Competitive Nonlinear Pricing

(preliminary and incomplete)

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1 Introduction

We study competitive nonlinear pricing in a model involving simultaneously horizontal and vertical product differentiation. It is a particular case of a more general model of optimal contracting with uncertain participation that we study elsewhere (Rochet-Stole (1997)).

2 The Model

Our model can be viewed as a synthesis of the well known models of Hotelling (or Salop) and Mussa-Rosen: two firms compete for consumers who are uniformly distributed on a line $[0, \Delta]$, while each firm is located at one extremity of the line: firm 1 at $x = 0$, and firm 2 at $x = \Delta$. These firms offer competing product lines Q_i and nonlinear pricing schedules $P_i : Q_i \to I\!R$. Consumers differ not only by their location x but also by their marginal valuation for quality t. The two parameters x and t are independently distributed. We assume linear preferences: $U = tq - p - \gamma d$, where the transportation cost parameter γ is normalized to one.

Similarly, the individual rationality level is normalized to zero. Let $u_i(t)$ denote

$$\max_{q \in Q_i} \{tq - P_i(q)\},$$

the indirect utility obtained by a consumer of type t when he buys from firm i. We assume that t is uniformly distributed on $[\underline{t}, \bar{t}]$. The marginal customer of firm i among consumers of type t is located at a distance

$$x_i(t) = \min(u_i(t), \frac{1}{2}(\Delta + u_i(t) - u_j(t)).$$

Indeed there are two regimes:

- Local monopolies when $u_1(t) + u_2(t) < \Delta$ and $x_i(t) = u_i(t)$.
- Direct competition when $u_1(t) + u_2(t) > \Delta$ and $x_i(t) = \frac{\Delta + u_i(t) - u_j(t)}{2}$.

Finally, we assume that cost functions are identical: the unit cost of producing a good of quality q is $C(q) = \frac{1}{2}q^2$. Adopting the dual approach (Mirrlees), one can express the profit of each firm as a function of the indirect utility functions $u_1(\cdot), u_2(\cdot)$. This is made possible by the following implementation lemma:

Lemma 1 *: For a given indirect utility function $u(\cdot) : [\underline{t}, \bar{t}] \to I\!R$, and a given quality assignment, $q(\cdot) : [\underline{t}, \bar{t}] \to Q$, there exists a price schedule, $P(\cdot) : Q \to I\!R$ such that $u(t) = \max_{q \in Q}\{tq - P(q)\}$ (the maximum being obtained for $q = q(t)$), if and only if: u is convex and $q(t) = \dot{u}(t)$ for a.e.t.*

The total profit of firm i is equal to:

$$B_i = \int_{\underline{t}}^{\bar{t}} \frac{x_i(t)}{\Delta} \left(P_i(q_i(t)) - \frac{1}{2} q_i^2(t) \right) dt.$$

Using the above remarks we can express everything in terms of u_i and u_j:

$$\begin{cases} x_i(t) & = \min\left(u_i(t), \frac{1}{2}\left(\Delta + u_i(t) - u_j(t)\right)\right) \\ q_i(t) & = \dot{u}_i(t) \\ P_i(q_i(t)) & = t\dot{u}_i(t) - \frac{1}{2}\dot{u}_i^2(t). \end{cases}$$

The analysis of competition in nonlinear schedules is thus represented by a normal form game:

$$B_i(u_i, u_j) = \frac{1}{\Delta} \int_{\underline{t}}^{\bar{t}} \min\left[u_i(t), \frac{1}{2}\left(\Delta + u_i(t) - u_j(t)\right)\right] \times \left\{ t\dot{u}_i(t) - \frac{1}{2}\dot{u}_i^2(t) - u_i(t) \right\} dt \qquad (1)$$

where u_i, u_j the strategies of firms i and j are convex functions from $[\underline{t}, \bar{t}]$ to $\mathbb{R}$.

3 The Monopolistic Regime

When Δ is large enough, the equilibrium market shares do not overlap for any t so that each firm is in fact in a monopoly position: B_i only depends on u_i. Because of our symmetry assumptions we can drop the index i. (1) becomes:

$$B(u) = \frac{1}{\Delta} \int_{\underline{t}}^{\bar{t}} u(t) \left\{ t\dot{u}(t) - \frac{1}{2}\dot{u}^2(t) - u(t) \right\} dt. \qquad (2)$$

The optimal u is characterized by the Euler equation:

$$\frac{d}{dt}\left[(t - \dot{u})u\right] = t\dot{u} - \frac{1}{2}\dot{u}^2 - 2u, \qquad (3)$$

and the transversality conditions:

$$\dot{u}(t) = t \quad \text{for} \quad t = \underline{t} \quad \text{and} \quad \bar{t}. \qquad (4)$$

Rearranging terms in (3) we obtain a second order differential equation in u:

$$\ddot{u} = 3 - \frac{\dot{u}^2}{2u}. \qquad (5)$$

Although the solution of (4), (5) cannot be expressed analytically, its graph $(t, u(t))$ can be parameetrized as a function of the quality choice q of consumer t.

The convexity condition does not bind when $\underline{t}/\bar{t}$ is larger than some threshold γ.

Lemma 2 *: When $\underline{t}/\bar{t}$ is large enough, the unique solution of (4), (5) is strictly convex. It is defined implicitly by $u[t_m(q,c)] = U_m(q,c)$, where*

$$U_m(q,c) = \frac{1}{6}\left\{q^2 + \sqrt{q^4 - c^4}\right\}, \quad (6)$$

$$t_m(q,c) = \frac{1}{3}\left\{2\underline{t} + q + \int_{\underline{t}}^{q} \frac{x^2 dx}{\sqrt{x^4 - c^4}}\right\}, \quad (7)$$

and c is the unique solution of the equation:

$$\int_{\underline{t}}^{\bar{t}} \frac{x^2 dx}{\sqrt{x^4 - c^4}} = 2(\bar{t} - \underline{t}). \quad (8)$$

Proof: See the Appendix.

Comments:

- It is easily seen that $\frac{\partial U_m}{\partial q} = q\frac{\partial t_m}{\partial q}$, so that q is indeed equal to $\dot{u}(t_m(q,c))$, the quality chosen by consumer $t_m(q,c)$. Therefore formula (7) gives the marginal price schedule $P'_m(q)$ faced by consumers in the monopolistic regime.

- The transversality condition imposes that $q_m(t)$ be efficient not only at the top ($t = \bar{t}$) but also at the bottom ($t = \underline{t}$), contrarily to the Mussa-Rosen case. This comes from the uncertain participation of consumers (x is unknown to the seller) which makes the market share vary continuously with $u(t)$ (in our example, it is exactly $\frac{u(t)}{\Delta}$). An important consequence is that the range of qualities offered by the firms is the efficient range $[\underline{t}, \bar{t}]$, even though firms are local monopolies!

- The convexity property of u is indeed satisfied whenever $t_m(\cdot)$ increases with q on $[\underline{t}, \bar{t}]$. Differentiating (7) with respect to q we get:

$$\frac{\partial t_m}{\partial q}(q,c) = \frac{1}{3}\left[1 + \frac{q^2}{\sqrt{q^4 - c^4}}\right],$$

  which is clearly positive.

- The condition $\underline{t} \geq \gamma\bar{t}$ is needed to ensure that equation (8) has indeed a solution $c \leq \underline{t}$ (so that the integral is well defined). This is true if only if $\underline{t} \geq \gamma\bar{t}$, where $\gamma \sim 0.76$. This condition will be assumed in the sequel. When it is not satisfied, there is bunching at the bottom: $\dot{u}(t)$ is constant on some interval $[\underline{t}, t^*]$ (see Rochet-Stole (1997)).

- Finally, we have to check that the market shares of the two firms do not overlap.

Since u is increasing, this is satisfied if and only if

$$\Delta \geq \Delta_m \equiv 2u(\bar{t}) = \frac{1}{3}\left\{\bar{t}^2 + \sqrt{\bar{t}^4 - c^4}\right\}.$$

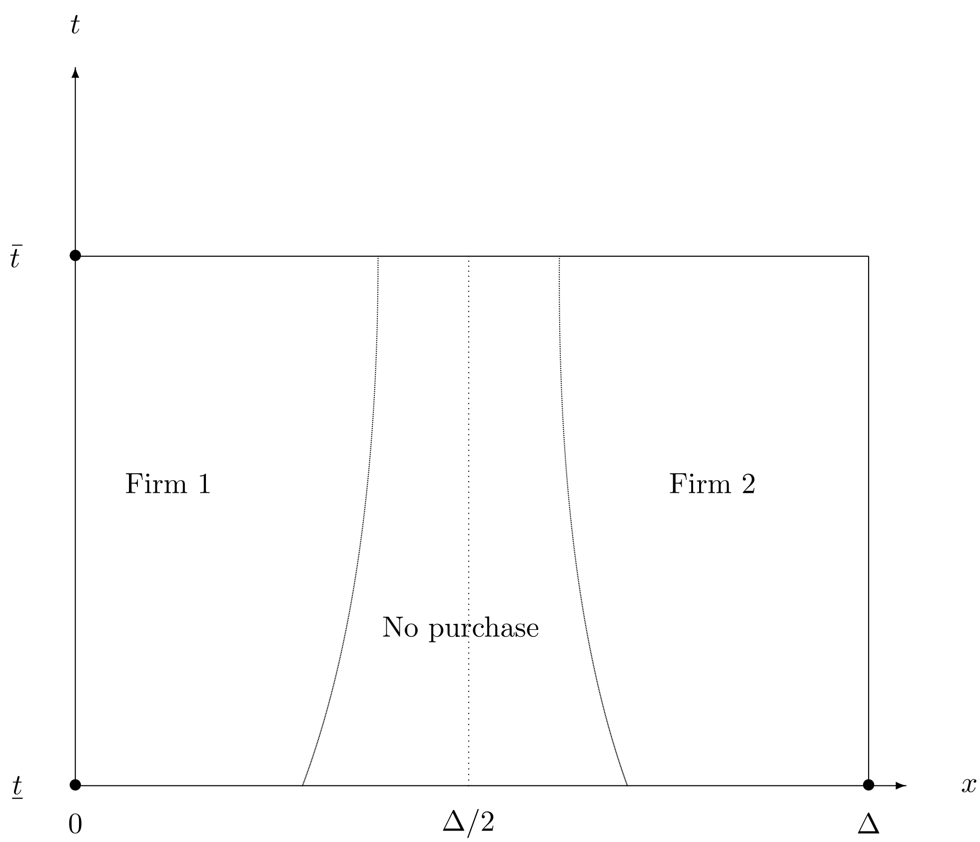
The analysis of the monopolistic regime is summarized in the following proposition:

Proposition 1 *: For Δ larger than a threshold value Δ_m, competition in nonlinear schedules yields a regime of local monopolies. Each firm offers the efficient product line $Q = [\underline{t}, \bar{t}]$, but marginal price always exceeds marginal cost (negative distortion) except at the top and at the bottom, where they are equal.*

Comments:

Figure 1: The quality allocation for $\underline{t} = 4, \bar{t} = 5, \Delta \geq \Delta_m$

Figure 2: The market shares



4 The Competitive Regime

When Δ is small (and $\underline{t}$ large enough) it is intuitive that all consumers are served at equilibrium, so that (1) becomes:

$$B_i(u_i, u_j) = \int_{\underline{t}}^{\bar{t}} \frac{1}{2}\left(1 + \frac{u_i(t) - u_j(t)}{\Delta}\right)\left\{t\dot{u}_i(t) - \frac{1}{2}\dot{u}_i^2(t) - u_i(t)\right\} dt. \tag{2}$$

When the convexity condition does not bind (this will be checked ex post) the Nash equilibrium of our competition game is characterized by a system of Euler equations:

$$\frac{d}{dt}\left[(t - \dot{u}_i)(\Delta + u_i - u_j)\right] = t\dot{u}_i - \frac{1}{2}\dot{u}_i^2 - u_i - \Delta, \tag{9}$$

and transversality conditions:

$$\dot{u}_i(t) = t \quad \text{for} \quad t = \underline{t} \quad \text{and} \quad \bar{t}. \tag{10}$$

The solution of this system is symmetric ($u_i = u_j = u$) and (9), (10) can be simplified into:

$$\Delta(2 - \ddot{u}) = t\dot{u} - \frac{1}{2}\dot{u}^2 \quad \text{on} \quad]\underline{t}, \bar{t}[, \tag{11}$$

$$\dot{u}(t) = t \quad \text{for} \quad t = \underline{t}, \bar{t}. \tag{12}$$

It turns out that the unique solution of (11), (12) is easy to find:

Lemma 3 *: The equation (11) has a unique solution that satisfies the boundary conditions (12). It is defined by:*

$$u(t) = \frac{1}{2}t^2 - \Delta. \tag{13}$$

The corresponding price schedule involves a constant mark-up above unit cost:

$$P(q) = C(q) + \Delta. \tag{14}$$

As a result, the choice of qualities is everywhere efficient:

$$\forall t, \quad q(t) \equiv t. \tag{15}$$

Comments: Armstrong-Vickers; non robustness.

proof: By identification, it is immediate that (13) defines a solution u to (11), (12). Suppose that there exists another solution v, and define

$$w(t) \equiv \dot{v}(t) - \dot{u}(t) = \dot{v}(t) - t.$$

The boundary condition imposes that $w(\underline{t}) = w(\bar{t}) = 0$. Moreover by assumption

$$\Delta(2 - \ddot{v}) = t\dot{v} - \frac{1}{2}\dot{v}^2 - v.$$

By differentiation of this equation, we get:

$$-\Delta \dddot{v} = (t - \dot{v})\ddot{v}.$$

But $\dot{v} - t = w, \ \ddot{v} = 1 + \dot{w}, \ \ \dddot{v} = \ddot{w}$. Thus w solves:

$$\begin{aligned} \Delta\ddot{w} &= w(\dot{w}+1) \qquad \text{on } \ ]\underline{t}, \bar{t}[\\ w(\underline{t}) &= w(\bar{t}) = 0. \end{aligned} \tag{16}$$

We now prove that $w \equiv 0$, which will establish that v and u differ by a constant, and thus that $u = v$, given the form of (11).

We first establish that $w \leq 0$. By continuity of w, it has a maximum on $[\underline{t}, \bar{t}]$. If this maximum is attained at the boundary then it is equal to zero, which proves the desired result. If it is attained in an interior point t_0 then one has $\dot{w}(t_0) = 0$ and $\ddot{w}(t_0) \leq 0$. Equation (16) then implies that $w(t_0) \leq 0$, which proves $w \leq 0$.

By exactly symmetric arguments, it is easy to prove that $w \geq 0$, so that in fact $w \equiv 0$.

■

It remains to check that all consumers are indeed served at equilibrium. This is true if and only if:

$$u(\underline{t}) \geq \frac{\Delta}{2}, \quad \text{or} \quad \Delta \leq \frac{2}{3}\underline{t}^2.$$

Proposition 2 *: For $\Delta \leq \Delta_c \equiv \frac{2}{3}\underline{t}^2$, competition in nonlinear schedules yields a competitive regime. The price schedules offered by the firms consist of a constant mark-up above costs: $P(q) = C(q) + \Delta$ and the quality choice is fully efficient: $q(t) \equiv t$.*

5 The Mixed Regime

The more interesting (but more difficult) case corresponds to intermediate values of Δ: $\Delta \in]\Delta_c, \Delta_m[$. In this case the symmetric equilibrium $u_1 = u_2 = u$ is mixed: There is a unique type

$\hat{t} \in]\underline{t}, \bar{t}[$ such that $u(\hat{t}) = \frac{\Delta}{2}$. Then:

- For $t < \hat{t}$, we are in the local monopoly regime and u satisfies:

$$\begin{aligned} \ddot{u} &= 3 - \frac{\dot{u}^2}{2u}, \quad t < \hat{t} &(17)\\ \dot{u}(\underline{t}) &= \underline{t}. &(18) \end{aligned}$$

- For $t > \hat{t}$, we are in the competitive regime and u satisfies:

$$\begin{aligned} \Delta(2 - \ddot{u}) &= t\dot{u} - \frac{1}{2}\dot{u}^2 - u, \quad t > \hat{t} &(19)\\ \dot{u}(\bar{t}) &= \bar{t}. &(20) \end{aligned}$$

Moreover there is smooth pasting[1] at $\hat{t}$:

$$\begin{aligned} \lim_{\substack{t \to \hat{t} \\ t < \hat{t}}} u(t) &= \lim_{\substack{t \to \hat{t} \\ t > \hat{t}}} u(t) = \frac{\Delta}{2}, &(21)\\ \lim_{\substack{t \to \hat{t} \\ t < \hat{t}}} \dot{u}(t) &= \lim_{\substack{t \to \hat{t} \\ t > \hat{t}}} \dot{u}(t). &(22) \end{aligned}$$

An immediate implication of lemma 2 shows that the solutions of (17), (18) can be parameterized as a function of q:

$$\begin{aligned} U_m(q, c) &= \frac{1}{6}\left\{q^2 + \sqrt{q^4 - c^4}\right\}, &(23)\\ t_m(q, c) &= \frac{1}{3}\left\{2\underline{t} + q + \int_{\underline{t}}^{q} \frac{x^2 dx}{\sqrt{x^4 - c^4}}\right\} &(24) \end{aligned}$$

where c is a constant to be determined.

A similar method can be applied to equations (19), (20). This time, the appropriate parameter is not the quality q but the distortion $d = t - q$:

Lemma 4 *: All the solutions of (19), (20) (except $u(t) = \frac{1}{2}t^2 - \Delta$, for which $d \equiv 0$) can be defined implicitly by $u[t_c(d, a)] = U_c(d, a)$ with*

$$\begin{aligned} t_c(d, a) &= \bar{t} - \int_0^d \frac{ds}{\phi\left(a + \frac{s^2}{2\Delta}\right) - 1}, &(25)\\ U_c(d, a) &= \frac{1}{2}t_c^2(d, a) - \frac{1}{2}d^2 - \Delta\left(2 - \phi\left(a + \frac{d^2}{2\Delta}\right)\right), &(26) \end{aligned}$$

[1] Smooth pasting is a consequence of the Erdmann-Weierstrass necessary condition.

where ϕ is the inverse function of $y \to y - \ln y$, a bijection from $[1, +\infty[$ into itself, and a is a constant to be determined.

Proof: See the Appendix.

Four parameters remain to be determined: $c, a, \hat{q}$ and $\hat{d} = \hat{t} - \hat{q}$. They are given by the smooth pasting conditions (21) and (22), which are equivalent to a system of four equations:

$$U_m(\hat{q}, c) = U_c(\hat{d}, a) = \frac{\Delta}{2}, \tag{27}$$

$$t_m(\hat{q}, c) = t_c(\hat{d}, a) = \hat{q} + \hat{d}. \tag{28}$$

As Δ varies, the solutions to this system describe a curve in the (t, U) plane. The limit values of Δ correspond to those for which $\hat{t}$ equals $\underline{t}$ and $\bar{t}$, respectively:

- **If $\hat{t} = \underline{t}$:** conditions (18) to (20) and lemma 3 imply that $u(t) \equiv \frac{1}{2}t^2 - \Delta$ (degenerate case where $d \equiv 0$). Condition (27) gives $\Delta = 2u(\underline{t}) = \Delta_c$.

- **If $\hat{t} = \bar{t}$:** condition (25) implies that $\hat{d} = 0$ so that $\hat{q} = \bar{t}$. Condition (24) then gives

  $$\bar{t} = \frac{1}{3}\left\{2\underline{t} + \bar{t} + \int_{\underline{t}}^{\bar{t}} \frac{x^2 dx}{\sqrt{x^4 - c^4}}\right\},$$

  or:

  $$2(\bar{t} - \underline{t}) = \int_{\underline{t}}^{\bar{t}} \frac{x^2 dx}{\sqrt{x^4 - c^4}},$$

  so that u coincides with the solution of the local monopoly regime, obtained in lemma 2. Now, condition (27) implies

  $$\Delta = 2u(\bar{t}) = \frac{1}{3}\left\{\bar{t}^2 + \sqrt{\bar{t}^4 - c^4}\right\} = \Delta_m.$$

Therefore the limit values of Δ, beyond which the mixed solution is not defined, corresponds exactly to Δ_c and Δ_m: this completes our characterization.

Proposition 3 *: For $\Delta \in]\Delta_c, \Delta_m[$, competition in nonlinear schedules yields a mixed regime. It is characterized by the following parameterizations:*

$t < \hat{t}$: $u = U_m(q, c)$ *(local monopoly region)*

$t = t_m(q, c)$

$t > \hat{t}$: $u = U_c(d, a)$ *(competitive region)*

$t = t_c(d, a)$

where $(c, a, \hat{q}, \hat{d})$ *solves the system (27), (28) and* $\hat{t} = \hat{d} - \hat{q} \in [\underline{t}, \bar{t}]$*. In particular, the quality choice is efficient at both extremes (*$q(t) = t$ *for* $t = \underline{t}$ *and* $\bar{t}$*) but nowhere else.*

Comments: (to be completed)

6 Conclusion

REFERENCES:

Appendix

Proof of lemma 2: We transform equation (4) by introducing a potential function (see Rochet-Stole (1997) for more details). Let us define:

$$K(t) = u(t)\dot{u}^2(t) - 3u^2(t).$$

We have:

$$\dot{K}(t) = \dot{u}^3(t) + 2u(t)\dot{u}(t)\ddot{u}(t) - 6u(t)\dot{u}(t).$$

By equation (4) this is identically zero, so that $K(t) \equiv K$, or $3u^2 - u\dot{u}^2 + K \equiv 0$. Thus we can write $\dot{u}$ as a function of u:

$$\dot{u}^2 = 3u + \frac{K}{u}$$

(this is the method we use in Rochet-Stole (1997)). Alternatively we can express u as a function of $q = \dot{u}$:

$$u = U(q) = \frac{1}{6}\left\{q^2 \pm \sqrt{q^4 - 12K}\right\}.$$

t can also be written as a function of q, since

$$U(q) \equiv u(t(q)), \quad \text{and} \quad q \equiv \dot{u}(t(q)).$$

Therefore:

$$\dot{t}(q) = \frac{\dot{U}(q)}{q} = \frac{1}{3}\left\{1 \pm \frac{q^2}{\sqrt{q^4 - 12K}}\right\}. \tag{29}$$

Moreover, the convexity of u requires that $\dot{t}(q) \geq 0$. This, together with $U(q) \geq 0$, implies $K \geq 0$ and rules out the "minus" solution.

If we set $c = \sqrt[4]{12K}$ we obtain condition (6):

$$u = U_m(q, c) = \frac{1}{6}\left\{q^2 + \sqrt{q^4 - c^4}\right\}.$$

Equation (7) is then obtained by integrating (29) between $\underline{t}$ (for which $t = \underline{t}$) and q. Condition (8) the results from the second boundary condition ($t = \bar{t}$ for $q = \bar{t}$), which completes the proof of lemma 2.

■

Proof of lemma 4: We first transform (19) into an autonomous equation by setting: $D(t) = \frac{1}{2}t^2 - u(t)$. Therefore $u(t) = \frac{1}{2}t^2 - D(t)$, $\dot{u}(t) = t - \dot{D}(t) \equiv d(t)$ and $\ddot{u} = 1 - \ddot{D}$, so that (19), (20) becomes

$$\begin{aligned}\Delta(1+\ddot{D}) &= D - \frac{1}{2}\dot{D}^2 \quad \text{on} \quad]\hat{t},\bar{t}[,\\ \dot{D}(\bar{t}) &= 0.\end{aligned}$$

We can reduce the order of this equation by defining

$$A(t) = \left(\frac{1}{2}\dot{D}^2(t) - D(t) + 2\Delta\right)\exp\frac{D(t)}{\Delta}.$$

We have:

$$\begin{aligned}\dot{A}(t) &= \left(\exp\frac{D(t)}{\Delta}\right)\left(\dot{D}\ddot{D} - \dot{D} + \frac{\dot{D}^3}{2\Delta} - \frac{D\dot{D}}{\Delta} + 2\dot{D}\right)\\ &= \frac{\dot{D}}{\Delta}\left(\exp\frac{D(t)}{\Delta}\right)\left(\Delta(\ddot{D}+1) - D + \frac{\dot{D}^2}{2}\right) \equiv 0.\end{aligned}$$

Therefore $A(t) \equiv A$ and $\dot{D}(t)$ can be expressed as a function of $D(t)$:

$$\frac{1}{2}\dot{D}^2(t) = D(t) - 2\Delta + A\exp -\frac{D(t)}{\Delta}. \tag{30}$$

Alternatively, we can express $D(t)$ as a function of $d = \dot{D}(t)$:

$$D(t) = \frac{1}{2}d^2 + 2\Delta - A\exp -\frac{D(t)}{\Delta}. \tag{31}$$

Let us introduce the auxiliary variable

$$y = \frac{A}{\Delta}\exp -\frac{D}{\Delta},$$

so that (31) becomes

$$-\Delta\ln = \frac{\Delta y}{A} = \frac{1}{2}d^2 + 2\Delta - \Delta y,$$

or:

$$y - \ln y = \frac{d^2}{2\Delta} + a, \tag{32}$$

with

$$a \equiv 2 + \Delta\ln\frac{\Delta}{A}.$$

Introducing the function ϕ, inverse of $y \to y - \ln y$, we obtain:

$$y = \phi\left(a + \frac{d^2}{2\Delta}\right).$$

(30) then implies:

$$D(t(d)) = \frac{1}{2}d^2 + \Delta\left(2 - \phi\left(a + \frac{d^2}{2\Delta}\right)\right). \tag{33}$$

The parameterization of t can be obtained from the identity $d = \dot{D}(t(d))$ so that by differentiating (33), we get:

$$\dot{t}(d) = 1 - \phi'\left(a + \frac{d^2}{2\Delta}\right) = \frac{-1}{\phi\left(a + \frac{d^2}{2\Delta}\right) - 1}. \tag{34}$$

Condition (25) is then obtained by integrating (34) between 0 (for which $t = \bar{t}$) and d. Condition (26) is an immediate consequence of (33), given that $D(t)$ is defined as

$$\frac{1}{2}t^2 - u(t).$$

■