

Tricks with Aggregate Games*

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1 Introduction

Aggregative games have been of interest ever since Selten's (1970) first description of the game as a generalization of Cournot. In the papers which followed over the four decades since, the class has been enlarged to include many applications beyond industrial organization, and the theory has been developed to provide a rich set of existence and comparative static results. Typically, an aggregative game is defined to be normal-form game with the property that each player's payoff is a function *only* of his own strategy and an aggregate function of the chosen strategy profile of all players. The Cournot quantity game is probably the best known example of such a game where the aggregate is the industry supply. More generally, this class includes a broad category of games including public goods games, common resource games, rent-seeking contests, cost sharing games, and patent races, to name a few.¹ Within this classification, strong comparative statics properties are often available once one makes payoff assumptions regarding the payoff interaction of each player's strategy with the aggregate.² In addition, powerful existence theorems easily establish pure-strategy equilibria in these games.³ That said, the class of games that is typically analyzed in these papers is restricted to finite dimensional spaces and, more often than not, the aggregate space is restricted to one-dimension.⁴

In the games that are of direct interest to the present authors (e.g. competitive agency games with nonlinear transfer functions), strategy spaces are infinite-dimensional and preferences fail to satisfy supermodularity or increasing differences. Nonetheless, there is a set of purely algebraic properties of aggregate games that, even when broadly defined to include large strategy spaces, can be used with great success to characterize equilibria. In short, aggregate games can be embedded in simpler aggregate spaces to

¹See Bergstrom, et al. (1986), Okuguchi (1993), Corchon (1994), Cornes and Hartley (2000, 2005), Jensen (2009) and Acemoglu and Jensen (2009), and the many references cited therein.

²See in particular, Corchon (1994), Dubey, et al. (2006), and Acemoglu and Jensen (2009).

³See, for example, Szidarovszky and Yakowitz (1977), Novshek (1984, 1985), Kukushkin (1994, 2004), Dubey, et al. (2006), and Jensen (2009).

⁴Jensen (2009) and Acemoglu and Jensen (2009) briefly consider extensions to larger finite-dimensional aggregates, but the powerful comparative statics and existence properties are compromised by the fact that assumptions of supermodularity and decreasing differences are less likely to be satisfied in many applications that come to mind.

which we can apply straightforward optimization techniques. Indeed, one can show that an aggregate game is equivalent to a self-generating maximization program which in turn is equivalent to a two-person, zero-sum game over aggregates. The proofs of these properties are quite straightforward, almost following from the definitions of a set of mappings derived from the game. Nevertheless, it is this simple insight that allows one to conveniently solve infinite-dimensional aggregate games (as well as their simpler cousins) in a straightforward and intuitive manner.

Consider the following game in normal form:

$$G = (X_i, u_i)_{i \in N},$$

where $N = \{1, \dots, n\}$ is the set of players and each player i has chooses a strategy x_i from X_i and has payoffs represented by a utility function, $u_i : \mathbf{X} \rightarrow \mathbb{R}$ with $\mathbf{X} = \prod_{j \in N} X_j$. We follow the convention that $\mathbf{x} = (x_i, \mathbf{x}_{-i}) \in X_i \times \mathbf{X}_{-i} = \mathbf{X}$ and denote player i 's payoff as $u_i(x_i, \mathbf{x}_{-i})$.

We say that G_ϕ is an **aggregate representation** of G with **aggregate function** $\phi : \mathbf{X} \rightarrow Y$ if there exist payoff functions $(\tilde{u}_i)_{i \in N}$ such that for each $i \in N$

$$u_i(\mathbf{x}) = \tilde{u}_i(x_i, \phi(\mathbf{x})), \quad \forall \mathbf{x} \in \mathbf{X}.$$

G_ϕ is also a normal-form game with the additional structure of an aggregate function, ϕ . Define $\mathcal{E}(G)$ to be the set of Nash equilibrium points for any normal-form game G . If G_ϕ is an aggregate representation of G , it follows that $\mathcal{E}(G) = \mathcal{E}(G_\phi)$.

Every normal-form game has a trivial aggregate representation – namely, the identity mapping $\phi(\mathbf{x}) = \mathbf{x}$ with $Y = \mathbf{X}$. Of course it is a misuse of language to call this an aggregate function. To be clear, when ϕ is not one-to-one on Y (in contrast to the identity), we will say that ϕ is a *proper* aggregate function. When G is representable by a proper aggregate function, we say that the game G_ϕ is an *aggregate-strategy game*. We are primarily interested in proper aggregate games and their associated aggregate functions.

Example - Cournot games: Consider the output game of Cournot. Each of the n firms simultaneously chooses an output, $q_i \geq 0$, to bring to market at a cost of $C_i(q_i) = c_i q_i$. The market price is set such that total demand, $D(p)$, is equal to the quantity supplied, $\sum_{i \in N} q_i$. In this case,

there are at least two aggregate representations. The trivial identity of $\phi(q_1, \dots, q_n) = \{q_1, \dots, q_n\} \in \mathbb{R}_+^n$ and the more familiar, proper aggregate, $\phi(q_1, \dots, q_n) = Q = \sum_{i \in N} q_i$, the total quantity supplied.

We consider additional structure on ϕ beyond properness. The most useful restriction is that ϕ is a multi-injection. We say that ϕ is a **multi-injection** if, for each i and $\mathbf{x}_{-i} \in \mathbf{X}_{-i}$, the restriction of ϕ to \mathbf{x}_{-i} denoted $\phi|_{\mathbf{x}_{-i}} : X_i \rightarrow \phi(X_i, \mathbf{x}_{-i})$ is one-to-one. Denote the inverse correspondence of $\phi|_{\mathbf{x}_{-i}}$ by

$$\phi_i^{-1}(y, \mathbf{x}_{-i}) \equiv \{x_i \in X_i \mid y = \phi(x_i, \mathbf{x}_{-i})\}.$$

When ϕ is multi-injective it follows that ϕ_i^{-1} is single-valued and can be treated as a function. If ϕ is multi-injective and also $\phi(X_i, \mathbf{x}_{-i}) = Y$ for all i and $\mathbf{x}_{-i} \in \mathbf{X}_{-i}$, then we say that ϕ is a **multi-bijection**. Multi-bijective aggregate games, therefore, have the property that any player i with knowledge of rival strategies, \mathbf{x}_{-i} , can implement any aggregate $y \in Y$.

In the above example of Cournot, the aggregate function $\phi(q_1, \dots, q_n) = \sum_i q_i$ is a multi-injection and the inverse mapping for each player i is $\phi_i^{-1}(Q, \mathbf{q}_{-i}) = Q - \sum_{j \neq i} q_j$. This aggregate is not a multi-bijection, however, because firm i 's output is nonnegative, $q_i \geq 0$, which implies that the aggregate is at least $\sum_{j \neq i} q_j$. Player i is limited to implementing an aggregate which exceeds the sum of all rival players' outputs. In the following two examples, however, the aggregates are multi-bijections.

Example - Supply function games: Consider the supply function game studied in Klemperer and Meyer (1989). Suppose that each of n firms chooses a supply function as its strategy, $S_i(p)$, where $S_i(p)$ is any bounded function from nonnegative prices to the number of units firm i commits to supply at the price p . Thus, the space of strategies is an infinite-dimensional vector space. We also allow that supply commitments may be negative, which translates into the firm committing to purchase some output. Each firm chooses its supply function, S_i , prior to knowing the state of demand. We denote demand in state θ as $D(p, \theta)$. After θ is realized, a market maker chooses the price to equate supply to demand: $\sum_{i \in N} S_i(p) = D(p, \theta)$.⁵ The interesting aggregate in this game is $S(p) = \sum_{i \in N} S_i(p)$. It is multi-

⁵To simplify the analysis, we assume that if no such price can be found, then no trade takes place.

bijjective because we have allowed each firm to choose negative supplies, though in equilibrium no firm makes use of this option.

Example - Intrinsic common agency game: Another example with infinite-dimensional strategy spaces is the case in which n principals offer nonlinear contracts, T_i , which are commitments to pay a common agent $T_i(q)$ if the agent chooses action q . The agent, however, has private information, θ , at the time of contracting, and so typically each principal will gain from distorting q in order to reduce the agent's information rents. We assume that the common agency of this game is *intrinsic*: that is, we assume the agent must either accept all contract offers, obtaining the aggregate transfer $T(q) = \sum_{i \in N} T_i(q)$, or reject all contracts and obtain a reservation payoff normalized to zero. Thus, the agent's optimal choice of q will be a function of θ and $\sum_i T_i$. Because transfers may be negative, the aggregate transfer function is a multi-bijection.

2 An aggregate representation theorem

Define $\mathcal{E}(G) \subseteq \mathbf{X}$ as the (possibly empty) set of Nash equilibria to the game G . The following theorem is a consequence of our notion of aggregation.

Theorem 1 *If $G_\phi = \{(X_i, u_i)_{i \in N}, \phi\}$ is an aggregate game, then*

$$\bar{\mathbf{x}} \in \mathcal{E}(G_\phi) \iff \phi(\bar{\mathbf{x}}) \in \bigcap_{i \in N} \arg \max_{y \in \phi(X_i, \bar{\mathbf{x}}_{-i})} \max_{x_i \in \phi_i^{-1}(y, \bar{\mathbf{x}}_{-i})} u_i(x_i, y). \quad (1)$$

The key insight for this result is that for each player i ,

$$x_i \in \arg \max_{x_i \in X_i} u_i(x_i, \phi(x_i, \bar{\mathbf{x}}_{-i})) \iff y \in \arg \max_{y \in \phi(X_i, \bar{\mathbf{x}}_{-i})} \left\{ \max_{x_i \in \phi_i^{-1}(y, \bar{\mathbf{x}}_{-i})} u_i(x_i, y) \right\}.$$

The right-hand side represents player i 's strategic problem after it has been translated to the space of aggregates. It can be rendered as a two-step optimization program. First, for any given target aggregate y , the player chooses the best x_i from the set of choices that implement y given $\bar{\mathbf{x}}_{-i}$. Thus, $x_i \in \phi_i^{-1}(y, \bar{\mathbf{x}}_{-i})$. If ϕ is multi-injective, then this step is trivial as ϕ_i^{-1} is a singleton. In the second stage, the player chooses the best

aggregate in the set of all implementable aggregates, given $\bar{\mathbf{x}}_{-i}$; that is, y is chosen from the set $\phi(X_i, \bar{\mathbf{x}}_{-i})$. Because equilibrium requires that no one can profitably change the aggregate, the equilibrium aggregate $\phi(\bar{\mathbf{x}})$ must lie in the intersection of the players' optimal aggregate choices.

This argument also suggests that whenever ϕ is not a multi-injection, we can project each player's strategy space onto aggregate-equivalence sets and analyze the game with the strategies chosen from the quotient set, X_i/\sim , embedding the first step of optimization into each player's payoff function. If we are only interested in the equilibrium aggregate, $\phi(\bar{\mathbf{x}})$, and not the component strategies $\bar{\mathbf{x}}$, this reduction is without any loss. Going forward, therefore, we assume explicitly that ϕ is a multi-injection. To this end, we have the corollary

Corollary 1 *If G_ϕ is a multi-injective aggregate game, then*

$$\bar{\mathbf{x}} \in NE(G_\phi) \iff \phi(\bar{\mathbf{x}}) \in \bigcap_{i \in N} \arg \max_{y \in \phi(X_i, \bar{\mathbf{x}}_{-i})} u_i(\phi_i^{-1}(y, \bar{\mathbf{x}}_{-i}), y).$$

An immediate consequence of the representation theorem is that in any equilibrium, all players must concur on the choice of the aggregate. We refer to this as the **principle of aggregate concurrence**; it provides a powerful perspective for analyzing equilibria in aggregate games.⁶ For example, aggregate concurrence implies that if payoff functions are differentiable and the aggregate is multidimensional, then between any two players who choose interior strategies, the marginal rates of substitution across any pair of y components must be identical. Marginal rates of substitution on the aggregate are equalized across the set of active players in any equilibrium.

One can present the same notion of aggregate concurrence by constructing arbitrary welfare functions (strictly increasing functions of the players' payoffs translated over aggregate strategies) and demonstrating that an equilibrium aggregate, $\bar{y} = \phi(\bar{\mathbf{x}})$, must also maximize any such artificial welfare function. It suffices for our purposes to consider linear welfare functions with weights $\lambda \in \Delta^n$; i.e., $\lambda_i \geq 0$ and $\sum_{i \in N} \lambda_i = 1$. To this end,

⁶While others have used this idea indirectly, to our knowledge Bernheim and Whinston (1986a) are the first to recognize explicitly the force of this principle in their study of moral hazard in a common agency game. "We underscore the need to make the principals' objectives congruent in equilibrium: since all principals can effect the same changes in the aggregate incentive scheme, none must find any such change worthwhile." (p. 929)

define the weighted payoff function

$$\Lambda(y, \bar{\mathbf{x}}, \lambda) \equiv \sum_{i \in N} \lambda_i u_i(\phi_i^{-1}(y, \bar{\mathbf{x}}_{-i}), y).$$

The welfare-function analogue to our previous theorem is

Theorem 2 *If $G_\phi = \{(X_i, u_i)_{i \in N}, \phi\}$ is a multi-injective aggregate game, then*

$$\bar{\mathbf{x}} \in \mathcal{E}(G_\phi) \iff \forall \lambda \in \Delta^n, \quad \phi(\bar{\mathbf{x}}) \in \arg \max_{y \in \bigcap_{\lambda_i > 0} \phi(X_i, \bar{\mathbf{x}}_{-i})} \Lambda(y, \bar{\mathbf{x}}, \lambda). \quad (2)$$

Stated in this way, the aggregate concurrence property is also an invariance property. Theorem 2 holds that the subset of optimal aggregates that are invariant to the welfare weights is exactly the set of equilibrium aggregates. The proof of the result follows again directly from (1). Indeed, because $\Lambda(y, \bar{\mathbf{x}}, \lambda)$ is linear in λ , one can reduce the invariance requirement to a basis of weights with $\lambda_i = 1$ for some i . Several additional remarks are in order.

Remark 1: It should be emphasized that the righthand side of (2) is an optimization program with a fixed point. Specifically, we are identifying self-generating maximizers of Λ . To see this more plainly, consider the simplest setting in which ϕ is multi-bijective and Λ depends upon $\bar{\mathbf{x}}$ only through its effect on the aggregate. Thus (with a slight abuse of notation) we obtain the necessary condition for a given λ

$$\bar{\mathbf{x}} \in \mathcal{E}(G_\phi) \implies \bar{y} \in \arg \max_{y \in Y} \Lambda(y, \bar{y}, \lambda).$$

We will refer to the right-hand side program as a **self-generating maximization (SGM)** program. If we know an equilibrium exists, it suffices to consider equilibrium outcomes that are self-generating maximizers of Λ at λ . If the SGM solution set has a unique element, then it represents the unique Nash equilibrium outcome. If the set contains multiple candidates, then we can take variations of λ to identify the invariant (equilibrium) components.

Remark 2: Note also that the solution to an SGM program can be restated as finding the minmax value of a fictitious two-person, zero-sum

game. Let player A have the payoff $\Phi(y_a, y_b) \equiv \Lambda(y_a, y_b, \lambda) - \Lambda(y_b, y_b, \lambda)$ and player B the negative of Φ . By construction, it follows that

$$\min_{y_b} \max_{y_a} \Phi(y_a, y_b) \geq 0 \geq \max_{y_a} \min_{y_b} \Phi(y_a, y_b).$$

The minmax solution corresponds to finding values of \bar{y}_a and \bar{y}_b such that the inequalities become equalities, $\Phi(\bar{y}, \bar{y}) = 0$ and $\Lambda(\bar{y}, \bar{y}) = \max_y \Lambda(y, \bar{y})$.⁷ In this sense, finding the equilibrium aggregates to an n -player aggregate game over \mathbf{X} is equivalent to solving a 2-person zero-sum game over the smaller aggregate space, Y .

Remark 3: It is also worth remarking that in the trivial aggregation case, $\phi = Id_{\mathbf{X}}$, that it suffices to find the set of self-generating maximizers for an arbitrary, strictly positive weighting vector. That is, for *any* $\lambda > 0$,

$$\bar{\mathbf{x}} \in \mathcal{E}(G_\phi) \iff \bar{\mathbf{x}} \in \arg \max_{\mathbf{x} \in \mathbf{X}} \Lambda(\mathbf{x}, \bar{\mathbf{x}}, \lambda).$$

This feature of equilibrium points was noted more than half a century ago by Nikaido and Isoda (1955), but the simple proof of sufficiency is worth repeating here. If $\bar{\mathbf{x}} \notin \mathcal{E}(G_\phi)$, then there exists some player i such that $u_i(x_i, \bar{\mathbf{x}}_{-i}) > u_i(\bar{\mathbf{x}})$. But x_i only appears in $\Lambda(\mathbf{x}, \bar{\mathbf{x}}, \lambda_0)$ through $\lambda_i u_i(x_i, \bar{\mathbf{x}}_{-i})$ component, which implies that $\Lambda((x_i, \bar{\mathbf{x}}_{-i}), \bar{\mathbf{x}}, \lambda_0) > \Lambda(\bar{\mathbf{x}}, \bar{\mathbf{x}}, \lambda_0)$, yielding a contradiction. When ϕ is a proper aggregate function, however, the fact that an Λ -improving variation over X_i exists, $u_i(x_i, \phi(x_i, \bar{\mathbf{x}}_{-i})) > u_i(\bar{x}_i, \phi(\bar{\mathbf{x}}))$, does not guarantee that there is a similar improving variation over Y . This is because a variation in y that raises the $\lambda_i u_i$ component cannot be constructed without simultaneously influencing the other components of Λ . Without more structure, one can only show that the set of equilibrium aggregates is a subset of the self-generating maximizers of Λ .

3 A useful class of aggregate-strategy games

The aggregate representation results presented thus far are based on an abstract notion of aggregation. We placed no additional algebraic or topological restrictions on the game. We now develop two stronger results that require additional layers of structure. The first result establishes conditions

⁷If Φ is quasi-concave in y_a and quasi-convex in y_b , then we are guaranteed such a solution exists. Sion (1958). These conditions, however, are not necessary conditions.

under which we can replace an SGM program over strategy profiles, $\mathbf{x} \in \mathbf{X}$, with one over the space of aggregates, $y \in Y$, thereby providing a simple necessary condition for equilibrium aggregates. The second result establishes conditions under which the solution set to this simpler aggregate SGM program is exactly the set of equilibrium aggregates.

3.1 Additive-affine aggregate games

In practice, many aggregate games have an additive-affine-symmetric structure that places them in the following class of aggregate games. We refer to the collective characteristics of such aggregate game as “additive-affine” although this should be understood as the following list of attributes:

Definition 1 G_ϕ is an **additive-affine aggregate (AAA) game** if

- (1) **(symmetric, linear strategy sets):**

$$X_1 = \dots = X_n = Y = X,$$

where X is a subset of a linear space;

- (2) **(additive aggregation):**⁸ $\phi(\mathbf{x}) = \sum_{i \in N} x_i$.

- (3) **(affine payoffs):** *payoffs (after possibly an order-preserving transformation) are affine in own strategies: for all $i \in N$, $y \in X$*

$$u(x_i, y) - u_i(0, y)$$

is a continuous linear functional in x_i .

- (4) **(symmetric cross-differences):** *for each $i, j \in N$ there exists a $\tau_{ij} \in \mathbb{R}$ such that for all $x, x', y, y' \in X$:*

$$\begin{aligned} & (u_i(x, y) - u_i(x', y)) - (u_i(x, y') - u_i(x', y')) \\ & = \tau_{ij} \{ (u_j(x, y) - u_j(x', y)) - (u_j(x, y') - u_j(x', y')) \}. \end{aligned} \quad (3)$$

The first condition imposes symmetry and linearity on the strategy spaces, the latter property allowing us to add and scale. Importantly, linear spaces include infinite-dimensional spaces of functions as well as the

⁸The aggregate function can easily be generalized to a linear function with distinct weights for each component x_i .

standard Euclidean space. We are not imposing any topological properties on the space of strategies. The second condition assumes aggregation is a simple additive function of strategies. This aggregate could easily be generalized to affine functions with asymmetric coefficients without any essential differences in the results below.⁹ The first two conditions, in tandem, imply that X is closed under the aggregation operator.¹⁰

The third requirement of affine preferences is perhaps the strongest of the four. Surprisingly, however, many aggregate games satisfy this property, including our three working examples. If payoff affinity is inappropriate (e.g., cost functions in Cournot games may be nonlinear), we can modify the definition to include linear approximations of preferences. The resulting necessary conditions will be less powerful but more generally available.

The fourth requirement imposes a form of payoff symmetry that may be misleading at first take. The essence of the condition is that, in combination with the payoff affinity condition, payoff functions can be separated into three components after suitable transformations:

$$u_i(x, y) = v(x, y) + w_i(x) + z_i(y),$$

where $v(\cdot, y)$ and $w_i(\cdot)$ are continuous linear functions of x . Stated in this form, the payoff component, $v(x, y)$, is common to every player. In the case of the Cournot game with $u_i(q, Q) = q_i(P(Q) - c_i) - F_i$, the condition requires only that

$$q(P(Q) - c_i) - q(P(Q) - c_j) = q(c_j - c_i)$$

must be independent of Q , which it is. It is tempting to think of the common component $v(x, y)$ as a form of common potential function (in the sense of Monderer and Shapley (1996)), but this interpretation is misleading. There is no direction connection between AAA games and potential games. A Cournot game with nonlinear demand and constant unit costs, for example, is a AAA game but not a potential game; a Cournot game with linear demand and nonlinear costs is a potential game but not a AAA

⁹Indeed, with some care at defining algebraic groups and operations, we can extend the class of aggregate games to include nonlinear functions (e.g., aggregates that are products of strategies, etc.) by considering homomorphic strategy spaces and corresponding payoff functions, but the increased generality is minor in our opinion.

¹⁰More generally, this property follows if the aggregate function is a composition of binary operations on a group, X .

game.¹¹

While it is not as immediate, supply-function games (with symmetric marginal costs) and intrinsic common agency games are also members of this class. We demonstrate this membership below. For now, however, we use Theorem 2 and the requirements of additive-affine aggregate games to develop a simple necessary condition for equilibrium aggregates.

Consider the strategic problem facing player i . Substituting $y - \phi(\bar{\mathbf{x}}) + \bar{x}_i$ for x_i , we have the equivalence

$$\bar{x}_i \in \arg \max_{x_i \in X_i} u_i(x_i, \phi(x_i, \bar{\mathbf{x}}_{-i})) \iff \bar{y} \in \arg \max_{y \in X_i + \bar{y}_{-i}} u_i(y - \bar{y} + \bar{x}_i, y).$$

Choose any subset of players $M \subseteq N$, $|M| = m$, and take a uniform average of the right-hand side over M (i.e., choosing the weights $\lambda = \lambda_{M,i} = 1/m$ if $i \in M$ and 0 otherwise).¹² Define $\bar{y}_S \equiv \sum_{j \in S} \bar{x}_j$. Using the fact that payoffs in AAA games can be represented as $u_i(x, y) = v(x, y) + w_i(x) + z_i(y)$ where v and w_i are linear functions in x , we obtain

$$\frac{1}{m} \sum_{i \in M} u_i(y - \bar{y} + \bar{x}_i, y) = \frac{1}{m} \sum_{i \in M} v(y - \bar{y} + \bar{x}_i, y) + w_i(y - \bar{y} + \bar{x}_i) + z_i(y).$$

Using the linearity of $v(\cdot, y)$ and $w_i(\cdot)$, the right hand side simplifies to

$$v\left(y - \bar{y} + \frac{1}{m} \bar{y}_M, y\right) + \frac{1}{m} \sum_{i \in M} (w_i(y - \bar{y}) + z_i(y)) + \frac{1}{m} \sum_{i \in M} w_i(\bar{x}_i).$$

The final term is independent of y , and so it has no effect on the maximization program over the aggregate. Thus, we may define

$$\Lambda_M(y, \bar{y}, \bar{y}_M) \equiv v\left(y - \bar{y} + \frac{1}{m} \bar{y}_M, y\right) + \frac{1}{m} \sum_{i \in M} w_i(y - \bar{y}) + z_i(y),$$

and conclude that for any given $M \subseteq N$ and arbitrary $\bar{Y} \subseteq X$

$$\arg \max_{y \in \bar{Y}} \frac{1}{m} \sum_{i \in M} u_i(y - \bar{y} + \bar{x}_i, y) = \arg \max_{y \in \bar{Y}} \Lambda_M(y, \bar{y}, \bar{y}_M).$$

We have succeeded in replacing the various $\bar{\mathbf{x}}_{-i}$ by a pair of aggregates, \bar{y} and \bar{y}_M . Indeed, for $M = N$, we may write the simple self-generating maximization program as

$$\bar{y} \in \arg \max_{y \in \bar{Y}} \Lambda_N(y, \bar{y}).$$

¹¹Of course, every potential game that has a normal form also has at least one aggregate representation. Such a representation is not necessarily part of the AAA class.

¹²If the aggregate function is linear but with non-uniform weights, we can use the same relative weights in constructing λ_M to generalize our proposition.

Applying Theorem 2 and summarizing the above argument, we have our first proposition for the class of AAA games.

Proposition 1 *If G_ϕ is an additive-affine affine aggregate game, then for any $M \subseteq N$*

$$\bar{x} \in \mathcal{E}(G_\phi) \implies \bar{y} \in \arg \max_{y \in \cap_{i \in M} X + \bar{y}_{-i}} \Lambda_M(y, \bar{y}, \bar{y}_M). \quad (4)$$

If ϕ is a multi-bijection, then

$$\bar{x} \in \mathcal{E}(G_\phi) \implies \bar{y} \in \arg \max_{y \in X} \Lambda_N(y, \bar{y}). \quad (5)$$

The set of self-generating maximizers for (4) or (5) is often quite tractable to calculate as we will see below where we return to our three examples.

The result in Proposition 1, while useful, does not tell us which solutions to the SGM program (if any) represent equilibrium aggregates. To obtain a sufficiency result, we add one more restriction on the class of AAA games and narrow our focus to settings in which strategies are functions.

3.2 A subclass of functional games

An important subclass of AAA games has strategy spaces restricted to measurable functions from Ω to \mathbb{R}^k . Of course, simple games in Euclidean spaces are contained within this restriction, but we also include games in which players choose ∞ -dimensional menus (e.g., nonlinear tariffs).

Definition 2 *A functional additive-affine aggregate game, G_ϕ , is an additive-affine aggregate game in which, for each $i \in N$,*

- *player i 's strategy set contains measurable functions from Ω to \mathbb{R}^k*
- *player i 's payoff is*

$$u_i(x_i, y) = \int_{\Omega} (\alpha_i(y, \omega) + (\beta(y, \omega) + \delta_i)' \cdot x_i(\omega)) g(\omega) d\omega, \quad (6)$$

where $\alpha_i : X \times \Omega \rightarrow \mathbb{R}$, $\beta : X \times \Omega \rightarrow \mathbb{R}^k$ and $\delta_i \in \mathbb{R}^k$.

NB: $\alpha_i(\cdot, \omega)$ and $\beta(\cdot, \omega)$ map from the function y , not the value $y(\omega)$.

It is straightforward to verify that the functional payoff above is affine in x_i and has symmetric cross-differences. Thus, the functional aggregate game is a member of the AAA class. In fact, it is difficult to think of an interesting AAA game that is not also in the functional subclass. When, for example, $|\Omega| = 1$ and $k = 1$, we have strategies $x_i, y \in \mathbb{R}$ and payoffs that of the form

$$u_i(x_i, y) = \alpha_i(y) + (\beta(y) + \delta_i)x_i,$$

which is easily seen to include our Cournot game with constant marginal costs.¹³ Games with ∞ -dimensional spaces require a bit more scrutiny.

Consider, for example, the case of intrinsic common agency. We let $\Omega = \Theta \times \mathcal{Q}$. Each principal is restricted to choosing a transfer function from

$$\mathcal{T} \equiv \{T_i \in \mathbb{R}^{\mathcal{Q}} \mid T_i(q_0) = 0, T_i \text{ u.s.c.}\},$$

where \mathcal{Q} is the space of allowable agent choices and the choice of outside option $q_0 \in \mathcal{Q}$ results in a zero transfer from principal i (equivalent to rejecting i 's contract). This is set of measurable functions from Ω to \mathbb{R} that is constant on Θ . Note that the aggregate $T = \sum_{i \in N} T_i$ is also in this allowable set (indeed, the set \mathcal{T} is a cone) and the aggregate mapping is multi-bijective. Consider the agent's best-response function for a given aggregate contract: $q_0(\theta, T)$.¹⁴ We can embed this choice function in a Dirac probability measure, $\mu(\theta, q|T)$, defined on the graph of $q_0(\theta, T)$ in $\Omega = \Theta \times \mathcal{Q}$. This construction allows us to write principal i 's payoff as

$$\int_{\Omega} (v_i(q) - T_i(q)) d\mu(\theta, q|T),$$

which has the format of (6).¹⁵ Hence, it is a functional AAA game. Supply function games can also be placed in the form of functional AAA games when unit costs are constant and symmetric as we will see below.

An important property of functional aggregate games arises when the games are multi-bijective and satisfy a further symmetry restriction. The additional symmetry we require is that for each player i , $\delta_i = 0$ and there exists an $\tilde{\alpha}_i$ such that $\alpha_i(y, \omega) = \beta(y, \omega)' \cdot \tilde{\alpha}_i(\omega)$. Any payoff asymmetry is contained entirely in these $\tilde{\alpha}_i$ factors. In the context of our Cournot game,

¹³Set $x_i = q_i$, $y = Q$, $\alpha_i(y) = 0$, $\beta(y) = P(Q)$ and $\delta_i = -c_1$.

¹⁴If it is a correspondence, then fix any single-valued selection.

¹⁵Specifically, we let $x_i = T_i$, $y = T$, $\omega = (\theta, q)$, $\alpha_i(T, (\theta, q)) = v_i(q)\mu(\theta, q|T)$, $\beta(T, (\theta, q)) = \mu(\theta, q|T)$, $\delta_i = 1$, $\gamma_i = 0$.

such symmetry is quite restrictive, requiring that $c_i = c_j$. Hence, one must be careful as the symmetry restriction may be wholly inappropriate in any given strategic setting. That said, the additional requirement is satisfied in both the supply-function game and the intrinsic common agency game with any additional restrictions beyond those assumed by the AAA class.

We are now prepared to present our sufficiency result.

Proposition 2 *Suppose that G_ϕ is a multi-bijective, functional aggregate game with payoffs given by*

$$u_i(x_i, y) = \int_{\Omega} \beta(y, \omega)' \cdot (\tilde{\alpha}_i(\omega) + x_i(\omega)) d\omega,$$

where $\tilde{\alpha}_i : \Omega \rightarrow \mathbb{R}^k$ and $\beta : X \times \Omega \rightarrow \mathbb{R}^k$. Then

$$\bar{y} \in \arg \max_{y \in Y} \Lambda_N(y, \bar{y}) \implies \exists \bar{\mathbf{x}} \in \mathcal{E}(G_\phi) \text{ with } \phi(\bar{\mathbf{x}}) = \bar{y}. \quad (7)$$

The proof of this fact is constructive. For any given SGM solution, \bar{y} , define \bar{x}_i pointwise for $\omega \in \Omega$ using

$$\bar{x}_i(\omega) = \frac{1}{n} \bar{y}(\omega) - \left(\tilde{\alpha}_i(\omega) - \frac{1}{n} \sum_{j \in N} \tilde{\alpha}_j(\omega) \right).$$

By construction, $\bar{\mathbf{x}}$ satisfies $\bar{y} = \sum_{i \in N} \bar{x}_i$. Substituting for $\bar{x}_i(\omega)$ in

$$u_i(y - \bar{y} + \bar{x}_i, y) = \int_{\Omega} \beta(y, \omega)' \cdot (\tilde{\alpha}_i(\omega) + y(\omega) - \bar{y}(\omega) + \bar{x}_i(\omega)) d\omega,$$

we have

$$u_i(y - \bar{y} + \bar{x}_i, y) = \int_{\Omega} \beta(y, \omega)' \cdot \left(y(\omega) - \frac{n-1}{n} \bar{y}(\omega) + \frac{1}{n} \sum_{j \in N} \tilde{\alpha}_j(\omega) \right) d\omega.$$

Observe, however, that this expression is identical to $\Lambda_N(y, \bar{y})$. Given \bar{y} is a solution to the SGM program (5) and that $y - \bar{y} + \bar{x}_i \in X$ (which follows because ϕ is multi-bijective), the choice $y = \bar{y}$ (equivalently, $x_i = \bar{x}_i$) is a best response for player i . Hence, the constructed $\bar{\mathbf{x}}$ is an equilibrium with aggregate \bar{y} . It suffices, therefore, to restrict attention to the solution set of the SGM in (7) if one is interested in characterizing the set of all equilibrium aggregates.

We return now to our three working examples of aggregate games to apply these propositions.

4 Applications

4.1 Cournot games

As argued above, the Cournot output game with constant unit costs of production is an additive-affine aggregate game. Ignore for the moment that Cournot aggregation is not multi-bijective. Then the application of (4) in Proposition 1 becomes

$$\bar{Q} \in \arg \max_{Q \geq 0} \frac{1}{n} (P(Q) - c_N) (nQ - (n-1)\bar{Q})$$

where $c_N = \frac{1}{n} \sum_i c_i$ is the average unit cost in the industry. More economically relevant, this simplifies to

$$\bar{Q} \in \arg \max_{Q \geq 0} Q(P(Q) - c_N) + (n-1)(Q - \bar{Q})(P(Q) - c_N). \quad (8)$$

The first term represents industry profit. The second term captures the strategic effects on output. Without any conditions on P , we know that in any equilibrium with $n > 1$ active firms, there will be over production relative to the monopoly level. This is because the second term is increasing in Q around any equilibrium point $Q = \bar{Q}$. Furthermore, providing that $P(Q)$ is concave (or at least not too convex), there exists a unique solution to the SGM program.

This analysis is predicated on the assumption that aggregation is multi-bijective. From Proposition 1, the appropriate SGM program to consider is in fact:¹⁶

$$\begin{aligned} \bar{Q} \in \arg \max_{Q \geq \max_{i \in M} \sum_{j \neq i} \bar{q}_j} & Q(P(Q) - c_M) + (m-1)(Q - \bar{Q})(P(Q) - c_M) \\ & + (P(Q) - c_M)(\bar{Q}_M - \bar{Q}). \end{aligned}$$

If exactly the firms $i \in M$ produce positive output in equilibrium, then $\bar{Q} = \bar{Q}_M$ and the inequality constraint on Q is slack. Furthermore, if the program is quasi-concave, we can replace the slack constraint with $Q \geq 0$ and obtain the variation

$$\bar{Q} \in \arg \max_{Q \geq 0} Q(P(Q) - c_M) + (m-1)(Q - \bar{Q})(P(Q) - c_M), \quad (9)$$

¹⁶We have multiplied by m and subtracted $c_M \bar{Q}_M$ to simplify the program further.

which is (8) above, applied to only a subset M of firms. Conditional on the equilibrium set of active firms, M , we can conclude that output is greater than that which maximizes the collective profits of the producing firms. In this sense, we can side step the assumption that aggregation is multi-bijective.

One can also readily see from the statement in (9) that a mean-preserving change in the distribution of industry unit costs that leaves the set of active firms unchanged can have no effect on the market price. Hence, the well-known neutrality of differential taxation in public finance follows.¹⁷ Of course, the neutrality result is as fragile as the condition that the set of active firms is invariant.

4.2 Supply function games

Consider again supply-function games. The market-maker's price can be modeled as a pointwise the solution to the functional equation, $D(p, \theta) = S(p)$. We assume that D is increasing in θ so that a unique, strictly increasing, price functional exists which we denote as $\bar{p}_0(\theta, S)$. Let $\Omega = \Theta \times \mathbb{R}_+$. Given the aggregate supply function, we can construct a Dirac probability measure on the graph of $\bar{p}_0(\theta, S)$ in Ω . We denote the measure $\mu(\theta, p|S)$.

In order to apply our propositions, we assume that each firm's cost of production is constant *and identical* and therefore the supply function game is in the class of functional aggregate games. To see this, note that firm i 's expected profit can be written as

$$\int_{\Theta \times \mathbb{R}_+} S_i(p)(p - c) d\mu(\theta, p|S).$$

Because $S = \sum_i S_i$ is multi-bijective, we can immediately apply Proposition 1 to obtain a necessary condition on the equilibrium aggregate supply function, \bar{S} :

$$\bar{S} \in \arg \max_{S \in \mathbb{R}^{\mathbb{R}_+}} \int_{\Theta \times \mathbb{R}_+} \left(S(p) - \frac{n-1}{n} \bar{S}(p) \right) (p - c) d\mu(\theta, p|S).$$

Because this supply-function game is satisfies the symmetry requirements in Proposition 2, any such solution to the above program must also be an equilibrium aggregate.

¹⁷Bergstrom, Blume and Varian (1985) first noted such an invariance in the context of public goods games; Bergstrom and Varian (1985a, 1985b) note a similar invariance for Cournot games.

Using this fact that there is a one-to-one mapping from the aggregate supply function defined over prices, $S(\cdot)$, and a strictly increasing price function defined over Θ satisfying $D(p(\theta), \theta) = S(p(\theta))$, we can translate the program into an equivalent one over the dual space of price functions:

$$\bar{p} \in \arg \max_{p \uparrow \in (\mathbb{R}_+)^{\Theta}} \int_{\Theta} (nD(p(\theta), \theta) - (n-1)\bar{S}(p(\theta)))(p(\theta) - c)dF(\theta), \quad (10)$$

where $F(\theta)$ is the distribution function for θ . We are able to further decompose this objective into parts with a similar interpretation as in the Cournot game:

$$\begin{aligned} \bar{p} \in \arg \max_{p \uparrow \in (\mathbb{R}_+)^{\Theta}} \int_{\Theta} D(p(\theta), \theta)(p(\theta) - c)dF(\theta) + \\ (n-1) \int_{\Theta} (D(p(\theta), \theta) - \bar{S}(p(\theta)))(p(\theta) - c)dF(\theta). \end{aligned} \quad (11)$$

The first term is industry profit; the second term represents the strategic effects present in the supply-function game.

Suppose that the demand function is twice differentiable and that $D_{p\theta} > 0$. Then for any \bar{S} function, the price function found by maximizing the integrand pointwise over θ is strictly increasing and represents a solution to the optimization program. To find the solution points to the SGM program, we need to find the self-generating fixed point, \bar{p} (or \bar{S} , alternatively). If \bar{p} is the price functional associated with \bar{S} , we can replace $\bar{S}(p)$ with $D(p, \bar{p}^{-1}(p))$ in the integrand and use pointwise maximization to obtain a differential equation for the market price. Because payoffs are strongly symmetric, every solution to the SGM program is a Nash equilibrium aggregate for the symmetric strategies $\bar{S}_i = \frac{1}{n}\bar{S}$.

Rather than finding explicit solutions, we investigate the character of the SGM solutions (and hence the equilibrium price functions) from the optimization program. Define $p^b(\theta) = c$ as the Bertrand (perfectly competitive) price function and $p^c(\theta)$ as the Cournot price function, which satisfies

$$D_p(p^c(\theta), \theta)(p^c(\theta) - c) + \frac{1}{n}D(p^c(\theta), \theta) = 0.$$

Suppose that we restrict attention to equilibria with a strictly increasing supply function. Thus, \bar{S} is differentiable almost everywhere. The pointwise derivative of the integrand in (10) with respect to price in equilibrium is

$$(nD_p(\bar{p}(\theta), \theta) - (n-1)\bar{S}_p(\bar{p}(\theta)))(\bar{p}(\theta) - c) + D(\bar{p}(\theta), \theta).$$

If, in addition, $D_{\theta\theta} \geq 0$, then the integrand is strictly quasi-concave, and so the first-order solution characterizes the optimum. Evaluated at $p^b(\theta) = c$, this derivative is $D \geq 0$, indicating $p(\theta) \geq p^b(\theta)$. Evaluated at $p^c(\theta)$, we instead obtain a derivative of $-(n-1)\bar{S}_p(p^c(\theta)) < 0$. Thus, in any equilibrium with increasing aggregate supply, the equilibrium price function must lie between the Bertrand and Cournot solutions:

$$p^b(\theta) \leq \bar{p}(\theta) < p^c(\theta).$$

If we restrict the strategy space of each firm to contain only nondecreasing supply schedules (as many papers in the literature have done), then any resulting equilibrium price function would satisfy these bounds. Absent such a restriction, a priori, we cannot rule out the possibility that aggregate supply is decreasing in some region, only that it cannot decrease too fast: $\bar{S}_p(p) - D_p \geq 0$. Wherever a non-increasing equilibrium exists (and this case is not vacuous), the upper bound on the price function must exceed $p^c(\theta)$ in the decreasing supply regions. Note that the derivative of (10) evaluated at the monopoly price, which we denote $p^m(\theta)$, is nonnegative, $(n-1)(D_p - \bar{S}_p)(p^m - c) \leq 0$. We conclude that $p^m(\theta)$ is an upper bound on $\bar{p}(\theta)$ for the general case.

For the case in which firms have different (but constant) marginal costs, c_i , solving explicitly for the set of Nash equilibria involves finding all the solutions to a system of differential equations. Unfortunately, the SGM program approach is not entirely satisfactory either. Evaluated at the Cournot price function, the marginal effect of price becomes

$$-(n-1)\bar{S}_p(p^c(\theta))(p^c(\theta) - c_N) - \sum_{i \in N} \bar{S}_{i,p}(p^c(\theta))(c_i - c_N),$$

where c_N is the average unit cost. Notice that the new term is the covariance of c_i and $S_{i,p}(p^c)$ across the firms. The aggregate concurrence principle tells us that any two players must have the same marginal returns with respect to the aggregate components. It follows that whenever high-cost firms have lower market share than low-cost firms, then high-cost firms must also exhibit greater supply responses. We may conclude, therefore, that if high cost firms have weakly smaller markets shares in equilibrium, then the covariance must be positive and $\bar{p}(\theta) < p^c(\theta)$.

4.3 Intrinsic common agency game

Our final example is the game which originally motivated this note: intrinsic common agency games with public contracts.¹⁸ Recall that each principal i has a direct payoff of $v_i(q)$ when the agent chooses the action $q \in \mathcal{Q}$, from a compact set. To motivate the agent, a principal may offer any bounded, upper-semicontinuous transfer schedule, $T_i : \mathcal{Q} \rightarrow \mathbb{R}$, which is an agreement to pay the agent $T_i(q)$ whenever q is chosen. Principal i 's net payoff is simply $v_i(q) - T_i(q)$ in such a case. We assume that the agent has private information, θ , distributed on Θ (compact) with distribution $F(\theta)$. An agent of type θ has a direct payoff of $u(q, \theta)$ when choosing q and a net payoff of

$$u(q, \theta) + \sum_{i \in N} T_i(q),$$

when accepting all of the contracts and receiving payments accordingly. Finally, we assume that the agent must either accept all contract offers or reject all offers: that is, if the agent is active, he must be a common agent to all n principals. We assume that when the agent rejects the contracts he must choose q_0 , and receives $u(q_0, \theta) = 0$, independent of type. Without loss of generality, we assume that $q_0 \in \mathcal{Q}$ and therefore principals are effectively required to offer $T_i(q_0) \geq 0$. Let player i 's strategy set therefore be given by $\mathcal{T} \equiv \{T_i \in \mathbb{R}^{\mathcal{Q}} \mid T_i(q_0) \geq 0\}$. We normalize the principal's payoffs with respect to the agent's rejection, so the principals' reservation payoff is also zero.

There are many equilibria to this game. One trivial equilibrium is for two of the principals to each charge the agent a sufficiently large sum regardless of action causing the agent to respond by rejecting every contract offer. There are also more interesting equilibria that do not appear to suffer from such coordination failures. We wish to characterize the entire set of equilibria, including possibly equilibria with discontinuous transfer functions. This is surprisingly simple to do using our aggregation results, once one establishes the conditions for Proposition 2.

The necessary and sufficient SGM program from Propositions 1 and 2

¹⁸A full treatment of the issues involved is outside the scope of this note and is developed in Martimort and Stole (2005,2009b). A closely related application is found in Bernheim and Whinston's (1986a) study of moral hazard in public agency contracts. They use a simplifying change of variables to convert their game into an aggregate program just as in our general methodology.

is

$$\bar{T} \in \arg \max_{T \in \mathcal{T}} \int_{\Theta \times \mathcal{Q}} \left(\sum_{i \in N} v_i(q) - nT(q) + (n-1)\bar{T}(q) \right) d\mu(\theta, q|T).$$

Just as in the case of supply function games, it is convenient to translate the program from a choice of T into a choice of (q, U) in the dual space. To this end, define the implementable set of function-pairs (q, U) as

$$\mathcal{IC} \equiv \left\{ (q, U) \in \mathcal{Q}^\Theta \times \mathbb{R}_+^\Theta \mid \exists T \in \mathcal{T} \text{ s.t. } q(\theta) \in \arg \max_{q \in \mathcal{Q}} u(q, \theta) + T(q), \right. \\ \left. U(\theta) = \max_{q \in \mathcal{Q}} u(q, \theta) + T(q) \right\}.$$

Note that the non-participation contract $(q_\emptyset, 0)_{\theta \in \Theta} \in \mathcal{IC}$ implies $U \geq 0$.

Returning to our SGM program, after substituting $(U - u)$ for T , we have our dual SGM program:

$$(\bar{q}, \bar{U}) \in \arg \max_{(q, U) \in \mathcal{IC}} \int_{\Theta} \left\{ \left(\sum_{i \in N} v_i(q) + u(q, \theta) \right) \right. \\ \left. + (n-1)(u(q, \theta) + \bar{T}(q)) - nU \right\} dF(\theta), \quad (12)$$

where \bar{T} implements (\bar{q}, \bar{U}) .

What is remarkable about the above argument and the program in (12) is that this characterizes the *entire* set of Nash equilibrium pairs (\bar{q}, \bar{U}) . The result holds regardless of the dimension of Θ , the presence of single-crossing, continuity of transfers, etc. Of course, without such properties, it is possible that the game has no equilibrium, in which case the SGM program will have no solution.

Consider, as an example, the equilibrium in which the agent rejects all contracts because two or more principals make undesirable offers. This corresponds to an equilibrium \bar{T} that is extremely large and negative for all $q \neq q_\emptyset$. The solution to the self-generating maximization program therefore is to implement q_\emptyset .

Next consider smooth equilibria with full participation. If they exist, the envelope theorem implies that (12) simplifies to

$$(\bar{q}, \bar{U}) \in \arg \max_{(q, U) \in \mathcal{IC}} \int_{\Theta} \left\{ \left(\sum_{i \in N} v_i(q) + u(q, \theta) \right) - nU \right\} dF(\theta).$$

Thus, a smooth equilibrium exists if and only if the equilibrium outcome solves the maximization program above which, importantly, is no longer self-generating. The program suggests that it is *as if* there is a single principal maximizing collective surplus less an information rent, but in this case the rent is magnified n times. It is tantamount to an n -fold marginalization of rents. A single firm collectively representing the interests of all principals would solve a similar program with n replaced with 1. It follows necessarily that competition among principals reduces payoffs in the smooth equilibria. It also follows that expected consumer rents are also reduced. This conclusion is derived entirely from the assumption that the program has a solution without any requirement that we can calculate the solution explicitly. When Θ is multi-dimensional or single-crossing fails, this is particularly valuable as closed-form, analytic solutions are typically elusive.

There are still other equilibria that are not smooth but which generate agent participation. Now the envelope theorem cannot be used in its standard form, but with some additional techniques developed in Martimort and Stole (2009a), much can be learned from using the SGM approach.

5 Concluding remarks

6 References

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