Manager Incentives for Channel Stuffing with Market-based Compensation

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Abstract. We study an extension of a two-period inventory management problem with positively correlated demands in which the manager’s compensation is partially based on an external, market-based assessment of the firm’s value. As typically the “real” demand is only observed internally in the firm, the manager may ship more than the real demand to downstream customers and report higher than real sales revenues to influence the external firm valuation, which is known as “channel stuffing.” As it is costly and does not reflect the real demand, channel stuffing destroys the firm’s value. We identify three factors that drive the manager’s incentives for channel stuffing: the marginal effect, the boundary effect and the carryover effect. The marginal effect, analogous to those earnings management incentives revealed in the literature (e.g., Stein, 1989), is independent of the inventory problem, while the boundary and carryover effects arise from the nature of the inventory problem. The boundary effect occurs when the real demand realization is high, but, still less than the available inventory: reporting a “sold out” situation censors the upper tail of the demand distribution, and hence, leads to an increase in market valuation that the manager would like to cash in with channel stuffing. The carryover effect occurs when the real demand realization is low. In this scenario, channel stuffing would make the firm’s future performance look more rosy because of positively correlated future demand and high future sales margin as the firm will be able to satisfy the future demand from the large current inventory. When examining the initial inventory decision, we find that under rational market valuation, both over- and under-investment may arise in presence of channel stuffing incentives. Based on our model analysis, we derive empirically testable hypotheses for channel stuffing.

Keywords: Channel Stuffing, Inventory Management, Market-based Compensation

1 Introduction

Operations management research usually assumes that the interests of firms’ decision makers are perfectly aligned with the interests of the firm. This however may not be the case in practice. For example, public firms are owned by shareholders but are run by their managers (CEOs and senior executives). The managers’ decisions are also motivated by the compensation they receive. The latter is typically tied to the firm’s

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performance measures, such as stock market prices. According to Morgenson (1998), in 1997, the 200 largest U.S. firms reserved more than 13% of their common stock for compensation awards to managers. If the stock price perfectly reflects the firm’s value, a compensation based on the stock price would perfectly align the manager’s decisions with the firm’s value. Unfortunately, that is not always the case. Managers have better information about the firm’s internal operating environment. In practice, the most common way for investors to retrieve information about a firm’s performance is through financial reports. As these reports indirectly impact the managers’ compensation, it is not surprising that managers may have incentives to intentionally manage the operation to influence the reports. For example, managers may be tempted to sell excess units of inventory and report higher sales in order to obtain a favorable short-term capital market reaction, even if it is costly for the firm in the long run. This is known as “channel stuffing.”

In the notorious case of Coca-Cola’s channel stuffing between 1997 and 1999, Coca-Cola offered downstream bottlers extended credit terms to induce them to purchase more than demanded, which turned into inventory at the bottlers’ places. These “padded” sales enhanced the firm’s capital market performance in those years, but, damaged the firm’s value in the long run. Managers in private firms may have channel stuffing incentives too. Channel stuffing is observed for private firms contemplating IPOs, since firms are tempted to influence the market price upward by boosting their sales revenues. In general, when the compensation of a manager is contingent on the assessment of the firm’s value by external parties, such as potential investors, non-managing shareholders, debt holders, etc., the manager may be tempted to pad the sales to influence the external assessment. For convenience, in the remaining of the paper, we refer to all of those external parties that may potentially value the firm as “investors.” Similarly, we refer to the valuation of the firm by the external parties as the “market price” of the firm, and the manager’s compensation contingent on the external valuation as the “market-based compensation.”

Channel stuffing is a form of “real earnings management,” in which managers distort “real” operations to

2Practices related to channel stuffing have been documented in the economics and accounting literature such as Oyer (1998), Graham et al. (2005), Roychowdhury (2006) and Cohen et al. (2008), as well as in the operations management literature such as Lee et al. (2004), Kapuscinski et al. (2004), Sohoni et al. (2005) and Lai (2008), which are sometimes referred to as trade loading, sales timing, gallon pushing, fiscal-year-end effect, or hockey-stick phenomenon.

There are also real business cases involving firms such as Coca-Cola, Bristol-Myers Squibb, McAfee, Lucent, Sunbeam, Cylink, Virbac, etc., that have been investigated by the U.S. Security and Exchange Commission (SEC) associated with channel stuffing. Some of those firms were prosecuted by SEC. For brevity, we refer readers to the SEC litigation database (http://www.sec.gov/) for those real business cases. Kieso and Weygandt (1998) have the following comments on channel stuffing: “Some companies record revenues at date of delivery with neither buyback nor unlimited return provisions. Although they appear to be following acceptable point of sale revenue recognition, they are recognizing revenues and earnings prematurely... Trade loading and channel stuffing are management and marketing policy decisions and actions that hype sales, distort operating results, and window dress financial statements. End-of-period accounting adjustments are not made to reduce the impact of these types of sales on operating results. The practices of trade loading and channel stuffing need to be discouraged. Business managers need to be aware of the ethical dangers of misleading the financial community by engaging in such practices to improve their financial statements.”

3For example, BigBand Networks, which is a company that sells video, voice and data solutions to major cable operators, was involved in a class-action shareholder lawsuit in October 2007. It is alleged that BigBand artificially boosted their revenues through channel stuffing before its IPO in March 2007. BigBand later failed to meet its guidance and its stock price declined from the post-IPO high of $21.43 to a low of $3.76.
influence the reported performance. In contrast, “accrual-based earnings management” involves management of discretionary accruals or creative accounting to influence the financial reports and may leave operational decisions untouched. The most famous examples of accrual-based earnings management include the cases of Enron and WorldCom, which directly led to the passage of the Sarbanes-Oxley Act (SOX) in 2002. As real earnings management is more difficult to detect by auditors or regulators, it significantly increased after the passage of SOX, while accrual-based earnings management decreased (Cohen et al. 2008). Hence, there is a pressing need to understand the determinants of real earnings management such as channel stuffing.

To evaluate a firm, rational investors must correctly interpret the performance report provided by the manager, taking the manager’s channel stuffing incentives into account. As the manager’s compensation depends on the market price of the firm, the rational manager will also anticipate how the market price will be influenced by channel stuffing activities. Hence, the manager’s channel stuffing activities and the firm’s market valuation need to be determined simultaneously. To the best of our knowledge, no theoretical study has jointly examined channel stuffing, performance report and firm market valuation.

In this paper, we develop a model that helps us to understand the key determinants of channel stuffing. To that end, we analyze a firm’s two-period inventory management problem with positively correlated demands. The demands are stochastic and must be satisfied from inventory. The firm is run by a self-interested manager. The manager’s payoff partially depends upon the market price of the firm and partially depends on the long-term value of the firm. The relative weights of both components of the manager’s payoff are exogenously given. We consider a typical source of information asymmetry: The investors only know the demand distribution but do not observe the realized real demand, while the manager privately observes the real demand. If there is excess inventory beyond the real demand, the manager may pad additional sales through downstream parties, which is costly in the long run, and recognize the revenue immediately in the firm’s accounting books. The manager then reports the (possibly padded) sales revenue to the investors. However, the investors will infer the potential padding and value the firm based on their expectation of the current and future demands for the firm’s products.

We characterize a market equilibrium on channel stuffing and firm valuation. We find that in equilibrium,

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4Empirically, it can be difficult to obtain a clean cut between accrual-based earnings management and real earnings management. Traditional empirical studies in accounting usually test the discretionary accruals as an indicator of earnings management. Some popular models assume a zero expected value of the first differences in total accruals and regard any non-zero change in total accruals as an indicator of earnings management. The discretionary accruals calculated in this way may include both accrual manipulation that will not affect operational decisions and real manipulation such as channel stuffing (channel stuffing may result in increase in accounts receivable when the padded sales are on credit, which is captured by total accruals). It is only recent that empirical studies in accounting try to separate real earnings management from accrual management (McVay 2006, Roychowdhury 2006, and Cohen et al. 2008). Though different studies may define real and accrual earnings management differently, most studies regard accrual management as abnormal operational practices motivated by managers’ own interests but do not contribute to the firms’ real value. According to the literature (e.g., Bartov 1993, Graham et al. 2005, Roychowdhury 2006, Cohen et al. 2008), activities that are identified as real earnings management include using price discounts, credit terms, flexible returns to boost sales (channel stuffing), cutting discretionary expenditures such as R&D, advertising, and SG&A (Selling, General and Administrative expenses), and timing of income recognition from disposal of long-lived assets.
the manager’s incentive to pad sales is driven by three factors: the *marginal effect*, the *boundary effect*, and the *carryover effect*. The marginal effect is an effect analogous to those reported in previous studies about earnings management (e.g., Stein 1989). When the market price of the firm plays a major role in the manager’s payoff such that the marginal value to the manager of padding sales is positive, channel stuffing will be observed. Besides the marginal effect, we characterize two more effects that influence channel stuffing: the boundary and carryover effects. These two effects are tied to the special properties of inventory problems. The boundary effect is linked to the demand filtering effect that finite inventories have: if the inventory is “sold out,” it is difficult in general to infer the true demand. The investors only know that the true demand is at least as large as the available inventory. Hence, their valuation may exhibit a discrete jump when the inventory is sold out. When the inventory is nearly sold out, the manager can be strongly tempted to pad the sales in order to “clear the shelves” and cash in on the jump in the valuation via his compensation. The carryover effect is also tied to the properties of inventory problems. When the realized demand is very small, the firm has a lot of inventory leftover so that no replenishment will be needed. This indicates that the margin of the future sales will be high as satisfied by the existing inventory. In such a situation, the manager has extra incentive to signal a stronger demand in the future by padding the current sales. This incentive decreases as the amount of leftover inventory decreases.

In equilibrium, however, the manager’s incentives are anticipated by the rational investors. They discount the reported performance of the firm according to their belief about the padded sales, which is consistent with the manager’s strategy. When examining the manager’s initial inventory decision, we find that the invested inventory level can be either higher or lower than the classical newsvendor solution; i.e., the first-best level. Facing rational investors, the manager takes the costly consequence of channel stuffing into account. While under-investment might be generally expected since channel stuffing generates an extra “overage” cost, it is especially interesting to see that over-investment may also occur. The manager deploys extra inventory to alleviate the channel stuffing incentive from the boundary effect. Our findings characterize the potential that managers might not only apply real earnings management to change the firm’s financial status for reporting, but also use operational levers, such as inventory investment, to mitigate the short-term incentives.

The remaining sections are organized as follows: The next section reviews the relevant literature. Section 3 describes the model. In Section 4, we analyze the equilibrium on channel stuffing and market pricing. Section 5 investigates the incentives on initial inventory investment. In Section 6, we discuss the empirical implications of our findings and we conclude in Section 7.
2 Related Literature

This paper is related to two streams of research: the literature on how a firm’s operations interact with its capital market performance, and the literature on earnings management. While the link between a firm’s operations and its capital market performance is fundamental, it has been an under-explored area for a long time in operations management research. Recent empirical studies are filling in this gap. Hendricks and Singhal (2001) show a positive linkage between the adoption of total quality management and a firm’s long-term stock performance. In contrast, Hendricks and Singhal (2005) find that a firm’s stock price can drop substantially upon an announcement of supply chain disruption. Generally, excess inventory can also drive down the stock price, as discussed in Chen et al. (2005) and Hendricks and Singhal (2008). More interestingly, Hendricks and Singhal (2008) reveal that investors may punish a firm more if excess inventory is announced at its customers. According to their findings, “inventory buildups at customers could suggest that the announcing firm may have engaged in ‘channel stuffing’ or ‘trade loading,’ which could raise concerns about earnings management and possibility of lawsuits and litigation.” This empirical observation is consistent with the motivation of our analytical study.

Fisher et al. (2002) set out the goal to understand the relationship between a firm’s accounting performance and stock return. By examining a sector of public retailers, they show that return on assets (ROA) and sales growth have a positive association, while standard deviation of ROA has a negative association, with long-term stock returns. Raman (2006) illustrates with real examples that the role of inventory management can be substantial to determine a firm’s expected stock return. By examining a sample of public retailers’ inventory turnover, an important measure for capital market valuation, Gaur et al. (2005) reveal that variation in inventory turnover can be largely explained by the factors of gross margin, capital intensity and sales surprise.

Different from the above papers, Lai (2006a,b, 2008) and Oyer (1998) document how firms may intentionally manage their operations anticipating the market reaction. Lai (2006a,b) shows that firms may intentionally under-invest in the inventory to signal their competence. In particular, Oyer (1998) and Lai (2008) reveal that fiscal-year-end effect widely exists, where firms’ sales figures could be substantially high while inventory levels could be substantially low at the end of fiscal year that may not necessarily be the calendar year end. They document strong evidence of sales pushing for various firms in different business sectors.

There are numerous studies on earnings management. However, most of these works address the discretion of accounting accruals and how a manager may misreport the earnings number, which has little to do with real operations. We therefore only discuss studies that are closely related to our work. Among the early
works is Stein (1989), which studies a market equilibrium on earnings management between rational investors and the manager of a firm. In Stein’s model, the manager consistently borrows from next period’s earnings (for instance, to forsake good investment) to inflate current earnings, even though the capital market takes this into account. The incentive from the short-term compensation drives the manager away from the first-best decision. Our paper shares the same interest in how the manager’s short-term interest impacts his decisions, but we explore a more detailed operational problem—inventory management. Moreover, we show that although the incentive of earnings management may always exist, the manager may alter the initial inventory investment to reduce this incentive facing rational investors. Dye and Sridhar (2004) and Liang and Wen (2007) are the other two papers closely related to ours. Both of them address the impact of earnings management on firms’ investment decisions. They show that with the chance of earnings management, a firm may deviate from the first-best decision by over- or under-investing. Similar to Stein’s study, they also assume a stylized capital investment problem, but they focus more on the exploration of accounting rules.

Our work is also related to several papers that address the agency problem in operations contexts. The studies by Chen (2000, 2005), Chen and Xiao (2009a,b) and Wang and Zipkin (2008) investigate inventory management problems where the demand for the product can be influenced by the sales or purchasing agent. They discuss the contracting issue of how to appropriately motivate the agent, while we examine the equilibrium behavior of rational external firm valuation and inventory decision with earnings management. Baiman et al. (2008) study an agency problem in a manufacturing context where the agent controls the processing time of the workstation. They explore the role of inventory buffer size in motivating the agent’s effort. In a context concerned with inventory decision under capital constraint, Xu and Birge (2008) discuss how a bonus scheme linked to the firm’s operating income may affect the manager’s inventory decision and capital financing. Finally, Debo et al. (2008) address the agency issue associated with expert services. Analogous to ours, in their model, the service expert observes the amount of work that is necessary for the customer, but the customer does not. This creates an incentive to pad the work.

3 The Model

Inventory Model. We consider a firm that sells a product in two periods \( t \in \{1, 2\} \). The “real” demand in period \( t \) is \( \xi_t \). We assume that: \( \xi_1 = \eta_1 \) and \( \xi_2 = a \xi_1 + \eta_2 \), where \( \eta_1 \) and \( \eta_2 \) are independent random variables. We use bold \( \mathbf{\eta}_t \) and \( \mathbf{\xi}_t \) to represent random variables, while plain \( \eta_t \) and \( \xi_t \) represent real variables. We assume that \( \eta_t \) follows a continuous distribution function \( F_t(\cdot) \) (density \( f_t(\cdot) \)) over \([0, \eta_t]\) with finite mean, where \( \eta_t \) may be infinitely large. \( a \) is a constant that captures the correlation of the demands in the two periods. If \( a = 0 \), the demands are independent, while \( a \gtrless 0 \) reflects a positive (negative) correlation. In
particular, for the business motivations we are interested in, we focus on the case where \( a > 0 \).

In both periods, the selling price per unit is fixed at \( p \). The inventory purchasing cost per unit is \( c < p \). The firm needs to install inventory \( q_t \) in each period \( t \in \{1, 2\} \) before the real demand of that period, \( \xi_t \), is realized. The unmet demand in each period is lost. This is appropriate in consumer settings where an out of stock demand leads to a substitution among competing products. We assume that the firm needs to pay a holding cost \( h < c \) per unit for carrying the leftover inventory from the first period to the second period. At the end of the second period, all remaining inventory is salvaged at a value that we normalize to zero. For the sake of clarity, we assume the firm has initial liquid assets \( A_0 \) at the beginning of the first period which are sufficient to cover any cost during the two periods.\(^6\)

Notice that without any further considerations, we have described a classical two-period inventory management problem with correlated and lost demands, whose objective is to determine the optimal inventory levels \( q_t \) for \( t \in \{1, 2\} \). When the manager’s interest is perfectly aligned with the firm’s long-term value, the manager would optimize the inventory decision according to classical inventory theory.

**Firm Structure.** We consider a firm that is run by a self-interested manager whose compensation partially depends on an external valuation (i.e., market price) of the firm. That valuation is the investors’ expectation of the firm’s long-term value, conditional on the financial report released by the firm and all other publicly available information. In particular, the investors do not observe the exact demand realization of the current reporting period. They only know the distribution of the demand.

To model the manager’s objective function, we adopt, as in Stein (1989), the most simple model for the manager’s compensation structure: A fraction \( \beta \) is based on the market price of the firm and a fraction \( 1 - \beta \) is based on the firm’s true value. In other words, the manager does care about the true value of the firm, as an owner-manager would do, but, not about 100% since a part of his compensation is based on the investors’ belief about the firm’s value. More complex managerial compensation mechanisms do exist in reality. However, the structure that we adopt captures, in the simplest way, the key feature that we are interested in. We assume that the information about the manager’s compensation (i.e., \( \beta \)) is known to the investors.\(^7\)

**Channel Stuffing.** In our model, given that the investors not only care about the current sales realization but also, due to the correlation, regard the first-period sales realization as a predictor of the firm’s future performance, the manager may be tempted to pad the real demand in order to obtain a more favorable short-term compensation. More precisely, we assume that the manager may “sell” an excess amount of \( x (\geq 0) \)\(^5\) our model is general and can be interpreted for \( a < 0 \) too. As positively correlated demand has more interesting business motivations (for instance, the market is growing), we focus on the case with \( a > 0 \).

\(^6\) \( A_0 \) does not play a key role in the analysis, but avoids having to deal with the complexity of such as financing or bankruptcy which may arise otherwise.

\(^7\) For example, regarding stock option compensation, the companies should provide forecasts about when the stock options will be exercised by the employees, according to SFAS No. 123.

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units to the downstream parties after the real first-period demand, $\xi_1$, is privately observed. Obviously, the manager can choose not to pad the sales at all ($x = 0$). On the other hand, the maximum possible padding amount is determined by the available inventory ($x = (q_1 - \xi_1)^+$), because the manager cannot pad more than the available inventory. The manager may pad any amount between 0 and $(q_1 - \xi_1)^+$ and reports revenue $pz$ (where $z = \xi_1 + x$). For simplicity, we assume that the $x$ units that are padded sales return to the firm at the beginning of the second period.\(^8\) The downstream parties then need to be reimbursed (i.e., the firm incurs a cost $p$ per unit) and the warehouse is credited by $c$ per unit. In addition, an extra cost $\gamma(x)$ is incurred, representing the channel stuffing cost including any physical cost and incentives for the downstream parties. We assume that $\gamma(x)$ is a convex function over $x \in [0, \infty)$, satisfying: $\gamma(0) = 0$, $\gamma'(x) > 0$ and $\gamma''(x) > 0$.

**Reporting.** The firm releases a financial report to the investors at the end of the first period. This report includes the sales revenue $pz$, cost of goods sold $cz$ and leftover inventory $cI$ (where $I = q_1 - z$). Except for the real demand $\xi_1$ and the amount of padded sales $x$, all other information is observable to the investors. In other words, provided with the sales revenue $pz$, the investors can infer the amount of sales $z$ (but not $\xi_1$ or $x$). Similarly, provided with the leftover inventory $cI$, the investors will know the quantity $I$, and thus because of $I$ and $z$ can be inferred, the investors will know the first-period inventory investment, $q_1$, as well. For convenience, we use $z$ and $q_1$ to represent the financial report.

**Timeline.** Figure 1 details the timeline of the model. At the beginning of the first period, the manager makes the initial inventory decision, $q_1$. The manager observes the actual demand $\xi_1$ and decides how many sales $x$ to pad beyond $\xi_1$. The firm releases a financial report at the end of the first period. The investors react to the information and value the firm based on the available information. The manager obtains his market-based compensation. At the beginning of the second period, the padded sales are returned and the firm invests additional inventory $q_2$. Then, the second-period demand and sales are realized. The firm is liquidated after satisfying the demand and the manager receives his remaining compensation. Notice that the firm is priced at each point in time, but only at $t = 1$ the manager has information advantage.

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\(^8\)In practice, padded sales may not be physically returned. However, as we observed from the real business cases investigated by SEC, in order to induce the downstream parties to participate in channel stuffing, firms effectively need to compensate them for most of the costs incurred irrespective of what approach is used for channel stuffing. The analysis of our model holds to a setting in which padded sales are not returned physically to the firm, but, the firm compensates the downstream parties for the holding cost and for any leftover inventory from the padded sales. For simplicity, we assume in this paper that the padded sales are returned.

In this paper we assume that there is no explicit buyback agreement between the firm and its downstream parties. According to SFAS No. 49, if a company sells a product in one period and agrees to buy it back in the next period, no sales revenue should be recognized since the risks of ownership are retained by the seller. However, if there is no explicit buyback agreement, the seller can recognize revenue in the current period and recognize returns in “sales returns,” which is a contra-revenue account, in the next period when the inventory is returned. Even if the seller expects more than usual returns in the future, the seller’s incentive is unobservable and the revenue is recognized in the current period without breaking the accounting rules. Another accounting rule used to prevent overstatement of revenue is to ask the company to set up a separate account called sales returns and allowance, which is a reduction in sales revenue, to estimate the potential returns (SFAS No. 48). However, since this allowance is a subjective estimation by the manager, the manager can understatement the potential returns. For simplicity, in this paper we do not directly model the estimation of sales returns and allowance.
The manager makes the initial inventory investment decision, $q_1$;
(1.2) The firm reports initial asset level, sales revenue, cost of goods sold and leftover inventory;
(1.3) Investors price the firm based on the report and their belief;
(1.4) The manager receives the market-based compensation;
(0.1) The manager makes the initial inventory investment decision, $q_1$;
(1.1) The manager observes the real demand and decides how many sales to pad;
(1.2) The firm reports initial asset level, sales revenue, cost of goods sold and leftover inventory;
(1.3) Investors price the firm based on the report and their belief;
(1.4) The manager receives the market-based compensation;
(2.1) The padded sales are returned;
(2.2) The manager makes the second-period inventory investment decision, $q_2$;
(2.3) The second-period demand and sales are realized;
(2.4) The firm is liquidated and the manager receives his remaining compensation.

Figure 1: The timeline of the model.

The Equilibrium. We introduce an equilibrium concept on channel stuffing and market pricing formally as follows: Define $v_2(\xi_1, q_1)$ to be the firm’s expected second-period profit when the first-period real demand is $\xi_1$ and initial inventory is $q_1$. Suppose the manager pads the sales by an amount $x$. Then, the firm’s true expected value at the end of the first period is:

$$v_1(x; \xi_1, q_1) = A_0 + p \min(q_1, \xi_1) - cq_1 - \gamma(x) - h(q_1 - \xi_1)^+ + v_2(\xi_1, q_1)$$

where $0 \leq x \leq (q_1 - \xi_1)^+$. Note that as padding is costly and does not increase the real value, it is obvious that if there is no market-based compensation, the manager will never pad the sales; $\frac{dv_1(x; \xi_1, q_1)}{dx} = -\gamma'(x) < 0$ for $x > 0$. However, when the manager’s compensation depends on the firm’s market price, he may have the incentive to boost the market price by padding the sales given that the investors cannot observe the actual demand, $\xi_1$. Let $P(z, q_1)$ be the market price of the firm as a function of the initial inventory level $q_1$ and the reported sales amount $z$. Provided with such a market pricing function, the manager will choose a padding amount that maximizes his expected payoff, $\pi$:

$$\max_{0 \leq x \leq (q_1 - \xi_1)^+} \pi(x; \xi_1, q_1) = \beta P(\xi_1 + x, q_1) + (1 - \beta) v_1(x; \xi_1, q_1).$$

(1)

On the other hand, the investors will try to infer the demand from the financial report, which will lead to the market pricing function $P(z, q_1)$. We define the following market equilibrium concept. Similar concepts have also been defined in the literature (e.g., Stein 1989, Dye and Sridhar 2004, Liang and Wen 2007).

**Definition 1** Given $q_1$, a perfect Bayesian market equilibrium $(x^o(\xi_1, q_1), P^o(z, q_1))$ is reached on channel stuffing and market pricing if:

(1) Given $P^o(z, q_1)$, $x^o(\xi_1, q_1)$ maximizes the manager’s payoff $\pi(x; \xi_1, q_1)$ for any $\xi_1$;
(2) \( P^o (z, q_1) \) satisfies \( P^o (0, q_1) = v_1 (0; 0, q_1) \) and
\[
P^o (z, q_1) = \mathbb{E}_{\eta_1} [v_1 (z - \eta_1; \eta_1, q_1) \mid z = \eta_1 + x^o (\eta_1, q_1)] \text{ for } z \in (0, q_1].
\]

This definition outlines the following process. Anticipating the investors’ reaction, the manager makes a padding decision that maximizes his payoff over two periods. The investors, knowing the manager’s incentive, value the firm based on the available information and their conjecture about the manager’s behavior. In particular, since we assume that the demand distribution of \( \eta_1 \) is over \([0, \eta_1]\) and the manager pads sales by \( x \in [0, (q_1 - \xi_1)^+]\), Definition 1 imposes an initial condition on the market pricing function that if the reported sales amount \( z \) is zero, the investors will directly believe the real demand is zero. However, for any reported sales amount \( z > 0 \), the investors will have to apply Bayes rule to infer the real demand since such sales could be potentially padded. In equilibrium the manager’s optimal padding decision should match the investors’ belief for any \( \xi_1 \). At the beginning of the first period, the manager selects \( q_1^o \) that maximizes his expected payoff \( \mathbb{E}_{\eta_1} [\pi (x^o (\eta_1, q_1); \eta_1, q_1)] \) taking the subsequent padding into account. In the following, we examine the market equilibrium for any given \( q_1 \) in Section 4, and in Section 5 we investigate the first-period inventory decision, \( q_1^o \).

4 Analysis of Channel Stuffing Incentives

Since the demand in the first period is correlated to the demand in the second period \((a > 0)\), investors update the firm’s value after obtaining information about the first-period sales. To investigate the market equilibrium, we first analyze the second-period decision with given first-period inventory \( q_1 \) and demand \( \xi_1 \).

4.1 Second Period Decision

Let \( I = (q_1 - \xi_1 - x)^+ \) denote the leftover inventory from the first period. Then, the firm’s second-period inventory investment is determined by:
\[
\max_{q_2 \geq 0} \mathbb{E}_{\eta_2} [p \min (a \xi_1 + \eta_2, I + x + q_2)] - cq_2.
\]

Note that the padded amount, \( x \), does not play a role for the second-period optimal investment as the padded demand is not real and will be returned at the beginning of the second period. Hence, only \( q_1 - \xi_1 \) matters for the optimal second-period inventory investment, which is, according to the classical newsvendor solution: \( q_2^o (\xi_1, q_1) = 0 \) when \( 0 \leq \xi_1 \leq \bar{\xi} (q_1) \) and \( q_2^o (\xi_1, q_1) = k_2 + a \xi_1 - (q_1 - \xi_1)^+ \) when \( \xi_1 > \bar{\xi} (q_1) \), where
the potential of channel stuffing, it is not impossible that the initial inventory level may exceed

In this subsection we characterize the equilibrium of channel stuffing and market pricing. Note that due to

4.2 Equilibrium Channel Stuffing and Market Pricing

\[ v_2(\xi_1, q_1) = \begin{cases} 
  pa_1 + E_n [p \min(n_2, q_1 - (1 + a)\xi_1)], & 0 \leq \xi_1 \leq \bar{\xi}(q_1), \\
  (p - c) a_1 + c (q_1 - \xi_1)^+ + E_n [p \min(n_2, k_2)] - ck_2, & \xi_1 > \bar{\xi}(q_1). 
\end{cases} \] (2)

The profit depends on both the inventory level \( q_1 \) and the actual demand \( \xi_1 \) in the first period. We use \( v'_2(\xi_1, q_1) \) to denote \( \frac{\partial v_2(\xi_1, q_1)}{\partial \xi_1} \) throughout the paper:

\[ v'_2(\xi_1, q_1) = \begin{cases} 
  ap - (1 + a)pF_2(q_1 - (1 + a)\xi_1), & 0 \leq \xi_1 \leq \bar{\xi}(q_1), \\
  a(p - c) - c, & \bar{\xi}(q_1) < \xi_1 < q_1, \\
  a(p - c), & \xi_1 \geq q_1. 
\end{cases} \] (3)

Equation (3) will be useful to determine the manager’s incentive to pad the first-period sales. If \( \xi_1 \) is low (below \( \bar{\xi}(q_1) \)), there will be a lot of inventory leftover at the end of the first period and the firm will not need to invest in new inventory in the second period. It is clear from the first branch of Equation (2): An increase of \( \xi_1 \) directly leads to a net margin \( ap \) in the second period due to the correlation between the first- and second-period demands. However, an increase of \( \xi_1 \) leads to the decrease of second-period inventory with a ratio \( (1 + a) \) (“1” captures the first-period consumption and “a” captures the second-period consumption).

When the inventory decreases, the marginal loss from the unsatisfied second-period demand amounts to \( (1 + a)pF_2(q_1 - (1 + a)\xi_1) \). When \( \xi_1 \) is higher than \( \bar{\xi}(q_1) \) but less than \( q_1 \), the firm has some leftover inventory from the first-period but needs to replenish it in the second period. In this case, an increase of \( \xi_1 \) will lead to an increase of the expected profit by only \( a(p - c) \) per unit. Moreover, it also brings an additional purchasing cost, \( c \), in the second period, which the firm otherwise does not need to incur. This leads to the second branch of Equation (3). If \( \xi_1 \) is higher than \( q_1 \), the firm uses up the inventory in the first period and needs to replenish all the inventory that is needed. Therefore, an increase of the first-period demand signals an increase of the second-period profit by \( a(p - c) \) through the demand correlation. Finally, based on Equation (3), we can easily see that the derivative \( v'_2(\xi_1, q_1) \) is continuous and decreasing in \( \xi_1 \) when \( \xi_1 < q_1 \), which is concluded by the following lemma.

**Lemma 1** \( v'_2(\xi_1, q_1) \) is continuous and decreases in \( \xi_1 \) for \( \xi_1 < q_1 \).

### 4.2 Equilibrium Channel Stuffing and Market Pricing

In this subsection we characterize the equilibrium of channel stuffing and market pricing. Note that due to

the potential of channel stuffing, it is not impossible that the initial inventory level may exceed \( \bar{\pi}_1 \), the upper bound of the first-period demand distribution. We therefore analyze the equilibrium for any given \( q_1 \).

As discussed in the model, the manager maximizes his expected payoff, \( \pi(x; \xi_1, q_1) \), on the amount of sales to pad in the domain \( x \in [0, (q_1 - \xi_1)^+] \). Notice that there is a difference between the cases where
the manager chooses a padding amount $x$ in $[0, (q_1 - \xi_1)^+]$ and where the manager chooses the amount $x = (q_1 - \xi_1)^+$. If the latter padding amount is taken, the manager will report $z = q_1$ even when the real demand is less than the inventory ($\xi_1 < q_1$). Such a report cannot be differentiated from the pool where the real demand is indeed higher than $q_1$ (for $\xi_1 \geq q_1$, the manager has no choice but to report $z = q_1$). The investors will be unsure whether it is a real sold-out situation or the sales are padded. In contrast, if a padding amount in $[0, (q_1 - \xi_1)^+]$ is selected and a report with $z < q_1$ is released, the investors can exclude the region $\xi_1 > z$ from their consideration.

To characterize the equilibrium padding strategy and market price, we conjecture a structural form and prove that it satisfies all equilibrium conditions of Definition 1. Our conjecture of an equilibrium is that given $q_1$, there exists a threshold, $\xi(q_1) \in [0, \bar{\eta}_1]$ such that for realizations of the first-period demand below the threshold, the reported sales monotonically increase in the real demand, while for realizations above that threshold, the manager always reports sales equal to the initial inventory.

In the following, we validate this conjecture. We first introduce a function $\varphi(\xi_1, q_1)$ that is specified by the following three conditions: (a) $\varphi(0, q_1) = 0$; (b) $\frac{\partial \varphi(0, q_1)}{\partial \xi_1} = \frac{\beta(p + h + v_z'(0, q_1)) - (1 - \beta)\gamma'(0)}{\gamma'(0)}$ if $\frac{\beta(p + h + v_z'(0, q_1)) - (1 - \beta)\gamma'(0)}{\gamma'(0)} > 0$, otherwise, $\frac{\partial \varphi(0, q_1)}{\partial \xi_1} = 0$; and (c) for $\xi_1 > 0$,

$$\frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1} = \begin{cases} \frac{\beta(p + h + v_z'(\xi_1, q_1)) - (1 - \beta)\gamma'(\varphi(\xi_1, q_1))}{\gamma'(\varphi(\xi_1, q_1))}, & \text{if } \varphi(\xi_1, q_1) > 0, \\ 0, & \text{o/w.} \end{cases}$$

(4)

$\varphi(\xi_1, q_1)$ will act as the padding function in our conjecture when the real demand $\xi_1$ is less than the conjectured threshold $\hat{\xi}(q_1)$. The above three conditions specify $\varphi(\xi_1, q_1)$ as a non-negative function that starts from zero. In particular, condition (c) in Equation (4) specifies an autonomous ordinary differential equation (ODE). With this ODE, it is challenging to express $\varphi(\xi_1, q_1)$ explicitly (in Appendix D, we provide an approach to derive implicit equations for the solution of the ODE with uniform demand distributions and quadratic penalty cost functions). However, $\varphi(\xi_1, q_1)$ exhibits several structural properties as shown in Lemma 2.

**Lemma 2** Given $q_1$, (i) $\xi_1 + \varphi(\xi_1, q_1)$ monotonically increases in $\xi_1$.

(ii) Let $\theta(q_1) = \beta(p + h + v_z'(0, q_1)) - (1 - \beta)\gamma'(0)$.

If $\theta(q_1) \leq 0$, then $\varphi(\xi_1, q_1) = 0$ for all $\xi_1$;

If $\theta(q_1) > 0$ then $\varphi(\xi_1, q_1) > 0$ for some or all $\xi_1 > 0$.

Lemma 2(i) states that if the manager pads according to $\varphi(\xi_1, q_1)$, the reported sales $z$ will increase monotonically in the real demand $\xi_1$ as long as $z < q_1$. Furthermore, $\theta(q_1)$ in Lemma 2(ii) has a meaningful

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9Notice that this is because of our assumption that no unsatisfied demand can be backlogged. This assumption is appropriate when products are commoditized and a stock-out will lead to a substitution.
intuition. Suppose the investors always trust the manager, taking the reported sales number as the real demand to value the firm for any \( z < q_1 \). In that case, the marginal increase of the firm’s market price with respect to \( z \) coincides with the marginal increase of the firm’s real value with respect to \( \xi_1 \), which is \( p + h^o_1 (\xi_1, q_1) \). Recall from Lemma 1, \( v_1^o (\xi_1, q_1) \) decreases in \( \xi_1 \). Therefore, the marginal market price is the highest at \( \xi_1 = 0 \). On the other hand, the penalty cost \( \gamma(x) \) is convex, whose derivative is the lowest at \( \xi_1 = 0 \). As a result, \( \theta (q_1) \) corresponds to the largest marginal benefit that the manager can achieve from channel stuffing in that case. \( \theta (q_1) \) will be useful when we explain the insights of the equilibrium.

Based on the properties of \( \varphi(\xi_1, q_1) \), the following proposition holds.

**Proposition 1** Given \( q_1 \), for \( \xi_1 \in [0, \bar{\eta}_1] \), when \( \xi_1 < \xi_1^e (q_1) \), \( \varphi(\xi_1, q_1) \) satisfies

\[
\varphi(\xi_1, q_1) = \arg \sup_{0 \leq z < (q_1 - \xi_1)^+} \pi(x; \xi_1, q_1) \quad (5)
\]

when \( P(z, q_1) = v_1 (z - \xi_1^e (z, q_1); \xi_1^e (z, q_1), q_1) \) for \( z \in [0, q_1) \).

\( \xi_1^e (q_1) \) is the solution of \( \xi_1^e + \varphi(\xi_1^e, q_1) = q_1 \), and \( \xi_1^e (z, q_1) \) is the solution of \( z = \xi_1 + \varphi(\xi_1, q_1) \) if such a solution exists in \( [0, \bar{\eta}_1] \), otherwise \( \xi_1^e (z, q_1) = \bar{\eta}_1 \).

Proposition 1 has the following implication: If the manager follows the function \( \varphi(\xi_1, q_1) \) to pad the sales, then the reported sales amount, \( z = \xi_1 + \varphi(\xi_1, q_1) \), will monotonically increase as \( \xi_1 \) increases (see Lemma 2(ii)) before it reaches the inventory boundary (reflected by the condition \( \xi_1 < \xi_1^e (q_1) \)). This implies that the investors will be able to perfectly infer the real demand from the reported sales when \( z < q_1 \). In other words, there is a unique solution, \( \xi_1^e (z, q_1) \), of \( z = \xi_1 + \varphi(\xi_1, q_1) \), which monotonically increases in \( z \). As a result, the market price of the firm, \( P(z, q_1) \), proposed in Equation (5) will reflect the true value of the firm. On the other hand, if the investors value the firm by the function \( P(z, q_1) \) in Equation (5), then the strategy to pad an amount of \( x = \varphi(\xi_1, q_1) \) sales indeed provides the largest payoff to the manager among all the choices in the set \( 0 \leq x < (q_1 - \xi_1)^+ \). This payoff equals the firm’s true value, \( v_1 (\varphi(\xi_1, q_1); \xi_1, q_1) \).

Proposition 1 imposes a constraint on the manager’s optimization problem with \( 0 \leq x < (q_1 - \xi_1)^+ \); i.e., we excluded the possibility that the manager “clears the shelves” by taking \( x = (q_1 - \xi_1)^+ \). In the following, we investigate the choice \( x = (q_1 - \xi_1)^+ \) and validate the existence of the threshold \( \hat{x} (q_1) \) in our conjecture. Let

\[
V_1(\xi, q_1) = \mathbb{E}_{\eta_1} \left[ v_1 \left( (q_1 - \eta_1)^+; \eta_1, q_1 \right) | \eta_1 \geq \hat{x} \right] \quad \text{for} \quad \xi \in [0, \bar{\eta}_1].
\]

\( V_1(\xi, q_1) \) is the firm’s expected value conditional on that the manager pads all remaining inventory for any first-period demand realization higher than \( \xi \) (and consequently reports \( z = q_1 \)). Then, the threshold \( \hat{x} (q_1) \)
in our conjecture, given the result in Proposition 1, must satisfy that: for \( \xi_1 \in [0, \overline{\xi}_1] \),

\[
\begin{align*}
&v_1(\varphi(\xi_1, q_1); \xi_1, q_1) > \beta V_1(\hat{\xi}(q_1), q_1) + (1 - \beta) v_1(q_1 - \xi_1; \xi_1, q_1), \quad 0 \leq \xi_1 < \hat{\xi}(q_1); \\
&v_1(\varphi(\xi_1, q_1); \xi_1, q_1) < \beta V_1(\hat{\xi}(q_1), q_1) + (1 - \beta) v_1(q_1 - \xi_1; \xi_1, q_1), \quad \hat{\xi}(q_1) < \xi_1 < \xi'_1(q_1).
\end{align*}
\]

(6)

The left hand side in Equation (6), \( v_1(\varphi(\xi_1, q_1); \xi_1, q_1) \), represents the manager’s payoff that corresponds to the padding strategy \( \varphi(\xi_1, q_1) \) which does not clear the shelves. The right hand side is the manager’s payoff when clearing the shelves. The investors observe a report \( z = q_1 \) and hence obtain a value \( V_1(\hat{\xi}(q_1), q_1) \) of the firm (based on our conjecture). This is the first part of the manager’s payoff (with weight \( \beta \)). The second part (with weight \( 1 - \beta \)) is the true value of the firm when clearing the shelves. The first case of Equation (6) states that for demand realizations lower than the threshold, not clearing the shelves yields a higher payoff to the manager than clearing the shelves. The second case of Equation (6) states the opposite for demand realizations that are higher than the threshold.

Proposition 2 confirms the existence of such a threshold.

**Proposition 2** Given \( q_1 \), there exists a unique \( \hat{\xi}(q_1) \in [0, \overline{\xi}_1] \) that satisfies Equation (6). In particular, if \( \xi'_1(q_1) < \overline{\xi}_1 \), then \( \hat{\xi}(q_1) < \xi'_1(q_1) \); if \( \xi'_1(q_1) \geq \overline{\xi}_1 \), then \( \hat{\xi}(q_1) = \overline{\xi}_1 \).

Figure 2 illustrates the manager’s payoff function with respect to the padding amount. When generating this figure, we assume the investors hold the belief that: if the reported sales amount \( z \) is less than \( q_1 \), then the real demand is \( \xi'_1(z, q_1) \); if the reported sales amount \( z \) equals \( q_1 \), then the real demand is within the domain \( \xi_1 \in [\hat{\xi}(q_1), \overline{\xi}_1] \) and they apply Bayes rule to derive the expected firm value. With such a belief, we observe in the provided experiments that the manager’s payoff function exhibits concavity in the region \([0, q_1 - \xi_1]\) and there is a jump at \( x = q_1 - \xi_1 \). In the left panel, the real demand is less than \( \hat{\xi}(q_1) \) and the maximum payoff is achieved at exactly \( x = \varphi(\xi_1, q_1) \). In contrast, in the right panel, the real demand is larger than \( \hat{\xi}(q_1) \) and the maximum payoff is achieved at \( x = q_1 - \xi_1 \).

With the above two propositions, we characterize the structure of a market equilibrium in Theorem 1 that is the main result of our paper.

**Theorem 1** There exists a market equilibrium that has the structure of semi-separating and semi-pooling.

(i) When \( \xi_1 > \hat{\xi}(q_1) \), the manager pushes the sales to the inventory boundary by padding \( x^o = (q_1 - \xi_1)^+ \). The manager reports \( z = q_1 \).

(ii) When \( \xi_1 < \hat{\xi}(q_1) \), the manager’s sales padding strategy is \( x^o = \varphi(\xi_1, q_1) \). The manager reports \( z = \xi_1 + \varphi(\xi_1, q_1) \).

(iii) When \( \xi_1 = \hat{\xi}(q_1) \), if \( \xi'_1(q_1) \leq \overline{\xi}_1 \), the manager follows \( x^o = (q_1 - \xi_1)^+ \), otherwise the manager follows
Manager Payoff

Padding Amount $x$

The boundary market price

In contrast, if manager is either indifferent between the two choices $\gamma = 1$, $h = 1$, $\gamma(x) = 4x + 2x^2$, $\eta_{1,2} \sim Beta(5,5)$ on $[0,10]$, $A_0 = 0$. With this set of parameters, $\xi(q_1) \approx 6.475$. In the left panel, $\xi_1 < \xi(q_1)$ and an intermediate padding amount about $x = 0.127$ (i.e., $\phi(\xi_1, q_1)$) achieves the maximum payoff; in the right panel, $\xi_1 > \xi(q_1)$ and the boundary padding amount $x = 0.4$ (i.e., $q_1 - \xi_1$) achieves the maximum payoff.

$$x^o = \varphi(\xi_1, q_1).$$

(iv) The equilibrium market pricing function follows

$$P^o(z, q_1) = \begin{cases} P(z, q_1) = v_1(z - \xi_1(z, q_1); \xi_1(z, q_1), q_1), & z \in [0, q_1); \\ \mathcal{P}(q_1) = V_1(\xi(q_1), q_1), & z = q_1. \end{cases} \tag{7}$$

**Corollary 1** If $\theta(q_1) \leq 0$ and $q_1 \geq \eta_1$, there is no channel stuffing, otherwise channel stuffing occurs.

**General Discussion.** Theorem 1 describes that the equilibrium has a pooling and separating co-existence structure: For a range of high demand realizations, $\xi_1 > \xi(q_1)$, the manager reports $q_1$; i.e., all initial inventory is sold. The investors cannot differentiate the real demand and thus average the firm’s value over all possible demand realizations higher than $\hat{\xi}(q_1)$, assuming that $(q_1 - \xi_1)^+ \leq \xi_2$ sales are padded. This is the “pooling” part of the equilibrium.

However, to intentionally ship all the leftover inventory to the downstream parties could be very costly. For a range of low demand realizations, $0 \leq \xi_1 < \hat{\xi}(q_1)$, the manager instead follows a strategy to pad an intermediate amount, $\varphi(\xi_1, q_1)$. The reported sales strictly increase in $\xi_1$. As a result, the investors can infer

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10 In the special case if the realized demand $\xi_1 = \xi(q_1)$, the manager’s decision depends on $\xi_1(q_1)$. If $\xi_1(q_1) \leq \eta_1$, the manager is either indifferent between the two choices $\varphi(\xi_1, q_1)$ and $q_1 - \xi_1$, or strictly prefers the latter. We therefore combine this scenario with the “pooling” part. The boundary market price $\mathcal{P}(q_1)$ in Equation (7) is consistent with this specification. In contrast, if $\xi_1(q_1) > \eta_1$, the manager strictly prefers the choice $\varphi(\xi_1, q_1)$. In this scenario, the reported sales amount $z$ in equilibrium is always less than $q_1$, and thus $\mathcal{P}(q_1)$ in Equation (7) reflects an off-equilibrium path belief where the investors would believe $\eta_1$ was the demand if a report with $z = q_1$ arose.
the real demand from the report and the market price of the firm matches the firm’s true value. This is the “separating” part of the equilibrium.

Notice from Corollary 1 that even if \( \theta(q_1) \) is negative, channel stuffing can still occur in equilibrium. When \( q_1 < \eta_1 \), the threshold \( \hat{\xi}(q_1) \) is lower than \( \eta_1 \). Channel stuffing arises for high demand realizations where the manager pushes out the left-over inventory and reports that the first-period inventory has been cleared. We refer to this effect as the “boundary effect.” For low demand realizations, padding which is governed by \( \varphi(\xi_1, q_1) \) only occurs if \( \theta(q_1) > 0 \) (see Lemma 2). In particular, its derivative \( \varphi'(\xi_1, q_1) \) may be related to the initial inventory level \( q_1 \), the demand \( \xi_1 \), and the fact whether a replenishment is needed in the second period, through the component \( \psi'(\xi_1, q_1) \). From Equation (3), when the initial inventory is small, \( q_1 \leq k_2 \) (i.e., \( \bar{\xi}(q_1) = 0 \)), a replenishment is always needed. Therefore, \( \psi'(\xi_1, q_1) = a(p - c) - c \) for any \( \xi_1 < q_1 \). In this case, the padding strategy, \( \varphi(\xi_1, q_1) \), does not depend on the initial inventory, \( q_1 \). We refer to this as the “marginal effect.” In contrast, when the initial inventory is large, \( q_1 > k_2 \), a replenishment may not be needed if the demand realization is low (\( \xi_1 < \bar{\xi}(q_1) \)). In this situation, \( \psi'(\xi_1, q_1) = ap - (1 + a)p\bar{F}_2(q_1 - (1 + a)\xi_1) \). It follows that the padding strategy, \( \varphi(\xi_1, q_1) \), will depend on the initial inventory, \( q_1 \). We refer to this effect as the “carryover effect.”

Finally, notice that within the above equilibrium structure, the reported sales amount \( z \) as a function of the first-period demand realization contains a jump at the threshold \( \hat{\xi}(q_1) \) from \( \hat{\xi}(q_1) + \varphi(\hat{\xi}(q_1), q_1) \) to \( \xi_1 \) when \( \hat{\xi}(q_1) < \xi'_1(q_1) \). In other words, a financial report with \( z \in [\hat{\xi}(q_1) + \varphi(\hat{\xi}(q_1), q_1), q_1] \) will not appear in equilibrium. Since Bayes’ theorem does not specify the off-equilibrium path beliefs, we can choose a belief structure over this zone such that the equilibrium is sustained. We therefore specify the market price for this off-equilibrium path following the belief consistent with the separating equilibrium path that the manager would pad, still according to the function \( \varphi(\xi_1, q_1) \). Furthermore, in the case if a report arose such that the solution \( \xi_1(z, q_1) \) solved from \( z = \xi_1 + \varphi(\xi_1, q_1) \) exceeded the upper bound of the demand distribution, \( \eta_1 \), the investors would believe \( \eta_1 \) was the real demand as specified in Proposition 1.

### 4.3 Determinants of Channel Stuffing: Boundary, Marginal and Carryover Effects

In the following, we discuss the identified three effects on channel stuffing according to the scenarios that highlight them.

**Boundary Effect** \([\theta(q_1) < 0 \text{ and } q_1 < \eta_1] \). For the specific characteristics of the inventory problem in our model, even when \( \theta(q_1) < 0 \), padding can take place when the demand realization is close to the inventory boundary if \( q_1 < \eta_1 \). A sold-out sales report will not convey perfect demand information to the investors, as any \( \xi_1 \geq q_1 \) will lead to such a report. If the investors averaged the firm’s value over the region \( \xi_1 \geq q_1 \), the
market price of the firm might contain a large upward jump at $\xi_1 = q_1$ due to the positive correlation of the demands ($a > 0$), and the manager would have strong incentive to pad the sales to this level when there is not much inventory leftover. Anticipating the manager’s incentive, the investors will adjust their belief. In particular, if a “sold-out” news is announced, they will average the firm’s value over a wider zone $\xi_1 \geq \hat{\xi}(q_1)$ instead of $\xi_1 \geq q_1$, which leads to a lower boundary market price. This boundary market price discourages the manager to ship all the inventory to the downstream if the real demand is less than $\hat{\xi}(q_1)$ when there is too much leftover. Therefore, padding occurs when the leftover is little. It is remarkable but intuitive because of the real inventory constraint. This constraint “filters” the demand information and creates an upward jump in the market price of the firm that the manager wants to cash in.

**Marginal Effect** \([\theta(q_1) > 0 \text{ and } q_1 \leq k_2]\). Notice that if $q_1 \leq k_2$, the magnitude of $\varphi(\xi_1, q_1)$ does not depend on $q_1$. In other words, the padding that arises for low demand realizations is fully determined by the economic factors of the underlying business. Different from the boundary effect, in this case the manager wants to balance the marginal gain of pushing up the market price of the firm versus the marginal loss of the destruction to the firm’s real value. The manager will pad the sales only when the marginal gain is higher than the marginal loss. In particular, from Equation (4), we can find that the derivative $\varphi'(\xi_1, q_1)$ will never reach zero if $\theta(q_1) > 0$ and $q_1 \leq k_2$, which implies that the padded amount is increasing in $\xi_1$. This marginal effect is analogous to those incentives reported in previous studies on earnings management (e.g., Stein 1989).

**Carryover Effect** \([\theta(q_1) > 0 \text{ and } q_1 > k_2]\). In contrast, if $q_1 > k_2$, the magnitude of padding for low demand realizations becomes dependent on the inventory level. Similar to the boundary effect, this occurs because of the specific characteristics of the inventory problem. When $q_1 > k_2$, it may happen that, if the demand in the first period is small, a replenishment will not proceed in the second period. In other words, the firm may just use the leftover inventory to satisfy the potential second-period demand.

To explain the intuition of the carryover effect, suppose the given initial inventory is large. When the realized demand is small, the firm will have a lot of inventory leftover. In such a situation, if the demand could increase by one unit, it would immediately add up a current revenue by $p$ and save a holding cost by $h$. Moreover, it would also signal an increase of the future demand with a multiplier of $a$. Since the firm does not need to replenish inventory in the second period, that would bring in a future value of $ap$. Given the firm has a lot of leftover inventory, the decrease of the inventory by one unit would not affect much the chance of satisfying the stochastic demand in the second period. This implies that the marginal increase of the firm’s value with respect to the demand realization is large, and thus the manager has a strong incentive to “make up” the demand. However, when the real demand increases, the leftover inventory level will decrease. With less inventory, an additional drop of the inventory level may affect much more the chance of satisfying
Figure 3: The equilibrium padding strategy and the corresponding market price of the firm for $(\beta, q_1) = (0.25, 6)$ (left panels), $(\beta, q_1) = (0.45, 4.5)$ (middle panels) and $(\beta, q_1) = (0.4, 7)$ (right panels). The other parameters are: $p = 10, c = 6, h = 1, a = 0.4, \gamma(x) = 4x + 2x^2, \eta_{1,2} \sim Beta(5, 5)$ on $[0, 10]$, $A_0 = 0$. The left panels highlight the boundary effect ($\theta(q_1) < 0$); the middle panels highlight the marginal effect ($\theta(q_1) > 0$ and $q_1 \leq k_2$); the right panels highlight the carryover effect ($\theta(q_1) > 0$ and $q_1 > k_2$).

In the bottom panels, the solid lines represent the market price in the separating equilibrium part which the second-period demand. Therefore, the marginal benefit of additional demand will decrease as the real demand increases. This indicates that the manager’s incentive on channel stuffing decreases in the demand realization. It is interesting because the boundary effect on the opposite becomes stronger if the demand is higher, but it is also intuitive because of the role of the inventory value.

**Numerical Illustration.** Figure 3 illustrates in the best way the three main padding effects in our real earnings management setting and are due to the nature of the inventory problem. The top three panels show the equilibrium padding strategies and the bottom three panels show the corresponding market prices.

On the top, the left panel highlights the boundary effect where padding appears only in a high demand region close to the inventory level. The middle panel highlights the marginal effect in which we set $q_1 \leq k_2$. The padding monotonically increases as the demand increases until it reaches the threshold $\hat{\xi}(q_1)$ where it jumps to $q_1 - \hat{\xi}(q_1)$. The right panel highlights the carryover effect. The padding first increases, but when the demand exceeds some level the padded amount starts to decrease. This is driven by the increase of the inventory marginal value.

In the bottom panels, the solid lines represent the market price in the separating equilibrium part.
reflects the true value of the firm, and the black dots represent the boundary market price in the pooling equilibrium part which averages the firm’s value conditional on the belief that $\xi_1 \geq \hat{\xi}(q_1)$. The dashed lines represent the off-equilibrium market price which corresponds to the jumps of the padding in the top panels. Notice that those dashed lines directly follow the extensions of the solid lines due to the way we specify the off-equilibrium belief. The market pricing function contains small jumps at the inventory boundary whose size depends on the strength of the boundary effect. For example, in the middle panel, the boundary effect is strong and the jump size is relatively large.

4.4 Comparative Statics

In this subsection we discuss the determinants for $\varphi(\xi_1, q_1)$ and $\hat{\xi}(q_1)$. Given $q_1$, we have the following results:

**Proposition 3** (i) When $\beta$, $h$ or $p$ increases or $c$ decreases, $\varphi(\xi_1, q_1)$ increases for a given $\xi_1$. When $\gamma'(x)$ is parametrized such that it increases for all $x$, $\varphi(\xi_1, q_1)$ decreases for a given $\xi_1$. If $\bar{\xi}(q_1) = 0$ (i.e., $q_1 \leq k_2$), when $a$ increases, $\varphi(\xi_1, q_1)$ increases for a given $\xi_1$.

(ii) If $\varphi(\hat{\xi}(q_1), q_1)$ is equal to zero for small variations of $q_1$, when $\beta$, $a$, $h$ or $p$ increases or $c$ decreases, $\hat{\xi}(q_1)$ decreases; when $\gamma'(x)$ is parametrized such that it increases for all $x$, $\hat{\xi}(q_1)$ increases.

Proposition 3(i) shows that if the manager’s compensation depends more on the market price of the firm ($\beta$ increases), his incentive to pad the sales increases. The holding cost reduces the benefit of keeping the inventory. As a result, if $h$ increases, the manager will become more willing to “sell” additional units. If the margin of the business increases, either the revenue $p$ increases or the inventory purchasing cost $c$ decreases, then the return from padding the sales will become more attractive. It is the opposite for the padding cost, $\gamma(x)$. An increase of the penalty cost will obviously reduce padding. For the correlation factor $a$, we show that if the carryover effect does not exist (i.e., $q_1 \leq k_2$), the padding amount $\varphi(\xi_1, q_1)$ increases in $a$ for any given $\xi_1$. A larger $a$ enhances the manager’s incentive to signal a better future performance. When the carryover effect exists, however, it becomes more complicated because the term $v_2'(\xi_1, q_1) = ap - (1 + a)\bar{pF}_2(q_1 - (1 + a)\xi_1)$ may not be monotone in $a$. It is difficult to determine the impact analytically. From our numerical experiments, we observe that a larger $a$ generally increases channel stuffing even with the carryover effect.

The threshold $\hat{\xi}(q_1)$ is determined by comparing the manager’s separating payoff with the pooling payoff (see Equation (6)). To investigate the determinants of $\hat{\xi}(q_1)$ in the general situation, we need to be able to obtain $\varphi(\xi_1, q_1)$ to derive the manager’s payoff in the separating case. This is not tractable analytically. Therefore, we analyze the behavior of $\hat{\xi}(q_1)$ in a special case where $\varphi(\hat{\xi}(q_1), q_1)$ is zero for small variations on
This occurs, for example, if \( \theta(q_1) < 0 \) (see Lemma 2). In other words, we evaluate \( \xi(q_1) \) in the situation where padding around \( \xi(q_1) \) is purely driven by the boundary effect. The results for \( \xi(q_1) \) in Proposition 3(ii) are intuitive. When the market-based compensation becomes more important for the manager (i.e., \( \beta \) increases), the boundary effect will be stronger and thus the threshold \( \xi(q_1) \) from which the manager will push all the inventory out will be lower. The implications are similar for the change of the other parameters.

As we have discussed, the manager’s padding strategy may also depend on the inventory level \( q_1 \). The following proposition reveals how padding changes as the inventory level varies.

**Proposition 4**  
(i) When \( q_1 \) increases, \( \varphi(\xi_1, q_1) \) increases for a given \( \xi_1 \);  
(ii) If \( \varphi(\hat{\xi}(q_1), q_1) \) is equal to zero for small variations of \( q_1 \) and \( \hat{\xi}(q_1) > \xi(q_1) \), when \( q_1 \) increases, \( \hat{\xi}(q_1) \) increases.

Proposition 4(i) shows that padding for the separating part of the equilibrium, \( \varphi(\xi_1, q_1) \), increases in the initial inventory level, \( q_1 \). This is because when the initial inventory level is larger, for the same demand realization, the firm will have more inventory leftover which has lower marginal value. Therefore, the manager will have incentive to pad some additional inventory to the sales.

However, for the pooling part of the equilibrium, it could be the opposite. When the initial inventory increases, for the same demand realization, in order to clear the shelves, the manager needs to ship more leftover inventory to the downstream parties. This increases the padding cost for reaching the boundary. As a result, the threshold \( \xi(q_1) \) may increase in \( q_1 \), as we prove for some particular scenarios shown in Proposition 4(ii). This is interesting since it implies that an increase of initial inventory might be able to reduce the boundary effect and thus padding. Proposition 4 will be useful when we discuss the manager’s decision on \( q_1 \) in the next section.

### 5 Inventory Investment Incentives

In previous sections, we regarded the first-period inventory level, \( q_1 \), as given. In this section, we discuss the manager’s initial inventory investment decision. Our main question is whether channel stuffing may lead to a distortion of the initial inventory investment, compared to the classical newsvendor investment where the manager’s interest is perfectly aligned with the firm’s real value (we denote this case as our *benchmark*). Incorporating the manager’s equilibrium padding strategy as a function of \( q_1 \), the initial inventory investment problem can be written as follows (see Appendix C for the derivation):

\[
\max_{q_1 \geq 0} A_0 - cq_1 + E_{\eta_1} \left[ p \min(\eta_1, q_1) - h(q_1 - \eta_1)^+ + v_2(\eta_1, q_1) \right] - E_{\eta_1} \left[ \gamma \left( x^o(\eta_1, q_1) \right) \right].
\]  

(8)
The objective function is a combination of the classical newsvendor terms and an extra negative term that reflects the expected cost from channel stuffing. As padding does not create any value to the firm, the manager’s short-term incentive to pad always reduces the firm’s value. This effect has been documented earlier in Stein (1989), “...the excessive capital market pressure may have adverse effects on firm performance...”

In our model, the manager is aware of this perverse but inescapable incentive and may partially control for this negative impact by deliberately selecting the first-period inventory level. In other words, the manager may set up an initial inventory level which limits his incentives on channel stuffing afterwards and thus makes the report more credible to the investors.

Obtaining analytical insights about the optimal first-period inventory investment is challenging. The reason is that the padding incentive depends on \( q_1 \) via the threshold \( \hat{\xi}(q_1) \) and via the padding amount \( \varphi(\xi_1, q_1) \), the solution of an autonomous ODE. Even though for special cases implicit analytical expressions for the solution of the ODE can be obtained, the first order condition depends on the integral of the solution (see Appendix C), which is analytically intractable. Hence, we numerically compute the objective function of Equation (8) and find close-optimal initial inventory investment. Here, we focus on the most important insights. In Figure 4, we illustrate how the optimal inventory level \( q^*_1 \) changes with \( \beta \); i.e., the weight on the short-term market-based compensation. Notice that over-investment occurs when \( \beta \) is small. The inventory level first increases in \( \beta \), reaches a maximum, and then drops. Under-investment appears when the market-based compensation has a large weight. As \( \beta \) keeps increasing, the inventory level has a sharp decrease and then stays at a fixed level below the benchmark.

To explain the intuition, we write the expected channel stuffing cost as:

\[
E_{q_1} [\gamma (x^*(\eta_1, q_1))] = \int_{\hat{\xi}(q_1)}^{\xi(q_1)} \gamma (\varphi(\eta_1, q_1)) \, dF_1(\eta_1) + \int_{\hat{\xi}(q_1)}^{\min(q_1, \eta_1)} \gamma (q_1 - \eta_1) \, dF_1(\eta_1),
\]

where the first term on the right hand side is the expected cost from the separating part of the equilibrium and the second term is from the pooling part of the equilibrium. When \( \beta \) is small such that \( \theta(q_1) \) is negative, \( \varphi(\xi_1, q_1) \) will stay at zero (see Lemma 2) and thus the cost of channel stuffing is exclusively determined by the second term in Equation (9); i.e., the boundary effect. Following Proposition 4(ii), the threshold \( \hat{\xi}(q_1) \) in such situations may increase when \( q_1 \) increases. Indeed, in our numerical experiments, we observe that for such situations, the threshold \( \hat{\xi}(q_1) \) increases slightly faster than \( q_1 \). In other words, the range where padding occurs, \( [\hat{\xi}(q_1), q_1] \), is shrinking, which leads to the decrease of the expected channel stuffing cost when the initial inventory investment, \( q_1 \), increases. This drives the over-investment in initial inventory.

However, when the firm’s market price becomes a relatively more important component of the manager’s compensation (i.e., \( \beta \) increases), \( \theta(q_1) \) may become positive for a given \( q_1 \), and channel stuffing occurs in the separating part of the equilibrium. The first term on the right hand side of Equation (9) then becomes
Inventory Level

Benchmark

Figure 4: Equilibrium inventory level with respect to $\beta$. The parameters are: $p = 10$, $c = 6$, $h = 1$, $a = 1$, $\gamma(x) = 4x + 2x^2$, $\eta_{1,2} \sim Beta(5, 5)$ on $[0, 10]$, $A_0 = 0$. The manager over-invests in the inventory when $\beta$ is small to limit the boundary effect. The over-investment amount first increases in $\beta$ as the boundary effect becomes stronger, but when $\beta$ exceeds some level, the carryover and marginal effects appear. The manager then starts to under-invest to limit the total expected padding cost. When $\beta$ is significantly large, the manager is tempted to always push the sales to the inventory boundary in the subgame, which leads to a substantially low initial inventory investment. The investment level stays at this level for any larger $\beta$.

Proposition 4(i) shows that the padding function $\phi(\xi_1, q_1)$ increases in $q_1$; i.e., there will be more padding in the separating region of the equilibrium, $[0, \hat{\xi}(q_1)]$. This indicates that as $q_1$ increases, the region where channel stuffing occurs, $[0, q_1]$, expands and the magnitude of channel stuffing can be stronger. The manager therefore will have incentive to reduce the inventory investment. Notice that if $\beta$ is sufficiently large, the manager may be induced to pad all the leftover inventory for any demand realization in the subgame equilibrium (i.e., $\hat{\xi}(q_1) = 0$). As a result, the manager may find it is in his interest to sharply reduce the initial inventory investment. Once $\beta$ exceeds such a threshold level, the subgame equilibrium and the optimal inventory level do not change any more as $\beta$ increases.

These observations provide an interesting insight that the inventory level in presence of channel stuffing incentives can be either higher or lower than the classical newsvendor solution:

**Observation 1** The manager may over- or under-invest in initial inventory, compared to the classical newsvendor solution.

Under-investment is because of the loss due to channel stuffing. To put it another way, the existence of channel stuffing increases the “overage” cost: When there is inventory leftover, the manager is tempted to pad the sales. The costs associated with channel stuffing hence are costs associated with leftovers. These costs reduce the investment in the initial inventory. More interestingly, the manager may also find it efficient...
to over-invest. Anticipating the “myopia” of playing a costly game with rational investors, a rational manager may use an operational lever, investment in the inventory, to limit the boundary effect (since it is exclusively costly to pad a lot of sales) and thus make the report more credible. As discussed in Abegglen and Stalk (1985), many U.S. executives are clearly aware of the cost of this gaming to temporarily push the firm’s market price high. Therefore, the above results not only explore how a firm’s inventory decision in presence of channel stuffing incentives may differ from the classical inventory theory, but also characterize the potential that managers might leverage operations to mitigate the earnings management friction.

**Summary of Numerical Results.** In Appendix A, we present two sets of numerical results, corresponding to two cases where the manager’s payoff function has a relatively low and high weight on the market-based compensation (for $\beta = 0.25$ and $\beta = 0.45$ respectively). In these experiments, we show the impact of the change of the system parameters on the first-period inventory investment and the total expected profit of the firm. We present the results relative to our benchmark under the classical newsvendor setting.

In summary, we observe that if the change of a parameter enhances the manager’s incentives for channel stuffing, the firm underperforms more (i.e., obtains a relatively lower profit) compared with the benchmark, for instance, when the inventory purchasing cost $c$ decreases, the holding cost $h$ increases, the penalty cost $\gamma(x)$ decreases (when it is parameterized), or the correlation factor of the demands $a$ increases. This is intuitive because if the temptation from the market-based compensation increases, the manager would pad more sales. Moreover, he might also over- or under-invest more in the inventory at the beginning to limit his incentives for channel stuffing, which leads to a higher friction. However, we find that the comparison on the optimal inventory decision is more complicated due to the overage and underage trade-off embedded in the newsvendor problem. The inventory investment in the benchmark case changes as some of the parameters change, such as, $c$, $h$ and $a$. In particular, in these experiments, we observe that if the inventory purchasing cost $c$ decreases, the penalty cost $\gamma(x)$ decreases, or the correlation factor of the demands $a$ increases, there will be relatively more over-investment when the firm’s market price has a low weight in the manager’s compensation ($\beta = 0.25$) and more under-investment when the market price has a high weight ($\beta = 0.45$); if the holding cost $h$ increases, there will be relatively more over-investment when $\beta = 0.25$ but relatively less under-investment when $\beta = 0.45$.

6 Empirical Implications

While there is anecdotal evidence that channel stuffing occurs, and the U.S. Security Exchange Committee (SEC) has prosecuted companies in the past for such misrepresentation of sales to potential investors, the accounting literature has mostly focused on empirically testing the existence of earnings management which
may include both accrual earnings management and real earnings management. It is only recent that some accounting empirical studies try to separate real earnings management from accrual management. Among these recent studies, Roychowdhury (2006) develops hypotheses and finds empirical evidence that firms commit sales manipulation besides other real earnings management approaches. Roychowdhury defines sales manipulation as “accelerating the timing of sales and/or generating additional unsustainable sales through increased price discounts or more lenient credit terms.” We may use his empirical method to test channel stuffing based on the analysis of our study.

From our model, we find that the manager’s incentives for channel stuffing are stronger when the executives’ compensation depends more on market-based performance measures ($\beta$ is larger). We predict that channel stuffing can be more pronounced in the firms that motivate their management largely by market-based compensations such as stock options and common shares. Our results also show that the manager’s incentives for channel stuffing increase when the gross margin of the firm is higher ($p$ increases or $c$ decreases) or the holding cost is higher. This implies that channel stuffing can be possibly more pronounced in the industries such as information technology, pharmaceutical, fashion, or, industries with pronounced seasonal demands. Moreover, we observe from our results that a higher correlation between the current and future demands ($a$ is larger) generally increases the manager’s incentives for channel stuffing. These factors of margin, holding cost and demand correlation are factors tightly connected to specific industries. Finally, our results reveal that if the firm has more initial inventory, the carryover effect will be stronger when the demand turns out to be low and the manager may pad more sales in such a scenario. In contrast, if the firm has less initial inventory, it might be easier for the manager to deliberately create “sold-out” situations through channel stuffing when the sales are relatively strong. These two factors connect to operations and can arise in different industries at different time. Note that to test these two factors it may not be appropriate to directly use a firm’s reported sales as an indicator of the real demand since the sales may have been padded. However, a measure based on industry-wise sales could be used as a proxy for the firm’s real demand. Based on the above findings, we provide the following hypotheses about channel stuffing: All else equal,

**H1:** There are more channel stuffing activities if the executives’ incentive packages have a larger proportion of market-based compensation.

**H2:** There are more channel stuffing activities if the firm’s business has a higher margin, a higher holding cost, or a higher correlation between the current and future performance.

**H3:** There are more channel stuffing activities if the firm has more initial inventory while the industry-wise sales turn out to be low, or, if the firm has less initial inventory while the industry-wise sales turn out to be high.
While testing the above hypotheses is outside the scope of our paper, it is helpful to see how accounting research typically tests earnings management hypotheses as a similar method could be followed. First, accounting researchers develop a proxy for earnings management like discretionary accruals, production costs or cash flows from operations.\(^\text{11}\) Next, they identify a subset of firms “suspected” by the researcher of earnings manipulation during a particular year out of a large set of yearly firm observations. This could be a subset of firms that have been prosecuted by the SEC during a particular year (Dechow et al. 1995), or a subset that report marginally positive earnings (Roychowdhury 2006), or any other subset of firms that have characteristics that make the researcher believe that are conducive to earnings management (for instance, following Hendricks and Singhal (2008), Oyer (1998) and Lai (2008), the “suspected” firms could be those firms who announce inventory write-off at their customers, or those firms whose sales and inventory data exhibit strong fiscal-year-end effect). Then, the researcher determines a relevant (accounting) measure that would reveal earnings management and estimates whether the coefficient of a dummy variable that indicates a suspected firm year is significantly positive when explaining the proxy for earnings management. A similar study could be envisioned in order to test the existence and magnitude of channel stuffing.

7 Conclusion

In this paper, we study how a manager whose interest is not perfectly aligned with the long-term firm value makes different operational decisions than an owner-manager. The manager may “pad” the sales in order to influence the firm valuation on which his compensation is based. The incentives to pad the sales distort the incentives to invest in the inventory. As with the increased attention after the Enron and WorldCom scandals and the Sarbanes-Oxley Act, accrual-based earnings management becomes more difficult and real earnings management becomes more attractive for managers. With real earnings management, the manager faces “real” constraints, like bounds on physical inventory. We identify three effects that drive the manager’s incentives for channel stuffing: the marginal effect, the boundary effect and the carryover effect. The marginal effect has been analogously identified in the literature (e.g., Stein 1989). The boundary and carryover effects are tied to the nature of inventory management. When the demand realization is high, the boundary effect is the incentive to pad the leftover inventory and report a sold-out situation to the investors. When the demand realization is low, the carryover effect is the incentive to make up sales to increase revenue, reduce holding cost and improve the outlook of future performance, while avoiding triggering a possible replenishment.

Furthermore, we find that facing rational investors, the manager, anticipating the negative consequences

\(^{11}\)Instead of examining only cost of goods sold (COGS), Roychowdhury includes change in inventory during the period into the proxy to eliminate the accrual manipulation to lower COGS. For example, the firm may postpone obsolete inventory write-offs to lower COGS, but this will not change the sum of COGS and change in inventory. In addition, it eliminates the effects on COGS from FIFO/LIFO inventory flow assumptions.
of the subsequent channel stuffing game, may either under- or over-invest in the initial inventory compared
to the classical newsvendor solution. Channel stuffing introduces an extra “overage” cost, which acts as the
incentive for the manager to under-invest in the inventory. It is surprising to see that over-investment also
exists. Over-investment is a consequence of the manager’s desire to mitigate the boundary effect. With
a higher inventory level, the chance that the realized demand will fall into the boundary effect zone will
be lower which leads to less negative boundary effect. These findings show how interest conflict between a
manager and a firm’s long-term value may result in deviations from classical optimal operations management
solutions.

Our paper contributes to both the operations management literature and the earnings management
literature. The traditional operations management literature usually assumes that the interests of firms’
decision makers are perfectly aligned with the long-term interest of the firm. However, for those firms, such
as public firms, that are owned by shareholders but are run by their managers (CEOs and senior executives),
the managers’ interests are more short termed and their decisions may deviate from the optimal strategies
for the firms’ long-term value. Capital market interaction has been widely documented which has substantial
impact on firms’ operational decisions. Similarly, firms’ operations also influence the external valuation. The
ongoing pioneer research in operations management by Hendricks and Singhal (2001, 2005, 2008), Fisher
et al. (2002), Gaur et al. (2005), Raman (2006), Chen et al. (2005), Lai (2006a,b, 2008), etc., tries
to answer questions regarding how operation and capital market performance are exactly connected. Our
study contributes to this stream of literature by studying the problem analytically in a two-period inventory
management context as a game between the manager of a firm and the rational investors that value the
firm. Our theoretical analysis is, to the best of our knowledge, new and provides meaningful and empirically
testable insights. We believe that our way of introducing the market-based compensation, adopted from the
economics and accounting literature, is novel and relevant in the operations contexts, and it enriches the
agency literature in operations management, such as, Chen (2000, 2005), Chen and Xiao (2009a,b), Wang
and Zipkin (2008), Baiman et al. (2008), Xu and Birge (2008) and Debo et al. (2008).

Our paper also contributes to the earnings management literature. The existing studies in this stream
include empirical research to document the phenomena of earnings management (e.g., Dechow et al. 1995,
Graham et al. 2005, Roychowdhury 2006, Cohen et al. 2008) and analytical studies on related agency
problems (e.g., Stein 1989, Dye and Sridhar 2004, Liang and Wen 2007). These studies usually focus on
accrual-based earnings management or stylized capital investments. We enrich this literature by addressing
a detailed operational problem and relating it with widely reported real earnings management activities that
are known as channel stuffing, trade loading, or sales timing.

We conclude by discussing the assumptions in our model and suggesting directions for future research.
Generally, when the ownership and the management of a firm are separated, agency problems may arise. The information advantage of inside managers cultivates the incentive to game. In this paper, we investigate the information asymmetry about the realized actual demand. However, the games between inside managers and outside investors can have more applications, for example, if the efficiency (e.g., the internal management cost, the creativity, the forecasting capability) of the firm is private knowledge instead of the actual demand. Furthermore, our paper addresses only one instrument of real earnings management. As discussed in Roychowdhury (2006), managers in practice may have alternative choices to change their real operation to influence the market price, for instance, managers can intentionally reduce expenditures such as cost on R&D activities, managers can sometimes overproduce to report lower cost of goods sold, etc.

Second, we have assumed that the initial inventory is perfectly observable and hence, investors know exactly how much can be satisfied from inventory. In practice, there will be noise on the observation of the initial inventory (e.g., multiple products, pipeline inventory). Hence, the boundary effect will not be so sharp that reporting the initial inventory will lead to a discrete jump in the firm’s market price, while the market price may be rather steeply increasing around the expected initial inventory. Differently, the carryover effect is somewhat less sensitive to the initial inventory, as it occurs for low realizations of the demand. Irrespective of what the initial inventory is, if no replenishment is expected in the next period, it will drive the steeper increase in the firm’s value as a function of the real demand.

Finally, our paper assumes that the financial report only contains information of the past operation. However, in practice, a financial report may also contain information of the ongoing business and future forecast. For example, firms may report information of the newly installed facility or inventory along with the future forecast. In such situations, managers may have incentive to intentionally change the initial investment to signal the potential of the business to investors. We address this forecasting and signaling effect in future research. The extension of our two-period model to a multi-period model is challenging because of many modeling assumptions that need to be made about the interaction between the manager and the investors (e.g., a sales report may be issued every period or bi-period, the investors may not know all the parameters of the firm’s underlying business but learn and update their belief every period). However, we think this as an interesting direction for future research. We believe understanding the incentives for real earnings management is a rich area for further research in operations management.

References


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Notation Summary

Table 1 summarizes the notation used in the main body of this paper.

<table>
<thead>
<tr>
<th>Notations</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>Firm’s initial asset level</td>
</tr>
<tr>
<td>$q_{1,2}$</td>
<td>Inventory decision at the beginning of period 1 and 2</td>
</tr>
<tr>
<td>$p$</td>
<td>Unit sales price</td>
</tr>
<tr>
<td>$c$</td>
<td>Unit inventory purchasing cost</td>
</tr>
<tr>
<td>$h$</td>
<td>Unit holding cost for carrying inventory from period 1 to 2</td>
</tr>
<tr>
<td>$a$</td>
<td>Demand correlation factor</td>
</tr>
<tr>
<td>$\xi_{1,2}$</td>
<td>Stochastic demands in period 1 and 2</td>
</tr>
<tr>
<td>$\hat{\xi}_{1,2}$</td>
<td>Demand realizations in period 1 and 2</td>
</tr>
<tr>
<td>$\eta_{1,2}$</td>
<td>Random variables generating demands</td>
</tr>
<tr>
<td>$\eta_{1,2}$</td>
<td>Realization of the random variables $\eta_{1,2}$</td>
</tr>
<tr>
<td>$f_{1,2}(\cdot)$</td>
<td>Density functions of $\eta_{1,2}$</td>
</tr>
<tr>
<td>$F_{1,2}(\cdot)$</td>
<td>Cumulative distribution functions of $\eta_{1,2}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Fraction of manager’s compensation on the market price of the firm</td>
</tr>
<tr>
<td>$1 - \beta$</td>
<td>Fraction of manager’s compensation on the long-term value of the firm</td>
</tr>
<tr>
<td>$x$</td>
<td>Padded sales amount</td>
</tr>
<tr>
<td>$\gamma(x)$</td>
<td>Channel stuffing cost function</td>
</tr>
<tr>
<td>$z$</td>
<td>Reported sales number, $z = \xi_1 + x$</td>
</tr>
<tr>
<td>$P(z, q_1)$</td>
<td>Market pricing function</td>
</tr>
<tr>
<td>$v_2(\xi_1, q_1)$</td>
<td>Firm’s expected profit in period 2</td>
</tr>
<tr>
<td>$v_1(x; \xi_1, q_1)$</td>
<td>Firm’s real expected value at the end of period 1</td>
</tr>
<tr>
<td>$\pi(x; \xi_1, q_1)$</td>
<td>Manager’s payoff function at the end of period 1</td>
</tr>
<tr>
<td>$k_2$</td>
<td>Newsvendor ordering level in period 2, $k_2 = F_{2}^{-1}\left(\frac{p-c}{p}\right)$</td>
</tr>
<tr>
<td>$\bar{\xi}(q_1)$</td>
<td>Threshold demand in period 1 above which a replenishment is desirable in period 2 based on the newsvendor theory</td>
</tr>
<tr>
<td>$\varphi(\xi_1, q_1)$</td>
<td>Sales padding function in the separating part of the equilibrium</td>
</tr>
<tr>
<td>$\varphi^s(z, q_1)$</td>
<td>Investors’ belief of the real demand for given $z$ and $q_1$</td>
</tr>
<tr>
<td>$\zeta_1(q_1)$</td>
<td>Threshold where $\zeta_1(q_1) + \varphi(\zeta_1(q_1), q_1) = q_1$</td>
</tr>
<tr>
<td>$\hat{\xi}(q_1)$</td>
<td>Threshold demand in period 1 dividing the separating and pooling parts of the equilibrium</td>
</tr>
<tr>
<td>$\theta(q_1)$</td>
<td>Condition function for the occurrence of the marginal and carryover effects</td>
</tr>
</tbody>
</table>
Appendix A: Numerical Experiments

In this section, we discuss two sets of numerical experiments which correspond to an over-investment scenario ("Case I") and an under-investment scenario ("Case II"); i.e., the two representative phenomena we reveal in this study. In these experiments, we evaluate the impact of the system parameters on the first-period inventory level as well as the total expected profit of the firm (or equivalently, the manager’s expected payoff) in equilibrium. We compare these results with the benchmark.

The base parameters of the experiments are shown by bold numbers in Table 2. We vary one parameter at a time and examine its impact, keeping the other parameters under the base setting. In particular, we use a quadratic penalty cost function, $\gamma(x) = \gamma_1 x + \frac{1}{2}\gamma_2 x^2$, and symmetrical Beta distributions for the first- and second-period stochastic demands. In the tables, we use "o" ("∗") to denote the association with the market equilibrium in presence of channel stuffing (benchmark), and use "Π" to denote the total expected profit of the firm in the two periods.

According to Tables 3-5, we observe that if the inventory purchasing cost $c$ decreases, the holding cost $h$ increases, the penalty cost $\gamma(x)$ decreases (i.e., $\gamma_1$ or $\gamma_2$ decreases), or the correlation factor of the demands $\rho$ increases, then the impact of channel stuffing on the firm’s profit becomes more significant. The firm’s profit in the case with channel stuffing underperforms relatively more compared to the benchmark case.

The comparison on the inventory decision however is more complicated because the overage and underage trade-off embedded in the newsvendor problem will change as the parameters, such as, $c$, $h$ and $\rho$, change. The benchmark inventory level will change as those parameters change. In these experiments, we observe that if the inventory purchasing cost $c$ decreases, the penalty cost $\gamma(x)$ decreases, or the correlation factor of the demands $\rho$ increases, there will be relatively more over-investment in inventory in Case I and more under-investment in Case II; if the holding cost $h$ increases, there will be relatively more over-investment in inventory in Case I but less under-investment in Case II.

Table 2: The parameters in the experiments. The bold numbers represent the parameters for the base case. When one parameter varies, the others are fixed at the values of the base case.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>(I) 0.25 and (II) 0.45</td>
</tr>
<tr>
<td>$\eta_{1,2}$</td>
<td>Beta(5, 5) on [0, 10]</td>
</tr>
<tr>
<td>$p$</td>
<td>10</td>
</tr>
<tr>
<td>$\rho$</td>
<td>{0.4, 0.7, 1}</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>{4, 6, 8}</td>
</tr>
<tr>
<td>$h$</td>
<td>{0.2, 0.6, 1}</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>{3, 4, 5}</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>{4, 12, 20}</td>
</tr>
</tbody>
</table>
Table 3: The impact of the inventory purchasing cost, \( c \), on the equilibrium inventory investment and firm profit.

<table>
<thead>
<tr>
<th>( c )</th>
<th>( q_0^* )</th>
<th>( q_1^* )</th>
<th>( \Pi^o )</th>
<th>( \Pi^* )</th>
<th>( \frac{\Pi^* - \Pi^o}{\Pi^o} )</th>
<th>( q_0^* )</th>
<th>( q_1^* )</th>
<th>( \Pi^o )</th>
<th>( \Pi^* )</th>
<th>( \frac{\Pi^* - \Pi^o}{\Pi^o} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6.740</td>
<td>6.680</td>
<td>72.634</td>
<td>72.685</td>
<td>-0.898%</td>
<td>0.070%</td>
<td>5.704</td>
<td>6.680</td>
<td>66.987</td>
<td>72.685</td>
</tr>
<tr>
<td>6</td>
<td>6.374</td>
<td>6.338</td>
<td>45.904</td>
<td>45.931</td>
<td>-0.568%</td>
<td>0.059%</td>
<td>5.845</td>
<td>6.338</td>
<td>43.305</td>
<td>45.931</td>
</tr>
<tr>
<td>8</td>
<td>5.705</td>
<td>5.693</td>
<td>21.075</td>
<td>21.085</td>
<td>-0.211%</td>
<td>0.047%</td>
<td>5.608</td>
<td>5.693</td>
<td>20.934</td>
<td>21.085</td>
</tr>
</tbody>
</table>

Table 4: The impact of the holding cost, \( h \), on the equilibrium inventory investment and firm profit.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( q_0^* )</th>
<th>( q_1^* )</th>
<th>( \Pi^o )</th>
<th>( \Pi^* )</th>
<th>( \frac{\Pi^* - \Pi^o}{\Pi^o} )</th>
<th>( q_0^* )</th>
<th>( q_1^* )</th>
<th>( \Pi^o )</th>
<th>( \Pi^* )</th>
<th>( \frac{\Pi^* - \Pi^o}{\Pi^o} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>7.490</td>
<td>7.479</td>
<td>47.441</td>
<td>47.447</td>
<td>-0.147%</td>
<td>0.013%</td>
<td>6.570</td>
<td>7.479</td>
<td>45.110</td>
<td>47.447</td>
</tr>
<tr>
<td>0.6</td>
<td>6.787</td>
<td>6.759</td>
<td>46.579</td>
<td>46.594</td>
<td>-0.414%</td>
<td>0.032%</td>
<td>6.172</td>
<td>6.759</td>
<td>44.143</td>
<td>46.594</td>
</tr>
<tr>
<td>1.0</td>
<td>6.374</td>
<td>6.338</td>
<td>45.904</td>
<td>45.931</td>
<td>-0.568%</td>
<td>0.059%</td>
<td>5.845</td>
<td>6.338</td>
<td>43.305</td>
<td>45.931</td>
</tr>
</tbody>
</table>

Table 5: The impact of the correlation factor of the two demands, \( a \), on the equilibrium inventory investment and firm profit.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( q_0^* )</th>
<th>( q_1^* )</th>
<th>( \Pi^o )</th>
<th>( \Pi^* )</th>
<th>( \frac{\Pi^* - \Pi^o}{\Pi^o} )</th>
<th>( q_0^* )</th>
<th>( q_1^* )</th>
<th>( \Pi^o )</th>
<th>( \Pi^* )</th>
<th>( \frac{\Pi^* - \Pi^o}{\Pi^o} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>6.351</td>
<td>6.337</td>
<td>39.921</td>
<td>39.931</td>
<td>-0.221%</td>
<td>0.025%</td>
<td>5.971</td>
<td>6.337</td>
<td>38.362</td>
<td>39.931</td>
</tr>
<tr>
<td>0.7</td>
<td>6.374</td>
<td>6.338</td>
<td>45.904</td>
<td>45.931</td>
<td>-0.568%</td>
<td>0.059%</td>
<td>5.845</td>
<td>6.338</td>
<td>43.305</td>
<td>45.931</td>
</tr>
<tr>
<td>1.0</td>
<td>6.398</td>
<td>6.339</td>
<td>51.880</td>
<td>51.931</td>
<td>-0.931%</td>
<td>0.098%</td>
<td>5.598</td>
<td>6.339</td>
<td>48.252</td>
<td>51.931</td>
</tr>
</tbody>
</table>

Table 6: The impact of the penalty cost coefficient, \( \gamma_1 \), on the equilibrium inventory investment and firm profit.

<table>
<thead>
<tr>
<th>( \gamma_1 )</th>
<th>( q_0^* )</th>
<th>( q_1^* )</th>
<th>( \Pi^o )</th>
<th>( \Pi^* )</th>
<th>( \frac{\Pi^* - \Pi^o}{\Pi^o} )</th>
<th>( q_0^* )</th>
<th>( q_1^* )</th>
<th>( \Pi^o )</th>
<th>( \Pi^* )</th>
<th>( \frac{\Pi^* - \Pi^o}{\Pi^o} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6.382</td>
<td>6.338</td>
<td>45.890</td>
<td>45.931</td>
<td>-0.694%</td>
<td>0.089%</td>
<td>5.652</td>
<td>6.338</td>
<td>42.774</td>
<td>45.931</td>
</tr>
<tr>
<td>4</td>
<td>6.374</td>
<td>6.338</td>
<td>45.904</td>
<td>45.931</td>
<td>-0.568%</td>
<td>0.059%</td>
<td>5.845</td>
<td>6.338</td>
<td>43.305</td>
<td>45.931</td>
</tr>
<tr>
<td>5</td>
<td>6.363</td>
<td>6.338</td>
<td>45.912</td>
<td>45.931</td>
<td>-0.394%</td>
<td>0.041%</td>
<td>5.981</td>
<td>6.338</td>
<td>44.063</td>
<td>45.931</td>
</tr>
</tbody>
</table>
Table 7: The impact of the penalty cost coefficient, $\gamma_2$, on the equilibrium inventory investment and firm profit.

<table>
<thead>
<tr>
<th>$\gamma_2$</th>
<th>Case I</th>
<th>Case II</th>
<th>$\Pi^*$ Difference</th>
<th>$\Pi^* - \Pi^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^*_1$</td>
<td>$q^*_1$</td>
<td>$\Pi^o$</td>
<td>$\Pi^*$</td>
<td>$\frac{\Pi^* - \Pi^o}{\Pi^o}$</td>
</tr>
<tr>
<td>4</td>
<td>6.374</td>
<td>6.338</td>
<td>45.904</td>
<td>45.931</td>
</tr>
<tr>
<td>12</td>
<td>6.359</td>
<td>6.338</td>
<td>45.912</td>
<td>45.931</td>
</tr>
<tr>
<td>20</td>
<td>6.353</td>
<td>6.338</td>
<td>45.916</td>
<td>45.931</td>
</tr>
</tbody>
</table>
Appendix B: Proof of the Equilibrium Structure

Important Note. Throughout Appendix B, we assume the initial inventory level $q_1$ is given. The inventory decision is explored in Appendix C.

Technical Lemmas

Before we move to the proofs for the results in the main text, we first show the structural properties of the functions $\varphi(\xi_1, q_1)$ and $\xi_1^*(z, q_1)$ that we define in Section 4.2. Moreover, based on $\varphi(\xi_1, q_1)$ and $\xi_1^*(z, q_1)$, we introduce another function $\phi(z, q_1)$ in this appendix that we will use to represent the investors’ belief of the padded sales for given reported sales $z$ and inventory $q_1$. The introduction of $\phi(z, q_1)$ will provide the convenience for the proof of Proposition 1. We provide three technical lemmas that discuss the properties of $\varphi(\xi_1, q_1)$, $\xi_1^*(z, q_1)$ and $\phi(z, q_1)$ and their relationships. These lemmas will be useful to prove the results in the main text. In particular, in these lemmas we need to apply the results in Lemma 1 which we will prove later. The proof of Lemma 1 does not depend on $\varphi(\xi_1, q_1)$, $\xi_1^*(z, q_1)$ or $\phi(z, q_1)$.

Recall from Proposition 1, $\xi_1^*(z, q_1)$ is defined as the solution of $z = \xi_1 + \varphi(\xi_1, q_1)$ if such a solution exists in the region $[0, \bar{\eta}_1]$, otherwise, $\xi_1^*(z, q_1) = \bar{\eta}_1$. Let $z(\xi_1, q_1) = \xi_1 + \varphi(\xi_1, q_1)$ and $\bar{z} = \bar{\eta}_1 + \varphi(\bar{\eta}_1, q_1)$. The following lemma discusses the properties of $\varphi(\xi_1, q_1)$, $z(\xi_1, q_1)$ and $\xi_1^*(z, q_1)$.

**Lemma 3** Given $q_1$, (i) $\varphi(\xi_1, q_1)$ is non-negative and continuous.

(ii) $z(0, q_1) = 0$, and $z(\xi_1, q_1)$ monotonically increases in $\xi_1$.

(iii) $\xi_1^*(0, q_1) = 0$, and $\xi_1^*(z, q_1)$ monotonically increases in $z$.

(iv) $\xi_1^*(z(\xi_1, q_1), q_1) = \xi_1$ for $\xi_1 \leq \bar{\eta}_1$.

(v) $z(\xi_1^*(z, q_1), q_1) = z$ for $z \leq \bar{z}$.

**Proof:** (i) By the three conditions in the definition of $\varphi(\xi_1, q_1)$, it is direct to see that $\varphi(\xi_1, q_1)$ is non-negative. We confirm the continuity of $\varphi(\xi_1, q_1)$ as follows:

- If $\frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1} = 0$ when $\xi_1 = 0$, then $\varphi(\xi_1, q_1) = 0$ for any $\xi_1$ according to condition (c) in the definition of $\varphi(\xi_1, q_1)$.

- If $\frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1} > 0$ when $\xi_1 = 0$, then $\varphi(\xi_1, q_1)$ increases from zero. As long as $\varphi(\xi_1, q_1) > 0$, $\varphi(\xi_1, q_1)$ is governed by $\frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1} = \frac{\varphi(\xi_1, q_1) - (1-\beta)\gamma(\varphi(\xi_1, q_1))}{\gamma'(\varphi(\xi_1, q_1))}$. Since $v_2'(\xi_1, q_1)$ is bounded and $\gamma'(x)$ is a continuous and positive function for all $x$, $\frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1}$ is bounded for finite $\xi_1$. Therefore, $\varphi(\xi_1, q_1)$ is continuous when $\varphi(\xi_1, q_1) > 0$. If there is a $\xi_1 < q_1$ such that $\varphi(\xi_1, q_1)$ reaches zero at $\xi_1$, then according to condition (c) in the definition of $\varphi(\xi_1, q_1)$, we know that for all $\xi_1 > \xi_1$, $\varphi(\xi_1, q_1) = 0$. Then, $\varphi(\xi_1, q_1)$ is continuous at $\xi_1 = \xi_1$ as well as $\xi_1 > \xi_1$.  

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(ii) By definition, \( \varphi(0, q_1) = 0 \), and thus \( z(0, q_1) = 0 \). In the following, we verify that \( z(\xi_1, q_1) \) monotonically increases in \( \xi_1 \).

- If \( \frac{\partial z(\xi_1, q_1)}{\partial \xi_1} = 0 \) when \( \xi_1 = 0 \), then \( \varphi(\xi_1, q_1) = 0 \) for any \( \xi_1 \). Thus, \( \frac{\partial z(\xi_1, q_1)}{\partial \xi_1} = \frac{\partial (\xi_1 + \varphi(\xi_1, q_1))}{\partial \xi_1} = 1 \) for any \( \xi_1 \).

- If \( \frac{\partial z(\xi_1, q_1)}{\partial \xi_1} > 0 \) when \( \xi_1 = 0 \), then \( \varphi(\xi_1, q_1) \) increases from zero. As long as \( \varphi(\xi_1, q_1) > 0 \), we have

\[
\frac{\partial z(\xi_1, q_1)}{\partial \xi_1} = \frac{\partial (\xi_1 + \varphi(\xi_1, q_1))}{\partial \xi_1} = 1 + \frac{\beta (p + h + v_2(q_1, \xi_1)) - (1 - \beta)\gamma'(\varphi(\xi_1, q_1))}{\gamma'(\varphi(\xi_1, q_1))} > 0.
\]

If there is a \( \xi_1 \) such that \( \varphi(\xi_1, q_1) \) reaches zero at \( \xi_1 \), then for \( \xi_1 > \xi_1 \), \( \varphi(\xi_1, q_1) = 0 \) which leads to \( \frac{\partial (\xi_1 + \varphi(\xi_1, q_1))}{\partial \xi_1} = 1 \). Hence, \( z(\xi_1, q_1) = \xi_1 + \varphi(\xi_1, q_1) \) monotonically increases in \( \xi_1 \).

(iii) Given (i) and (ii), we can construct a one-to-one mapping between \( \xi_1 \) and \( z \). Given that \( z(\xi_1, q_1) \) monotonically increases in \( \xi_1 \), the function \( \xi_1^0(z, q_1) \) which is the solution of \( z = \xi_1 + \varphi(\xi_1, q_1) \) is also monotone and increasing in \( z \) before it reaches \( \bar{z} \). When \( z \) reaches \( \bar{z} \), by the definition, \( \xi_1^0(z, q_1) \) will remain at \( \bar{z} \) for any larger \( z \). Therefore, \( \xi_1^0(z, q_1) \) monotonically increases in \( z \).

(iv) and (v) are straightforward based on the results of (ii) and (iii). When \( z \leq \bar{z} \), \( \xi_1^0(z, q_1) \) acts as the inverse function of \( z(\xi_1, q_1) \). Equivalently, for \( \xi_1 \in [0, \bar{z}] \), \( z(\xi_1, q_1) \) acts as the inverse function of \( \xi_1^0(z, q_1) \). Furthermore, if we expand \( z(\xi_1^0(z, q_1), q_1) \), we can obtain \( \xi_1^0(z, q_1) + \varphi(\xi_1^0(z, q_1), q_1) = z \).

Now, we introduce the function: \( \phi(z, q_1) \equiv z - \xi_1^0(z, q_1) \), which will provide the convenience for the proof of Proposition 1. According to Lemma 3(v), when \( 0 \leq z \leq \bar{z} \), \( \phi(z, q_1) \) is equivalent to \( \varphi(\xi_1^0(z, q_1), q_1) \) (since \( z(\xi_1^0(z, q_1), q_1) = z \) implies that \( \xi_1^0(z, q_1) + \varphi(\xi_1^0(z, q_1), q_1) = z \)). When \( z < \bar{z} \), \( \phi(z, q_1) = z - \bar{z} \) by the definition of \( \xi_1^0(z, q_1) \). Based on the relationship between \( \phi(z, q_1) \) and \( \xi_1^0(z, q_1) \), Lemma 4 provides some important properties of \( \phi(z, q_1) \) that will be used later in this Appendix.

**Lemma 4** Given \( q_1 \), (i) when \( 0 \leq z \leq \bar{z} \), \( \phi(z, q_1) \) satisfies the following three conditions: (a) \( \phi(0, q_1) = 0 \); (b) \( \frac{\partial \phi(0, q_1)}{\partial z} = \frac{\beta [p + h + v_2(q_1)] - (1 - \beta)\gamma'(0)}{\beta [p + h + v_2(q_1)] + \beta\gamma'(0)} > 0 \), otherwise, \( \frac{\partial \phi(0, q_1)}{\partial z} = 0 \); and (c) for \( 0 < z \leq \bar{z} \) (we measure the left derivative at \( \bar{z} \)),

\[
\frac{\partial \phi(z, q_1)}{\partial z} = \begin{cases} 
\frac{\beta [p + h + v_2(z - \phi(z, q_1), q_1)] - (1 - \beta)\gamma'(\phi(z, q_1))}{\beta [p + h + v_2(z - \phi(z, q_1), q_1)] + \beta\gamma'(\phi(z, q_1))}, & \text{if } \phi(z, q_1) > 0, \\
0, & \text{otherwise}.
\end{cases}
\] (10)

(ii) For any \( \bar{z} > 0 \) that satisfies \( \bar{z} \leq \bar{z} \) and \( \bar{z} < q_1 \), if \( \phi(\bar{z}, q_1) = 0 \), then \( \frac{\beta [p + h + v_2(z, q_1)] - (1 - \beta)\gamma'(0)}{\beta [p + h + v_2(z, q_1)] + \beta\gamma'(0)} \leq 0 \) at \( \bar{z} \).
Proof: (i) Condition (a) is straightforward, since $\xi_1^0(0, q_1) = 0$ and then $\phi(0, q_1) = 0$. For condition (b), notice from Equation (3) that $\beta [p + h + v_2' (0, q_1)] + \beta \gamma' (0) > 0$. Therefore, $\frac{\beta [p + h + v_2' (0, q_1)] - (1 - \beta) \gamma' (0)}{\gamma (0)} > 0$ implies that $\frac{\beta [p + h + v_2' (0, q_1)] - (1 - \beta) \gamma' (0)}{\gamma (0)} > 0$ and thus $\frac{\partial \phi (0, q_1)}{\partial \xi_1} = \frac{\beta [p + h + v_2' (0, q_1)] - (1 - \beta) \gamma' (0)}{\gamma (0)} > 0$. When $z \leq \xi_1$, $\xi_1^0(z, q_1)$ is the inverse function of $z(\xi_1, q_1) = \xi_1 + \varphi(\xi_1, q_1)$. As a result,

$$
\frac{\partial \phi (0, q_1)}{\partial z} = 1 - \frac{\partial \xi_1^0 (0, q_1)}{\partial z} = 1 - \frac{1}{\frac{\partial z (0, q_1)}{\partial \xi_1}} = 1 - \frac{1}{\frac{\partial \phi (0, q_1)}{\partial \xi_1}}.
$$

In contrast, if $\frac{\beta [p + h + v_2' (0, q_1)] - (1 - \beta) \gamma' (0)}{\gamma (0)} \leq 0$, then $\frac{\beta [p + h + v_2' (0, q_1)] - (1 - \beta) \gamma' (0)}{\gamma (0)} \leq 0$ and thus $\frac{\partial \phi (0, q_1)}{\partial \xi_1} = 0$, from which we can obtain $\frac{\partial \phi (0, q_1)}{\partial z} = 0$.

We verify condition (c) as follows. Since $\xi_1^0(z, q_1)$ is the inverse function of $z(\xi_1, q_1) = \xi_1 + \varphi(\xi_1, q_1)$ where $0 \leq z \leq \xi_1$, we have

$$
\frac{\partial \phi (z, q_1)}{\partial z} = 1 - \frac{\partial \xi_1^0 (z, q_1)}{\partial z} = 1 - \frac{1}{\frac{\partial z (z, q_1)}{\partial \xi_1}} = 1 - \frac{1}{\frac{\partial \phi (z, q_1)}{\partial \xi_1}}.
$$

As $\phi(z, q_1)$ is equivalent to $\varphi(\xi_1^0(z, q_1), q_1)$ for $0 \leq z \leq \xi_1$, if $\phi(z, q_1) > 0$, then $\varphi(\xi_1^0(z, q_1), q_1) > 0$. From the definition of $\varphi(\xi_1, q_1)$, the derivative of $\varphi(\xi_1, q_1)$ in this case follows

$$
\frac{\partial \varphi (\xi_1, q_1)}{\partial \xi_1} = \frac{\beta (p + h + v_2' (\xi_1, q_1)) - (1 - \beta) \gamma' (\varphi(\xi_1, q_1))}{\gamma (\varphi(\xi_1, q_1))}.
$$

As a result, we have

$$
\frac{\partial \phi (z, q_1)}{\partial z} = 1 - \frac{1}{\frac{\beta (p + h + v_2' (\xi_1, q_1)) - (1 - \beta) \gamma' (\varphi(\xi_1, q_1))}{\gamma (\varphi(\xi_1, q_1))}} = \frac{\beta (p + h + v_2' (\xi_1, q_1)) - (1 - \beta) \gamma' (\varphi(\xi_1, q_1))}{\beta (p + h + v_2' (\xi_1, q_1)) - (1 - \beta) \gamma' (\varphi(\xi_1, q_1))}.
$$

Replacing $\varphi(\xi_1^0(z, q_1), q_1)$ by $\phi(z, q_1)$ and $\xi_1^0(z, q_1)$ by $z - \phi(z, q_1)$, we obtain

$$
\frac{\partial \phi (z, q_1)}{\partial z} = \frac{\beta (p + h + v_2' (z - \phi(z, q_1), q_1)) - (1 - \beta) \gamma' (\phi(z, q_1))}{\beta (p + h + v_2' (z - \phi(z, q_1), q_1)) + \beta \gamma' (\phi(z, q_1))}.
$$

If $\phi(z, q_1) = 0$, then $\varphi(\xi_1^0(z, q_1), q_1) = 0$ given that they are equivalent. In this case, $\frac{\partial \varphi (\xi_1, q_1)}{\partial \xi_1} = 0$. Therefore, when $\phi(z, q_1) = 0$,

$$
\frac{\partial \phi (z, q_1)}{\partial z} = 1 - \frac{1}{\frac{\beta (p + h + v_2' (z - \phi(z, q_1), q_1)) - (1 - \beta) \gamma' (\phi(z, q_1))}{\beta (p + h + v_2' (z - \phi(z, q_1), q_1)) + \beta \gamma' (\phi(z, q_1))}} = 0.
$$

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This completes the proof for (i). From these three conditions, it is easy to see that \( \phi(z, q_1) \) is non-negative and starts from zero.

(ii) We show the result in (ii) by contradiction. Suppose at some \( \hat{z} > 0 \) that satisfies \( \hat{z} \leq \bar{z} \) and \( \hat{z} < q_1 \), both \( \phi(\hat{z}, q_1) = 0 \) and \( \frac{\beta [p + h + v_2'(\hat{z}, q_1)]}{\beta [p + h + v_2'(\hat{z}, q_1)] + \beta \gamma'(0)} > 0 \) hold. We discuss two situations. First, suppose \( \hat{z} \) is such a point that \( \phi(z, q_1) > 0 \) for \( 0 < z < \hat{z} \) and \( \phi(z, q_1) = 0 \) for \( z \geq \hat{z} \). Then, for some small \( \varepsilon \), we have

\[
\lim_{\varepsilon \to 0} \frac{\partial \phi(\hat{z} - \varepsilon, q_1)}{\partial z} = \frac{\beta [p + h + v_2'(\hat{z} - \varepsilon - \phi(\hat{z} - \varepsilon, q_1), q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v_2'(\hat{z} - \varepsilon - \phi(\hat{z} - \varepsilon, q_1), q_1)] + \beta \gamma'(0)} > 0.
\]

When \( z < q_1 \), we have \( z - \phi(z, q_1) < q_1 \). According to Lemma 1, \( v_2'(z - \phi(z, q_1), q_1) \) is continuous when \( z < q_1 \). Therefore, \( \frac{\partial \phi(z - \varepsilon, q_1)}{\partial z} \) is continuous. Given that \( \lim_{\varepsilon \to 0} \frac{\partial \phi(z - \varepsilon, q_1)}{\partial z} > 0 \), there must be some region of \( \varepsilon \) as \( \varepsilon \) goes to zero in which \( \frac{\partial \phi(z - \varepsilon, q_1)}{\partial z} \geq 0 \). However, as assumed \( \phi(z, q_1) \) is positive for \( z < \hat{z} \), if such a region of \( \varepsilon \) exists, \( \phi(z, q_1) \) will not reach zero at \( z = \hat{z} \). This contradicts with the assumption that \( \phi(\hat{z}, q_1) = 0 \).

Second, suppose there is a \( 0 < z < \hat{z} \) such that \( \phi(z, q_1) > 0 \) for \( 0 < z < \hat{z} \) and \( \phi(z, q_1) = 0 \) for \( z \geq \hat{z} \). Given that the function \( v_2'(\xi, q_1) \) is decreasing in \( \xi \) (according to Lemma 1), we have

\[
\frac{\beta [p + h + v_2'(\hat{z}, q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v_2'(\hat{z}, q_1)] + \beta \gamma'(0)} > 0.
\]

We can show the contradiction with the same argument as in the above. Therefore, \( \phi(\hat{z}, q_1) = 0 \) and \( \frac{\beta [p + h + v_2'(\hat{z}, q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v_2'(\hat{z}, q_1)] + \beta \gamma'(0)} > 0 \) cannot hold simultaneously. This completes the proof for (ii). \[ \blacksquare \]

Lemma 5 obtains an important relationship between \( \varphi(\xi_1, q_1) \) and \( \phi(z, q_1) \).

**Lemma 5** Given \( q_1 \), for \( \xi_1 \in [0, \bar{\eta}_1] \), \( \varphi(\xi_1, q_1) \) is the unique solution of \( x = \phi(\xi_1 + x, q_1) \).

**Proof:** We first show that \( \varphi(\xi_1, q_1) \) is a solution of \( x = \phi(\xi_1 + x, q_1) \). For \( \xi_1 \in [0, \bar{\eta}_1] \), we have \( z(\xi_1, q_1) \leq \bar{z} \) since \( z(\xi_1, q_1) \) monotonically increases in \( \xi_1 \) according to Lemma 3 and \( \bar{z} = z(\bar{\eta}_1, q_1) \). By definition, when \( z \leq \bar{z} \), \( \phi(z, q_1) = \varphi(\xi_1(z, q_1), q_1) \). Therefore, we have

\[
\phi(\xi_1 + \varphi(\xi_1, q_1), q_1) = \varphi(z(\xi_1, q_1), q_1) = \varphi(\xi_1(z, q_1), q_1) = \varphi(\xi_1, q_1).
\]

The last equality holds as \( \xi_1(z(\xi_1, q_1), q_1) = \xi_1 \) for \( \xi_1 \in [0, \bar{\eta}_1] \) according to Lemma 3.

In the following, we verify that \( \varphi(\xi_1, q_1) \) is the unique solution. Given any \( \xi_1 \), suppose there is another solution \( \hat{x} \neq \varphi(\xi_1, q_1) \) that satisfies \( \hat{x} = \phi(\xi_1 + \hat{x}, q_1) \). By definition, \( \phi(z, q_1) = z - \xi_1(z, q_1) \). Therefore, we
obtain simultaneously:

$$\varphi(\xi_1, q_1) = \phi(\xi_1 + \varphi(\xi_1, q_1), q_1) = \xi_1 + \varphi(\xi_1, q_1) - \xi_1(\xi_1 + \varphi(\xi_1, q_1))$$

and

$$\dot{x} = \phi(\xi_1 + \dot{x}, q_1) = \xi_1 + \dot{x} - \xi_1(\xi_1 + \dot{x}, q_1).$$

In other words,

$$\xi_1 = \xi_1(\xi_1 + \varphi(\xi_1, q_1), q_1) = \xi_1(\xi_1 + \dot{x}, q_1).$$

However, according to Lemma 3(iii), the function $\xi_1(z, q_1)$ is monotone, which contradicts with the assumption that $\dot{x} \neq \varphi(\xi_1, q_1)$. This completes the proof for this lemma. ■

7.1 Proofs for the Results in the Main Text

Proof of Lemma 1: When $\xi_1 < \xi(q_1)$, $v_2'(\xi_1, q_1) = ap - (1 + a)pF_2(q_1 - (1 + a)\xi_1)$ which is continuous and decreasing in $\xi_1$; when $\xi(q_1) < \xi_1 < q_1$, $v_2'(\xi_1, q_1) = ap - (1 + a)c$ which is a constant. We can directly derive

$$\lim_{\xi_1 \rightarrow \xi(q_1)^-} v_2'(\xi_1, q_1) = \lim_{\xi_1 \rightarrow \xi(q_1)^+} v_2'(\xi_1, q_1) = ap - (1 + a)c.$$

This confirms the continuity of $v_2'(\xi_1, q_1)$ at $\xi_1 = \xi(q_1)$. Therefore, $v_2'(\xi_1, q_1)$ is continuous and decreasing in $\xi_1$ when $\xi_1 < q_1$. This completes the proof. ■

Proof of Lemma 2: (i) has been shown in Lemma 3(ii).

(ii) Notice that $\theta(q_1)$ is the numerator of $\frac{\beta[p + h + v_2'(0, q_1)] - (1 - \beta)\gamma(0)}{\gamma(0)}$. If $\theta(q_1) \leq 0$, then $\frac{\partial\varphi(0, q_1)}{\partial\xi_1} = 0$ according to the definition of $\varphi(\xi_1, q_1)$. Therefore, $\varphi(\xi_1, q_1)$ will remain at zero for any $\xi_1 > 0$ since its first derivative $\frac{\partial\varphi(0, q_1)}{\partial\xi_1} = 0$ according to the definition.

In contrast, if $\theta(q_1) > 0$, then $\frac{\partial\varphi(0, q_1)}{\partial\xi_1} = \frac{\beta[p + h + v_2'(0, q_1)] - (1 - \beta)\gamma(0)}{\gamma(0)} > 0$ and thus $\varphi(\xi_1, q_1)$ increases from zero. When $\varphi(\xi_1, q_1) > 0$, $\varphi(\xi_1, q_1)$ is governed by the ODE

$$\frac{\partial\varphi(\xi_1, q_1)}{\partial\xi_1} = \frac{\beta(p + h + v_2'(\xi_1, q_1)) - (1 - \beta)\gamma'(\varphi(\xi_1, q_1))}{\gamma'(\varphi(\xi_1, q_1))}.$$

Since $v_2'(\xi_1, q_1)$ decreases in $\xi_1$ according to Lemma 1 and $\gamma'(x)$ is increasing as $\gamma(x)$ is convex, it is easy to see that $\frac{\partial\varphi(\xi_1, q_1)}{\partial\xi_1}$ decreases in $\xi_1$ as long as $\frac{\partial\varphi(\xi_1, q_1)}{\partial\xi_1} > 0$. It is possible that $\frac{\partial\varphi(\xi_1, q_1)}{\partial\xi_1}$ may become negative when $\xi_1$ reaches some level and then $\varphi(\xi_1, q_1)$ decreases. However, according to the definition of $\varphi(\xi_1, q_1)$,
indeed match the investors’ belief.

and inventory level (ii), we verify that if the investors hold such a belief following the firm’s market price will follow:

one-to-one mapping between the reported sales and the inferred demand under this belief. Therefore, the definition of

Substituting the market pricing function and the firm’s real value function, we obtain

Based on this market pricing function, we can write the manager’s problem as

\[
\sup_{x \in [0, q_1 - \xi_1]} \pi(x; \xi_1, q_1) = \beta P(\xi_1 + x, q_1) + (1 - \beta) v_1 (x; q_1, \xi_1)
\]

Substituting the market pricing function and the firm’s real value function, we obtain

\[
\pi(x; \xi_1, q_1) = \beta \left[ A_0 + p (\xi_1 + x - \phi(\xi_1 + x, q_1)) - c q_1 - \gamma (\phi(\xi_1 + x, q_1)) - h(q_1 - z + \phi(z, q_1)) + v_2 (z - \phi(z, q_1), q_1) \right] + (1 - \beta) [A_0 + p \xi_1 - c q_1 - \gamma (x - h(q_1 - \xi_1)) + v_2 (\xi_1, q_1)].
\]

For convenience of notation, we momentarily replace \(\xi_1 + x\) by \(z\). We take the first derivative of \(\pi(x; \xi_1, q_1)\)

\[
\frac{\partial \pi(x; \xi_1, q_1)}{\partial x} = \beta \left[ A_0 + p (\xi_1 + x - \phi(\xi_1 + x, q_1)) - c q_1 - \gamma (\phi(\xi_1 + x, q_1)) - h(q_1 - z + \phi(z, q_1)) + v_2 (z - \phi(z, q_1), q_1) \right]
\]

Proof of Proposition 1: Throughout this proof, we assume that the real demand \(\xi_1 < q_1\) (otherwise, the inventory is sold out and the manager has no decision to make on channel stuffing). As discussed in the main text, there is a difference between the action to pad the sales by an amount \(x \in [0, q_1 - \xi_1]\) and the action to pad the sales by an amount \(x = q_1 - \xi_1\). In this proposition, we limit the manager’s optimization problem in the region \(x \in [0, q_1 - \xi_1]\). In other words, the manager is not able to reach \(x = q_1 - \xi_1\). The proof of this proposition is constructed with two steps. In step (i), we suppose the investors hold a belief following the function \(\phi(z, q_1)\); i.e., the amount of padded sales is \(\phi(z, q_1)\) in their belief given the reported sales amount \(z\) and inventory level \(q_1\). We then find the condition for the manager’s optimal padding strategy. In step (ii), we verify that if the investors hold such a belief following \(\phi(z, q_1)\), the manager’s optimal strategy will indeed match the investors’ belief.

(i) Given the reported sales amount \(z\) and inventory level \(q_1\), suppose the investors hold the belief that the amount of padded sales is \(\phi(z, q_1)\) and the real demand is \(z - \phi(z, q_1)\) (which is identical to \(\xi_1^*(z, q_1)\) by the definition of \(\phi(z, q_1)\)). Since \(\xi_1^*(z, q_1)\) is monotone according to Lemma 3, the investors can construct a one-to-one mapping between the reported sales and the inferred demand under this belief. Therefore, the firm’s market price will follow:

\[
P(z, q_1) = v_1 (\phi(z, q_1); z - \phi(z, q_1), q_1)
\]

\[
= A_0 + p (z - \phi(z, q_1)) - c q_1 - \gamma (\phi(z, q_1)) - h(q_1 - z + \phi(z, q_1)) + v_2 (z - \phi(z, q_1), q_1)
\]

Based on this market pricing function, we can write the manager’s problem as

\[
\sup_{x \in [0, q_1 - \xi_1]} \pi(x; \xi_1, q_1) = \beta P(\xi_1 + x, q_1) + (1 - \beta) v_1 (x; q_1, \xi_1)
\]
with respect to $x$,

$$
\frac{d\pi(x; \xi_1, q_1)}{dx} = \beta \left[ p \left( 1 - \frac{\partial \phi(z, q_1)}{\partial z} \right) dx - \gamma'(\phi(z, q_1)) \frac{\partial \phi(z, q_1)}{\partial z} \right] + h \left( 1 - \frac{\partial \phi(z, q_1)}{\partial z} \right) + v'_2 (z - \phi(z, q_1), q_1) \left( 1 - \frac{\partial \phi(z, q_1)}{\partial z} \right) \\
- (1 - \beta) \gamma'(x).
$$

The optimal strategy $x^\text{opt}(\xi_1, q_1)$ shall satisfy the first-order condition (FOC), $\frac{d\pi(x; \xi_1, q_1)}{dx} = 0$, whenever it is achievable. If the FOC cannot be satisfied, then there will be some optimal decision at the boundary. We organize the FOC, $\frac{d\pi(x; \xi_1, q_1)}{dx} = 0$, to

$$
\beta \left[ p \left( 1 - \frac{\partial \phi(z, q_1)}{\partial z} \right) dx + \gamma'(\phi(z, q_1)) \left( 1 - \frac{\partial \phi(z, q_1)}{\partial z} \right) \right] + h \left( 1 - \frac{\partial \phi(z, q_1)}{\partial z} \right) + v'_2 (z - \phi(z, q_1), q_1) \left( 1 - \frac{\partial \phi(z, q_1)}{\partial z} \right) = (1 - \beta) \gamma'(x) + \beta \gamma'(\phi(z, q_1))
$$

and then

$$
\frac{\partial \phi(z, q_1)}{\partial z} \left[ dx = 1 - \frac{(1 - \beta) \gamma'(x) + \beta \gamma'(\phi(z, q_1))}{\beta [p + h + v'_2 (z - \phi(z, q_1), q_1)] + \beta \gamma'(\phi(z, q_1))}. 
$$

Notice that $\frac{dz}{dx} = 1$ given $z = \xi_1 + x$. Therefore, the optimal strategy $x^\text{opt}(\xi_1, q_1)$ shall satisfy:

$$
\frac{\partial \phi(z, q_1)}{\partial z} = \frac{\beta [p + h + v'_2 (z - \phi(z, q_1), q_1)] - (1 - \beta) \gamma'(x)}{\beta [p + h + v'_2 (z - \phi(z, q_1), q_1)] + \beta \gamma'(\phi(z, q_1))}.
$$

whenever the FOC can be achieved.

(ii) In this step, we verify that for any $\xi_1 \in [0, \eta_1]$, when $\xi_1 < \xi'_1(q_1)$ ($\xi'_1(q_1)$ satisfies $z(\xi'_1(q_1), q_1) = q_1$), if the investors hold the belief following $\phi(z, q_1)$, then the optimal strategy for the manager is to take a padding amount $x^\text{opt}$ that solves $x = \phi(\xi_1 + x, q_1)$. The constraint $\xi_1 < \xi'_1(q_1)$ (i.e., $z(\xi_1, q_1) < q_1$) guarantees that the reported sales will not reach the inventory boundary $q_1$ by taking the strategy $x^\text{opt}$. For $\xi_1 \in [0, \eta_1]$, there exists a unique solution of $x = \phi(\xi_1 + x, q_1)$ which is $\varphi(\xi_1, q_1)$ according to Lemma 5. We first prove the result for the case when $0 < \xi_1 < \xi'_1(q_1)$. We divide the proof into two steps. In (ii.a), we prove the result for the scenario where the solved $x^\text{opt}$ is larger than 0 for the given $\xi_1$, and in (ii.b), we prove the result for the scenario where the solved $x^\text{opt} = 0$ for the given $\xi_1$. After that, we show the result for the special case when $\xi_1 = 0$ in (ii.c).

(ii.a) Given $\xi_1 \in [0, \eta_1]$ that satisfies $0 < \xi_1 < \xi'_1(q_1)$, if $x^\text{opt}$ that solves $x = \phi(\xi_1 + x, q_1)$ is larger than
0, then \( \phi(x_1 + x_{opt}, q_1) = x_{opt} > 0 \). Let \( z_{opt} = x_1 + x_{opt} \). According to Lemma 4, when \( \phi(z_{opt}, q_1) > 0 \),

\[
\frac{\partial \phi(z_{opt}, q_1)}{\partial z} = \frac{\beta [p + h + v'_2 (z_{opt} - \phi(z_{opt}, q_1), q_1)] - (1 - \beta) \gamma'(\phi(z_{opt}, q_1))}{\beta [p + h + v'_2 (z_{opt} - \phi(z_{opt}, q_1), q_1)] + \beta \gamma'(\phi(z_{opt}, q_1))}.
\]

Comparing this ODE with the FOC in Equation (12) in step (i), we can easily find that if the manager takes an action \( x_{opt} = \phi(z_{opt}, q_1) \), this action satisfies the FOC of his optimization problem. Therefore, we only need to verify whether this is the unique solution of the FOC and whether it corresponds to the maximum of the manager’s problem:

- The solution of \( \frac{dn(z_{\xi_1, q_1})}{dx} = 0 \) is unique. Suppose there is another solution which can be expressed by \( \hat{x} \) different from \( x_{opt} \). Let \( \hat{z} = \xi_1 + \hat{x} \). In other words, \( \hat{z} \) and \( \hat{x} \) satisfy the FOC in Equation (12) as:

\[
\frac{\partial \phi(\hat{z}, q_1)}{\partial z} = \frac{\beta [p + h + v'_2 (\hat{z} - \phi(\hat{z}, q_1), q_1)] - (1 - \beta) \gamma'(\hat{x})}{\beta [p + h + v'_2 (\hat{z} - \phi(\hat{z}, q_1), q_1)] + \beta \gamma'(\phi(\hat{z}, q_1))}.
\]  

We verify whether Equation (13) can hold. There are three situations according to: (a) \( \phi(\hat{z}, q_1) > 0 \) and \( \hat{z} \leq \hat{z} \); (b) \( \phi(\hat{z}, q_1) = 0 \) and \( \hat{z} \leq \hat{z} \); and (c) \( \phi(\hat{z}, q_1) = \hat{z} - \bar{\tau}_1 \) and \( \hat{z} > \hat{z} \). For the first situation, based on Lemma 4, we know that the derivative \( \frac{\partial \phi(\hat{z}, q_1)}{\partial z} \) for \( \phi(\hat{z}, q_1) > 0 \) follows:

\[
\frac{\partial \phi(\hat{z}, q_1)}{\partial z} = \frac{\beta [p + h + v'_2 (\hat{z} - \phi(\hat{z}, q_1), q_1)] - (1 - \beta) \gamma'(\phi(\hat{z}, q_1))}{\beta [p + h + v'_2 (\hat{z} - \phi(\hat{z}, q_1), q_1)] + \beta \gamma'(\phi(\hat{z}, q_1))}.
\]

As a result, by comparing Equations (13) and (14), we can find that Equation (13) can hold only if \( \hat{x} = \phi(\hat{z}, q_1) \) given that the function \( \gamma'(x) \) is monotone. This contradicts with Lemma 5 that there is a unique solution of \( x = \phi(x_1 + x, q_1) \) for \( x_1 \in [0, \bar{\tau}_1] \).

For the second situation with \( \phi(\hat{z}, q_1) = 0 \) and \( \hat{z} \leq \hat{z} \), we have \( \frac{\partial \phi(\hat{z}, q_1)}{\partial z} = 0 \) from Lemma 4. Because \( \phi(z_{opt}, q_1) > 0, \phi(\hat{z}, q_1) = 0 \) implies that \( \hat{z} \) must be larger than \( z_{opt} \); otherwise, \( \phi(z_{opt}, q_1) \) would be zero as well based on the properties of \( \phi(z, q_1) \) shown in Lemma 4. Therefore, \( \hat{x} > x_{opt} = \phi(z_{opt}, q_1) > 0 \). With the above information, we can write the FOC in Equation (13) as

\[
0 = \frac{\beta [p + h + v'_2 (\hat{z} - \phi(\hat{z}, q_1), q_1)] - (1 - \beta) \gamma'(\hat{x})}{\beta [p + h + v'_2 (\hat{z} - \phi(\hat{z}, q_1), q_1)] + \beta \gamma'(\phi(\hat{z}, q_1))} = \frac{\beta [p + h + v'_2 (\hat{z}, q_1)] - (1 - \beta) \gamma'(\hat{x})}{\beta [p + h + v'_2 (\hat{z}, q_1)] + \beta \gamma'(0)}
\]

This will hold only if

\[
\beta [p + h + v'_2 (\hat{z}, q_1)] - (1 - \beta) \gamma'(\hat{x}) = 0.
\]  

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Since \( \gamma'(x) \) is increasing and \( \hat{x} > 0 \), if Equation (15) holds, it implies that both

\[
\frac{\beta [p + h + v'_2 (\hat{z}, q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v'_2 (\hat{z}, q_1)] + \beta \gamma'(0)} > 0
\]

and \( \phi(\hat{z}, q_1) = 0 \) hold, which contradicts with Lemma 4(ii).

For the third situation with \( \phi(\hat{z}, q_1) = \hat{z} - \eta_1 \) and \( \hat{z} > \hat{z} \), we have \( \frac{\partial \phi(\hat{z}, q_1)}{\partial z} = 1 \). We can write the FOC in Equation (13) as

\[
1 = \frac{\beta [p + h + v'_2 (\hat{z} - \phi(\hat{z}, q_1), q_1)] - (1 - \beta) \gamma'(\hat{x})}{\beta [p + h + v'_2 (\hat{z} - \phi(\hat{z}, q_1), q_1)] + \beta \gamma'(\phi(\hat{z}, q_1))}.
\]

It is obvious that the FOC cannot hold given that the function \( \gamma'(x) \) is always positive. Hence, \( \hat{z} \) cannot be another solution of \( \frac{d\pi(x; \xi_1, q_1)}{dx} = 0 \).

- The solution of \( \frac{d\pi(x; \xi_1, q_1)}{dx} = 0 \) corresponds to the maximum of \( \pi(x; \xi_1, q_1) \). To show this result, it suffices to verify that \( \frac{d\pi(x; \xi_1, q_1)}{dx} \geq 0 \) given that \( \frac{d\pi(x; \xi_1, q_1)}{dx} \) is continuous and there is a unique solution of the FOC. In other words, we only need to verify whether the manager’s payoff function is increasing in the padding amount at the beginning. We reorganize \( \frac{d\pi(x; \xi_1, q_1)}{dx} \) as (given that \( \frac{dz}{dx} = 1 \)

\[
\frac{d\pi(x; \xi_1, q_1)}{dx} = \beta \left[ p \left( 1 - \frac{\partial \phi(\xi_1 + x, q_1)}{\partial z} \right) \frac{d\phi(\xi_1 + x, q_1)}{dx} \right] - \gamma'(\phi(\xi_1 + x, q_1)) \frac{\partial \phi(\xi_1 + x, q_1)}{dx} + \gamma'(\phi(\xi_1 + x, q_1)) \frac{\partial \phi(\xi_1 + x, q_1)}{dx}
\]

When \( x = 0 \), we have

\[
\frac{d\pi(0; \xi_1, q_1)}{dx} = \beta [p + h + v'_2 (\xi_1 - \phi(\xi_1, q_1), q_1)] + \gamma'(\phi(\xi_1, q_1)) \frac{\partial \phi(\xi_1, q_1)}{dx}
\]

From Lemma 4, we have (replacing \( z \) by \( \xi_1 \) given \( x = 0 \))

\[
\frac{\partial \phi(\xi_1, q_1)}{dx} = \frac{\beta [p + h + v'_2 (\xi_1 - \phi(\xi_1, q_1), q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v'_2 (\xi_1 - \phi(\xi_1, q_1), q_1)] + \gamma'(\phi(\xi_1, q_1))}
\]

Because \( \gamma'(x) \) is increasing, it is straightforward that

\[
\frac{\beta [p + h + v'_2 (\xi_1 - \phi(\xi_1, q_1), q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v'_2 (\xi_1 - \phi(\xi_1, q_1), q_1)] + \gamma'(\phi(\xi_1, q_1))} > \frac{\beta [p + h + v'_2 (\xi_1 - \phi(\xi_1, q_1), q_1)] - (1 - \beta) \gamma'(\phi(\xi_1, q_1))}{\beta [p + h + v'_2 (\xi_1 - \phi(\xi_1, q_1), q_1)] + \gamma'(\phi(\xi_1, q_1))}
\]
where the equality holds only when \( \phi(\xi_1, q_1) = 0 \). Note that the term \( p + h + v'_2(\xi_1 - \phi(\xi_1, q_1), q_1) + \gamma'(\phi(\xi_1, q_1)) \) is always positive (from Equation (3) we can easily see \( p + h + v'_2(\xi_1 - \phi(\xi_1, q_1), q_1) > 0 \)). As a result, \( \frac{d\tau(0; \xi_1, q_1)}{dx} \geq 0 \).

The above procedure proves that if the investors hold the belief following \( \phi(z, q_1) \) and if \( x^{opt} \) that solves \( x = \phi(\xi_1 + x, q_1) \) is larger than 0, then such an \( x^{opt} \) indeed is the manager’s optimal strategy for given \( \xi_1 \).

In this case, the FOC of the manager’s problem is satisfied.

(ii.b) Given \( \xi_1 \in [0, \bar{\eta}] \) that satisfies \( 0 < \xi_1 < \xi'_1(q_1) \), if \( x^{opt} \) that solves \( x = \phi(\xi_1 + x, q_1) \) is equal to 0, then it means \( \phi(\xi_1 + x^{opt}, q_1) = x^{opt} = 0 \) and thus \( \phi(\xi_1, q_1) = 0 \). According to Lemma 4, we will have \( \frac{\partial \phi(\xi_1, q_1)}{\partial x} = 0 \); moreover, for any \( x > 0 \), the following is true: \( \phi(\xi_1 + x, q_1) = 0 \) and \( \frac{\partial \phi(\xi_1 + x, q_1)}{\partial z} = 0 \) (according to Lemma 4, once the function \( \phi(z, q_1) \) reaches zero, it will stay at zero afterwards). Now, the FOC of the manager’s problem in Equation (12) may not be achieved, which means that a boundary optimal decision may arise. Therefore, we directly explore the first derivative of the manager’s problem. Given that \( \phi(\xi_1 + x, q_1) = 0 \) and \( \frac{\partial \phi(\xi_1 + x, q_1)}{\partial z} = 0 \), the first derivative of the manager’s problem becomes

\[
\frac{d\tau(x; \xi_1, q_1)}{dx} = \beta [p + h + v'_2(\xi_1 + x - \phi(\xi_1 + x, q_1), q_1) + \gamma'(\phi(\xi_1 + x, q_1))] \times \\
\left[ \frac{\beta [p + h + v'_2(\xi_1 + x - \phi(\xi_1 + x, q_1), q_1)] - (1 - \beta) \gamma'(x)}{\beta (p + h + v'_2(\xi_1 + x - \phi(\xi_1 + x, q_1), q_1)) + \gamma'(\phi(\xi_1 + x, q_1))} - \frac{\partial \phi(\xi_1 + x, q_1)}{\partial z} \right]
\]

\[
= \beta (p + h + v'_2(\xi_1 + x, q_1)) - (1 - \beta) \gamma'(x)
\]

Because the function \( v'_2(\xi_1, q_1) \) is decreasing and the function \( \gamma'(x) \) is increasing, \( \frac{d\tau(x; \xi_1, q_1)}{dx} \) is decreasing in \( x \). In order to prove \( x^{opt} = 0 \) is the optimal strategy, it suffices to show that \( \frac{d\tau(0; \xi_1, q_1)}{dx} \leq 0 \). We prove this by contradiction. Suppose \( \frac{d\tau(0; \xi_1, q_1)}{dx} > 0 \); i.e.,

\[
\beta (p + h + v'_2(\xi_1, q_1)) - (1 - \beta) \gamma'(0) > 0,
\]

which indicates

\[
\frac{\beta [p + h + v'_2(\xi_1, q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v'_2(\xi_1, q_1)] + \beta \gamma'(0)} > 0.
\]

In other words, we would have \( \phi(\xi_1, q_1) = 0 \) and \( \frac{\beta [p + h + v'_2(\xi_1, q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v'_2(\xi_1, q_1)] + \beta \gamma'(0)} > 0 \) simultaneously. This however contradicts with Lemma 4(ii), which states that any \( \bar{z} > 0 \) that satisfies \( \bar{z} \leq \bar{z} \) and \( \bar{z} < q_1 \), if \( \phi(\bar{z}, q_1) = 0 \), then

\[
\frac{\beta [p + h + v'_2(\bar{z}, q_1)] - (1 - \beta) \gamma'(0)}{\beta [p + h + v'_2(\bar{z}, q_1)] + \beta \gamma'(0)} \leq 0
\]

(notice that for any \( \xi_1 \in [0, \bar{\eta}] \), if it satisfies \( 0 < \xi_1 < \xi'_1(q_1) \), then it also satisfies: \( \xi_1 \leq \bar{z} \) and \( \xi_1 < q_1 \), since \( \bar{z} = \bar{\eta}_1 + \varphi(\bar{\eta}_1, q_1) \) and \( \xi'_1(q_1) + \varphi(\xi'_1(q_1), q_1) = q_1 \) by definitions). Therefore, \( x^{opt} = 0 \) indeed is the optimal strategy.

(ii.c) In this part, we show that if \( \xi_1 = 0 \), then \( x^{opt} = 0 \) is always the optimal strategy. When \( \xi_1 = 0 \),
the first derivative of the manager’s problem follows

\[
\frac{d\pi(x; \xi_1, q_1)}{dx} = \beta [p + h + v'_2 (\xi_1 + x - \phi(\xi_1 + x, q_1), q_1) + \gamma' (\phi(\xi_1 + x, q_1))] \times \left( \frac{\beta (p + h + v'_2 (\xi_1 + x - \phi(\xi_1 + x, q_1), q_1)) - (1 - \beta) \gamma'(x)}{\beta ((p + h + v'_2 (\xi_1 + x - \phi(\xi_1 + x, q_1), q_1)) + \gamma'(\phi(\xi_1 + x, q_1)))} - \frac{\partial \phi(\xi_1 + x, q_1)}{\partial z} \right)
\]

(16)

As long as \( \phi(x, q_1) > 0 \), we have

\[
\frac{\partial \phi(x, q_1)}{\partial z} = \frac{\partial \phi(x + 0, q_1)}{\partial z} = \frac{\beta (p + h + v'_2 (x - \phi(x, q_1), q_1) - (1 - \beta) \gamma'(\phi(x, q_1))}{\beta ((p + h + v'_2 (x - \phi(x, q_1), q_1)) + \beta \gamma'(\phi(x, q_1)))}
\]

and thus

\[
\frac{d\pi(x; \xi_1, q_1)}{dx} = (1 - \beta) (\gamma'(\phi(x, q_1)) - \gamma'(x)).
\]

It is obvious that \( \frac{d\pi(0; \xi_1, q_1)}{dx} \leq 0 \) given that \( \phi(0, q_1) = 0 \) by definition and \( \gamma'(0) > 0 \). By definition, \( \phi(z, q_1) = z - \xi_1^*(z, q_1) \) where \( \xi_1^*(z, q_1) \) monotonically increases in \( z \). This implies that \( x - \phi(x, q_1) \) monotonically increases in \( x \). Therefore, given that \( \gamma'(x) \) is an increasing function, \( \frac{d\pi(x; \xi_1, q_1)}{dx} \) will never become positive as \( x \) increases.

If \( \phi(x, q_1) \) reaches 0 at some \( \hat{x} \), then we have both \( \phi(x, q_1) = 0 \) and \( \frac{\partial \phi(x, q_1)}{\partial z} = 0 \) for any \( x > \hat{x} \). Substituting \( \phi(x, q_1) = 0 \) and \( \frac{\partial \phi(x, q_1)}{\partial z} = 0 \) into \( \frac{d\pi(x; \xi_1, q_1)}{dx} \) in Equation (16), we obtain

\[
\frac{d\pi(x; \xi_1, q_1)}{dx} = \beta (p + h + v'_2 (x, q_1)) - (1 - \beta) \gamma'(x)
\]

for \( x \geq \hat{x} \). Given that the function \( v'_2 (x, q_1) \) is decreasing and \( \gamma'(x) \) is increasing, \( \frac{d\pi(x; \xi_1, q_1)}{dx} \) is decreasing for \( x \geq \hat{x} \). Hence, \( x^{opt} = 0 \) is the manager’s optimal strategy.

**Summary:** We have shown that for any \( \xi_1 \in [0, \bar{\pi}_1] \), when \( \xi_1 < \xi_1^*(q_1) \), if the investors hold the belief following \( \phi(z, q_1) \), then the optimal strategy for the manager constrained by \( x \in [0, q_1 - \xi_1] \) is to take a padding amount \( x^{opt} \) that solves \( x = \phi(\xi_1 + x, q_1) \). This solution is equivalent to \( \varphi(\xi_1, q_1) \) as we have shown in Lemma 5. The constraint \( \xi_1 < \xi_1^*(q_1) \) is necessary because it guarantees the reported sales amount will not reach \( q_1 \) (or equivalently, \( x^{opt} \) will not reach \( q_1 - \xi_1 \)), given that we have yet defined the boundary market price. This completes the proof of Proposition 1. 

**Proof of Proposition 2:** Throughout this proof, we assume the demand \( \xi_1 < q_1 \) (otherwise, the manager has no decision to make).
**Overview:** In Proposition 1, we identify a function \( \phi(z, q_1) \) (and \( \varphi(\xi_1, q_1) \)). If the investors hold the belief that the amount of padded sales is \( \phi(z, q_1) \) given reported sales \( z \) and inventory level \( q_1 \), then for the manager, to pad the sales by \( x = \varphi(\xi_1, q_1) \) provides the highest payoff among all the choices in the set \( x \in [0, q_1 - \xi_1] \). The manager can achieve a payoff:

\[
v_1(\varphi(\xi_1, q_1); \xi_1, q_1) = A_0 + p\xi_1 - h(q_1 - \xi_1) + v_2(\xi_1, q_1) - cq_1 - \gamma(\varphi(\xi_1, q_1)).
\]

Proposition 2 investigates the manager’s problem by comparing the choice \( x = \varphi(\xi_1, q_1) \) with the choice \( x = q_1 - \xi_1 \); i.e.,

\[
\max_{\beta \in \eta_1} \left[ \begin{array}{l}
\beta \in \eta_1 \left[ v_1((\xi_1, q_1); \xi_1, q_1) \right]
\end{array} \right]
\]

if the investors believe that the manager will take the strategy \( x = \varphi(\xi_1, q_1) \) to pad the sales and report \( z = \xi_1 + \varphi(\xi_1, q_1) \) for any demand \( \xi_1 < \hat{\xi}(q_1) \), but take the strategy to pad \( x = q_1 - \xi_1 \) and report \( z = q_1 \) for any demand \( \xi_1 \geq \hat{\xi}(q_1) \). Proposition 2 aims to reveal the existence of such a threshold \( \hat{\xi}(q_1) \) and show that the manager will indeed follow the strategy consistent with the investors’ belief.

In order for such a scenario to arise, the threshold \( \hat{\xi}(q_1) \in [0, \eta_1] \) and the two payoff functions must satisfy: for \( \xi_1 > \hat{\xi}(q_1) \),

\[
v_1(\varphi(\xi_1, q_1); \xi_1, q_1) < \beta \in \eta_1 \left[ v_1((\xi_1, q_1); \xi_1, q_1) \right]
\]

and for \( \xi_1 < \hat{\xi}(q_1) \),

\[
v_1(\varphi(\xi_1, q_1); \xi_1, q_1) > \beta \in \eta_1 \left[ v_1((\xi_1, q_1); \xi_1, q_1) \right]
\]

If there is an interior solution for such a threshold (i.e., \( 0 \leq \hat{\xi}(q_1) \leq \eta_1 \)), we shall have that when \( \xi_1 = \hat{\xi}(q_1) \),

\[
v_1(\varphi(\xi_1, q_1); \xi_1, q_1) = \beta \in \eta_1 \left[ v_1((\xi_1, q_1); \xi_1, q_1) \right]
\]

We start the proof by investigating whether there can exist such an interior threshold. By rearranging the terms in Equation (17), we rephrase our objective as

\[
\hat{\xi}(q_1) : LHS(\xi, q_1) = RHS(\xi, q_1)
\]
\[
LHS(\xi, q_1) = A_0 + p\xi - h(q_1 - \xi) + v_2(\xi, q_1) + \frac{1-\beta}{\beta}\gamma(q_1 - \xi) - \frac{1}{\beta}\gamma(\varphi(\xi, q_1)) - cq_1
\]
\[
RHS(\xi, q_1) = A_0 + \mathbb{E}_\eta \left[ p\eta_1 - h(q_1 - \eta_1)^+ + v_2(\eta_1, q_1) - \gamma((q_1 - \eta_1)^+)\right] \mathbb{I}[\eta_1 \geq \xi] - cq_1
\]

In the following, we prove the results in this proposition in four steps. In all of those steps, we implicitly assume \(\xi \in [0, \eta_1]\) since the real demand will not exceed this region. In step (i), we show that \(LHS(\xi, q_1)\) decreases in \(\xi\) when \(\xi < \xi_1^*(q_1)\) (such that the inventory boundary is not reached; \(\xi_1^*(q_1)\) is the solution of \(\xi_1^* + \varphi(\xi_1^*, q_1) = q_1\)). In step (ii), we show that \(RHS(\xi, q_1)\) increases in \(\xi\). In step (iii), we show that if \(\xi_1(q_1) < \eta_1\), then \(LHS(\xi, q_1) < RHS(\xi, q_1)\) when \(\xi \rightarrow \xi_1(q_1)\); if \(\xi_1(q_1) > \eta_1\), then \(LHS(\xi, q_1) > RHS(\xi, q_1)\) when \(\xi \rightarrow \eta_1\); and if \(\xi_1(q_1) = \eta_1\), then \(LHS(\xi, q_1) = RHS(\xi, q_1)\) when \(\xi \rightarrow \eta_1\). In step (iv), we verify the results provided in this proposition.

(i) We prove that \(LHS(\xi, q_1)\) decreases in \(\xi\) when \(\xi < \xi_1^*(q_1)\). First, suppose \(\varphi(\xi, q_1) > 0\). Taking the first derivative of \(LHS(\xi, q_1)\), we obtain

\[
\frac{dLHS(\xi, q_1)}{d\xi} = p + h + v_2'(\xi, q_1) - \frac{1-\beta}{\beta}\gamma'(q_1 - \xi) - \frac{1}{\beta}\gamma'(\varphi(\xi, q_1)) \frac{\partial \varphi(\xi, q_1)}{\partial \xi}
\]
\[
= \frac{1}{\beta} \left[ \beta(p + h + v_2'(\xi, q_1)) - (1-\beta)\gamma'(q_1 - \xi) - \gamma'(\varphi(\xi, q_1)) \frac{\partial \varphi(\xi, q_1)}{\partial \xi} \right]
\]
\[
\leq \frac{1}{\beta} \left[ \beta(p + h + v_2'(\xi, q_1)) - (1-\beta)\gamma'(\varphi(\xi, q_1)) \frac{\partial \varphi(\xi, q_1)}{\partial \xi} - \gamma'(\varphi(\xi, q_1)) \frac{\partial \varphi(\xi, q_1)}{\partial \xi} \right]
\]
\[
= \frac{1}{\beta} \left[ \gamma'(\varphi(\xi, q_1)) \frac{\partial \varphi(\xi, q_1)}{\partial \xi} - \gamma'(\varphi(\xi, q_1)) \frac{\partial \varphi(\xi, q_1)}{\partial \xi} \right]
\]
\[
= 0
\]

The third inequality holds because \(q_1 - \xi > \varphi(\xi, q_1)\) when \(\xi < \xi_1^*(q_1)\) and the function \(\gamma'(x)\) is increasing. The fourth equality holds because when \(\varphi(\xi, q_1) > 0\),

\[
\frac{\partial \varphi(\xi, q_1)}{\partial \xi} = \frac{\beta(p + h + v_2'(\xi, q_1)) - (1-\beta)\gamma'(\varphi(\xi, q_1))}{\gamma'(\varphi(\xi, q_1))}
\]

Second, suppose \(\varphi(\xi, q_1) = 0\). Then, we have

\[
\frac{dLHS(\xi, q_1)}{d\xi} = p + h + v_2'(\xi, q_1) - \frac{1-\beta}{\beta}\gamma'(q_1 - \xi)
\]
\[
\leq \frac{1}{\beta} \left[ \beta(p + h + v_2'(\xi, q_1)) - (1-\beta)\gamma'(0) \right]
\]
\[
\leq 0
\]

The second inequality holds because \(q_1 - \xi \geq 0\) and \(\gamma'(x)\) is increasing. The third inequality holds based on the result of Lemma 4(ii). In particular, \(\varphi(\xi, q_1) = 0\) implies that \(z(\xi, q_1) = \xi\). Since \(\xi_1^*(z(\xi, q_1), q_1) = \xi\),
\[ \phi(z(\xi, q_1), q_1) = z(\xi, q_1) - \xi_1'(z(\xi, q_1), q_1) = 0. \] Based on Lemma 4(ii), we have \[ \frac{\beta[p + h + \varepsilon z(\xi, q_1) - (1 - \beta)\gamma'(0)]}{\beta[p + h + \varepsilon z(\xi, q_1) + \beta \gamma'(0)]} \leq 0; \] i.e., \[ \frac{\beta[p + h + \varepsilon z(\xi, q_1) - (1 - \beta)\gamma'(0)]}{\beta[p + h + \varepsilon z(\xi, q_1) + \beta \gamma'(0)]} \leq 0. \] Therefore, the third inequality holds.

(ii) We prove that \( RHS(\xi, q_1) \) increases in \( \xi \). Taking the first derivative, we obtain

\[
\frac{dRHS(\xi, q_1)}{d\xi} = d \left[ f_{\xi}^{\min(q_1, \eta_1)} \left( \frac{\eta_1 - h(\xi_1 - \eta_1)}{p \eta_1 - h(\xi_1 - \eta_1)^+ + v_2(\eta_1, q_1)} \right) f_{\xi}^{\eta_1} + f_{\xi}^{\eta_1} \left( \frac{\eta_1 - h(\xi_1 - \eta_1)^+ + v_2(\eta_1, q_1) - \gamma((q_1 - \eta_1)^+))}{f_{\xi}^{\eta_1}} \right) \right] \frac{f_{\xi}^{\eta_1}}{f_{\xi}^{\eta_1}} d\eta_1
\]

The last inequality holds since in the domain of integration, for any \( \eta_1 \) from \( \xi^+ \) to \( \eta_1 \), it is always true that \( p \eta_1 - h(\xi_1 - \eta_1)^+ + v_2(\eta_1, q_1) > p \xi - h(\xi_1 - \xi) + v_2(\xi, q_1) \) and \(-\gamma((q_1 - \eta_1)^+) > -\gamma(q_1 - \xi)\).

(iii) In this step, we prove that if \( \xi_1'(q_1) < \eta_1 \), then \( LHS(\xi, q_1) < RHS(\xi, q_1) \) when \( \xi \to \xi_1'(q_1) \); if \( \xi_1'(q_1) > \eta_1 \), then \( LHS(\xi, q_1) > RHS(\xi, q_1) \) when \( \xi \to \eta_1 \); and if \( \xi_1'(q_1) = \eta_1 \), then \( LHS(\xi, q_1) = RHS(\xi, q_1) \) when \( \xi \to \eta_1 \) (or identically, \( \xi_1'(q_1) \)).

For the case where \( \xi_1'(q_1) < \eta_1 \), when \( \xi \to \xi_1'(q_1) \), \( \varphi(\xi, q_1) \to \varphi(\xi_1'(q_1), q_1) = q_1 - \xi_1'(q_1) \). As a result, we have

\[
\lim_{\xi \to \xi_1'(q_1)} LHS(\xi, q_1) = A_0 + p \xi_1'(q_1) - h(\xi_1(q_1)) + v_2(\xi_1(q_1), q_1) - \gamma(q_1 - \xi_1'(q_1)) - c q_1.
\]

Comparing it with

\[
\lim_{\xi \to \xi_1'(q_1)} RHS(\xi, q_1) = A_0 + \int_{\xi_1'(q_1)}^{\eta_1} \left( \frac{\eta_1 - h(\xi_1 - \eta_1)^+ + v_2(\eta_1, q_1) - \gamma((q_1 - \eta_1)^+))}{f_{\xi_1'(q_1)}^{\eta_1}} \right) \frac{f_{\xi_1'(q_1)}^{\eta_1}}{f_{\xi_1'(q_1)}^{\eta_1}} d\eta_1 - c q_1
\]

we can obtain

\[
\lim_{\xi \to \xi_1(q_1)} RHS(\xi, q_1) - LHS(\xi, q_1)
\]

\[
= \int_{\xi_1'(q_1)}^{\eta_1} \left[ \left( \frac{\eta_1 - h(\xi_1 - \eta_1)^+ + v_2(\eta_1, q_1) - \gamma((q_1 - \eta_1)^+))}{f_{\xi_1'(q_1)}^{\eta_1}} \right) \right] \frac{f_{\xi_1'(q_1)}^{\eta_1}}{f_{\xi_1'(q_1)}^{\eta_1}} d\eta_1
\]

\[
> 0
\]

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For the case where \( \xi_1'(q_1) > \eta_1 \), we have \( \varphi(\eta_1, q_1) < q_1 - \eta_1 \); i.e., the inventory boundary is not reached at \( \eta_1 \) by taking the strategy \( x = \varphi(\xi_1, q_1) \). When \( \xi \to \eta_1 \), we obtain
\[
\lim_{\xi \to \eta_1} LHS(\xi, q_1) = A_0 + p\eta_1 - h(q_1 - \eta_1) + v_2(\eta_1, q_1) + \frac{1 - \beta}{\beta} \gamma(q_1 - \eta_1) - \frac{1}{\beta} \gamma(\varphi(\eta_1, q_1)) - cq_1.
\]

On the other hand, \( \xi \to \eta_1 \) implies that the investors believe that only when \( \xi_1 = \eta_1 \), it is possible for the manager to pad the sales by \( q_1 - \xi_1 \) and report \( z = q_1 \). When they receive a report with \( z = q_1 \), they map the inferred demand to \( \eta_1 \). As a result, we will have
\[
\lim_{\xi \to \eta_1} RHS(\xi, q_1) = \lim_{\xi \to \eta_1} A_0 + E_{\eta_1} [p\eta_1 - h(q_1 - \eta_1) + v_2(\eta_1, q_1) - \gamma((q_1 - \eta_1)^+)\eta_1 \geq \xi] - cq_1 = A_0 + p\eta_1 - h(q_1 - \eta_1) + v_2(\eta_1, q_1) - \gamma(q_1 - \eta_1) - cq_1.
\]

Given that \( \varphi(\eta_1, q_1) < q_1 - \eta_1 \) and \( \gamma(x) \) is convex, we obtain
\[
\lim_{\xi \to \eta_1} RHS(\xi, q_1) - LHS(\xi, q_1) = \frac{1}{\beta} \gamma(\varphi(\eta_1, q_1)) - \frac{1}{\beta} \gamma(q_1 - \eta_1) < 0.
\]

For the case where \( \xi_1'(q_1) = \eta_1 \), we have \( \varphi(\eta_1, q_1) = q_1 - \eta_1 \). It is straightforward that \( \lim_{\xi \to \eta_1} RHS(\xi, q_1) - LHS(\xi, q_1) = 0 \).

Thus far, we have revealed that \( LHS(\xi, q_1) \) is decreasing in \( \xi \) when \( \xi < \xi_1'(q_1) \), while \( RHS(\xi, q_1) \) is increasing in \( \xi \). We have shown that if \( \xi_1'(q_1) < \eta_1 \), then \( LHS(\xi, q_1) < RHS(\xi, q_1) \) as \( \xi \to \xi_1'(q_1) \). Therefore, when \( \xi_1'(q_1) < \eta_1 \), if there is a solution \( \hat{\xi}(q_1) \) such that \( LHS(\xi, q_1) = RHS(\xi, q_1) \) at \( \xi = \hat{\xi}(q_1) \), such a \( \hat{\xi}(q_1) \) is unique and less than \( \xi_1'(q_1) \). If there does not exist such a solution, it implies that \( LHS(\xi, q_1) < RHS(\xi, q_1) \) for any \( \xi \in [0, \xi_1'(q_1)] \). In this situation, set \( \hat{\xi}(q_1) = 0 \) and verify in the following step (iv) that such a \( \hat{\xi}(q_1) \) satisfies Equation (6) that we want to prove.

When \( \xi_1'(q_1) > \eta_1 \), we have revealed that \( LHS(\xi, q_1) > RHS(\xi, q_1) \) as \( \xi \to \eta_1 \). As a result, given \( LHS(\xi, q_1) \) is decreasing while \( RHS(\xi, q_1) \) is increasing, we will have \( LHS(\xi, q_1) > RHS(\xi, q_1) \) for any \( \xi \in [0, \eta_1] \) and thus there does not exist a solution of \( LHS(\xi, q_1) = RHS(\xi, q_1) \). In this situation, we set \( \hat{\xi}(q_1) = \eta_1 \) and verify in the following step (iv) that such a \( \hat{\xi}(q_1) \) satisfies Equation (6).

When \( \xi_1'(q_1) = \eta_1 \), there is a unique solution of \( LHS(\xi, q_1) = RHS(\xi, q_1) \) which is at \( \xi = \eta_1 \), and \( LHS(\xi, q_1) > RHS(\xi, q_1) \) for any \( \xi \in [0, \eta_1] \). In this case, \( \hat{\xi}(q_1) = \eta_1 \). We verify in the following step (iv) that it satisfies Equation (6).

(iv) In this step, we verify all the results in this proposition based on the above steps (i-iii).

First, suppose we find a solution \( \hat{\xi}(q_1) \) at which \( RHS(\xi, q_1) = LHS(\xi, q_1) \). It implies that the two
In the following, we compare these two payoffs for favorable than the other payoff function when $RHS (\xi, q_1)$ and $LHS (\xi, q_1)$ are derived from reorganizing these two payoffs. In the following, we compare these two payoffs for $\xi_1 < \hat{\xi} (q_1)$ and $\xi_1 > \hat{\xi} (q_1)$. To reveal the relationship, we take the derivative of the difference between these two payoffs with respect to $\xi_1$. Notice that the term $\beta E_{\eta_1} \left[ v_1 ((q_1 - \eta_1)^+; \eta_1, q_1) | \eta_1 \geq \hat{\xi} (q_1) \right] + (1 - \beta) v_1 (q_1 - \xi_1; \xi_1, q_1)$ holds because $RHS (\xi, q_1)$ and $LHS (\xi, q_1)$ are derived from reorganizing these two payoffs. When we take the derivative of the difference between these two payoffs with respect to $\xi_1$, we can easily see that $\hat{\xi} (q_1)$ is a constant once $\hat{\xi} (q_1)$ is given. We obtain that when $\xi_1 < \xi'_1 (q_1)$,

$$\frac{d}{d\xi_1} [v_1 (\varphi(\xi_1, q_1); \xi_1, q_1) - (1 - \beta) v_1 (q_1 - \xi_1; \xi_1, q_1)] = \beta (p + h + v'_2 (\xi_1, q_1)) - (1 - \beta) (p + h + v'_2 (\xi_1, q_1) + \gamma' (q_1 - \xi_1))$$

$$= \beta (p + h + v'_2 (\xi_1, q_1)) - (1 - \beta) \gamma' (q_1 - \xi_1) - \gamma' (\varphi(\xi_1, q_1)) \frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1}$$

$$< \beta (p + h + v'_2 (\xi_1, q_1)) - (1 - \beta) \gamma' (\varphi(\xi_1, q_1)) - \gamma' (\varphi(\xi_1, q_1)) \frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1}$$

$$= \gamma' (\varphi(\xi_1, q_1)) \frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1} - \gamma' (\varphi(\xi_1, q_1)) \frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1}$$

$$= 0$$

The third inequality holds because $q_1 - \xi_1 > \varphi(\xi_1, q_1)$ and $\gamma' (x)$ is increasing.

This result reveals that the difference of the payoff $v_1 (\varphi(\xi_1, q_1); \xi_1, q_1)$ over the other decreases in $\xi_1$ when $\xi_1 < \xi'_1 (q_1)$. As a result, when $\xi_1 < \hat{\xi} (q_1)$, $v_1 (\varphi(\xi_1, q_1); \xi_1, q_1)$ is more favorable; and when $\hat{\xi} (q_1) < \xi_1 < \xi'_1 (q_1)$, $v_1 (\varphi(\xi_1, q_1); \xi_1, q_1)$ is less favorable. Equation (6) is satisfied under such a threshold.

Second, suppose a solution of $RHS (\xi, q_1) = LHS (\xi, q_1)$ does not exist. We have two situations as discussed in step (iii). When $\xi'_1 (q_1) < \varpi_1$, we have shown in step (iii) that if there does not exist a solution of $RHS (\xi, q_1) = LHS (\xi, q_1)$, it implies that $LHS (\xi, q_1) < RHS (\xi, q_1)$ for any $\xi \in [0, \xi'_1 (q_1))$. In other words, even by setting $\hat{\xi} (q_1) = 0$, the payoff $v_1 (\varphi(\xi_1, q_1); \xi_1, q_1)$ is still less favorable compared to the other when $\xi_1 = 0$, which implies that $v_1 (\varphi(\xi_1, q_1); \xi_1, q_1)$ is less favorable for any $\xi_1 > 0$. By examining Equation (6), we can easily see that $\hat{\xi} (q_1) = 0$ satisfies Equation (6) in this scenario. Moreover, it is the unique solution that can make Equation (6) satisfied.

In the other situation when $\xi'_1 (q_1) > \varpi_1$, as we have shown in step (iii), $RHS (\xi, q_1) < LHS (\xi, q_1)$ for any $\xi \in [0, \varpi_1]$. In other words, even by setting $\hat{\xi} (q_1) = \varpi_1$, the payoff $v_1 (\varphi(\xi_1, q_1); \xi_1, q_1)$ is still more favorable than the other payoff function when $\xi_1 = \varpi_1$, which implies that the payoff $v_1 (\varphi(\xi_1, q_1); \xi_1, q_1)$ is more favorable for any $\xi_1 < \varpi_1$. By examining Equation (6), we can find that $\hat{\xi} (q_1) = \varpi_1$ satisfies Equation (6) in this situation and it is the unique solution that can make Equation (6) satisfied.
Summarizing the above three scenarios, we can conclude that there is a unique solution $\hat{\xi}(q_1) \in [0, \eta_1]$ that satisfies Equation (6). In particular, if $\xi_1'(q_1) < \eta_1$, then $\hat{\xi}(q_1) \leq \xi_1'(q_1) < \eta_1$, and if $\xi_1'(q_1) \geq \eta_1$, $\hat{\xi}(q_1) = \eta_1$ based on the results in step (iii) and (iv). This completes the proof for the results in Proposition 2.

Proof of Theorem 1: Based on Propositions 1 and 2, the proof of this Theorem is straightforward. We just confirm that the proposed equilibrium satisfies the conditions in Definition 1.

(i) Given the proposed market pricing function $P(z, q_1)$ in Equation (7) for $z < q_1$, we have shown in Proposition 1 that when $0 \leq \xi_1 < \xi_1'(q_1)$, the strategy $x = \varphi(\xi_1, q_1)$ provides the highest payoff for the manager among all the choices in the set $x \in [0, q_1 - \xi_1]$. With the proposed boundary market price $\bar{P}(q_1)$ for $z = q_1$, we have proved in Proposition 2 that there is a unique threshold $\hat{\xi}(q_1) \leq \xi_1'(q_1)$ such that when the realized demand is less than $\hat{\xi}(q_1)$, the payoff for the manager by following the strategy $x = \varphi(\xi_1, q_1)$ is higher than the payoff to follow the strategy $x = q_1 - \xi_1$, and it is the opposite for demand realizations higher than $\hat{\xi}(q_1)$. This confirms that the manager does not have an incentive to deviate from the proposed sales padding strategy for either $\xi_1 < \hat{\xi}(q_1)$ or $\xi_1 > \hat{\xi}(q_1)$. In the case if the real demand is equal to the threshold, we have shown in Proposition 2 that when $\xi_1'(q_1) = \eta_1$, the two padding strategies become equal since $\eta_1 + \varphi(\eta_1, q_1) = q_1$. When $\xi_1'(q_1) > \eta_1$, the manager strictly prefers $x = \varphi(\xi_1, q_1)$. When $\xi_1'(q_1) < \eta_1$, the manager is indifferent between the two strategies if there is an interior solution of $\hat{\xi}(q_1)$ where $0 \leq \hat{\xi}(q_1) < \eta_1$, or strictly prefers $x = q_1 - \xi_1$ if there does not exist an interior solution of $\hat{\xi}(q_1)$ and thus $\hat{\xi}(q_1)$ is set to 0. We therefore organize the manager’s strategy as: to follow $x = \varphi(\xi_1, q_1)$ when $\xi_1'(q_1) > \eta_1$, and to follow $x = q_1 - \xi_1$ when $\xi_1'(q_1) \leq \eta_1$.

(ii) The market price $P(z, q_1)$ at $z = 0$ matches the firm’s real value for $\xi_1 = 0$ since $\xi_1''(0, q_1) = 0$. When the manager follows the sales padding strategy $\varphi(\eta_1, q_1)$ for $\xi_1 < \hat{\xi}(q_1)$ and reports a corresponding sales amount $z < q_1$, the separating market price in Equation (7) provides the true value of the firm. When the manager follows the padding strategy for $\xi_1 > \hat{\xi}(q_1)$ and reports a sales amount $z = q_1$, the boundary market price in Equation (7) provides the fair expected value of the firm contingent on the outcome that for any $\xi_1 > \hat{\xi}(q_1)$ the manager will report $z = q_1$. Note that any specification of the manager’s strategy at $\xi_1 = \hat{\xi}(q_1)$ will not affect the boundary market price since the probability measure at $\xi_1 = \hat{\xi}(q_1)$ is zero by the assumption that the demand distribution is continuous. Therefore, the conditions in Definition 1 are all satisfied.

Proof of Corollary 1: When $\theta(q_1) \leq 0$, $\varphi(\xi_1, q_1) = 0$ according to Lemma 2. In this situation, $\xi_1'(q_1) = q_1$ based on the definition of $\xi_1'(q_1)$. Therefore, if $q_1 \geq \eta_1$, then $\xi_1'(q_1) \geq \eta_1$. According to Proposition 2,
when \( \xi_1' (q_1) \geq \eta_1 \), \( \dot{\xi} (q_1) = \eta_1 \). From Theorem 1, we know that for \( \xi_1 < \dot{\xi} (q_1) = \eta_1 \), the manager follows \( \varphi (\xi_1, q_1) = 0 \) and there is no channel stuffing. For \( \xi_1 = \dot{\xi} (q_1) = \eta_1 \), according to the specification in Theorem 1, if \( \xi_1' (q_1) > \eta_1 \), the manager follows \( \varphi (\xi_1, q_1) = 0 \); if \( \xi_1' (q_1) = \eta_1 \), the manager follows \( q_1 - \xi_1 \) which is also zero (given \( \varphi (\xi_1, q_1) = 0 \), if \( \xi_1' (q_1) = \eta_1 \), we have \( q_1 = \eta_1 \); as a result, when \( \xi_1 = \dot{\xi} (q_1) = \eta_1 \), we have \( q_1 - \xi_1 = 0 \). Hence, there is no channel stuffing in this case.

When either of these two conditions does not hold (i.e., either \( \theta (q_1) > 0 \) or \( q_1 < \eta_1 \), or both of them arise), then we will have \( \varphi (\xi_1, q_1) > 0 \) and/or \( \dot{\xi} (q_1) < \eta_1 \). As a result, channel stuffing will occur.

**Proof of Proposition 3:** (i) To show this result, we only need to examine the derivative of \( \varphi (\xi_1, q_1) \) for \( \varphi (\xi_1, q_1) > 0 \) (once \( \varphi (\xi_1, q_1) \) reaches zero, it stays at zero afterwards):

\[
\frac{\partial \varphi (\xi_1, q_1)}{\partial \xi_1} = \frac{\beta (p + h + v_2' (\xi_1, q_1)) - (1 - \beta) \gamma' (\varphi (\xi_1, q_1))}{\gamma' (\varphi (\xi_1, q_1))}
\]

Recall

\[
v_2' (\xi_1, q_1) = \begin{cases} 
  ap - (1 + a)p F_2 (q_1 - (1 + a) \xi_1), & 0 \leq \xi_1 < q_1, \\
  a \left( p - c \right) - c, & q_1 \leq \xi_1 < q_1.
\end{cases}
\]

Therefore,

\[
\frac{\partial \varphi (\xi_1, q_1)}{\partial \xi_1} = \begin{cases} 
  \frac{\beta (1 + a) p F_2 (q_1 - (1 + a) \xi_1) + h}{\gamma' (\varphi (\xi_1, q_1))} - (1 - \beta), & 0 \leq \xi_1 < \bar{\xi} (q_1), \\
  \frac{\beta (1 + a) (p - c) + h}{\gamma' (\varphi (\xi_1, q_1))} - (1 - \beta), & \bar{\xi} (q_1) \leq \xi_1 < q_1.
\end{cases}
\]

It is clear that for any given \( \xi_1 \) and \( \varphi (\xi_1, q_1) \), the derivative \( \frac{\partial \varphi (\xi_1, q_1)}{\partial \xi_1} \) will be higher if \( \beta \), \( p \) or \( h \) is larger, \( c \) is smaller, or \( \gamma' (x) \) is smaller for all \( x \) when it is parameterized. When \( \bar{\xi} (q_1) = 0 \) (i.e., when there is no carryover effect), \( \frac{\partial \varphi (\xi_1, q_1)}{\partial \xi_1} \) will also be higher if \( a \) is larger. (When there is carryover effect, we are however not able to analytically prove this result associated with \( a \) due to the term \( v_2' (\xi_1, q_1) = ap - (1 + a)p F_2 (q_1 - (1 + a) \xi_1) \) which contains a distribution function. As found in our numerical experiments, a larger \( a \) generally increases the padding amount.)

In the following, we prove \( \varphi (\xi_1, q_1) \) increases for any given \( \xi_1 \) if \( p \) increases. The proofs for the other results are analogous and thus omitted. This can be shown as follows. Suppose there are two prices, \( p_a < p_b \). Let \( \varphi_a (\xi_1, q_1) \) and \( \varphi_b (\xi_1, q_1) \) denote the two padding functions according to these two prices. Both \( \varphi_a (\xi_1, q_1) \) and \( \varphi_b (\xi_1, q_1) \) are equal to zero when \( \xi_1 = 0 \), but \( \frac{\partial \varphi_a (0, q_1)}{\partial \xi_1} < \frac{\partial \varphi_b (0, q_1)}{\partial \xi_1} \). As a result, \( \varphi_b (\xi_1, q_1) \) increases immediately above \( \varphi_a (\xi_1, q_1) \) as \( \xi_1 \) increases. Since both \( \varphi_a (\xi_1, q_1) \) and \( \varphi_b (\xi_1, q_1) \) are continuous, we can verify that there does not exist a second point besides zero where \( \varphi_a (\xi_1, q_1) \) and \( \varphi_b (\xi_1, q_1) \) cross. If \( \varphi_a (\xi_1, q_1) \) can cross \( \varphi_b (\xi_1, q_1) \) from the below to the above at some point \( \xi_1 > 0 \) where \( \varphi_a (\xi_1, q_1) = \varphi_b (\xi_1, q_1) \), then
the derivative $\frac{\partial \varphi_a(\hat{\xi}_1, q_1)}{\partial q_1}$ must be larger than $\frac{\partial \varphi_b(\hat{\xi}_1, q_1)}{\partial q_1}$. This however does not hold according to Equation (19). Given $p_a < p_b$, we always have $\frac{\partial \varphi_a(\hat{\xi}_1, q_1)}{\partial q_1} > \frac{\partial \varphi_b(\hat{\xi}_1, q_1)}{\partial q_1}$ when $\varphi_a(\hat{\xi}_1, q_1) = \varphi_b(\hat{\xi}_1, q_1)$. Therefore, $\varphi(\hat{\xi}_1, q_1)$ increases as $p$ increases for any $\hat{\xi}_1$. This completes the proof for (i).

(ii) Recall from Proposition 2 that the threshold $\hat{\xi}(q_1)$ is determined by equating the two payoffs following the separating and pooling equilibrium paths. In this proposition, we apply the same approach. In particular, with the assumption that $\varphi(\hat{\xi}(q_1), q_1)$ remains at zero, we have:

$$\hat{\xi}(q_1) : \text{LHS}(\xi, q_1) = \text{RHS}(\xi, q_1)$$

where

$$\text{LHS}(\xi, q_1) = A_0 + p\xi - h(q_1 - \xi) + v_2(\xi, q_1) + \frac{1 - \beta}{\beta} \gamma(q_1 - \xi) - cq_1$$

$$\text{RHS}(\xi, q_1) = A_0 + \mathbb{E}_{\eta_1} [p\eta_1 - h(q_1 - \eta_1)^+ + v_2(\eta_1, q_1) - \gamma((q_1 - \eta_1)^+)\eta_1 \geq \xi] - cq_1.$$ 

In Proposition 2, we have shown that $\text{RHS}(\xi, q_1)$ monotonically increases in $\xi$, and $\text{LHS}(\xi, q_1)$ monotonically decreases in $\xi$. Therefore, the difference of the two payoffs, $\text{RHS}(\xi, q_1) - \text{LHS}(\xi, q_1)$, monotonically increases in $\xi$. Given this result, to show the impact of the parameters on the threshold $\hat{\xi}(q_1)$, we can directly evaluate the derivative of $\text{RHS}(\xi, q_1) - \text{LHS}(\xi, q_1)$ with respect to each parameter. If this payoff difference increases as one parameter increases, then it implies that $\text{RHS}(\xi, q_1)$ becomes more attractive and the threshold $\hat{\xi}(q_1)$ will decrease; it is the opposite if this payoff difference decreases. Note that this analysis is general whether an interior solution of the threshold $\hat{\xi}(q_1)$ exists or does not exist. In the latter case, the threshold $\hat{\xi}(q_1)$ is either at zero or at $\eta_1$ according to Proposition 2.

We first show the impact of the sales revenue $p$. We derive:

$$\frac{\partial \left(\text{RHS}(\xi, q_1) - \text{LHS}(\xi, q_1)\right)}{\partial p} = \mathbb{E}_{\eta_1} \left[ \eta_1 + \frac{\partial v_2(\eta_1, q_1)}{\partial p} | \eta_1 \geq \xi \right] - \xi - \frac{\partial v_2(\xi, q_1)}{\partial p}$$

$$= \int_{\xi}^{\eta_1} \left( \eta_1 + \frac{\partial v_2(\eta_1, q_1)}{\partial p} - \xi - \frac{\partial v_2(\xi, q_1)}{\partial p} \right) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1$$

We can easily show that $\xi + \frac{\partial v_2(\xi, q_1)}{\partial p}$ increases in $\xi$ because its derivative with respect to $\xi$ is positive:

$$\frac{\partial \left(\xi + \frac{\partial v_2(\xi, q_1)}{\partial p}\right)}{\partial \xi} = \begin{cases} 
(1 + a)F_2(q_1 - (1 + a)\xi), & 0 \leq \xi < \bar{\xi}(q_1), \\
(1 + a), & \bar{\xi}(q_1) \leq \xi < q_1. 
\end{cases}$$

Therefore, $\frac{\partial \left(\text{RHS}(\xi, q_1) - \text{LHS}(\xi, q_1)\right)}{\partial p} > 0$, and thus $\hat{\xi}(q_1)$ decreases in $p$. In other words, when $p$ increases, the
payoff following the pooling equilibrium path becomes more attractive compared to the payoff following the separating equilibrium path.

Similarly, for $\beta$, $h$, $c$ and $a$, we can derive

$$\frac{\partial (RHS(\xi, q_1) - LHS(\xi, q_1))}{\partial \beta} = \frac{1}{\beta^2} \gamma(q_1 - \xi) > 0;$$

$$\frac{\partial (RHS(\xi, q_1) - LHS(\xi, q_1))}{\partial h} = \int_\xi^{\eta_1} (- (q_1 - \eta_1)^+ + (q_1 - \xi)) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1 > 0;$$

$$\frac{\partial (RHS(\xi, q_1) - LHS(\xi, q_1))}{\partial c} = \int_\xi^{\eta_1} \left( \frac{\partial v_2(\eta_1, q_1)}{\partial c} - \frac{\partial v_2(\xi, q_1)}{\partial c} \right) f_1(\eta_1) \frac{F_1(\xi)}{F_1(\xi)} d\eta_1 < 0,$$

where $\frac{\partial v_2(\xi, q_1)}{\partial c} = -a\xi + (q_1 - \xi)^+ - k_2 + c \frac{1}{p f_2(k_2)}$;

$$\frac{\partial (RHS(\xi, q_1) - LHS(\xi, q_1))}{\partial a} = \int_\xi^{\eta_1} \left( \frac{\partial v_2(\eta_1, q_1)}{\partial a} - \frac{\partial v_2(\xi, q_1)}{\partial a} \right) f_1(\eta_1) \frac{F_1(\xi)}{F_1(\xi)} d\eta_1 > 0,$$

where $\frac{\partial v_2(\xi, q_1)}{\partial a} = \begin{cases} p\xi F_2(q_1 - (1 + a)\xi), & 0 \leq \xi < \xi(q_1), \\ (p - c)\xi, & \xi(q_1) \leq \xi. \end{cases}$

Therefore, $\hat{\xi}(q_1)$ decreases, if $\beta$, $h$ or $a$ increases, or $c$ decreases.

For the penalty cost $\gamma(x)$, we cannot use the above approach since it is not parameterized. However, we can directly compare the two payoffs:

$$RHS(\xi, q_1) - LHS(\xi, q_1) = -E_{\eta_1} \left[ \gamma((q_1 - \eta_1)^+) | \eta_1 \geq \xi \right] - \frac{1 - \beta}{\beta} \gamma(q_1 - \xi) + E_{\eta_1} \left[ p\eta_1 - b(q_1 - \eta_1)^+ + v_2(\eta_1, q_1) | \eta_1 \geq \xi \right] - (p\xi - b(q_1 - \xi) + v_2(\xi, q_1)).$$

In particular, we only need to examine the first two terms which depend on $\gamma(x)$. Since they are both negative, it is easy to see that if $\gamma(x)$ increases for all $x$ (the condition that $\gamma'(x)$ increases for all $x$ will be stronger), then $RHS(\xi, q_1) - LHS(\xi, q_1)$ decreases, and thus $\hat{\xi}(q_1)$ increases. This completes the proof for (ii).■

**Proof of Proposition 4:** In this proof, we show the impact of $q_1$ on $\varphi(\xi_1, q_1)$ and $\hat{\xi}(q_1)$.

(i) The impact of $q_1$ on $\varphi(\xi_1, q_1)$ is straightforward. We have the derivative of $\varphi(\xi_1, q_1)$ with respect to
\[ \xi_1 \text{ as:} \]

\[
\frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1} = \begin{cases} 
\beta \frac{(1 + a)pF_2(q_1 - (1 + a)\xi_1) + h}{\gamma(\varphi(\xi_1, q_1))} - (1 - \beta), & 0 \leq \xi_1 < \bar{\xi}(q_1), \\
\beta \frac{(1 + a)(p - c) + h}{\gamma(\varphi(\xi_1, q_1))} - (1 - \beta), & \bar{\xi}(q_1) \leq \xi_1 < q_1.
\end{cases}
\]

We can observe that for any given \( \xi_1 \) and \( \varphi(\xi_1, q_1) \), the derivative \( \frac{\partial \varphi(\xi_1, q_1)}{\partial \xi_1} \) is higher if \( q_1 \) is larger due to the term \((1 + a)pF_2(q_1 - (1 + a)\xi_1)\). Therefore, based on a similar argument as in the proof of Proposition 3(i), we can show \( \varphi(\xi_1, q_1) \) increases in \( q_1 \) for any \( \xi_1 < q_1 \).

(ii) With the assumption that \( \varphi(\xi_1, q_1) \) remains at zero, we can apply the same approach as in the proof of Proposition 3(ii) to prove this result. The threshold \( \bar{\xi}(q_1) \) is determined by:

\[ \bar{\xi}(q_1) : LHS(\xi, q_1) = RHS(\xi, q_1) \]

where

\[
LHS(\xi, q_1) = A_0 + p\xi - h(q_1 - \xi) + v_2(\xi, q_1) + \frac{1 - \beta}{\beta} \gamma(q_1 - \xi) - c q_1
\]

\[
RHS(\xi, q_1) = A_0 + \mathbb{E}_{\eta_1}[p\eta_1 - h(q_1 - \eta_1)^+ + v_2(\eta_1, q_1) - \gamma(q_1 - \eta_1^+)\eta_1 \geq \xi] - c q_1
\]

Therefore, we derive

\[
\frac{\partial (RHS(\xi, q_1) - LHS(\xi, q_1))}{\partial q_1} = \int_{\xi}^{\min(q_1, \eta_1)} \left(-h + \frac{\partial v_2(\eta_1, q_1)}{\partial q_1} - \gamma'(q_1 - \eta_1)\right) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1
\]

\[
+ \int_{\min(q_1, \eta_1)}^{\eta_1} \left(p + \frac{\partial v_2(\eta_1, q_1)}{\partial q_1}\right) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1 + h - \frac{\partial v_2(\xi, q_1)}{\partial q_1} - \frac{1 - \beta}{\beta} \gamma'(q_1 - \xi)
\]

\[
= \int_{\min(q_1, \eta_1)}^{\eta_1} \left(p + h\right) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1 + \int_{\xi}^{\min(q_1, \eta_1)} \frac{\partial v_2(\eta_1, q_1)}{\partial q_1} \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1 - \frac{\partial v_2(\xi, q_1)}{\partial q_1}
\]

\[
- \int_{\xi}^{\min(q_1, \eta_1)} \gamma'(q_1 - \eta_1) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1 - \frac{1 - \beta}{\beta} \gamma'(q_1 - \xi)
\]

Note that

\[
\frac{\partial v_2(\xi, q_1)}{\partial q_1} = \begin{cases} 
pF_2(q_1 - (1 + a)\xi), & 0 \leq \xi < \bar{\xi}(q_1), \\
c, & \bar{\xi}(q_1) \leq \xi < q_1, \\
0, & \xi \geq q_1.
\end{cases}
\]
When $\bar{\xi}(q_1) \leq \xi < q_1$, we can obtain

$$
\frac{\partial (RHS(\xi, q_1) - LHS(\xi, q_1))}{\partial q_1} = (p + h - c) \frac{F_1(\min(q_1, \eta_1))}{F_1(\xi)} - \int_{\xi}^{\min(q_1, \eta_1)} \gamma'(q_1 - \eta_1) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1
- \frac{1}{\beta} \gamma'(q_1 - \xi).
$$

$$
= \frac{1}{\beta} \left[ \beta (p + h - c) \frac{F_1(\min(q_1, \eta_1))}{F_1(\xi)} - (1 - \beta) \gamma'(q_1 - \xi) - \beta \int_{\xi}^{\min(q_1, \eta_1)} \gamma'(q_1 - \eta_1) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1 \right]
< \frac{1}{\beta} \left[ \beta ((1 + a) (p - c) + h) - (1 - \beta) \gamma'(0) - \beta \int_{\xi}^{q_1} \gamma'(q_1 - \eta_1) \frac{f_1(\eta_1)}{F_1(\xi)} d\eta_1 \right]
< 0
$$

The third inequality holds because $F_1(\min(q_1, \eta_1)) < F_1(\xi)$ given that $\xi < \min(q_1, \eta_1)$, $a > 0$ and $\gamma'(q_1 - \xi) > \gamma'(0)$. The fourth inequality holds because of the assumption $\varphi(\xi, q_1) = 0$ which implies $\frac{\beta[(p + h + v_2(q_1, \xi))] - (1 - \beta) \gamma'(0)}{\gamma'(0)} \leq 0$ based on the results of Lemma 4. Therefore, when $q_1$ increases, the above payoff difference decreases, which implies that the payoff from the pooling equilibrium path becomes less attractive, and thus $\bar{\xi}(q_1)$ increases in $q_1$. This completes the proof for (ii).
Appendix C: First Period Inventory Investment

In this section, we provide the derivation of the manager’s expected payoff function with respect to the first-period inventory decision $q_1$ and its first derivative.

**Derivation of Equation (8):** When the manager makes the first-period inventory decision $q_1$, the manager takes the equilibrium result of the subgame into consideration. For the part of the separating equilibrium (i.e., $\xi_1 < \hat{\xi} (q_1)$), the investors can perfectly infer the real demand, and thus the market price equals the real value of the firm. For the part of the pooling equilibrium (i.e., $\xi_1 > \hat{\xi} (q_1)$), the market price of the firm equals the expected true value of the firm over this region. Therefore, the manager’s expected payoff with respect to $q_1$ follows

$$
\mathbb{E}_{\eta_1} [ \pi (x; \eta_1, q_1)] = \mathbb{E}_{\eta_1} [ \beta P^o (\eta_1 + x, q_1) + (1 - \beta) v_1 (x; \eta_1, q_1)] \\
= A_0 - cq_1 + \int_0^{\xi(q_1)} \left[ \min (\eta_1, q_1) - h(q_1 - \eta_1)^+ + v_2 (\eta_1, q_1) - \gamma (x^o(\eta_1, q_1)) \right] f_1(\eta_1) d\eta_1 \\
+ (1 - \beta) \int_{\hat{\xi}(q_1)}^{\eta_1} \left[ \min (\eta_1, q_1) - h(q_1 - \eta_1)^+ + v_2 (\eta_1, q_1) - \gamma (x^o(\eta_1, q_1)) \right] f_1(\eta_1) d\eta_1 \\
+ \beta \int_{\hat{\xi}(q_1)}^{\eta_1} \int_{\xi(q_1)}^{\eta_1} \left[ \min (\eta_1, q_1) - h(q_1 - \eta_1)^+ + v_2 (\eta_1, q_1) - \gamma (x^o(\eta_1, q_1)) \right] f_1(\eta_1) f_1(\eta_2) d\eta_1 d\eta_2
$$

with

$$
x^o(\xi_1, q_1) = \begin{cases} 
\varphi(\xi_1, q_1), & 0 \leq \xi_1 < \hat{\xi} (q_1), \\
(q_1 - \xi_1)^+, & \xi_1 > \hat{\xi} (q_1).
\end{cases}
$$

This leads to Equation (8).

We derive the first order derivative with respect to $q_1$:

$$
0 = -c + p F_1 (q_1) - h F_1 (q_1) + p \mathbb{E}_{\eta_1 \in [0, \hat{\xi}(q_1))] [F_2 (q_1 - (1 + a) \eta_1)] + c \left(F_1 (q_1) - F_1 (\hat{\xi} (q_1)) \right) \\
- \frac{d}{dq_1} \mathbb{E}_{\eta_1 \in [0, \hat{\xi}(q_1))] \left[ \gamma (\varphi(\xi_1, q_1)) \right] - \frac{d}{dq_1} \mathbb{E}_{\eta_1 \in [\hat{\xi}(q_1), q_1]} \left[ \gamma (q_1 - \xi_1) \right]
$$

Classical Newsvendor

Channel Stuffing Friction

However, it is difficult to analytically determine the optimal inventory level $q_1^*$ since we do not have an explicit integral form for $\varphi(\xi_1, q_1)$. Generally, beyond the classical newsvendor trade-offs; i.e., the expected overage and underage costs, the manager also needs to balance the expected channel stuffing cost which is
driven by the marginal effect, carryover effect and the boundary effect. As we revealed in Propositions 4, the padding amount $\varphi(\xi_1, q_1)$ following the separating equilibrium path increases in $q_1$ but the boundary effect may be alleviated in some scenarios if $q_1$ slightly increases. As a result, it is possible that the manager may want to invest lower inventory to limit the incentives in the region $[0, \hat{\xi}(q_1))$ or over-invest to limit the incentives in the region $(\hat{\xi}(q_1), q_1)$. Due to the complexity of the analysis, we show the key insights by numerical examples in which we numerically integrate $\varphi(\xi_1, q_1)$ to find close optimal $q_1$. 
Appendix D: Solutions to the ODE with Uniform Demand and Quadratic Penalty Cost

In this Appendix, we derive an implicit solution of \( \phi(z, q_1) \) under a uniform demand distribution and a quadratic channel stuffing cost function. Recall, \( \phi(z, q_1) \) defined in Appendix B represents the investors’ belief of the padded sales in equilibrium given reported sales \( z \) and initial inventory \( q_1 \). \( \phi(z, q_1) \) satisfies the three properties specified in Lemma 4. In particular, if \( \frac{\partial \phi(0,q_1)}{\partial z} = 0 \), then the function \( \phi(z, q_1) \) is trivial which stays at zero for all \( z \). Therefore, we assume \( \frac{\partial \phi(0,q_1)}{\partial z} > 0 \) in the following. Once we obtain an implicit function of \( \phi(z, q_1) \), we can further obtain an implicit solution of \( \varphi(\xi_1, q_1) \) given that \( \varphi(\xi_1, q_1) \) is the unique solution of \( x = \phi(x + \xi_1, q_1) \) as shown in Lemma 5 in Appendix B.

Suppose the random variable \( \eta_2 \) follows a uniform distribution \( U[D,D] \) and the channel stuffing cost follows a quadratic function \( \gamma(x) = \gamma_1 x + \frac{1}{2} \gamma_2 x^2 \). Without loss of generality, we set \( D = 0 \). The procedure for \( D > 0 \) is analogous. Also in this derivation procedure, we assume \( q_1 \leq D \); i.e., the provided initial inventory is no higher than the upper bound of the demand distribution. For \( q_1 > D \), this procedure can be appropriately adjusted by assuming an additional condition on the probability distribution, which we point it out in the following.

By Equation (10) in Appendix B, we have

\[
\frac{\partial \phi(z, q_1)}{\partial z} = \frac{\beta (p + h + v_2' (z - \phi(z, q_1), q_1)) - (1 - \beta) \gamma'(\phi(z, q_1))}{\beta (p + h + v_2' (z - \phi(z, q_1), q_1)) + \beta \gamma'(\phi(z, q_1))}
\]

for \( \phi(z, q_1) > 0 \). To solve \( \phi(z, q_1) \), the constraint \( \phi(z, q_1) > 0 \) can be first relaxed and imposed later by simply taking the positive part of the solution solved without this constraint. Based on the uniform distribution, we have \( F_2(\eta_2) = \frac{\eta_2 - D}{D} \) for \( 0 \leq \eta_2 \leq D \), and thus:

\[
v_2' (\xi_1, q_1) = \begin{cases} 
    ap - (1 + a)p \frac{\xi_1 - (1 + a) \xi_1}{D}, & 0 \leq \xi_1 < \bar{\xi}(q_1), \\
    a(p - c) - c, & \bar{\xi}(q_1) \leq \xi_1 < q_1.
\end{cases}
\]

(Note that if \( q_1 > D \), we would need to adjust the above equation as \( v_2' (\xi_1, q_1) = ap - (1 + a)p \max(\frac{\xi_1 - (1 + a) \xi_1}{D}, 0) \) when \( 0 \leq \xi_1 < \bar{\xi}(q_1) \). For simplicity, we do not discuss this case in detail.)

Based on the quadratic penalty cost function and the above \( v_2' (\xi_1, q_1) \), the ODE of \( \frac{\partial \phi(z, q_1)}{\partial z} \) becomes:

\[
\frac{\partial \phi(z, q_1)}{\partial z} = \begin{cases} 
    1 - \frac{\frac{\gamma_1}{2} + \phi(z, q_1)}{\gamma_1 ((1 + a)p + h + \frac{a}{2} p - h + \frac{a}{2} p \gamma_2 z + \frac{a}{2} p \gamma_2 \phi(z, q_1) + \beta(\frac{\gamma_1}{2} + \phi(z, q_1))}, & 0 \leq z < z_f(q_1), \\
    1 - \frac{\frac{\gamma_1}{2} + \phi(z, q_1)}{\gamma_1 ((1 + a)(p - c) + h + \beta(\frac{\gamma_1}{2} + \phi(z, q_1))}, & z_f(q_1) \leq z < q_1.
\end{cases}
\]
where \( z_I(q_I) \) solves \( \phi(z_I, q_I) = z_I - \xi(q_I) \) according to the definition of \( \phi(z, q_I) \) (i.e., \( \phi(z, q_I) = z - \xi(q_I) \)).

For notational convenience, we drop \( q_I \) from the arguments as it is given when the manager is making the channel stuffing decision. We use \( \phi'(z) \) to denote \( \frac{\partial \phi(z, q_I)}{\partial z} \) and write \( \frac{\partial \phi(z, q_I)}{\partial z} \) as:

\[
\phi'(z) = \begin{cases} 
\Phi_I(\phi(z), z), & z \leq z_I, \\
\Phi_{II}(\phi(z), z), & z_I < z.
\end{cases}
\]  

(20)

Because of the linearity of \( F_2(\cdot) \) and \( \gamma'(\cdot) \), we can express the two sub-ODEs in the following form:

\[
\begin{align*}
\Phi_I(\phi, z) &= 1 - \frac{d + \phi}{a_1 - b_1 z + c_1 (d + \phi)} \\
\Phi_{II}(\phi, z) &= 1 - \frac{d + \phi}{a_2 + c_2 (d + \phi)}
\end{align*}
\]

where

\[
a_1 = \frac{\beta}{\gamma_2} \left( h + p (1 + a) \frac{q_I}{D} - p \frac{(1 + a)^2}{D} \frac{\gamma_1}{\gamma_2} \right), 
\quad b_1 = \frac{\beta}{\gamma_2} \frac{p (1 + a)^2}{D} + \gamma_2, 
\quad c_1 = \frac{\beta}{\gamma_2} \frac{p (1 + a)^2}{D} + \gamma_2, 
\quad d = \frac{\gamma_1}{\gamma_2}
\]

and

\[
a_2 = \frac{\beta}{\gamma_2} \left( (1 + a)(p - c) + h \right) \quad \text{and} \quad c_2 = \beta.
\]

As \( \phi(z) \) has two sub-ODEs, \( \phi(z) \) can be solved in two steps corresponding to \( \Phi_I(\phi, z) \) and \( \Phi_{II}(\phi, z) \) over the intervals \( z \in [0, z_I] \) and \( z \in [z_I, q_I] \). Recall from Lemma 4 in Appendix B that the function \( \phi(z) \) has an initial condition \( \phi(0) = 0 \). Therefore, we can solve the first part of \( \phi(z) \) over the interval \( z \in [0, z_I] \) based on \( \phi'(z) = 1 - \Phi_I(\phi(z), z) \) with the initial condition \( \phi(0) = 0 \). Then, let \( \phi_I = \phi(z_I) \), and we can solve the second part of \( \phi(z) \) over the interval \( z \in [z_I, q_I] \) based on \( \phi'(z) = 1 - \Phi_{II}(\phi(z), z) \) with the initial condition \( \phi(z) = \phi_I \) at \( z = z_I \). The following two subsections detail the procedure to solve \( \phi(z) \).

**Solution of \( \phi'(z) = 1 - \Phi_I(\phi(z), z) \) with \( \phi(0) = 0 \) over \( z \in [0, z_I] \)**

In the following, we solve

\[
\phi'(z) = 1 - \Phi_I(\phi(z), z) \quad \text{with} \quad \phi(0) = 0.
\]  

(21)

As the ODE is of the first degree, we can first solve \( \phi'(z) = 1 - \Phi_I(\phi(z), z) \) without the initial condition; then, the solution will contain one constant, \( K \), that can be later specified from the initial condition, \( \phi(0) = 0 \).

We define

\[
A = 2c_1 b_1 - c_1^2 + 2c_1 - (1 - b_1)^2.
\]

We consider two situations depending on the sign of \( A \):
**First case, $A > 0$:** Consider now the situation in which $A > 0$ and define:

$$
\delta = \frac{b_1 - c_1 + 1}{\sqrt{A}}, \quad \psi_0 = \frac{1 - b_1 - c_1}{\sqrt{A}} \quad \text{and} \quad \psi_1 = \frac{2b_1c_1}{\sqrt{A}}.
$$

Notice that $\psi_0$, $\psi_1$ and $\delta$ are real numbers. We prove that the solution to Equation (21) is given by:

$$
\begin{align*}
\exp(K) & \frac{\exp(\delta(\arctan(\frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0)))}{\exp(\delta(\arctan(\frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0)))} = \frac{1}{a_1} (a_1 - b_1 z) \sqrt{1 + (b_1 + c_1 - 1) \frac{d + \phi(z)}{a_1 - b_1 z} + c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2}, \quad z < a_1/b_1, \\
\phi(z) & = \frac{\exp(K + \delta \psi_1)}{a_1}, \quad z = a_1/b_1, \\
\exp(-\pi \delta + K) & \frac{\exp(\delta(\arctan(\frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0)))}{\exp(\delta(\arctan(\frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0)))} = -\frac{1}{a_1} (a_1 - b_1 z) \sqrt{1 + (b_1 + c_1 - 1) \frac{d + \phi(z)}{a_1 - b_1 z} + c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2}, \quad z > a_1/b_1.
\end{align*}
$$

(22)

We consider the three cases, $z < a_1/b_1$, $z > a_1/b_1$ and $z = a_1/b_1$ separately:

1. First, consider the case $a_1 - b_1 z > 0$. We show that the implicit solution of the first case of Equation (22) satisfies the ODE specified for $\phi(z)$. We derive the first derivative with respect to $z$ (note that $\frac{d}{dz} (\arctan(z)) = \frac{1}{1 + z^2}$):

$$
-\delta \exp(K - \delta \arctan(\psi_1 \frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0))) \frac{\psi_1 \frac{d + \phi(z)}{a_1 - b_1 z}}{1 + (\psi_1 \frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0)^2} = \frac{b_1}{a_1} \sqrt{1 + (b_1 + c_1 - 1) \frac{d + \phi(z)}{a_1 - b_1 z} + c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2} \\
+ \frac{1}{a_1} (a_1 - b_1 z) \frac{(b_1 + c_1 - 1) + 2c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2}{2 \sqrt{1 + (b_1 + c_1 - 1) \frac{d + \phi(z)}{a_1 - b_1 z} + c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2}} \frac{d}{dz} \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right).
$$

Organizing all terms with $\frac{d}{dz} \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)$ and replacing $\exp(K - \delta \arctan(\psi_1 \frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0)))$ by the right hand side of the first case of Equation (22), we obtain

$$
\frac{d}{dz} \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)
$$

$$
= \frac{-\frac{b_1}{a_1} \left( 1 + (b_1 + c_1 - 1) \frac{d + \phi(z)}{a_1 - b_1 z} + c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2 \right)}{\frac{1}{a_1} (a_1 - b_1 z) \left( -\psi_1 \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2 1 + (\psi_1 \frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0)^2} - \frac{1}{2} \left( b_1 + c_1 - 1 + 2c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2 \right)
$$

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Now, consider the case \( z < a_1 \), hence we have characterized with the first case of Equation (22) the solution to the ODE for \( a_1 - b_1 z \).

Therefore, the implicit solution of the first case of Equation (22) satisfies \( \phi'(z) = 1 - \Phi_I(\phi(z), z) \).

Hence, we have characterized with the first case of Equation (22) the solution to the ODE for \( z < a_1/b_1 \).

2. Now, consider the case \( a_1 - b_1 z < 0 \). We show in this scenario the implicit solution of the third
Finally, consider Equation (22): 

\[
\phi_{H}
\]

Hence, we obtain that the implicit solution of the third branch of Equation (22) also satisfies 

\[
-\delta \exp \left( -\pi \delta + K - \delta \left( \arctan \left( \psi_1 \frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0 \right) \right) \right) \frac{\psi'_1 \frac{d}{dz} \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)}{1 + \left( \psi'_1 \frac{d}{dz} \left( \frac{d + \phi(z)}{a_1 - b_1 z} - \psi_0 \right) \right)^2} = \frac{b_1}{a_1} \sqrt{1 + (b_1 + c_1 - 1) \frac{d + \phi(z)}{a_1 - b_1 z} + c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2}
\]

Gathering all terms with \( \frac{d}{dz} \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right) \) and substituting the original relationship of the second case of Equation (22):

\[
\frac{d}{dz} \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right) = \frac{b_1}{a_1} \left( 1 + (b_1 + c_1 - 1) \frac{d + \phi(z)}{a_1 - b_1 z} + c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2 \right)
\]

or

\[
\phi'(z) + \frac{b_1}{a_1} \frac{d + \phi(z)}{a_1 - b_1 z} = \frac{d}{dz} \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)
\]

from which follows that

\[
\phi'(z) + \frac{b_1}{a_1} \frac{d + \phi(z)}{a_1 - b_1 z} = \frac{1 + (b_1 + c_1 - 1) \frac{d + \phi(z)}{a_1 - b_1 z} + c_1 b_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)^2}{1 + c_1 \left( \frac{d + \phi(z)}{a_1 - b_1 z} \right)}
\]

or

\[
\phi'(z) = 1 - \frac{d + \phi(z)}{a_1 - b_1 z + c_1 (d + \phi(z))}
\]

Hence, we obtain that the implicit solution of the third branch of Equation (22) also satisfies \( \phi'(z) = 1 - \Phi_f(\phi(z), z) \).

3. Finally, consider \( z = a_1/b_1 \). Let \( FB \) denote the first branch of Equation (22) and \( TB \) denote the
third branch of Equation (22). It is easy to see that \( \lim_{z \to a_1/b_1 -} \arctan \left( \frac{\phi(z) + d}{a_1 - b_1 z} - \psi_0 \right) = -\frac{\pi}{2} \) and \( \lim_{z \to a_1/b_1 +} \arctan \left( \frac{\phi(z) + d}{a_1 - b_1 z} - \psi_0 \right) = -\frac{\pi}{2} \). Therefore,

\[
\begin{align*}
\lim_{z \to a_1/b_1 -} FB : \quad & \frac{\exp(K)}{\exp(\delta \frac{\pi}{2})} = \frac{1}{a_1} \sqrt{c_1 b_1^2} \phi(z), \\
\lim_{z \to a_1/b_1 +} TB : \quad & \frac{\exp(-\delta + K)}{\exp(\delta \frac{\pi}{2})} = \frac{1}{a_1} \sqrt{c_1 b_1^2} \phi(z).
\end{align*}
\]

The first and third branches of Equation (22) converge at \( z = a_1/b_1 \). Consequently, we obtain

\[
\phi \left( \frac{a_1}{b_1} \right) = \frac{\exp \left( K - \delta \frac{\pi}{2} \right)}{\frac{1}{a_1} \sqrt{c_1 b_1}}
\]

and hence the implicit solution solved from Equation (22) is continuous and satisfies the original ODE specified for \( \phi(z) \).

4. We determine the constant \( K \) as follows: for \( z = 0 \), we need to have that \( \phi(z) = 0 \), or

\[
\frac{\exp(K)}{\exp(\delta \arctan(\psi_1 - \psi_0))} = 1
\]

from which follows that the solution to the ODE of Equation (21) is:

\[
\left\{ \begin{array}{ll}
\frac{\exp(\delta(\arctan(\psi_1 - \psi_0)))}{\exp(\delta(\arctan(\psi_1/a_1 - b_1 z - \psi_0)))} = \frac{1}{a_1} (a_1 - b_1 z) \sqrt{1 + (b_1 + c_1 - 1) \frac{\phi(z) + d}{a_1 - b_1 z} + c_1 b_1 \left( \frac{\phi(z) + d}{a_1 - b_1 z} \right)^2}, & z < a_1/b_1 \\
\phi(z) = \frac{\exp(\delta(\arctan(\psi_1 - \psi_0) - \frac{\pi}{2}))}{\exp(\delta(\arctan(\psi_1/a_1 - b_1 z - \psi_0)))} = -\frac{1}{a_1} (a_1 - b_1 z) \sqrt{1 + (b_1 + c_1 - 1) \phi(z) \frac{\phi(z) + d}{a_1 - b_1 z} + c_1 b_1 \left( \frac{\phi(z) + d}{a_1 - b_1 z} \right)^2}, & z > a_1/b_1
\end{array} \right.
\]

(24)

• Second case, \( A < 0 \): Now, consider the situation in which \( A < 0 \) and define:

\[
\delta = \frac{b_1 - c_1 + 1}{\sqrt{|A|}}, \quad \psi_0 = \frac{1 - b_1 - c_1}{\sqrt{|A|}} \quad \text{and} \quad \psi_1 = \frac{2b_1 c_1}{\sqrt{|A|}}.
\]

Notice that \( \psi_0, \psi_1 \) and \( \delta \) are real numbers, but, Equation (22) now contains complex numbers. As \( A < 0 \), the solution is given by

\[
\frac{\exp(K)}{\exp \left( \frac{\delta}{\sqrt{-1}} \left( \arctan \left( \frac{\psi_1}{\sqrt{-1} a_1 - b_1 z} - \frac{\psi_0}{\sqrt{-1}} \right) \right) \right)} = \frac{1}{a_1} (a_1 - b_1 z) \sqrt{1 + (b_1 + c_1 - 1) \frac{\phi(z) + d}{a_1 - b_1 z} + c_1 b_1 \left( \frac{\phi(z) + d}{a_1 - b_1 z} \right)^2}
\]

(25)

With the identity

\[
\exp \left( \frac{1}{\sqrt{-1}} \left( \arctan \left( \frac{x}{\sqrt{-1}} \right) \right) \right) = \left( \frac{1 - x}{1 + x} \right)^{\frac{1}{2}}
\]

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we can rewrite Equation (25) as follows:

\[
\frac{\exp(K)}{\left(1 - \psi_1 \frac{\phi(z) + d}{a_1 - b_1 z} + \psi_0 \right)^2} = \frac{1}{a_1 (a_1 - b_1 z)} \sqrt{1 + (b_1 + c_1 - 1) \phi(z) + \frac{d}{a_1 - b_1 z} + c_1 b_1 \left(\frac{\phi(z) + d}{a_1 - b_1 z}\right)^2}
\]

With

\[1 + (b_1 + c_1 - 1) x + c_1 b_1 x^2 = 0 \iff x = \frac{\psi_0 \pm 1}{\psi_1}\]

or equivalently,

\[1 + (b_1 + c_1 - 1) x + c_1 b_1 x^2 = c_1 b_1 \left(x - \frac{\psi_0 - 1}{\psi_1}\right) \left(x - \frac{\psi_0 + 1}{\psi_1}\right)
\]

we can write Equation (25) as follows:

\[
\frac{\exp(K)}{\left(1 - \psi_1 \frac{\phi(z) + d}{a_1 - b_1 z} + \psi_0 \right)^2} = \frac{1}{a_1 (a_1 - b_1 z)} \sqrt{c_1 b_1 \left(\frac{\phi(z) + d}{a_1 - b_1 z} - \psi_0 - 1\right) \left(\frac{\phi(z) + d}{a_1 - b_1 z} - \psi_0 + 1\right)}.
\]

From the initial condition \(\phi(z) = 0\) when \(z = 0\), we obtain that the constant \(K\) satisfies:

\[
\frac{\exp(K)}{\left(1 - \psi_1 + \psi_0 \right)^2} = \frac{c_1 b_1 \psi_0 - 1}{\psi_1}.
\]

As a result, the solution to the ODE of Equation (21) is:

\[
\frac{\left(1 - \psi_1 + \psi_0 \right)^2}{\left(1 - \psi_1 \frac{\phi(z) + d}{a_1 - b_1 z} + \psi_0 \right)^2} = \left(1 - \frac{b_1}{a_1} z\right) \sqrt{\frac{c_1 b_1 \left(\frac{\phi(z) + d}{a_1 - b_1 z} - \psi_0 - 1\right) \left(\frac{\phi(z) + d}{a_1 - b_1 z} - \psi_0 + 1\right)}{\psi_0 - 1 \psi_0 + 1 \psi_1}}.
\]  \hspace{1cm} (26)

- Given the above implicit solutions, for a given \(z\), we can compute \(\phi(z)\) based on Equations (24) and (26) according to the scenario whether \(A > 0\) or \(A < 0\). Based on those solutions, let \(\phi_I = \phi(z_I)\).

In the following subsection, we solve the second part of \(\phi(z)\) over the interval \(z \in [z_I, q_I]\) from the sub-ODE \(\psi' = 1 - \Phi_{II}(\phi(z), z)\) with the initial condition \(\phi(z_I) = \phi_I\).

**Solution of \(\phi'(z) = 1 - \Phi_{II}(\phi(z), z)\) with \(\phi(z_I) = \phi_I\) over \(z \in [z_I, q_I]\)**

We now solve \(\phi'(z) = 1 - \Phi_{II}(\phi(z), z)\) with \(\phi(z_I) = \phi_I\) \hspace{1cm} (27)
over $z \in [z_I, q_1]$. In the following, we prove that the solution to Equation (27) is given by:

$$\frac{a_2 + (\phi + d) (c_2 - 1)}{a_2 + (\phi_I + d) (c_2 - 1)} = \exp \left( \frac{(c_2 - 1)^2 (z_I - z) + c_2 (\phi - \phi_I) (c_2 - 1)}{a_2} \right).$$

(28)

Given any $z$, Equation (28) has a unique solution for $\phi$ (the left-hand side is linear in $\phi$ while the right-hand side is exponential in $\phi$). It is easy to see that the initial condition $(\phi_I, z_I)$ satisfies Equation (28).

Moreover, we can derive the first derivative of Equation (28) with respect to $z$:

$$\frac{\phi' (z)}{a_2 + (\phi_I + d) (c_2 - 1)} = \left( - \frac{(c_2 - 1) + c_2^2 \phi (z)}{a_2} \right) \exp \left( \frac{(c_2 - 1)^2 (z_I - z) + c_2 (-\phi_I + z \phi (z)) (c_2 - 1)}{a_2} \right).$$

We can replace $\exp \left( \frac{(c_2 - 1)^2 (z_I - z) + c_2 (-\phi_I + z \phi (z)) (c_2 - 1)}{a_2} \right)$ by $\frac{a_2 + (\phi (z) + d) (c_2 - 1)}{a_2 + (\phi_I + d) (c_2 - 1)}$ based on Equation (28), and then obtain

$$\frac{\phi' (z) (c_2 - 1)}{a_2 + (\phi_I + d) (c_2 - 1)} = (c_2 - 1) \left( - \frac{(c_2 - 1) + c_2^2 \phi (z)}{a_2} \right) \frac{a_2 + (\phi (z) + d) (c_2 - 1)}{a_2 + (\phi_I + d) (c_2 - 1)}.$$

Reorganizing the terms, we can obtain

$$\phi' (z) = \left( - \frac{(c_2 - 1)}{a_2} + \frac{c_2^2 \phi' (z)}{a_2} \right) (a_2 + (\phi (z) + d) (c_2 - 1)),$$

or

$$\phi' (z) = 1 - \frac{d + \phi (z)}{a_2 + c_2 (d + \phi (z))}.$$

Therefore, this derivative matches the ODE of Equation (27). Hence, the solution of Equation (28) satisfies the ODE of Equation (27).

To summarize, for a given $z$, Equations (24), (26) and (28) allow computing $\phi (z)$ that is the solution of the ODE in Equation (20). Furthermore, given the relationship that $\varphi (\xi_1, q_1)$ is the unique solution of $x = \phi (x + \xi_1, q_1)$ (shown in Lemma 5 in Appendix B), we can derive the corresponding implicit solutions for $\varphi (\xi_1, q_1)$. In particular, we can replacing $\phi (z)$ by $x$ and $z$ by $x + \xi_1$ simultaneously in Equations (24), (26) and (28), which leads to the corresponding implicit solutions for $\varphi (\xi_1, q_1)$. 

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