which has \( a \preceq u \) for all \( a \in A \). In other words, \( x \) is “higher” in the ordering than all the elements in \( A \). For instance \( C \) in the previous example has an upper bound of \( \{a, b, c\} \). Note that the upper bound of \( A \) need not be in \( A \). Not all chains have upper bounds:

**Example A.1.4.** Consider again the poset \((\mathbb{R}^2, \leq)\) where \( \leq \) is the usual order on \( \mathbb{R}^2 \) and consider the chain \( C = \{(a, a) : a \in R\} \). Clearly, there is no upper bound on the set \( C \).

An element \( x \) of a partially ordered set \((S, \preceq)\) is called maximal of set \( A \subseteq S \) if there does not exist a \( y \in S \) such that \( x \preceq y \). A set may have more than one maximal element:

**Example A.1.5.** Consider the set \( A = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0 \} \) in \( \mathbb{R}^2 \) with its usual ordering. Then all elements of \( A \) are maximal since they are all incomparable.

Also a subset may have no maximal element even though it is bounded:

**Example A.1.6.** Consider the set \( A = \{ x \in \mathbb{Q} : x^2 \leq 2, x \geq 0 \} \), where \( \mathbb{Q} \) denotes the rationals. This is a subset of the partially ordered set \((\mathbb{R}, \leq)\). Clearly the set has an upper bound of \( \sqrt{2} \) and in fact \( \sqrt{2} \) is the least upper bound of the set, that is for any element \( x \in A \) where \( x < \sqrt{2} \) there exists another element of \( A \), call it \( y \), with \( x \preceq y \). Since \( \sqrt{2} \notin A \), this implies \( A \) has no maximal elements.

The following lemma, called Zorn’s lemma, gives conditions for the existence of maximal elements of a poset:

**Lemma A.1.7 (Zorn’s Lemma).** If every chain in a partially ordered set \( X \) has an upper bound, then \( X \) has a maximal element.
An example of the use of Zorn’s Lemma see Theorem 2.1.1 and Theorem 2.5.2.

Zorn’s lemma turns out to be equivalent to something called the Axiom of Choice, which is a potentially objectionable assumption needed to make much of modern set theory (and thus much of modern mathematics) run. We take Zorn’s lemma without proof.

There is another useful equivalent of the Axiom of Choice which we will have occasion to use in these notes. It starts with a definition. A set $X$ is well-ordered by a total order $\preceq$ if every non-empty subset of $X$ has a least element. An element of $A$ is least if $x \preceq y$ for all $y \in A$. Here is a cool fact due to Zermelo:

**Theorem A.1.8 (Zermelo’s Theorem).** Every nonempty set can be well ordered.

This theorem essentially says, no matter how wild the set there exists an ordering which allows you to describe a least element in that set. It may be that the ordering is equally as wild as the set, but it is guaranteed to exist by Zermelo. See Theorem 2.4.18 and Proposition 2.5.12 for examples of its use.

### A.2 Matrix algebra

#### A.2.1 Vector basics

Let $x$ be an $n$-dimensional real vector (or in this appendix an $n$-vector) then we write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $x_i \in \mathbb{R}$ for $i = 1, \ldots, n$. That is, we think of vectors as a column of real numbers. The set of all $n$-dimensional real vectors is denoted $\mathbb{R}^n$, which is a vector space over $\mathbb{R}$.

Let $y$ be another $n$-vector. Then the inner product, denoted $x^\top y$ is defined as

$$x^\top y = \sum_{i=1}^{n} x_i y_i.$$

The Euclidean norm or simply the norm of a vector $x$ in $\mathbb{R}^n$ is $||x|| = \sqrt{x^\top x}$. Properties of norms are described in Chapter 6.

Two vectors $x$ and $y$ are said to be orthogonal if $x^\top y = 0$. The name orthogonal comes from the following geometry:

$$x^\top y := \sum_{i=1}^{n} x_i y_i = ||x|| ||y|| \cos \delta \tag{A.2.1}$$

where $\delta$ is the angle formed between vectors $x$ and $y$ (when seen lying in a plane which contains both vectors, see Figure A.1).
A.2.2 Matrix basics

Let $A$ be an $m \times n$ real matrix (every matrix we consider here has real entries) which is a two-dimensional array of real numbers with $m$ rows and $n$ columns. We denote the rows of $A$ by $A_i$ for $i = 1, \ldots, m$ and the columns of $A$ by $A_j$ for $j = 1, \ldots, n$. We denote the entries by $A_{ij}$. A regular old $n$-vector is often thought of as an $n \times 1$ matrix, that is a single column with $n$ entries.

Let $A$ be an $m \times n$ matrix. The transpose of $A$, denoted $A^\top$, is an $n \times m$ matrix with entries $(A^\top)_{ij} = A_{ji}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. In other words, $A^\top$ is the matrix $A$ with the rows swapped for columns and the columns swapped for rows.

Let $A$ be an $m \times n$ matrix, $x$ be an $n$-vector and $y$ an $m$-vector. The vector matrix product $Ax$ may be written as $Ax = \sum_{j=1}^{n} x_j A_j$, that is, a weighted sum of the columns of $A$. Similarly, the vector matrix product $A^\top y$ is a weighted sum of the columns of $A^\top$, but since the columns of $A^\top$ are the rows of $A$ this amounts to: $A^\top y = \sum_{i=1}^{m} y_i A_i$; that is, a weighted sum of the rows of $A$.

Let $A$ be an $m \times n$ matrix and let $B$ be a $n \times p$ matrix. Then the matrix-matrix product $AB$ is an $m \times p$ matrix with entries $(AB)_{ij} = A_i B_j$. In general, matrices are not commutative under this operation, that is, $AB \neq BA$. One can also verify the following behavior with respect to taking transposes: $(AB)^\top = B^\top A^\top$.

The set of all $m \times n$ real matrices form a vector space, denoted $\mathbb{R}^{m \times n}$, with the following operations: (i) vector addition of two matrices $A$ and $B$ defined by $(A + B)_{ij} = A_{ij} + B_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. and (ii) scalar multiplication of a real number $\alpha$ with a matrix $A$ defined by $(\alpha A)_{ij} = \alpha A_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

A matrix is square if the number of its rows equal the number of its columns. We say a square matrix $A$ is symmetric if $A = A^\top$. It is easy to verify that the set of all $n \times n$ symmetric matrices form a subspace of the vector space of $n \times n$ square matrices $\mathbb{R}^{n \times n}$. We call this subspace $\mathcal{S}^n$. 
2.3 Quadratic forms and definiteness

A quadratic form is a functional on $\mathbb{R}^n$ of the form:

$$F(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is an $n$-vector and the $a_{ij}$ are real numbers for $i, j = 1, \ldots, n$. This quadratic form can be expressed in terms of a unique symmetric matrix $A$ as follows:

$$F(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & \frac{1}{2}a_{21} & \cdots & \frac{1}{2}a_{n1} \\ \frac{1}{2}a_{21} & a_{22} & \cdots & \frac{1}{2}a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{n1} & \frac{1}{2}a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x^\top Ax.$$

In other words, there is a bijection between the quadratic forms $F$ and the symmetric matrices $A$. We make the following definitions, which classify types of quadratics form, and thus classify types of symmetric matrices.

**Definition.** Let $A$ be $n \times n$ symmetric matrix and let $F(x) = x^\top Ax$ be its associated quadratic form. Then we say $A$ is

(i) **positive definite** if $F(x) > 0$ for $x \neq 0$ in $\mathbb{R}^n$;

(ii) **positive semidefinite** if $F(x) \geq 0$ for all $x \in \mathbb{R}^n$.

(iii) **negative definite** if $F(x) < 0$ for $x \neq 0$ in $\mathbb{R}^n$.

(iv) **negative semidefinite** if $F(x) \leq 0$ for all $x \in \mathbb{R}^n$.

(v) **indefinite** if there exists an $x \in \mathbb{R}^n$ such that $F(x) > 0$ and there exists an $x' \in \mathbb{R}^n$ such that $F(x') < 0$.

Let $S^n_\mathbb{R}$ denote the set of positive semidefinite (PSD) matrices. It forms of a pointed convex cone in the vector space $S^n$. One common way to express that $A$ is PSD is to write $A \succeq 0$. If $A$ is positive definite we write $A \succ 0$. The set of positive definite matrices is denoted $S^n_{++}$ and forms the interior of the PSD cone in the Frobenius norm topology.
A.2.4 Trace

In matrix algebra, the *trace* function takes as input a square matrix and gives as output the sum of the entries along the diagonal. More precisely, for an \( n \times n \) matrix define \( \text{Tr}(A) = \sum_{i=1}^{n} A_{ii} \). Here are some properties of the trace:

**Proposition A.2.1.** Let \( A, B, C \in \mathbb{R}^{n \times n} \) and \( \alpha, \beta \in \mathbb{R} \). Then the following hold:

(i) \( \text{Tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ij} \)

(ii) \( \text{Tr}(\alpha A + \beta B) = \alpha \text{Tr}(A) + \beta \text{Tr}(B) \).

(iii) \( \text{Tr}(AB) = \text{Tr}(BA) \).

(iv) \( \text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA) \)

(v) If \( A \) is a symmetric matrix then it can be written as

\[
A = U \Lambda U^\top
\]

where \( U \) is a unitary matrix and \( \Lambda \) is a diagonal matrix with diagonal elements equal to the eigenvalues of \( A \). That is, the trace of \( \Lambda \) is the sum of the eigenvalues. If \( A \) is positive semidefinite the eigenvalues are nonnegative.

(vi) If \( A \) is positive semidefinite, then there is a unique positive semidefinite matrix \( A^{\frac{1}{2}} \) such that \( A^{\frac{1}{2}}A^{\frac{1}{2}} = A \). The matrix \( A^{\frac{1}{2}} \) is called the square root of \( A \).

(vii) the trace is equal to the sum of the eigenvalues

**Proposition A.2.2.** (Fejer’s Theorem) If \( A \in S^n \), then \( A \) is positive semidefinite if and only if \( \text{Tr}(AB) \geq 0 \) for all \( B \in S^n_+ \)

**Proof.** See Horn and Johnson [17] page 459. \qed

**Proposition A.2.3.** If \( z \in \mathbb{R}^n \), then

(i) the matrix \( zz^\top \) is positive semidefinite

(ii) the matrix \( zz^\top \) has rank 1

(iii) if \( Z = zz^\top \), and \( A \) is \( n \times n \), then

\[
z^\top Az = \text{Tr}(AZ) = \text{Tr}(ZA) \quad (A.2.2)
\]
Proof. Show (i): trivial.

Show (ii): let \( Z = z^* z^\top \). Row \( i \) of \( Z \) is \( z_i^* z \) and row \( j \) of \( Z \) is \( z_j^* z \). So multiply row 1 of \( Z \) by \(-z_i\) and row \( i \) by \( z_1 \). Adding the two results in a row of all zeros.

Show (iii): Since \( z^\top A z \) is a one by one matrix, by definition of trace, \( z^\top A z = \text{Tr}(z^\top A z) \). Repeated application of Proposition A.2.1 gives

\[
z^\top A z = \text{Tr}(z^\top A z) = \text{Tr}(z(z^\top)A) = \text{Tr}(z(Az)) = \text{Tr}(AZ) = \text{Tr}(ZA)
\]

□

We extend the result in (A.2.2). Let \( q^1, \ldots, q^k \) be column vectors in \( \mathbb{R}^n \) and define the \( n \times k \) matrix \( Q = [q^1 \cdots q^k] \). Observe that

\[
QQ^\top = \sum_{i=1}^{k} q^i (q^i)^\top
\]

Then

\[
\text{Tr}(QQ^\top A) = \text{Tr}(AQQ^\top) = \sum_{i=1}^{k} (q^i)^\top A q^i \tag{A.2.3}
\]

Lemma A.2.4. If \( A \) and \( B \) are real \( n \times n \) positive semidefinite matrices then

\[
0 \leq \text{Tr}(AB) \leq \text{Tr}(A)\text{Tr}(B) \tag{A.2.4}
\]

Proof. The proof is from Coope [9]. Since \( A \) is positive semidefinite there is a matrix \( A^{\frac{1}{2}} \) such that \( A^{\frac{1}{2}} A^{\frac{1}{2}} = A \). Then

\[
\text{Tr}(AB) = \text{Tr}(A^{\frac{1}{2}} A^{\frac{1}{2}} B) = \text{Tr}(A^{\frac{1}{2}} (A^{\frac{1}{2}} B) A^{\frac{1}{2}}) = \text{Tr}((A^{\frac{1}{2}} B) A^{\frac{1}{2}}) = \text{Tr}((B^{\frac{1}{2}} A^{\frac{1}{2}})^\top (B^{\frac{1}{2}} A^{\frac{1}{2}}))
\]

\[
= \|B^{\frac{1}{2}} A^{\frac{1}{2}}\|_F^2
\]

By definition a norm is nonnegative, so \( \text{Tr}(AB) \geq 0 \). Now show \( \text{Tr}(AB) \leq \text{Tr}(A)\text{Tr}(B) \)

\[
\text{Tr}(AB) = \langle A, B \rangle \leq ||A||_F ||B||_F \leq \sqrt{\text{Tr}(AA)} \sqrt{\text{Tr}(BB)} \leq \text{Tr}(A)\text{Tr}(B) \tag{A.2.5}
\]

Equation (A.2.5) is Cauchy-Schwarz. Equation (A.2.6) is definition of Frobenius norm. Equation (A.2.7) follows from the fact that \( \text{Tr}(AA) \leq (\text{Tr}(A))^2 \).
Alternate Proof: We give an alternate proof that if $A$ and $B$ are real $n \times n$ positive semidefinite matrices then

$$\text{Tr}(AB) \geq 0$$  \hspace{1cm} (A.2.8)

This proof is from the notes of Chiang. See [8]. By property 4. above, since $A$ is positive semidefinite we can write $A = U\Lambda U^\top$ or stated another way

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^\top$$

where $u_i$ is column $i$ of $U$. Then

$$\text{Tr}(AB) = \text{Tr}(BA) = \text{tr}(B \sum_{i=1}^{n} \lambda_i u_i u_i^\top) = \text{Tr}(\sum_{i=1}^{n} \lambda_i B u_i u_i^\top) = \sum_{i=1}^{n} \lambda_i u_i^\top B u_i \geq 0$$

We are using the fact that $\lambda_i \geq 0$, $i = 1, \ldots, n$ and the definition of positive semidefinite matrix applied to the $u_i$, $i = 1, \ldots, n$. □

A.2.5 The operations diag and Diag

The diagonal of a square matrix $A$ refers to the entries that run along the diagonal of the array of numbers, starting in the top-left corner to the bottom right corner; that is, the elements $A_{ii}$ for $i = 1, \ldots, n$. A square matrix is diagonal if all of its off-diagonal elements are 0; that is, $A_{ij} = 0$ for $i \neq j$.

We can define some linear mappings between the space of $n \times n$ matrices and the vector space $\mathbb{R}^n$ based on the notion of diagonals. First we define a mapping diag from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^n$ as follows:

$$\text{diag}: \mathbb{R}^{n \times n} \to \mathbb{R}^n$$

$$A \mapsto \begin{bmatrix} A_{11} \\ A_{22} \\ \vdots \\ A_{nn} \end{bmatrix}$$

In words, diag takes a matrix $A$ into a vector consisting of its diagonal elements. Next define

$$\text{Diag}: \mathbb{R}^n \to \mathbb{R}^{n \times n}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_n \end{bmatrix}$$
In words, Diag takes an $n$-vector $x$ and puts it along the diagonal of a $n \times n$ diagonal matrix $\text{Diag}(x)$.

The following proposition demonstrates a key relationship between the Diag and diag mappings:

**Proposition A.2.5.** Let $v$ be an $n$-vector and $X$ an $n \times n$ matrix. Then:

$$\text{Tr}(\text{Diag}(v)X) = v^\top (\text{diag}(X)) \tag{A.2.9}$$

This proposition can be used to show that Diag and diag are adjoint maps.

### A.2.6 Eigenvalues and Spectral decomposition

Let $A$ be $n \times n$ a square matrix. Any vector which satisfies $Ax = \lambda x$ for some real $\lambda$ is called an eigenvector of $A$. The scalar $\lambda$ is known as an eigenvalue of $A$.

Let $\lambda_i(A)$ denote the $i$th eigenvalue of $A$. Then we have the following relation: let $r$ be a scalar and $I$ the $n \times n$ identity matrix

$$\lambda_i(A + rI) = r + \lambda_i(A).$$

Let $A^n$ denote the $n$-fold matrix product of $A$ with itself. It is straightforward to observe that $\lambda_i(A^n) = (\lambda_i(A))^n$.

In general, eigenvalues may be complex numbers, and the space spanned of the eigenvalues may have dimension less than $n$. This is not true for symmetric matrices, whose eigenvalues are real and always have $n$ linearly independent eigenvectors. In fact, these eigenvectors can be chosen to be mutually orthogonal.

For real symmetric matrices $A$, there exists a factorization called the spectral decomposition defined by

$$A = Q\Lambda Q^\top$$

where $Q$ is the orthogonal matrix whose columns consist of $n$ orthogonal eigenvectors (normalized to have norm 1) and $\Lambda$ is a diagonal matrix whose entries are the eigenvalues of $A$ (written in the order of the eigenvectors they are associated with in $Q$). Another way to express the spectral decomposition is:

$$A = \sum_{i=1}^{n} \lambda_i q^i (q^i)^\top \tag{A.2.10}$$

where $\lambda_i$ is the $i$th eigenvalue and $q^i$ its associated eigenvector. Using this expression, we
can view positive definiteness in terms of eigenvalues as follows:

\[
x^\top Ax = x^\top \left( \sum_{i=1}^{n} \lambda_i q_i (q_i)^\top \right) x
\]

\[
= \sum_{i=1}^{n} \lambda_i x^\top q_i (q_i)^\top x
\]

\[
= \sum_{i=1}^{n} \lambda_i ((q_i)^\top x)((q_i)^\top x)
\]

\[
= \sum_{i=1}^{n} \lambda_i ((q_i)^\top x)^2.
\]

Observe that \( A \) is positive semidefinite if and only if \( \lambda_i \geq 0 \) for all \( i = 1, \ldots, n \). Otherwise if \( \lambda_k < 0 \) we could take \( x = q_k \), and \( x^\top Ax = \lambda_k < 0 \).

The spectral decomposition and Property (ii) of the trace operation in Proposition A.2.1 allow us to show that trace of a matrix is equal to the sum of its eigenvalues; that is, 

\[
\text{Tr}(A) = \sum_{i=1}^{n} \lambda_i(A).
\]

One final relation connects the Frobenius norm of a matrix \( A \) with its eigenvalues:

\[
||A||_F^2 = \text{Tr}(AA) = \sum_{i=1}^{n} \lambda_i(AA) = \sum_{i=1}^{n} (\lambda_i(A))^2.
\]

That is, the norm of matrix is equal to the square root of the sum of the squared eigenvalues.

If \( Q \in S^n \) and \( Q \) has rank \( k \leq n \) then \( Q \) can be written as

\[
Q = \sum_{i=1}^{k} \lambda_i q_i q_i^\top
\]

where the \( \lambda_k \) are the nonzero eigenvalues of \( Q \) and the \( q_i \) are the orthonormal eigenvectors of \( Q \). If \( Q \) were of full rank, and therefore invertible, then the eigenvalues of \( Q^{-1} \) would be \( \lambda_i^{-1} \), \( i = 1, \ldots, n \). To see this observe that if \( \lambda_i \) is an eigenvalue of \( Q \) with eigenvector \( q_i \), then

\[
Qq_i = \lambda_i q_i
\]

but \( Q \) is full rank so

\[
q_i = \lambda_i Q^{-1} q_i
\]

and then dividing by the eigenvalue \( \lambda_i \) gives

\[
\lambda_i^{-1} q_i = Q^{-1} q_i
\]
and $q_i$ is an eigenvector of $Q^{-1}$ with eigenvalue $\lambda_i^{-1}$. The generalization of this for when $Q$ has rank $k < n$ is

$$Q^\dagger = \sum_{i=1}^{k} \lambda_i^{-1} q_i q_i^\top$$

where $Q^\dagger$ is the pseudo-inverse of $Q$. Given $Q \in S^n$ we define $\mathcal{R}(Q)$ by

$$\mathcal{R}(Q) = \{ x \mid x = Qy, \text{ for some } y \in \mathbb{R}^n \}$$

So $\mathcal{R}(Q)$ is the set of vectors in $\mathbb{R}^n$ that can be written as a linear combination of the columns of $Q$. If $Q$ were of full rank, then $\mathcal{R}(Q)$ would equal $\mathbb{R}^n$ since $Q$ would have an inverse and $x$ would be a linear combination of the columns of $Q$ given by $y = Q^{-1}x$. We follow this analogy using the pseudo-inverse and set $y = Q^\dagger x$. We then define

$$x_R = Q(Q^\dagger x)$$

Thus we take a linear combination of the columns of $Q$ using the vector of multipliers $Q^\dagger x$. This gives a decomposition of $x \in \mathbb{R}^n$ into

$$x = x_R + x_N$$
where \( x_R \in \mathcal{R}(Q) \). In addition \( x_N \) is in the null space of \( Q \). To see this observe that

\[
Qx_N = Q(x - x_R) = Qx - Qx_R = Qx - QQ(Q^\top x) = Qx - Q(QQ^\top)x = Qx - Q(\sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j^{-1}(q_i q_i^\top q_j q_j^\top))x
\]

\[
= Qx - Q(\sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j^{-1} q_i(q_i^\top q_j q_j^\top))x
\]

\[
= Qx - Q(\sum_{i=1}^{k} q_i q_i^\top)x
\]

\[
= Qx - \sum_{j=1}^{k} \lambda_j q_j q_j^\top(\sum_{i=1}^{k} q_i q_i^\top)x
\]

\[
= Qx - (\sum_{j=1}^{k} \sum_{i=1}^{k} \lambda_j q_j q_j^\top q_i q_i^\top)x
\]

\[
= Qx - (\sum_{j=1}^{k} \sum_{i=1}^{k} \lambda_j q_j q_j^\top) x
\]

\[
= Qx - Qx = 0
\]

This also implies \( x_R \) and \( x_N \) are orthogonal vectors, that is, \( x_N^\top x_R = x_N^\top QQ^\top x = (x_N^\top Q)Q^\top x = 0 \).

Thus every \( x \) can be written as the orthogonal decomposition of a vector in \( \mathcal{R}(Q) \) and a vector in the null space of \( Q \). See Figure A.2.6. The following lemma shows that \( QQ^\top \) is the orthogonal projection of \( x \) onto \( \mathcal{R}(Q) \).

**Lemma A.2.6.** If \( x \in \mathbb{R}^n \) then \( x = x_R + x_N \) and \( QQ^\top x = x_R \).

**Proof of Lemma A.2.6**

\[
QQ^\top = \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j^{-1}(q_i q_i^\top)(q_j q_j^\top) = \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j^{-1} q_i(q_i^\top q_j q_j^\top) = \sum_{i=1}^{k} q_i q_i^\top
\]
this implies

\[ QQ^\top x = \sum_{i=1}^{k} q_i q_i^\top x \]

\[ = \sum_{i=1}^{k} q_i q_i^\top (x_R + x_N) \]

\[ = \sum_{i=1}^{k} q_i q_i^\top x_R + \sum_{i=1}^{k} q_i q_i^\top x_N \]

\[ = \sum_{i=1}^{k} q_i q_i^\top x_R \]

\[ = \sum_{i=1}^{k} q_i q_i^\top (\sum_{j=1}^{k} \beta_j q_j) \]

\[ = \sum_{i=1}^{k} \sum_{j=1}^{k} \beta_j q_i q_i^\top q_j \]

\[ = \sum_{j=1}^{k} \beta_j q_j \]

\[ = x_R \]

In the above derivation we use the fact that \( q_i^\top x_N = 0, i = 1 \ldots, k \) and since \( x_R \in \mathcal{R}(Q) \) we can write \( x_R = \sum_{j=1}^{k} \beta_j q_j \). □

Figure A.2: The projection of a vector onto \( \mathcal{R}(Q) \)

The \( QQ^\top \) operator “projects” the vector \( x \) onto \( \mathcal{R}(Q) \). It is an orthogonal projection in that \( x_R \) and \( x_N \) are orthogonal vectors.
A.2.7 Schur’s Lemma

This section is taken from Lemaréchal and Oustry. It is their Lemma 3.8.

**Lemma A.2.7. Schur’s Lemma** If $Q \in S^p$ and $P \in S^n$ and $S = [s_1, \ldots, s_n]$ an $p \times n$ real matrix, then the following are equivalent:

(i) The matrix
\[
\begin{bmatrix}
  P & S^\top \\
  S & Q
\end{bmatrix}
\]
is positive semidefinite.

(ii) The matrices $Q$ and $P - S^\top Q^\dagger S$ are positive semidefinite and $s_i \in \mathcal{R}(Q)$, for $i = 1, \ldots, n$

**Proof of Lemma A.2.7:** For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^p$ define the quadratic function for fixed $x$
\[
q_x(y) = \begin{pmatrix}
  x^\top & y^\top
\end{pmatrix}
\begin{bmatrix}
P & S^\top \\
S & Q
\end{bmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= x^\top Px + 2(Sx)^\top y + y^\top Qy
\]

Show (i) implies (ii). Since the matrix
\[
\begin{bmatrix}
P & S^\top \\
S & Q
\end{bmatrix} \succeq 0
\]
it follows that for all $x$, $q_x(y) \geq 0$. Therefore $\inf_{y \in \mathbb{R}^p} q_x(y) > -\infty$. Then by Lemma 7.5.1 we have $Q \succeq 0$ and $Sx \in \mathcal{R}(Q)$. This is true for all $x \in \mathbb{R}^n$ and in particular for $x = e^i$ for $i = 1, \ldots, n$ so $Se^i \in \mathcal{R}(Q)$ which implies $s_i \in \mathcal{R}(Q)$, for $i = 1, \ldots, n$. Also by Lemma 7.5.2 the optimal solution to $\inf_{y \in \mathbb{R}^p} q_x(y)$ is
\[
\bar{y}(x) = -\frac{1}{2}Q^\dagger(2Sx)
\]
which implies
\[
\inf_{y \in \mathbb{R}^p} q_x(y) = x^\top Px - \frac{1}{4}(2Sx)^\top Q^\dagger(2Sx)
\]
\[
= x^\top Px - (Sx)^\top Q^\dagger(Sx)
\]
\[
= x^\top P x - x^\top (S^\top Q^\dagger S)x
\]
\[
= x^\top \left( P - (S^\top Q^\dagger S) \right) x
\]
But for all \( x, q_x(y) \geq 0 \) so in particular the infimum is nonnegative so
\[
x^\top \left( P - (S^\top Q^\dagger S) \right) x \geq 0
\]
for all \( x \in \mathbb{R}^n \). This gives \( P - S^\top Q^\dagger S \succeq 0 \). Therefore (i) implies (ii).

Now show (ii) implies (i). If (ii) then \( Q \succeq 0 \) and \( s_i \in \mathcal{R}(Q) \) for \( i = 1, \ldots, n \). But \( s_i \in \mathcal{R}(Q) \) for \( i = 1, \ldots, n \) implies \( Sx \in \mathcal{R}(Q) \) for all \( x \). Thus \( \inf_{y \in \mathbb{R}^p} q_x(y) > -\infty \) for all \( x \) and again by Lemma 7.5.2, the optimal value of \( \inf_{y \in \mathbb{R}^p} q_x(y) \) as shown above is
\[
\inf_{y \in \mathbb{R}^p} q_x(y) = x^\top \left( P - (S^\top Q^\dagger S) \right) x
\]
But also by (ii) \( (P - (S^\top Q^\dagger S)) \succeq 0 \). Thus for all \( x \)
\[
\inf_{y \in \mathbb{R}^p} q_x(y) = x^\top \left( P - (S^\top Q^\dagger S) \right) x \geq 0
\]
for all \( x \). Therefore \( \inf_{y \in \mathbb{R}^p} q_x(y) \) is nonnegative for all \( x \). Then by definition of \( q_x(y) \) is nonnegative which implies
\[
\left[ \begin{array}{cc} P & S^\top \\ S & Q \end{array} \right] \succeq 0
\]
Thus (ii) implies (i). □

**Corollary A.2.8.** If \((X, x) \in \mathcal{S}^n \times \mathbb{R}^n\) then

1. \( X - xx^\top \succeq 0 \) if and only if
\[
\begin{bmatrix} X & x \\ x^\top & 1 \end{bmatrix} \succeq 0,
\]
2. the set \( \{(X, x) \in \mathcal{S}^n \times \mathbb{R}^n \mid X \succeq xx^\top \} \) is closed and convex.

**Proof of Corollary A.2.8:** To prove part 1. we apply Lemma A.2.7 with \( Q = 1, P = X, \) and \( S = x^\top \). To prove part 2. we define the following *affine mapping* \( \phi : \mathcal{S}^n \times \mathbb{R}^n \to \mathcal{S}^{n+1} \) by
\[
\phi(X, x) = \begin{bmatrix} X & x \\ x^\top & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
It is easy to show that \( \phi \) is an affine mapping. Then
\[
\phi^{-1}(\mathcal{S}^{n+1}) = \{(X, x) \in \mathcal{S}^n \times \mathbb{R}^n \mid \phi(X, x) \in \mathcal{S}^{n+1}\} = \{(X, x) \in \mathcal{S}^n \times \mathbb{R}^n \mid \begin{bmatrix} X & x \\ x^\top & 1 \end{bmatrix} \succeq 0\}
\]
where $\hat{S}_{+}^{n+1}$ is the subset of $S_{+}^{n+1}$ with component (1,1) fixed to 1. Since $S_{+}^{n+1}$ is closed and convex it follows that $\hat{S}_{+}^{n+1}$ is closed and convex. Then by Proposition A.3.2 and Proposition A.3.1, $\phi^{-1}(\hat{S}_{+}^{n+1})$ is closed and convex. Then by part 1, the set $\{(X,x) \in S^{n} \times \mathbb{R}^{n} | X \succeq x x^{\top}\}$ is closed and convex. □

Lemma A.2.9. Schur’s Lemma Variation

If $Q \in S^{p}$ and $P \in S^{n}$ and $S = [s_{1}, \ldots, s_{n}]$ an $p \times n$ real matrix, then the following are equivalent:

(i) The matrix

$$
\begin{bmatrix}
Q & S \\
S^{\top} & P
\end{bmatrix}
$$

is positive semidefinite.

(ii) The matrices $Q$ and $P - S^{\top}Q^{\dagger}S$ are positive semidefinite and $s_{i} \in \mathcal{R}(Q)$, for $i = 1, \ldots, n$.

Proof of Lemma A.2.9: For $(x,y) \in \mathbb{R}^{p} \times \mathbb{R}^{n}$ define the quadratic function for fixed $x$

$$
g_{y}(x) = (x^{\top} y^{\top}) \begin{bmatrix}
Q & S \\
S^{\top} & P
\end{bmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} = x^{\top}Qx + 2(Sy)^{\top}x + y^{\top}Py
$$

Now simply follow the proof in Lemma A.2.7 with the role of $x$ and $y$ reversed. □

Rewriting Lemma A.2.7 as Lemma A.2.9 one to reformulate the rank one Corollary A.2.8 into a form that is frequently seen in this literature. The proof of Corollary A.2.10 is identical to that of Corollary A.2.8 and therefore omitted.

Corollary A.2.10. If $(X,x) \in S^{n} \times \mathbb{R}^{n}$ then

1. $X - xx^{\top} \succeq 0$ if and only if

$$
\begin{bmatrix}
1 & x \\
x^{\top} & X
\end{bmatrix} \succeq 0,
$$

2. the set $\{(X,x) \in S^{n} \times \mathbb{R}^{n} | X \succeq xx^{\top}\}$ is closed and convex.

A.3 Appendix: Miscellaneous Results

Proposition A.3.1. Let $X$ and $Y$ be vector spaces with an affine mapping $\phi : X \to Y$. If $W \subseteq Y$ is convex, then $\phi^{-1}(W) \subseteq X$ is convex.

Proposition A.3.2. Let $X$ and $Y$ be vector spaces with a topology and with an affine mapping $\phi : X \to Y$. If $W \subseteq Y$ is closed, then $\phi^{-1}(W) \subseteq X$ is closed.
Bibliography


