Signaling by Price in a Congested Environment*

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Abstract

We consider the informational role of a queue when a firm can adjust its price to signal its quality to uninformed consumers. When the proportion of informed consumers is relatively high, increasing the price of a high-quality good is a superior signaling strategy. In this case, high- and low-quality firms choose different prices, so the queue has no role in communicating information about quality. Such separation is costly for a high-quality firm if, to prevent imitation by the low-quality firm, it has to raise its price beyond a monopoly price. When the proportion of informed consumers is low, there are also equilibria in which high- and low-quality firms pool on a low price. We exhibit equilibria in which, in the limit as waiting costs vanish, the queue almost perfectly reveals the quality. Here, short queues are associated with the low-quality firm and long queues with the high-quality one. We show that the high-quality firm earns a higher profit in the pooling equilibrium with congestion as compared to the case of costly separation. This explains why some producers prefer to signal quality via the queue length rather than by charging a high price. We demonstrate numerically that our findings are robust to positive waiting costs.

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1 Introduction

Many producers choose to impose significant waiting costs on consumers before they can access the good. For example, the Playstation 2 in 2000 and the iPod in 2004 had long queues in front of stores when they first came on to the market (Wall Street Journal, Dec 2, 2005). Even some industrial firms frequently announce their backlog of orders, which makes potential consumers aware of the queue for their product. Since it is straightforward to increase the price to clear the queue when demand is strong, it is puzzling that producers allow these queues to develop. In this paper, we demonstrate the circumstances under which a firm prefers to charge a low price and maintain a long queue for its product.

We model a signaling game in which the type of the firm corresponds with the quality of the good it sells, which can be high or low. Both goods have the same production cost. After observing the quality, the firm maximizes profit by choosing a price for the good. Consumers arrive at the market according to a Poisson arrival process. Both types of firm have the same exponential service process. A consumer wishing to buy the good must join a queue if another consumer is being served when she arrives at the market. The queue is cleared via a First-In-First-Out rule. Consumers are heterogeneous in two ways. First, keeping the quality of the good fixed, their consumption valuations differ. Second, a fraction of the consumers is informed; that is, it also observes the quality of the good. The remaining fraction is uninformed, and knows the prior distribution over quality but not the realized quality itself. All consumers observe price and the length of the queue when they arrive at the market. On arrival, each consumer chooses whether to purchase the good or balk. In the latter case, she leaves the market.

We consider perfect Bayesian equilibria of the game. Each type of firm chooses an optimal price, and consumers join or balk optimally. The decision of informed consumers is straightforward. Uninformed consumers update their prior belief about the quality of the firm once they have observed the price and queue length, and determine whether to join based on their posterior belief. We use the Intuitive Criterion of Cho and Kreps (1987) to restrict off-equilibrium beliefs. We describe the model in Section 2.

Analytic results in this model are hard to determine when the waiting cost is strictly positive. Instead, in Section 3 we consider the limiting case when the waiting cost per unit time goes to zero. When waiting costs are low, the congestion implied by queueing is most relevant. In a separating equilibrium, the high- and low-quality firms optimally choose dif-

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ferent prices. The price immediately reveals the quality to uninformed consumers, so there is no informational role for the queue. Separating equilibria exist when the consumption value of the low-quality good exceeds a relatively small threshold. If the proportion of informed consumers is high, there is no cost to separation: The high-quality firm can separate out even at its monopoly price. However, if the proportion of informed consumers is low, separation is costly. In particular, it entails charging a price beyond the monopoly level. In that case, in equilibrium the expected queue length is lower for the high-quality firm than the low-quality one.

Next, we turn to pooling equilibria in which both types of firm charge the same price. In this case, uninformed consumers learn nothing from the price of the good. However, informed consumers join at different rates for each type of firm, so the queue length communicates information about quality. We find that in equilibrium, as the waiting cost goes to zero, the queue length may become almost perfectly informative. Uninformed consumers correctly infer that the firm almost surely has low (high) quality if the queue is short (long). This result holds when the pooling price is low, the service rate is at an intermediate level compared to the arrival rate and the proportion of informed consumers is low.

We exhibit conditions under which the parameters support both such a pooling equilibrium and a costly separating equilibrium. The pooling equilibrium features a price lower and the separating equilibrium a price higher than the monopoly price of the high-quality firm. We find that, whenever both equilibria exist, the high-quality firm earns a higher profit in the pooling equilibrium, and so prefers to charge a low price and impose a long waiting time on its consumers. In other words, it prefers congestion to a high price as a means of signaling its quality.

Our analytic results obtain in the limit as the waiting cost goes to zero. By continuity, the intuition must go through when the waiting cost is small. In Section 4, we numerically compute and comment on the features of separating and pooling equilibria when the waiting cost is large. The intuition from the limiting case of no waiting cost goes through: whenever pooling and separating equilibria both exist, the high-quality firm prefers the pooling equilibrium. That is, it prefers to keep prices low and allow a longer queue to build up.

Overall, we demonstrate the circumstances under which firms may wish to impose waiting times on consumers rather than increase prices to clear the market. Our model applies both to new goods and to existing goods which have significantly improved by firms. If such improvements cannot be easily communicated to a broad consumer base, a firm may refrain from increasing its price, relying instead on a longer queue to signal its quality. That is, the firm may prefer to leverage its improvement via an innovation in the quantity demanded
rather than the price.

Our work builds on both the queueing and signaling literatures. Hassin and Haviv (2003) provide an excellent introduction to queueing games. On price-signaling, Bagwell and Riordan (1991) show that a high-quality firm can signal its quality by initially charging a high price that is subsequently reduced. To the best of our knowledge, ours is the first model to consider a price-setting firm subject to stochastic capacity (as in Naor, 1969) when the quality is not directly observed by some consumers. Casual observation suggests that these two features are present in many situations that are notorious for long queues. Our analysis, therefore, is important in understanding why firms allow long queues to persist.

A related paper to ours is Stock and Balachander (2005), who show that when the proportion of informed consumers is low, a high-quality firm may use scarcity rather than price alone as a signaling device to separate itself from the low type. There are two important differences between our paper and theirs. First, Stock and Balachander (2005) consider firms that choose both a price and a capacity, with scarcity resulting if consumer demand at that price exceeds the available capacity. It may be difficult for a firm to credibly communicate its capacity decision to a consumer base. In our model, the service rate is unobservable; instead, consumers observe the actual queue length upon arrival, which is more natural. Second, Stock and Balachander (2005) focus on separating equilibria; that is, equilibria in which the two types of firm choose different price-capacity pairs. Our focus is on equilibria which feature pooling on price. In particular, we show that even when the capacity cannot be directly used as a signaling instrument, the firm can nevertheless signal with endogenously-created congestion.

We add to the emerging literature on signaling games in different Operations Management environments. On the supply side, Cachon and Lariviere (2001), Gumus, Ray and Gurnani (2012), Schmidt, et al. (2012) and Lai, et al. (2011) study contracts and capacity investment as signals. On the demand side, Allon and Bassamboo (2011) consider a cheap talk game in which delay announcements are signals of congestion. Long queues as signals of high quality are considered by Debo, Parlour and Rajan (2011) in a model in which a firm chooses the service rate and by Veeraraghavan and Debo (2008, 2011) in models with and without waiting costs.

Our work departs from these papers in two ways. First, none of the demand-side work allows for prices to signal quality. Second, in much of the above work, there is a single signaling instrument (such as capacity or contract) that determines the information set of the uninformed agent. In our model, the firm has a single signaling instrument, the price. However, the information set of uninformed consumers contains a second element, the queue
length, which is endogenously generated. We exhibit situations in which the queue length may be informative about the firms type even when the price is not.

2 Model

We consider a market in which a single firm produces an experience good. The sequence of events in the game is as follows. At stage 0, nature chooses the type of the firm ($\theta$), which corresponds with the quality of the good produced by the firm. The type is high ($h$) with probability $p_0$ and low ($\ell$) with probability $1 - p_0$. The unit cost of production is constant across type; without loss of generality, we normalize it to zero. The firm privately observes its own type. At stage 1, the firm chooses a price for its product, $P$. At stage 2, consumers arrive according to a Poisson process with rate $\Lambda$. Consumers who wish to buy the product are serviced at a known exponential service rate $\mu$. When a new consumer arrives, if another consumer is being serviced, she must join a queue if she wishes to purchase the good. Alternatively, the newly-arrived consumer balks and leaves the game. The queue is served on a FIFO basis. The waiting cost for a consumer is $c$ per unit of time.

2.1 Consumer and Firm Payoffs

On arrival, a consumer observes both the price chosen by the firm, $P$, and the length of the queue, $n$. Each consumer has a consumption value $tv_\theta$ for a good of quality $\theta$. The parameter $t$ is independent across consumers and is uniformly distributed over $[0, 1]$. Further, $v_h > v_\ell$. Without loss of generality, we normalize $v_h$ to one. A proportion $q$ of the consumers are informed, and know the type of the firm. The remaining $1 - q$ of the consumers are uninformed consumers. These consumers know the prior $p_0 \in (0, 1)$ and can update it based on $P$ and $n$.

Consider first an informed consumer who knows the firm has type $\theta$ and arrives when the queue has length $n$. Her utility from purchasing the product at price $P$ is $tv_\theta - P - (n+1)\frac{c}{\mu}$. She buys the good only if her utility is non-negative; that is, if $t \geq \frac{P+(n+1)c/\mu}{tv_\theta}$. Denoting $x^+ = \max\{x, 0\}$, the demand from informed consumers at that queue length and price is then $\left(1 - \frac{P+(n+1)c/\mu}{tv_\theta}\right)^+$. Next, consider an uninformed consumer who sees the price $P$ and arrives when the queue has length $n$. Suppose this consumer has a posterior belief $b$ that the firm has high quality. Then, her expected utility from joining is $tv_h b + (1-b)tv_\ell - P - (n+1)\frac{c}{\mu}$. Therefore, the demand from uninformed consumers given $(P, n)$ is $\left(1 - \frac{P+(n+1)c/\mu}{tv_h b + (1-b)tv_\ell}\right)^+$. Let $s_\theta(P, n, b) \in [0, 1]$ denote the probability that a newly-arrived consumer will join when the
firm has type $\theta$ and charges a price $P$, the queue has $n$ consumers and uninformed consumers have posterior belief $b$ that the quality is high. Then, for each $\theta = h, \ell$,

$$s_\theta(P, n, b) = q \left( 1 - \frac{P + (n + 1)c/\mu}{v_\theta} \right)^+ + (1 - q) \left( 1 - \frac{P + (n + 1)c/\mu}{bv_h + (1 - b)v_\ell} \right)^+ . \quad (1)$$

Define $n_\theta(P) = \left\lfloor \frac{\mu c}{v_\theta - P} \right\rfloor$, where $\lfloor x \rfloor$ is the largest integer smaller than or equal to $x$. Then, if a firm of type $\theta$ charges a price $P$, there is zero demand from informed consumers at any length greater than or equal to $n_\theta(P)$. It follows that for $n \geq n_h(P)$, the demand from uninformed consumers is also zero. Let $\bar{N} = n_h(0) = \lfloor \mu/c \rfloor$; then, at or above queue length $\bar{N}$, the overall demand is zero at any $P \geq 0$.

The combination of the arrival rate $\Lambda$, the consumer joining process and the service rate $\mu$ induce a birth-death process over queue lengths between 0 and $\bar{N} - 1$. Let $s = (s(0), \ldots, s(\bar{N} - 1)) \in [0, 1]^\bar{N}$ be a specific vector of joining probabilities, and $\pi(n, s)$ be the probability of queue length $n \in \{0, \ldots, \bar{N} - 1\}$ under the stationary distribution of this birth-death process. As is standard from the PASTA property (Wolff, 1982), it follows that

$$\pi(0, s) = \frac{1}{1 + \sum_{n=1}^{\bar{N}} \left( \frac{\Lambda}{\mu} \right)^n \prod_{j=0}^{n-1} s(j) , \quad (2)$$

$$\pi(n, s) = \pi(0, s) \left( \frac{\Lambda}{\mu} \right)^n \prod_{j=0}^{n-1} s(j) \text{ for } n = 1, \ldots, \bar{N} . \quad (3)$$

In the steady state, the throughput per unit of time for the firm is $\mu(1 - \pi(0, s))$.

To define the payoff for a firm, its throughput must be defined for all values of $P$, including prices that are not chosen in equilibrium. The best response of informed consumers at any $P$ depends only on the type of the firm and the queue length. The best response of uninformed consumers, however, depends on their beliefs over the type of the firm. To ensure consistency of beliefs across different queue lengths, we update the beliefs of uninformed consumers as follows. Let $\psi(P)$ denote the posterior probability an uninformed consumer places on the firm having the high type if she only observes the price of the good, and has not yet seen the queue length. We require that for each $n$, the posterior beliefs be updated from a common $\psi$. Effectively, we consider a two-stage Bayesian updating process for each $(P, n)$, with the prior belief $p_0$ being updated to $\psi(P)$ once $P$ has been observed, and then further updated to a belief $\gamma(P, n)$ at a given $n$.

### 2.2 Equilibrium

We consider a perfect Bayesian equilibrium of the game. The equilibrium must describe an optimal price for each type of firm and consumers’ joining strategies as well as uninformed
consumers’ beliefs for every pair of price and queue length, \((P, n)\). Formally, we define a pure strategy equilibrium as follows:

**Definition 1** A pure strategy equilibrium of the game consists of prices for each type of firm, \(P_h, P_\ell\), a joining strategy profile for consumers for each type of firm \(\theta\), \(\alpha_\theta : [0,1] \rightarrow [0,1]^N\), an intermediate belief function for uninformed consumers \(\psi : [0,1] \rightarrow [0,1]\), and a posterior belief for uninformed consumers \(\gamma : [0,1] \rightarrow [0,1]^\bar{N}\) such that:

(i) For each \(P \in [0,1]\), \(\alpha_\theta(P) = (s_\theta(P,0,\gamma(P,0)), \ldots, s_\theta(P,\bar{N} - 1,\gamma(P,\bar{N} - 1)))\), where \(s_\theta(P,n,b)\) is as specified in equation (1).

(ii) For each \(P \in [0,1]\), \(\gamma(P,n) = (\gamma(P,0), \ldots, \gamma(P,\bar{N} - 1))\), with
\[
\gamma(P,n) = \frac{\psi(P)^{\bar{N}} \pi(n,\alpha_\ell(P))}{\psi(P)^{\bar{N}} \pi(n,\alpha_\ell(P)) + (1 - \psi(P))^{\bar{N}} \pi(n,\alpha_h(P))}
\]
whenever the denominator is positive, and where \(\pi(n,s)\) is as specified in equations (2) and (3).

(iii) For each \(P = P_h, P_\ell\), the value \(\psi(P)\) is updated from \(p_0\) using Bayes’ rule.

(iv) For each \(\theta = h, \ell\), \(P_\theta \in \arg \max_{P \in [0,1]} P_\mu(1 - \pi(0,\alpha_\theta(P)))\).

In equilibrium, on arrival each consumer observes a price \(P \in \{P_h, P_\ell\}\) and the number of consumers already in the queue, \(n\). Condition (i) requires that all consumers (informed and uninformed) play an optimal strategy; that is, they join if and only if they have a weakly positive payoff. Condition (ii) requires that for any price a firm may choose, the beliefs of uninformed consumers at different queue lengths are consistent with each other and depend on the actual joining rate given that price. Given the intermediate belief \(\psi\), Bayes’ rule is used to update the posterior belief at each queue length. Consistency across different queue lengths is important: For example, given some \(P\), it cannot be that uninformed consumers at queue length \(n\) believe the firm has high quality with probability 1 while those at length \(n + 1\) believe the firm has low quality with probability 1. Condition (iii) requires that the intermediate belief \(\psi\) is consistent with the actual strategies of firms. Finally, condition (iv) requires that each type of firm maximizes its profit, given the consumer joining strategy.

### 2.3 Off-equilibrium restrictions on \(\psi\): Intuitive Criterion

We adopt the widely-applied Intuitive Criterion of Cho and Kreps (1987) to restrict \(\psi\) off the equilibrium path. Suppose the observed price is \(P\) and uninformed consumers have an intermediate belief \(\psi\). For a fixed \(\psi\), we can solve for joining strategies for each type of firm, \(\alpha_h, \alpha_\ell\) and posterior beliefs at each queue length \(\gamma(P,n)\) that mutually satisfy conditions.
(i) and (ii) of Definition 1. Recall that the production cost of the good is zero. We can therefore write the profit of firm \( \theta \) as \( \Pi(P, \psi) = \mu(1 - \pi(0, \alpha_\theta))P \), where both \( \alpha_\theta \) and the overall profit depend on \( \psi \).

Fix a pair of equilibrium prices \( (P^*_h, P^*_\ell) \) and an associated pair of profits, \( (\Pi^*_h, \Pi^*_\ell) \). For any off-equilibrium price \( P \notin \{P^*_h, P^*_\ell\} \), define the maximal profit a firm of type \( \theta \) can make by deviating to \( P \) as \( \max_{\psi \in [0, 1]} \Pi_\theta(P, \psi) \). It is immediate that the maximal deviation profit at price \( P \) for either type of firm will be obtained when \( \psi = 1 \); i.e., when all uninformed consumers believe with probability one that the firm has high quality — at any given \( P \), this belief induces the greatest demand from uninformed consumers, and also weakly increases the maximal queue length at which they join.

The Intuitive Criterion requires that, if only one type \( \theta \) gains by deviating from \( P^*_\theta \) to \( P \), uninformed consumers believe that price \( P \) was posted by type \( \theta \). That is,

(i) If \( \Pi_\ell(P, 1) > \Pi^*_\ell \) and \( \Pi_h(P, 1) < \Pi^*_h \), then it must be that \( \psi(P) = 0 \).

(ii) If \( \Pi_\ell(P, 1) < \Pi^*_\ell \) and \( \Pi_h(P, 1) > \Pi^*_h \), then it must be that \( \psi(P) = 1 \).

(iii) In all other cases, no restriction is placed on \( \psi(P) \).

Observe that there cannot be an off-equilibrium price \( P \) such that \( \Pi_h(P, 1) > \Pi^*_h \) but \( \Pi_\ell(P, 1) < \Pi^*_\ell \). If such a price existed, firm \( h \) should charge \( P \) rather than \( P^*_h \), breaking the conjectured equilibrium. Therefore, in all our equilibria, we set \( \psi(P) = 0 \) for \( P \neq P^*_h, P^*_\ell \).

That is, at any price not chosen in equilibrium, uninformed consumers believe the firm has low quality. In each kind of equilibrium we identify, we confirm that this belief satisfies the Intuitive Criterion.

### 2.4 Separating and Pooling Equilibria

In our model, both the price and the queue length may communicate information about quality to an uninformed consumer. However, the firm directly chooses only the price of the good, with the price in turn affecting the queue length distribution. As is conventional, we use the terms “separating equilibrium” and “pooling equilibrium” to refer to different types of firm choosing the same or different strategies (i.e., prices), respectively.

In a separating equilibrium, each type of firm chooses a different price, so the price fully reveals the type. Let \( (P^*_h, P^*_\ell) \) be the equilibrium prices. Condition (iv) in the Definition 1 then requires that \( \psi(P^*_h) = 1 \) and \( \psi(P^*_\ell) = 0 \). As mentioned earlier, at any \( P \notin \{P^*_h, P^*_\ell\} \) we assign the off-equilibrium belief \( \psi(P) = 0 \). In a pooling equilibrium with price \( P^* \), condition (iii) of Definition 1 implies that \( \psi(P^*) = p_0 \). For all \( P \neq P^* \), we set \( \psi(P) = 0 \).
2.5 Assumptions on Service Rate

We assume that $\mu$ is smaller than $\Lambda$ (to allow for the possibility of congestion) but greater than $\Lambda/2$. In Lemma 1 below, we show that under full information and when the waiting costs vanish, the throughput for firm $\theta$ is $\frac{\Lambda v_\theta}{2}$. Recall that $v_\theta = 1$, so part (i) of Assumption 1 below ensures that at the full-information price for either firm, the throughput is limited by consumer demand rather than capacity. In part (ii) of the assumption, we require that $v_\ell > \frac{c}{\mu}$, which ensures that there exists a price at which the low-quality firm has a strictly positive profit, regardless of the beliefs of uninformed consumers.

**Assumption 1**  
(i) $\frac{\mu}{\Lambda} \in (1/2, 1)$  
(ii) $v_\ell > \frac{c}{\mu}$.

3 Limiting Case: Vanishing Waiting Cost

In this section, we derive analytic results for the limiting case in which the waiting cost $c$ goes to zero. We find that, in the limit, there are many parameter values at which both a separating and pooling equilibrium exist, but firm $h$ earns a higher profit in the pooling equilibrium. In the pooling equilibrium, both types of firm charge the same low price but there is near-separation via the queue length, with long queues signaling high quality.

3.1 Separating Equilibrium

Suppose $c \to 0^+$. In the limit, the demand from informed consumers at any queue length $n$ is independent of $n$. As we are considering a separating equilibrium, uninformed consumers have degenerate beliefs in equilibrium. As mentioned earlier, we assign the off-equilibrium belief that the firm has low quality. Suppose the true type of the firm is $\theta$ and uninformed consumers believe the type is $\theta'$ (possibly equal to $\theta$). Then, at any price $P$, the overall joining rate is $\Lambda \left[ q \left( 1 - \frac{P}{v_\theta} \right)^+ + (1 - q) \left( 1 - \frac{P}{v_{\theta'}} \right)^+ \right]$. Observe that in the limit as $c \to 0^+$, the consumer demand is independent of queue length.

Consider the throughput. If the service rate $\mu$ exceeds this quantity, the throughput must equal the demand, as consumers are serviced more quickly than they arrive. On the other hand, if $\mu$ is less than the demand, the queue becomes increasingly longer. In this case, at any point of time some consumer is being serviced, and the throughput must equal the service rate. Overall, therefore, the throughput must be the smaller of the service rate $\mu$ and the joining rate. Recall that the profit of firm $\theta$ is denoted $\Pi_\theta(P, \psi)$, where $\psi$ represents uninformed consumers’ posterior belief that the firm has high quality. In a
separating equilibrium, for any \( P \) the belief \( \psi \) is either equal to 0 or 1. Let \( 1_{\{\theta' = h\}} \) be an indicator function set to 1 if \( \theta' = h \) and zero otherwise. Then, in the limit as \( c \to 0^+ \),

\[
\Pi_\theta(P, 1_{\{\theta' = h\}}) = P \min\left\{ \mu, \Lambda\left[ q \left( 1 - \frac{P}{v_\theta}\right)^+ + (1 - q) \left( 1 - \frac{P}{v_\theta}\right)^+ \right] \right\}.
\] (4)

Before analyzing the separating equilibrium, we briefly present the optimal price and resultant profit for each firm if consumers have complete information about its type; that is, when \( q = 1 \). We use the superscript \( FI \) to denote the full-information case. For notational convenience, in describing the limit as \( c \to 0^+ \), for any quantity \( x \), we denote \( x = \lim_{c \to 0^+} x \).

All proofs are in Appendix 6.1.

**Lemma 1** Suppose \( q = 1 \) and \( \mu \geq \frac{\Lambda}{2} \). Then, in the limit as \( c \to 0^+ \), the optimal price for firm \( \theta \) is \( P_{FI}^{\theta} = \frac{v_\theta}{2} \) and its profit is \( \Pi_{FI}^{\theta} = \Lambda \frac{v_\theta}{4} \).

Next, we demonstrate the profit firm \( \theta \) can anticipate if it deviates from its separating equilibrium price \( P^*_\theta \). To ensure that firm \( \ell \) has no incentive to deviate to the price \( P^*_h \), we need to determine its profit if uninformed consumers believe it has high quality; that is, \( \Pi_\ell(P^*_h, 1) \). For completeness, we compute \( \Pi_\ell(P, 1) \) for all \( P \). Similarly, to ensure that firm \( h \) does not wish to deviate to some other price \( P \), we need to determine its maximum profit if uninformed consumers believe it has low quality, \( \Pi_{h}^{\max} = \max_{P \in [0, 1]} \Pi_{h}(P, 0) \).

**Lemma 2** As \( c \to 0^+ \),

(i) For any \( P \in [0, 1] \) the limiting profit of firm \( \ell \) if uninformed consumers believe it has high quality is given by:

\[
\Pi_{\ell}(P, 1) = \begin{cases} 
P \min\left\{ \Lambda(1 - P(q/v_\ell + 1 - q)), \mu \right\} & \text{if } 0 \leq P < v_\ell \\
\Pi_{\ell}(P, 1) - \frac{\Lambda v_\ell}{2} & \text{if } v_\ell \leq P \leq 1.
\end{cases}
\]

(ii) In the limit, the maximal profit of firm \( h \) if uninformed consumers believe it has low quality is \( \Pi_{h}^{\max} = \Lambda \max\{ \frac{\Lambda}{4}, \frac{1}{4q + (1 - q)/v_\ell} \} \).

In a separating equilibrium, uninformed consumers know the type of each firm. Therefore, it must be that \( P_{\ell}^* = \max_{P \in [0, 1]} \Pi_{\ell}(P, 0) \). By Lemma 1, it follows that in the limit as \( c \to 0^+ \), \( P_{\ell}^* = \frac{v_\ell}{2} \) and its equilibrium profit is \( \Pi_{\ell}^* = \Lambda \frac{v_\ell}{4} \). The conditions for a pair of prices \( (P_h^*, P_{\ell}^*) \) to comprise a separating equilibrium can now be stated as follows: (i) \( P_h^* \geq \Pi_{h}^{\max} \) and (ii) \( P_{\ell}^* \geq \Pi_{\ell}(P_h^*, 1) \).

The intuition of separation is that by charging a sufficiently high price, the high-quality firm can make it too costly for the low-quality firm to imitate. If the low-quality firm charges
$P^*_h$, it loses some informed consumers (who know the quality is low) while potentially gaining a higher profit from uninformed consumers (who now think the quality is high). The ability of firm $\ell$ to successfully mimic the price of firm $h$ is therefore greater when the proportion of informed consumers is low.

Potentially, firm $h$ may also be able to separate out by charging a sufficiently low price below $v_\ell /2$. However, at such a price, capacity will become an important constraint. As the throughput cannot exceed $\mu$, the firm prefers to separate out at a high price. Congestion therefore rules out a deviation by firm $h$ to low prices, playing a similar role in our model as a production cost plays in other models of signaling by price (see Riley, 2001).

We first show that if the proportion of uninformed consumers is sufficiently high, the high-quality firm can obtain separation simply by charging its own full-information price, $1/2$ (recall that $v_h = 1$). Separation is therefore costless. In describing the separating equilibria in the next two propositions, we focus on equilibrium prices and profits. As mentioned in Section 2.4, intermediate beliefs $\psi$ and posterior beliefs $\gamma$ are trivial in a separating equilibrium, so the optimal consumer strategy for each $P$ also follows immediately.

**Proposition 1** Suppose $q \geq \max\{v_\ell, 1 - v_\ell\}$. In the limit as $c \rightarrow 0^+$, there is a separating equilibrium in which each type of firm charges its full information prices, so that $P^*_\theta = \frac{v_\theta}{2}$ for each $\theta = h, \ell$, with associated profit $\Pi^*_\theta = \frac{\Lambda v_\theta}{4}$.

If $q < \max\{v_\ell, 1 - v_\ell\}$, separation is costly for firm $h$: it must charge a price strictly higher than its full-information price to distinguish itself from the firm $\ell$. We refer to the equilibria in Proposition 2 as costly separating equilibria. Define $w = \frac{v_\ell}{q + (1 - q)v_\ell} \in [v_\ell, 1]$.

**Proposition 2** As $c \rightarrow 0^+$,

(i) If (a) $q \in (0, \frac{1}{2}]$ and $v_\ell \in (q(1 - q), 1 - \frac{1}{4(1 - q)})$ or (b) $q \in (\frac{1}{2}, 1)$ and $v_\ell \in (q(1 - q), 1 - q)$, there is a separating equilibrium with limiting prices $P^*_h = \frac{w}{2} \left(1 + \sqrt{1 - \frac{v_\ell}{1 - q}}\right)$ and $P^*_\ell = \frac{v_\ell}{2}$. The profits of the firms in the limit are $\Pi^*_\ell = \frac{\Lambda v_\ell}{4}$ and $\Pi^*_h = \frac{\Lambda w}{4(1 - q)}$.

(ii) If $v_\ell \in \left(\max(q, 1 - \frac{1}{4(1 - q)}), 1\right)$, there is a separating equilibrium with limiting prices $P^*_h = \frac{w}{2} \left(1 + \sqrt{1 - \frac{v_\ell}{w}}\right)$ and $P^*_\ell = \frac{v_\ell}{2}$. The profits of the firms in the limit are $\Pi^*_\ell = \frac{\Lambda v_\ell}{4}$ and $\Pi^*_h = \frac{\Lambda w}{2} \left(1 - \frac{w}{2} \left(1 + \sqrt{1 - \frac{v_\ell}{w}}\right)\right) \left(1 + \sqrt{1 - \frac{v_\ell}{w}}\right)$.

If $v_\ell$ is too low (below $q(1 - q)$), there is no separating equilibrium. This is standard in the literature (see, for example, Bagwell and Riordan, 1991). When $v_\ell$ is low, to prevent firm $\ell$ from copying its price, firm $h$ must charge a price further away from its own full-information price $P^{FI}_h$. The departure from $P^{FI}_h$ represents a cost to separation, and this cost decreases as $v_\ell$ or $q$ become large.
Case (i) of the proposition considers intermediate values of $v_\ell$, at which the gap in firm qualities is relatively high. If firm $\ell$ adopts the same price as firm $h$, no informed consumers join its queue. Therefore, $P_h^*$ decreases in $q$ and $v_\ell$. In case (ii), $v_\ell$ is high and the gap in qualities is relatively low. Now, if firm $\ell$ deviates to $P_h^*$, some informed consumers are also willing to join. Here, for high values of $q$ (above 0.25), $P_h^*$ first increases and then decreases in $v_\ell$. However, when $q$ is small (below 0.25), again $P_h^*$ decreases in $q$.

![Figure 1: Separating Equilibria as $c \to 0^+$](image)

Figure 1 indicates the values of $q$ and $v_\ell$ for which separating equilibria exist in the limit as $c \to 0^+$. The curved line demarcating the region “No Separating Equilibrium” is $v_\ell = q(1-q)$; to the left of this line, there is no separating equilibrium. The inverted triangle represents $q = \max\{v_\ell, 1-v_\ell\}$. Above this frontier, there is a separating equilibrium with $P_h^* = \frac{1}{2}$, the optimal full-information price of firm $h$. The dashed line represents the frontier along which firm $h$ is indifferent between charging the price $\frac{1}{2} \left(1 + \sqrt{1 - \frac{v_\ell}{1-q}}\right)$ (to the left of the dashed line) and the price $\frac{w}{2} \left(1 + \sqrt{1 - \frac{w}{w}}\right)$ (to the right of the dashed line).

If the price chosen by firm $h$ strictly exceeds its optimal full-information price, it has a shorter queue than firm $\ell$. The increase in price, therefore, comes at the cost of reducing demand for its product. We show in the proof of Remark 2.1 that under full information firms $h$ and $\ell$ have identical queue distributions. Consider a separating equilibrium in which
the high-quality firm increases its price beyond \( \frac{1}{2} \). Then, the joining rate for the high-quality firm, and hence its expected queue length, is strictly smaller.

**Remark 2.1** In a separating equilibrium with \( P^*_h > \frac{1}{2} \), the expected queue length of the high-quality firm is shorter than that of the low-quality firm.

This is contrary to common intuition, and offers some insight into why pooling equilibria may yield a higher profit: as we will show, in a pooling equilibrium the high-quality firm has a strictly greater expected queue length than the low-quality firm.

### 3.2 Pooling Equilibrium

In a pooling equilibrium, \( P^*_\ell = P^*_h = (\text{say}) \ P^* \), so that \( \psi(P^*) = p_0 \). That is, the equilibrium prices convey no additional information about the quality of the firm. However, informed consumers know the quality of the firm, and join the queue for the high- and low-quality firms at different rates. As a result, the queue length communicates some information about quality to uninformed consumers.

In this section, we focus on pooling equilibria at prices less than \( v_\ell - c/\mu \). At such prices, informed consumers join the queue for firm \( \ell \) at \( n = 0 \). Fix \( \psi = p_0 \). We first show that for any \( P < v_\ell - c/\mu \), there exist beliefs \( \gamma \) and consumer joining strategies \( \alpha_h, \alpha_\ell \) that are consistent with each other in the sense of satisfying parts (i) and (ii) of Definition 1.

**Lemma 3** Suppose both firms charge some \( P \in (0, v_\ell - c/\mu) \). Then, there exist consumer joining strategies \( \alpha_h, \alpha_\ell \) and uninformed consumer beliefs \( \gamma \) such that \( \alpha_h, \alpha_\ell \) satisfy Definition 1 (i) given \( \gamma \), and \( \gamma \) satisfies Definition 1 (ii) given \( \alpha_h, \alpha_\ell \) and \( \psi(P) = p_0 \).

Given \( P \), choose a pair of consumer joining strategies \( \alpha_h, \alpha_\ell \) that satisfy Lemma 3. These strategies give rise to stationary probabilities over queue length as defined by equations (2) and (3). Consider the stationary probabilities at each queue length as \( c \to 0^+ \). Recall that \( n_h = \lfloor (1 - P) \frac{c}{\mu} \rfloor \). Define \( N = (1 - P) \frac{c}{\mu} \); then \( n_h = \lfloor N \rfloor \). We scale each queue length \( n \in \{0, \ldots, n_h\} \) as follows. Let \( \alpha = \frac{n+1}{N} \) denote the scaled queue length (i.e., a queue length relative to \( n_h \)). Note that \( N \to \infty \) as \( c \to 0^+ \). Further, the scaled empty queue (where \( n = 0 \)) tends to \( \alpha = 0 \) and the scaled Naor threshold for firm \( h \) (i.e., \( n_h \)) tends to 1. For any \( c > 0 \), all queue lengths in \( \{0, \ldots, n_h - 1\} \) are scaled into the range \( (0, 1) \).

In what follows, for notational convenience, we suppress the dependence of different variables on \( P \). At any price \( P < v_\ell \), let \( \alpha_\ell = \frac{v_\ell - P}{1 - P} \) denote the limit of the relative queue length corresponding to \( n_\ell \). Let \( f : [0, 1] \to [v_\ell, 1] \) be a function that denotes the posterior
expectation of uninformed consumers at each relative queue length. The corresponding belief can then be denoted as \( b(n) = \frac{f((n+1)/N) - v_\ell}{1 - v_\ell} \) for \( n \in \{0, \ldots, n_h - 1\} \).

Let \( \tilde{s}_\theta(a) = s_\theta(P, n, b(n)) \) be the overall consumer joining rate for firm \( \theta \) at relative queue length \( \alpha = \frac{n+1}{N} \). Then,

\[
\tilde{s}_h(a) = 1 - (P + (1 - P)a) \left( q + \frac{1 - q}{f(a)} \right) \quad \text{if } a \leq \alpha_\ell
\]

\[
\tilde{s}_\ell(a) = \begin{cases} 
1 - (P + (1 - P)a) \left( \frac{q}{v_\ell} + \frac{1 - q}{f(a)} \right) & \text{if } a \leq \alpha_\ell \\
(1 - q) \left( 1 - (P + (1 - P)a) \frac{1}{f(a)} \right) & \text{if } a > \alpha_\ell.
\end{cases}
\]

Observe that \( \frac{\Lambda}{\mu} \tilde{s}_\theta(a) \) can be interpreted as the effective arrival rate (scaled by capacity) of consumers at firm \( \theta \).

For each \( \theta = h, \ell \) define \( G_\theta(\alpha) = \exp \left( \int_0^\alpha \ln((\Lambda/\mu) \tilde{s}_\theta(a)) \, da \right) \) and define the maximizer of \( G_\theta(a) \) over \([0, 1]\) as \( \alpha^*_\theta \). In the limit as \( c \to 0^+ \), the stationary probabilities for each type of firm can be expressed in terms of \( G_\theta \).

**Lemma 4** Consider a sequence \( c \to 0^+ \). Suppose \( P < v_\ell \) and for all \( c \) and all \( m < n_h \), the uninformed consumers’ posterior expectation is \( f\left(\frac{m+1}{N}\right) \), where \( f(a) > P + (1 - P)a \) for all \( a < 1 \). Then, the stationary probabilities for each firm \( \theta \) satisfy the following condition:

\[
\lim_{c \to 0^+} \left( \pi_\theta(\lfloor \alpha n_h \rfloor) \right)^{\frac{1}{N}} = \frac{G_\theta(\alpha)}{G_\theta(\alpha^*_\theta)}.
\]

Consider any sequence of numbers \( x_1, x_2, \ldots, x_M \) such that \( x_m \in [0, 1] \) for each \( m \) and \( \lim_{M \to \infty} x_M^{1/M} \) exists. Then, if \( \lim_{M \to \infty} x_M^{1/M} = 1 \), it must be that \( \lim_{M \to \infty} x_M > 0 \), whereas if \( \lim_{M \to \infty} x_M^{1/M} < 1 \), then \( \lim_{M \to \infty} x_M = 0 \). Therefore, it follows from Lemma 4 that, in the limit as \( c \to 0^+ \), the probability at some queue length \( n_\alpha \) is strictly positive only if \( \alpha = \alpha^*_h \); i.e., if the scaled queue length \( \alpha \) is a maximizer of \( G_\theta(a) \).

We use this fact to characterize a pooling equilibrium with congestion as \( c \to 0^+ \). We first conjecture that there exists a \( \beta \) such that the uninformed consumers believe the firm has low quality if \( \alpha < \beta \) and high quality if \( \alpha > \beta \). That is, the posterior expectation of uninformed consumers is \( f_\beta(\alpha) = v_\ell \) if \( \alpha < \beta \) and \( f_\beta(\alpha) = 1 \) if \( \alpha > \beta \). Suppose uninformed consumers join optimally given these beliefs (i.e., as in Definition 1 (i)). The joining strategy in turn implies a posterior expectation of quality via the resulting queue length distributions (Definition 1 (ii)). Under the conditions in Proposition 3, in the limit as \( c \to 0^+ \) this posterior expectation is also a step function, equaling \( v_\ell \) below some threshold and 1 above the threshold. We show that the mapping from conjectured to posterior expectations has a fixed point \( \beta^* \) that satisfies the condition \( \frac{G_h(\beta^*_h)}{G_h(\alpha^*_h)} = \frac{G_h(\beta^*_h)}{G_h(\alpha^*_h)} \), where \( \alpha^*_h = 0 \) and \( \alpha^*_h > \beta^* \). Then, scaled queue lengths below and above \( \beta^* \) effectively separate the two firms even though they pool on price.
In Proposition 3, we focus on prices below $v_\ell$; this choice is discussed later. The assumption that $\frac{\mu}{\Lambda} \in (1 - P(q + (1 - q)/v_\ell), 1 - P)$ requires the service rate to be in an intermediate range, and ensures that at price $P$, if all uninformed consumers believe the quality is high (low), the arrival rate in the limit for a high-quality firm is higher (lower) than the service rate. This requires the fraction of uninformed consumers to be sufficiently high. Conditions (b-i) and (b-ii) are a joint technical restriction on $\Lambda, \mu$ and $v_\ell$ that ensure $\beta^* < \alpha_\ell$. For brevity, we do not discuss the case $\beta^* > \alpha_\ell$.

**Proposition 3** Suppose both firms charge the same price $P < v_\ell$ such that (a) $\frac{\mu}{\Lambda} \in (1 - P(q + (1 - q)/v_\ell), 1 - P)$, and either (b-i) $v_\ell \geq 1 - \mu/\Lambda$ or (b-ii) $(1 - v_\ell)\frac{1 - v_\ell}{1 - P/v_\ell} < (\mu/\Lambda) \exp\left(1 - \frac{\mu}{\Lambda(1 - P)}\right)$. Then, there exists a $\beta^* \in (0, 1)$ and a $\bar{c} > 0$ such that for every $c \in (0, \bar{c})$, there is a best response strategy for uninformed consumers with corresponding beliefs $\gamma$ such that $\gamma(P, n) < \epsilon_v$ if $n < (\beta^* - \epsilon_c)n_h$ and $\gamma(P, n) > 1 - \epsilon_v$ if $n > (\beta^* - \epsilon_c)n_h$, where $\epsilon_c$ and $\epsilon_v$ each go to zero as $c \to 0^+$.

We illustrate this proposition with a numerical example.

**Example 1**

Let $\Lambda = 1, \mu = 0.6, v_h = 1, v_\ell = \frac{1}{2}, p_0 = \frac{1}{2}$ and $q = 0.1$. We select a price $P = \frac{3}{16}$, less than $v_\ell/2$. It is immediate to see that condition (b-i) of Proposition 3 is satisfied: $\Lambda(1 - v_\ell) = 0.5 < \mu$. Condition (b-ii) is also satisfied. With these parameters, $\alpha_\ell = 0.385$. Solving for $\beta^*$ using equation (12) in the proof of Proposition 3, we find $\beta^* = 0.07125$.

We consider two values of waiting cost, $c = 0.001$ and $c = 0.0001$. In Figure 2, we exhibit the stationary distributions over (scaled) queue length for each type of firm and the posterior belief of uninformed consumers at different (scaled) queue lengths. The scaled queue length $\beta^*$ is marked by a dashed vertical line in each sub-figure. It translates to an actual queue length $\lfloor \beta^* N \rfloor = 34$ if $c = 0.001$ and $347$ if $c = 0.0001$.

As seen from the figure, at $c = 0.001$ (the left panels in the figure), the stationary distributions for both firms place substantial mass at (scaled) queue lengths below $\beta^*$. As a result, the posterior belief for uninformed consumers at these queue lengths remains strictly between 0 and 1, and climbs perceptibly with the queue length. The posterior belief reaches close to 1 at a scaled queue length just below $\alpha_\ell$. When the waiting cost falls to $c = 0.0001$ or lower, the (scaled) queue length distributions are visibly separated by $\beta^*$. The distribution for firm $\ell$ has mass close to 1 at the left of $\beta^*$, whereas the distribution for firm $h$ has mass close to 1 at the right of $\beta^*$. At scaled queue lengths immediately below and above $\beta^*$, the posterior belief for uninformed consumers increases rapidly from 0 to 1.
Although not shown in the figure, at \( c = 0.00001 \) the posterior belief at the scaled queue length almost perfectly approximates a step function going from 0 to 1 at \( \beta^* \).  

Proposition 3 delivers a striking result. The two types of firm choose the same price for the good, and the price is their only choice variable. Nevertheless, with small but strictly positive waiting costs, there is near-perfect revelation of the firm quality through the queue length. As informed consumers do not join the low-quality queue at large \( n \), the queue length distribution for firm \( \ell \) has the bulk of its mass at low queue lengths. When \( n \ll \beta^* N \), uninformed consumers correctly infer the firm has low quality. Conversely, the queue length distribution for firm \( h \) has the bulk of its mass at long queue lengths, and when \( n \gg \beta^* N \) uninformed consumers infer the firm has high quality. The threshold \( \lfloor \beta^* N \rfloor \) endogenously emerges as a queue length separating the low- and high-quality firms.

The top panel shows the stationary distributions over the scaled queue length and the bottom panel the posterior belief of uninformed consumers. The waiting costs are \( c = 0.001 \) (left panel) and \( c = 0.0001 \) (right panel).

Figure 2: Queue length distributions and posterior beliefs of uninformed consumers
When the queue length directly reveals the quality of the firm, uninformed consumers act as they would under full information. In any equilibrium, separating or pooling, firm $\ell$ must earn a profit at least equal to $\Pi(\theta, 0)$ (recall that $\Pi(\theta, 0)$ is the optimal price of firm $\theta$ under full information). Therefore, the only pooling price that supports separation on queue length in equilibrium is the low price $\Pi(\theta, 0)$. There may be pooling equilibria at prices above $\Pi(\theta, 0)$ in which separation based on queue length does not occur. The profit of firm $h$ is greater if pooling occurs at a price strictly greater than $\Pi(\theta, 0)$ than at the price $\Pi(\theta, 0)$. Since our interest is in comparing the profit of firm $h$ in a pooling equilibrium and a separating equilibrium, we adopt the conservative approach of considering the lowest price that supports pooling equilibria, $\Pi(\theta, 0)$.

As $c \to 0^+$, the price $\Pi(\theta, 0)$ approaches $v(\theta)/2$ (Lemma 1). When $P = v(\theta)/2$, the conditions in Proposition 3 identify restrictions on $\nu(\theta)$, $\Lambda$, $\mu$. We numerically evaluate both conditions (i) and (ii) of the proposition over values of $\nu(\theta)$ between 0 and 1 and $\mu(\Lambda)$ between 1 and 2. The results are shown in the Appendix in Section 6.2, in Figure 5. The figure shows that the conditions hold over a wide region for this range of parameters.

Using Proposition 3 and letting $c \to 0^+$, in the limit we obtain precise conditions such that the price $v(\theta)/2$ can sustain a pooling equilibrium. In the proof of Proposition 4 below, we show that the off-equilibrium belief $\psi(P) = 0$ for $P = v(\theta)/2$ satisfies the Intuitive Criterion.

Define a threshold $\bar{\nu}(\nu(\theta))$ as follows:

$$\bar{\nu}(\nu(\theta)) = \begin{cases} 1 - \frac{\Lambda}{2\nu(\theta)} & \text{if } \nu(\theta) < 1 - \frac{\mu}{\Lambda} \\ \frac{v(\theta)(2\nu(\theta) - 1)}{2\nu(\theta)(1-v(\theta)/\Lambda + \sqrt{1-2\nu(\theta)/\Lambda})} & \text{if } \nu(\theta) \in \left[1 - \frac{\mu}{\Lambda}, 2(1 - \frac{\mu}{\Lambda})\right]. \end{cases} \tag{7}$$

For notational convenience we consider $\Lambda$ and $\mu$ to be fixed and write $\bar{\nu}(\nu(\theta))$.

**Proposition 4** Suppose (i) $\nu(\theta) < \frac{2\nu(\theta) - 1}{1-v(\theta)}$, (ii) $\nu(\theta) < 2(1 - \mu/\Lambda)$, (iii) either $\nu(\theta) \geq 1 - \mu/\Lambda$ or Condition (b-ii) in Proposition 3 is satisfied at $P = v(\theta)/2$, and (iv) $\nu(\theta) \leq \min\{2\nu(\theta)/\Lambda, \bar{\nu}(\nu(\theta))\}$. Then, in the limit as $c \to 0^+$, there is a pooling equilibrium in which both types of firm set price $P = v(\theta)/2$ and there is separation based on queue length. The equilibrium profits are $\Pi(\theta) = \frac{\mu v(\theta)}{2}$ and $\Pi(h) = \frac{\Lambda v(\theta)}{4}$ and the off-equilibrium belief is $\psi(P) = 0$ if $P = P^*$. Conditions (i) through (iii) together ensure that we can apply Proposition 3. Condition (iv) further requires the proportion of informed consumers to be sufficiently low. The two components of the upper bound on $\nu(\theta)$ come from different requirements. If $\nu(\theta)$ exceeds $\frac{2\nu(\theta)}{\Lambda}$, firm $\theta$ may deviate to a high price and earn $\Pi(\theta)$ (see Lemma 2 (ii)). The profit it thereby makes off informed consumers compensates for the lower joining rate of uninformed
consumers. If \( q \) exceeds \( \bar{q} \), the off-equilibrium belief \( \psi(P) = 0 \) for all \( P \neq P^* \) does not survive the Intuitive Criterion. That is, we can then find a price \( P \) such that the maximal deviation profit of the high-quality firm (across any belief for the uninformed consumers) exceeds the maximal deviation profit for the low-quality firm. In that case, firm \( h \) should deviate to \( P \), breaking the conjectured equilibrium.

Although there is pooling on price, the high-quality firm achieves near separation via the queue length. For small queue lengths, uninformed consumers believe the firm is almost surely firm \( \ell \), and for large queue lengths uninformed consumers believe the firm is almost surely firm \( h \). An immediate observation is that:

**Remark 4.1** In a pooling equilibrium, the expected queue of the high-quality firm is longer than the expected queue of the low-quality firm.

Observe the contrast with Remark 2.1: In a separating equilibrium, on average firm \( h \) has a weakly shorter queue than firm \( \ell \).

In Example 2, we exhibit the parameter region over which Proposition 4 holds.

**Example 2**

We set \( \Lambda = 1, \mu = 0.6 \) and \( v_h = 1 \). We vary \( q \) and \( v_\ell \), and exhibit our results in Figure 3. Observe that with these values, \( 2(1 - \mu/\Lambda) = 0.8 \). Numerically, we find that condition (b-ii) of Proposition 3 is satisfied for all \( v_\ell \) between 0.106 and 0.999; that is for practically all \( v_\ell \) to the right of the vertical dashed line in the figure.

The pooling equilibria of Proposition 4 exist for values of \( v_\ell \) and \( q \) in the region of the figure bounded by the solid lines. For example, the following \((v_\ell, q)\) pairs satisfy condition (iv): \((v_\ell, q) = (0.2, 0.1)\) and \((0.77, 0.25)\). Observe that for pooling equilibria to exist, \( q \) must be neither too high nor too low. In particular, firm \( h \) experiences congestion when all consumers believe it has high quality, and slack capacity when all consumers believe it has low quality.


\[\]

### 3.3 Comparison of Pooling and Separating Profits for Firm \( h \)

We now compare both the regions in which separating and pooling equilibria exist and the profit of the high-quality firm in the region of overlap. Our main result is that whenever a pooling equilibrium and a costly separating equilibria both exist, firm \( h \) earns a higher profit in the pooling equilibrium. Because the low-quality firm’s profits are equal in both equilibria, the pooling equilibrium Pareto dominates the costly separating equilibrium.
Proposition 5 Suppose the conditions of Proposition 4 are satisfied, so that in the limit as \( c \to 0^+ \), there is a pooling equilibrium at price \( P^*_p = \frac{v}{2} \). Then:

(a) If \( v_l > q(1 - q) \), there is also a costly separating equilibrium in which firm \( h \) charges a price \( P^*_h > \frac{1}{2} \). Further, firm \( h \) earns a weakly higher profit in the pooling equilibrium than in the costly separating equilibrium.

(b) If \( v_l < q(1 - q) \), there is no separating equilibrium, either costly or costless.

Recall from Proposition 4 that for the pooling equilibrium to exist, we need the proportion of informed consumers to be sufficiently low (in particular, \( q \leq \bar{q} \)). We show in Proposition 5 that under those circumstances, if a separating equilibrium exists, the cost of separation is high for firm \( h \). The comparison of profits in separating and pooling equilibria depends on the relative costs of separation and pooling. Separation is costly when firm \( h \) must charge a higher price than \( P^*_{FI} \), because of the resulting fall in demand. The cost to firm \( h \) of pooling comes both from the low pooling price \( (v_l/2) \) and the fixed capacity which implies congestion at low queue lengths. When \( q \) is sufficiently low, the cost of separation is higher, so firm \( h \) prefers the pooling equilibrium.

In Figure 3, separating equilibria exist in almost the entire region for which pooling equilibria exist (i.e., in the region defined by the solid lines). There is a small section in

![Figure 3: Pooling equilibria in the limit as \( c \to 0^+ \)](image-url)
the north-east corner of the pooling region where \( v_\ell < q(1 - q) \) and there is no separating equilibrium. Through the rest of the region, separating and pooling equilibria both exist, and firm \( h \) makes a higher profit when it pools compared to when it separates out.

4 Positive Waiting Cost

We have shown that, in the limit as \( c \to 0^+ \), over a range of parameter values pooling and costly separating equilibria both exist and the high-quality firm earns a higher profit in the pooling equilibrium than in the separating equilibrium. We show numerically that these properties continue to hold even when the waiting costs are relatively high.

For our study, we fix the following parameters: \( v_h = 1, \Lambda = 1, \mu = 0.6, c = 0.05, p_0 = 0.25 \). With these parameters, the maximum queue length is \( \left\lfloor \frac{v_h \mu}{c} \right\rfloor = 12 \). Observe that the queues are therefore significantly constrained compared to the ones in Section 3, where they can become arbitrarily long as the waiting cost drops to zero. We have explored different values of \( \mu, c \) and \( p_0 \), and have obtained similar results.

We allow \( v_\ell \) and \( q \) to vary, as in Figures 1 and 3. We consider values of \( v_\ell \) and \( q \) each between 0.01 and 0.99, in steps of 0.01. That is, we consider 9,801 \((v_\ell, q)\) pairs. For each pair \((v_\ell, q)\), we determine the profit of each firm \( \theta = h, \ell \) when uninformed consumers believe (i) the firm has high quality and (ii) the firm has low quality. We compute these profits at prices between 0 and 1, in steps of \( 10^{-3} \). We also compute the profit of each firm if both firms pool at that price. We then determine whether a separating equilibrium exists, and if it does, whether the high-quality firm can separate out at its full-information price. We also determine whether a pooling equilibrium exists at that price.

Recall that in Section 3.2 we consider pooling equilibria that, in the limit as \( c \to 0^+ \), exhibit separation based on queue length. Among prices below \( v_\ell \), only the price \( v_\ell / 2 \) can support such an equilibrium. In our numerical study, the maximal queue length is 12, so there cannot be near-separation on queue length in a pooling equilibrium. Instead, even at low queue lengths, the probability the firm has high quality remains well above zero. As a result, prices strictly greater than \( \frac{v_\ell}{2} \) can also be sustained in pooling equilibrium. For any given set of parameters, whenever a pooling equilibrium exists at any price, we choose the pooling price that maximizes the profit of the high-quality firm.

Our results on when separating and pooling equilibria exist are shown in Figure 4. In Figure 4 (a), the region above the solid black line is where \( q > \max\{v_\ell, 1 - v_\ell\} \), and the region to the right of the dashed curve is where \( v_\ell > q(1 - q) \). As can be seen from the figure, separating equilibria do not exist for a wide range of parameters when \( v_\ell \) and \( q \) are
low. For example, for \( q \leq 0.7 \), there is no separating equilibrium when \( v_\ell \leq 0.4 \). Due to the positive waiting cost, at low values of \( v_\ell \) the full-information profit of firm \( \ell \) is low, leading to an incentive to deviate. Compared to the limiting case when \( c \to 0^+ \), the region in which firm \( h \) can separate out at its full-information price is also truncated.

Figure 4 (b) indicates values of \( v_\ell \) and \( q \) at which pooling equilibria at some price exist. For a given value of \( v_\ell \), these equilibria exist only if \( q \) is sufficiently low. If \( q \) is high, firm \( h \) prefers to deviate even though uninformed consumers think it is a low-quality firm. It is interesting to note that pooling equilibria exist even when \( v_\ell \) is close to zero, for \( q \leq 0.17 \). Under full information, given the waiting cost, no consumer would purchase good \( \ell \). However, in the pooling equilibrium firm \( \ell \) is able to earn a positive profit.

The optimal pooling price (i.e., the pooling price at which firm \( h \) earns the highest profit) is in general close to (but strictly below) half of the expected value of the good, \( \frac{p_0 + (1-p_0)v_\ell}{2} \). In Figure 4 (b), we also indicate values of \( q \) and \( v_\ell \) at which the price \( P_{FI}^\ell \) sustains a pooling equilibrium. These values are indicated by the black diamonds in the figure. It is immediate from the figure that the range of values for which pooling equilibria at the price \( P_{FI}^\ell \) exist is much smaller, typically low values of \( v_\ell \) (below 0.4) and low values of \( q \). Note that with a positive waiting cost, \( P_{FI}^\ell < \frac{v_\ell}{2} \).

Whenever a pooling and separating equilibrium both exist, we find that firm \( h \) earns a higher profit from the pooling equilibrium. The optimal pooling price is lower than the price at which firm \( h \) can separate out. Nevertheless, firm \( h \) prefers to pool rather than raising its price. Our analytic results in the limit as \( c \to 0^+ \) are therefore robust to larger
waiting costs.

5 Conclusion

We have shown that, when the quality of a good is unknown and the proportion of informed consumers is low, queues can be a more efficient signaling instrument than prices. That is, a high-quality firm is willing to set a low price (in particular, the same price charged by a low-quality firm) and use congestion as a device to communicate its quality to uninformed consumers. To credibly signal its quality by charging a high price, it needs to set a price so high that its profit is substantially reduced. As a result, congestion rather than price is the preferred signal of quality.

Our model can also be interpreted as follows. Consider a firm that sells a regular good with quality \( \ell \), priced at the full-information price \( P_{\ell}^{FI} \). Suppose the firm now develops a superior good with quality \( h \), but not all consumers understand that the quality has improved. How can the firm communicate the quality improvement to a large consumer base? It can increase price to \( P_{h}^{FI} \) (or higher), but uninformed consumers may balk, reducing demand. Alternatively, it can continue to price at \( P_{\ell}^{FI} \) and allow the increased demand from informed consumers to lead to congestion. The latter strategy is beneficial when the fraction of informed consumers is low and the capacity is at an intermediate level. The strategy leads to long queues at a high-quality firm, even when the firm has the flexibility to raise its price.

Our model integrates elements of price signaling in economics and equilibrium queueing in operations research. We hope that our research stimulates further interest in signaling games in operations environments.

6 Appendix

6.1 Proofs

Proof of Lemma 1

Consider a firm of type \( \theta \) and let \( c \to 0^+ \). In the limit, at any \( n \), informed agents who arrive join with probability \( 1 - \frac{P}{v_\theta} \). That is, \( \alpha_\theta(P, \gamma) \) approaches the vector \( \left( 1 - \frac{P}{v_\theta}, 1 - \frac{P}{v_\theta}, \ldots \right) \). Since \( q = 1 \), informed agents arrive at the rate \( \Lambda \). Therefore, the throughput approaches \( R(\alpha_\theta, P) = \min \left\{ \mu, \Lambda \left( 1 - \frac{P}{v_\theta} \right) \right\} \). If the throughput is unconstrained by \( \mu \), the profit of firm \( \theta \) is \( \Pi_\theta(P) = \Lambda \left( 1 - \frac{P}{v_\theta} \right) P \). It is immediate that the optimal price is \( P_{\theta}^{FI} = \frac{v_\theta}{2} \).
with associated profit $\Lambda_{\mu}$. The throughput in the limit at that price is $R(\alpha_\theta, \frac{v_\theta}{2}) = \frac{\alpha}{2}$. Therefore, for $\mu \geq \frac{\alpha}{2}$, the lemma stands.

**Proof of Lemma 2**

(i) Suppose $\theta = \ell$ but uninformed consumers believe the firm has high quality. Then, in the limit as $c \rightarrow 0^+$, its throughput is $\min\left\{\mu, \Lambda \left[ 1 - P \left( \frac{q}{v_\ell} + \frac{1-q}{v_\ell} \right) \right] \right\}$, and if $P > v_\ell$ it reduces to $\min\{\mu, \Lambda(1-q)(1-P/v_\ell)\}$. Substitute $v_\ell = 1$ to obtain part (i) of the Lemma.

(ii) Suppose $\theta = h$ but uninformed consumers believe the firm has low quality. Then, in the limit as $c \rightarrow 0^+$, its throughput is $\min\left\{\mu, \Lambda \left[ q \left( 1 - \frac{P}{v_h} \right) + (1-q) \left( 1 - \frac{P}{v_\ell} \right) \right] \right\}$ if $P \leq v_\ell$ and $\min\left\{\mu, \Lambda q \left( 1 - \frac{P}{v_h} \right) \right\}$ if $P \in [v_\ell, v_h]$. Suppose the throughput is unconstrained by $\mu$. Then, when $P \leq v_\ell$, the profit of firm $h$ may be written as $\Pi_h(P, 0) = \Lambda \left[ 1 - P \left( \frac{q}{v_\ell} + \frac{1-q}{v_\ell} \right) \right] P$, so that the optimal price is $P^* = \frac{1}{2} \frac{1}{\frac{q}{v_\ell} + \frac{1-q}{v_\ell}}$. At this price, the throughput is $\frac{\alpha}{2}$, so it follows under Assumption 1 that $\mu$ exceeds the throughput. Further, the profit is $\frac{\Lambda_\mu}{2} P = \frac{\Lambda}{4} \frac{1}{\frac{q}{v_\ell} + \frac{1-q}{v_\ell}}$. The expression in the statement of the Lemma is obtained on recalling that $v_\ell = 1$.

When $P \in [v_\ell, v_h]$, the profit of firm $h$ in the limit as $c \rightarrow 0^+$ is $\Lambda q \left( 1 - \frac{P}{v_h} \right) P$, so the optimal price is $P^* = \frac{v_h}{2}$. Now, the throughput is $\frac{\alpha \Lambda}{2}$, which is again less than $\mu$, and the firm’s profit is $\frac{\Lambda q}{4}$. The latter strictly exceeds $\frac{\Lambda}{2} P = \frac{\Lambda}{4} \frac{1}{\frac{q}{v_\ell} + \frac{1-q}{v_\ell}}$ whenever $v_\ell < \frac{q}{1+q}$.

**Proof of Proposition 1**

From Lemma 1, in the limit firm $h$ earns its highest possible profit in a separating equilibrium with $P^*_h = \frac{1}{2}$, so it has no incentive to deviate. Consider firm $\ell$. In the limit, if it charges a price $P_\ell = \frac{v_\ell}{2}$, its profit is $\frac{\Lambda v_\ell}{4}$. Suppose it deviates to $P = \frac{1}{2}$. As this price exceeds its full-information price $\frac{v_\theta}{2}$, the throughput is strictly less than $\frac{1}{2}$, and so less than $\mu$. Therefore, if $v_\ell > \frac{1}{2}$, by Lemma 2 (i), its profit is equal to $\frac{1}{2} \left( 1 - \frac{1}{2} [q/v_\ell + 1 - q] \right)$. The condition $\frac{\Lambda v_\ell}{4} \geq \frac{1}{2} \left( 1 - \frac{1}{2} [q/v_\ell + 1 - q] \right)$ reduces to $v_\ell \geq 1 - \frac{1}{v_\ell} + q$, which is satisfied if and only if $q \geq v_\ell$. Observe that as $v_\ell > \frac{1}{2}$, it follows that $q > 1 - v_\ell$. Similarly, if $v_\ell \leq \frac{1}{2}$, the profit of firm $\ell$ if it deviates to $P = \frac{1}{2}$ is equal to $\frac{1}{2} \left( 1 - q \right)$. The condition $\frac{\Lambda v_\ell}{4} \geq \frac{1}{2} \left( 1 - q \right)$ directly implies $q \geq 1 - v_\ell$. As $v_\ell \leq \frac{1}{2}$, it follows that $q \geq v_\ell$.

Therefore, if $q \geq \max\{v_\ell, 1 - v_\ell\}$, in the limit firm $\ell$ earns a higher profit by charging $P = \frac{v_\ell}{2}$ than by deviating to $P = \frac{1}{2}$. Since the off-equilibrium belief of uninformed consumers for any $P \notin \left( \frac{v_\ell}{2}, \frac{1}{2} \right)$ is that the firm has low quality, it must be that it earns a higher profit from $P = \frac{v_\ell}{2}$ than from any other price $P \neq \frac{1}{2}$.
As neither firm $\ell$ nor firm $h$ can gain by deviating, the prices $P^*_h = \frac{1}{2}$ and $P^*_\ell = \frac{v_\ell}{2}$ induce a separating equilibrium. The profit for each type of firm follows immediately.

**Proof of Proposition 2**

In any separating equilibrium, in the limit as $c \to 0^+$, it must be that $\Pi^*_h = \frac{\Lambda_v}{4}$, because firm $\ell$ can always guarantee itself that profit even when uninformed consumers recognize its type. Let $P^*_h$ be the price of the high firm. Then, $P^*_h = \arg \max_{P \in [0,1]} \Pi^*_h(P,1)$ subject to $\Pi^*_h(P,1) \leq \frac{\Lambda_v}{4}$, where the constraint ensures that firm $\ell$ has no incentive to deviate. At equilibrium, the constraint must bind, else firm $h$ can earn a higher profit by changing its price by a small amount.

Let $D_\theta(P,1)$ be the demand for firm $\theta$ in the limit as $c \to 0^+$ (observe that in the limit the demand does not depend on queue length $n$) and uninformed agents believe the firm has high quality. Then, $D_\theta(P,1) = \min \{\Lambda(1 - P), \mu\}$ and from Lemma 2,

$$D_\ell(P,1) = \begin{cases} \min \{\Lambda(1 - P(q/v_\ell + 1 - q)), \mu\} & \text{if } 0 \leq P < v_\ell \\ \min \{\Lambda(1 - q)(1 - P), \mu\} & \text{if } v_\ell \leq P \leq 1 \end{cases}$$

Now, consider any two prices $P', P''$ such that each satisfy $\Pi^*_h(P,1) = \frac{\Lambda_v}{4}$ and $P'' > P'$. It is straightforward to verify that $\frac{D_\ell(P,1)}{D_\theta(P,1)}$ is weakly decreasing in $P$, so it must be that $\Pi^*_h(P'',1) \geq \Pi^*_h(P',1)$. Therefore, $P''$ must be an optimal price for firm $h$.

We consider two cases, representing (i) and (ii) in the statement of the Proposition. For notational simplicity, denote $P = P^*_h$. First, suppose that $P \geq v_\ell$. As $v_\ell > \frac{\nu}{4(1-q)}$, it follows that if firm $\ell$ deviates to $P$, its profit is $\Pi(P,1) = \Lambda(1 - q)(1 - P)P$. In a separating equilibrium, this must equal $\Lambda \frac{\nu}{4}$. Consider the quadratic equation $P^2 - P + \frac{v_\ell}{4(1-q)} = 0$. Its larger root is $P = \frac{1}{2} \left( 1 + \sqrt{1 - \frac{v_\ell}{1-q}} \right)$. The condition $P \geq v_\ell$ then leads to $\sqrt{1 - \frac{v_\ell}{1-q}} \geq 2v_\ell - 1$, from which the conditions in (i) in the statement of the proposition follow. Finally, in a separating equilibrium the limiting profit of the high-quality firm is $\Pi^*_h = \Lambda P(1 - P)$; substituting in $P = \frac{1}{2} \left( 1 + \sqrt{1 - \frac{v_\ell}{1-q}} \right)$ yields the expression for profit in (i).

Second, suppose $P < v_\ell$. Suppose $\Lambda(1 - P/w) \geq \mu$. Then, from Lemma 2, the profit of firm $\ell$ if it deviates to $P$ is $\Lambda[P - P^2/w]$, where $w = \frac{v_\ell}{q + (1-q)v_\ell}$. In a separating equilibrium, this must equal $\Lambda \frac{\nu}{4}$, so that $(1/w)P^2 - P + \frac{\nu}{4} = 0$. Now, the larger of the two roots of this quadratic equation is $P = \frac{\nu}{2}(1 + \sqrt{1 - v_\ell/w})$. As $\mu > \frac{\nu}{4}$, it is straightforward to verify that, at this price, $\Lambda(1 - P/w) > \mu$.

We now need to find the conditions under which $P < v_\ell$, or $\frac{\nu}{2}(1 + \sqrt{1 - v_\ell/w}) < v_\ell$. The inequality reduces to $\frac{\nu}{2}(1 + \sqrt{1 - v_\ell/w}) < 2$. Let $y = \frac{v_\ell}{w} = q + (1-q)v_\ell$; we then have $\sqrt{1 - y} < 2y - 1$ or $y > \frac{3}{4}$, from which the conditions in Case (i) in the statement...
of the proposition follow. Finally, in a separating equilibrium the limiting profit of the high-quality firm is \( \Pi_h = \Lambda P(1 - P) \); substituting in \( P = \frac{w}{2}(1 + \sqrt{1 - v\ell/w}) \) yields the expression for profit in the second case.

**Proof of Remark 2.1**

Let \( q = 1 \) and consider \( c \to 0^+ \). In the limit, \( \frac{P_{FI}^h}{\gamma} = \frac{w}{2} \) and the joining rate for any queue length \( n \) is \( \frac{A}{2} \), independent of \( \theta \). As \( \mu \) is the same for both firms, the queue length distributions must be the same. Now, consider a separating equilibrium in which \( P_{h}^* > \frac{1}{2} \). The joining rate for firm \( h \) is now \( \Lambda(1 - P_{h}^*) < \Lambda/2 \), so the expected queue length must be smaller than at \( P_{FI}^h \), and hence smaller than the expected queue length for firm \( \ell \).

**Proof of Lemma 3**

Consider any \( P \in (0, v\ell - c/\mu) \). For any \( n \), let \( \phi(n) \) denote the likelihood that the firm has high quality. For notational simplicity, we suppress the dependence of \( \phi \) and \( \gamma \) on \( P \).

Let \( x \in [0, 1] \) and define \( \phi(0) = x \). For any \( n \in \{0, \ldots, \bar{N}\} \) such that \( \phi(n) \) is finite, let \( \psi(P) = p_0 \) and define \( \gamma(n) = \frac{p_0\phi(n)}{p_0\phi(n) + (1 - p_0)} \). Given \( \gamma(n) \), for each \( \theta \) determine the joining rate \( s_\theta \) from equation (1). Then, define \( \phi(n + 1) = \phi(n) \frac{s_\theta(P,n,\gamma(n))}{s_\theta(P,n,\gamma(n))} \) whenever \( s_\ell > 0 \). If \( s_\ell = 0 \), directly set \( \gamma(n) = 1 \). For each \( \theta = h, \ell \), the strategy \( \alpha_\theta = (s_\theta(P,0,\gamma(0)), \ldots, s_\theta(P,\bar{N} - 1, \gamma(\bar{N} - 1))) \) as defined induces a stationary distribution over the set \( \{0, \ldots, \bar{N}\} \), with stationary probabilities given by equations (2) and (3). Define \( \Phi(x, P) = \frac{\pi(0,\alpha_h)}{\pi(0,\alpha_\ell)} \). Then, for any \( x \in [0, 1] \) and any \( P > 0 \), the function \( \Phi(x, P) \) is well-defined.

Observe that when \( x = 1 \), \( \gamma(0) = p_0 \). However, informed consumers join firm \( h \) at a greater rate. Therefore, \( \gamma(1) < p_0 \), so that \( \alpha_h(1, P, \gamma(1)) > \alpha_\ell(1, P, \gamma(n)) \). It then follows that \( \gamma(2) \leq \gamma(1) \), so that if \( \alpha_h(2, P, \gamma(2)) > 0 \), it must be that \( \alpha_h(2, P, \gamma(2)) > \alpha_\ell(2, P, \gamma(2)) \).

In general, it will follow that either \( \alpha_h(n, P, \gamma(n)) = \alpha_\ell(n, P, \gamma(n)) = 0 \) or \( \alpha_h(n, P, \gamma(n)) > \alpha_\ell(n, P, \gamma(n)) \). As \( \alpha_h(\cdot) \) weakly exceeds \( \alpha_\ell(\cdot) \) for all \( n \) and strictly exceeds it for some \( n \), it follows that \( \Phi(1, P) > 1 \).

Next, consider \( x \to \infty \). Then \( \gamma(0) \to 0 \), in the limit, uninformed consumers join at the same rate as they would if the firm were known to have low quality. Hence, \( \Phi(x, P) \) remains finite as \( x \to \infty \). Now, by continuity of \( \Phi \) in \( x \), it follows that there exists at least one \( x^*(P) \) such that \( \Phi(x^*(P), P) = x^*(P) \).

Choose any such \( x^*(P) \), set \( \phi(0) = x^*(P) \), and construct \( \gamma \) and \( \alpha_h, \alpha_\ell \) as in the first part of the proof. By construction, \( \alpha_h, \alpha_\ell \) satisfy Definition 1 (i) given \( \gamma \), and \( \gamma \) satisfies Definition 1 (ii) given \( \alpha_h, \alpha_\ell \) and \( \psi(P) = p_0 \).

**Proof of Lemma 4**
Throughout this proof, denote $N = (1 - P)^n$, so that $n_h = \lfloor N \rfloor$ and $n_\ell = \lfloor \frac{v_\ell - P}{1 - v_\ell} N \rfloor$.

Note that $c/\mu = \frac{1 - P}{N}$. Let $b(n)$ denote the posterior expectation of an uninformed consumer at price $P$ and queue length $n$, so that $b(n) = v_\ell + \gamma(P, n)(1 - v_\ell)$ or $\gamma(P, n) = \frac{b(n) - v_\ell}{1 - v_\ell}$.

Then, the probability a consumer joins at queue length $n \in \{0, 1, \ldots, n_h - 1\}$ is

$$s_h(P, n, \frac{b(n) - v_\ell}{1 - v_\ell}) = q \left( 1 - \left( P + (n + 1)c/\mu \right) \right) + (1 - q) \left( 1 - P + (n + 1)c/\mu \right) b(n)$$

$$= 1 - \left( P + (1 - P)^{n + 1} \right) \left( q + \frac{1 - q}{b(n)} \right)$$

$$s_\ell(P, n, \frac{b(n) - v_\ell}{1 - v_\ell}) = \begin{cases} 
1 - P + (1 - P)^{n + 1} \left( \frac{q}{v_\ell} + \frac{1 - q}{b(n)} \right) & \text{for } n \in \{0, 1, \ldots, n_\ell - 1\} \\
(1 - q) \left( 1 - (P + (1 - P)^{n + 1}) \right) \frac{1}{b(n)} & \text{for } n \in \{n_\ell, \ldots, n_h - 1\}.
\end{cases}$$

By assumption, for each $n$, the posterior expectation is $b(n) = f \left( \frac{n + 1}{N} \right)$. Suppressing $P$, let $\bar{s}_\theta \left( \frac{n + 1}{N} \right) = s_\theta \left( P, n, \frac{b(n) - v_\ell}{1 - v_\ell} \right)$, to obtain equations (5) and (6) in the text.

Denote $\rho = \Lambda/\mu$. Observe that the stationary probability for firm $\theta$ at queue length $k$ is $\pi_\theta(k) = \frac{\rho^k}{1 + \sum_{n=1}^{\lfloor N \rfloor} \rho^n \bar{s}_\theta \left( \frac{n + 1}{N} \right)}$. Therefore, we can write

$$\left( \frac{\rho}{N} \right)^{\frac{k}{N}} = \frac{\rho^k \exp \left( \frac{1}{N} \sum_{m=0}^{k-1} \ln \left( \bar{s}_\theta \left( \frac{m + 1}{N} \right) \right) \right)}{1 + \sum_{n=1}^{\lfloor N \rfloor} \rho^n \exp \left( \frac{1}{N} \sum_{m=0}^{n-1} \ln \left( \bar{s}_\theta \left( \frac{m + 1}{N} \right) \right) \right)} N^{\frac{1}{N}}. \quad (8)$$

Consider the numerator of the RHS of equation (8) for some generic queue length $k = \alpha N$, where $\alpha \in \{ \frac{1}{N}, \frac{2}{N}, \ldots, \frac{n_h}{N} \}$. As $c \to 0^+$, it follows that $N \to \infty$. Therefore,

$$\lim_{c \to 0^+} \rho^k \exp \left( \frac{1}{N} \sum_{m=0}^{k-1} \ln \left( \bar{s}_\theta \left( \frac{m + 1}{N} \right) \right) \right) = \lim_{N \to \infty} \rho^\frac{\alpha N}{N} \exp \left( \frac{1}{N} \sum_{m=0}^{\lfloor \alpha N \rfloor - 1} \ln \left( \bar{s}_\theta \left( \frac{m + 1}{N} \right) \right) \right)$$

$$= \rho^\alpha \exp \left( \lim_{N \to \infty} \left( \frac{1}{N} \sum_{m=0}^{\lfloor \alpha N \rfloor - 1} \ln \left( \bar{s}_\theta \left( \frac{m + 1}{N} \right) \right) \right) \right) = \rho^\alpha \exp \left( \int_0^\alpha \ln (\bar{s}_\theta (a)) da \right) = G_\theta (\alpha),$$

where the last step uses the fact that $\exp \left( \int_a^b \ln (\rho \bar{s}_\theta (x)) dx \right) = \rho^{b-a} \exp \left( \int_a^b \ln (\bar{s}_\theta (x)) dx \right)$.

Applying similar algebra to the denominator, we obtain

$$\lim_{c \to 0^+} \left( \pi_\theta (\lfloor \alpha N \rfloor) \right)^{\frac{1}{N}} = \frac{G_\theta (\alpha)}{\lim_{N \to \infty} \left( \sum_{n=0}^{N} \left( G_\theta \left( \frac{n}{N} \right) \right)^N \right)^{\frac{1}{N}}}. \quad (9)$$

Now, observe that

$$\left( \sum_{n=0}^{N} \left( G_\theta \left( \frac{n}{N} \right) \right)^N \right)^{\frac{1}{N}} = N^{\frac{1}{N}} \left( \frac{1}{N} \sum_{n=0}^{N} \left( G_\theta \left( \frac{n}{N} \right) \right)^N \right)^{\frac{1}{N}}. \quad (9)$$
cause \( \lim_{N \to \infty} N^{-\frac{1}{N}} = 1 \), it follows that

\[
\lim_{N \to \infty} \left( \sum_{n=0}^{N} \left( G_\theta(n/N) \right)^N \right)^{\frac{1}{N}} = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=0}^{N} \left( G_\theta(n/N) \right)^N \right)^{\frac{1}{N}}
\]

\[
= \lim_{N \to \infty} \lim_{M \to \infty} \left( \frac{1}{M} \sum_{n=0}^{M} \left( G_\theta(n/M) \right)^N \right)^{\frac{1}{N}} = \lim_{N \to \infty} \left( \int_{0}^{1} [G(a)]^N da \right).
\]

Since \( G(\cdot) \) is continuous and defined over the closed interval \([0, 1]\), it follows that

\[
\lim_{N \to \infty} \left( \int_{0}^{1} [G(a)]^N da \right)^{\frac{1}{N}} = \max_{a \in [0, 1]} G(a) \text{ (this is the definition of the infinity norm)}.
\]

Therefore, \( \lim_{c \to 0^+} (\pi_\theta(\lfloor \alpha N \rfloor))^{\frac{1}{N}} = \frac{G_\theta(\alpha)}{\max_{a \in [0, 1]} G(a)} \). \( \square \)

**Proof of Proposition 3**

Conjecture that there exists a \( \beta^* \) such that, in the limit as \( c \to 0^+ \), the posterior belief of uninformed consumers is as follows: If \( n < \beta^* N \), then \( \gamma(P, n) = 1 \) and if \( n > \beta^* N \), then \( \gamma(P, n) = 0 \). We will show that the consumer best responses and the resulting stationary probabilities over queue lengths lead to a posterior belief that satisfies the conjecture.

Recall that \( N = (1 - P) \frac{\mu}{\ell} \). As in the proof of Lemma 4, let \( \tilde{s}_h \left( \frac{n+1}{N} \right) = s_\theta \left( P, n, \frac{\psi(n) - v_\ell}{1 - v_\ell} \right) \) denote the mass of consumers that join the queue of firm \( \theta \) at length \( n \). Given the conjectured beliefs, equations (5) and (6) imply that for each \( a = \frac{n+1}{N} \), in the limit as \( c \to 0^+ \),

\[
\tilde{s}_h(a) = \begin{cases} 
1 - (P + (1 - P)a)(q + \frac{1-q}{v_\ell}) & \text{if } a < \beta^* \\
(1-a)(1-P) & \text{if } a > \beta^*
\end{cases}
\]

\[
\tilde{s}_\ell(a) = \begin{cases} 
1 - (P + (1 - P)a)\frac{1}{v_\ell} & \text{if } a < \beta^* \\
(1-q)(1-a)(1-P) & \text{if } a > \beta^*
\end{cases}
\]

As before, denote \( \rho = \frac{\Lambda}{\mu} \). Observe that the derivative of \( G_\theta(\alpha) \) is \( G_\theta'(\alpha) = G_\theta(\alpha) \ln(\rho \tilde{s}_\theta(\alpha)) \).

Now, \( \mu/\Lambda > 1 - P(q + (1 - q)/v_\ell) \) implies that \( \rho(1 - P(q + (1 - q)/v_\ell)) < 1 \). Therefore, for \( \alpha < \beta^* \), we have \( \rho \tilde{s}_h(\alpha) < 1 \). Further, \( \tilde{s}_\ell(\alpha) < \tilde{s}_h(\alpha) \) for all \( \alpha \), so \( \rho \tilde{s}_\ell(\alpha) < 1 \) for \( \alpha \) in this range. Therefore, \( G_\ell(0) < 0 \) for each \( \theta \); that is, for each \( \theta \), the function \( G_\theta(\cdot) \) has a local maximum at \( 0 \).

Now, define \( \beta^* \) to satisfy the following equation:

\[
\frac{G_\ell(\beta)}{\max_{a \in [0, 1]} G_\ell(a)} = \frac{G_h(\beta)}{\max_{a \in [0, 1]} G_h(a)}.
\]

We proceed through a series of steps.

**Step 1: Characterizing \( \beta^* \).**
Suppose that (i) \( G_\ell \) has a global maximum at 0 and \( G_h \) has a global maximum at \( \hat{\alpha}_h > \beta^* \) and (ii) \( \beta^* \in (0, \alpha_\ell) \). These conjectures will be verified later. Note that \( G_\ell(0) = 1 \). Then, \( \beta^* \) satisfies the equation \( G_\ell(\beta) = \frac{G_h(\beta)}{G_h(\hat{\alpha}_h)} \).

Now, \( \hat{\alpha}_h \) satisfies \( G'_h(a) = 0 \). As mentioned before, \( G'_h(a) = G_h(a) \ln(\rho \tilde{s}_h(a)) \), so \( \hat{\alpha}_h \) satisfies \( \tilde{s}\hat{\alpha}_h = \frac{1}{\rho} \). For \( \alpha > \beta, \tilde{s}_h(\alpha) = (1 - \alpha)(1 - P) \), so that we have \( \hat{\alpha}_h = 1 - \frac{1}{\rho(1 - P)} \). It is immediate to verify that the second order condition \( G''(\hat{\alpha}_h) < 0 \) holds.

Observe that for \( \alpha > \beta \) we can write \( G_h(\alpha) = G_h(\beta) \rho^{\alpha - \beta} \exp \left( \int_\beta^\alpha \ln[1 - (1 - P)(1 - a)] da \right) \). Now, \( \int_\beta^\alpha \ln[1 - (1 - P)(1 - a)] da = -(1 - \alpha) \ln[(1 - P)(1 - \alpha)] - \alpha + (1 - \beta) \ln[(1 - P)(1 - \beta)] + \beta \), so \( \exp \left( \int_\beta^\alpha \ln[1 - (1 - P)(1 - a)] da \right) = [(1 - P)(1 - \alpha)]^{1 - \beta} [(1 - P)(1 - \beta)]^{1 - \beta} \exp(-\alpha + \beta) \). Further, we have shown \( (1 - P)(1 - \hat{\alpha}_h) = 1/\rho \), so

\[
G_h(\hat{\alpha}_h) = \frac{G_h(\beta)}{\exp(\hat{\alpha}_h - \beta)} \left[ \rho(1 - P)(1 - \beta) \right]^{1 - \beta}.
\]  
(10)

Now, consider \( G_\ell(\beta) \) when \( \beta < \alpha_\ell \). In this region, \( \tilde{s}_\ell(a) = 1 - (P + (1 - P)a)/v_\ell = 1 - P/v_\ell - (1 - P)a/v_\ell \). This expression is of the form \( y - za \), where \( y = 1 - P/v_\ell \) and \( z = (1 - P)/v_\ell \). Using \( \int (y - za) da = (a - \frac{y^2}{2}) \ln(y - za) - a \) and simplifying, we obtain

\[
G_\ell(\beta) = \rho^\beta \exp(-\beta)(1 - \beta/\alpha_\ell)^{\beta - \alpha_\ell}(1 - P/v_\ell)^{\beta}.
\]  
(11)

Putting together equations (10) and (11), we obtain that \( \beta^* \) is the solution to:

\[
\rho [(1 - P)(1 - \beta)]^{1 - \beta} (1 - P/v_\ell)^{\beta} (1 - \beta/\alpha_\ell)^{1 - \beta/\alpha_\ell} = \exp \left( 1 - \frac{1}{\rho(1 - P)} \right).
\]  
(12)

**Step 2:** Verifying that \( \beta^* \in (0, \alpha_\ell) \).

Consider equation (12) when \( \beta = 0 \). The left-hand side is \( \rho(1 - P) > 1 \) by assumption in the statement of the proposition. The right-hand side is \( \exp \left( 1 - \frac{1}{\rho(1 - P)} \right) \). Now, for any \( x > 1 \), we have \( x > \exp(1 - 1/x) \), so at \( \beta = 0 \), the LHS of equation (12) exceeds the RHS.

Now, there are two cases to consider:

**Case 1:** Suppose \( \hat{\alpha}_h \leq \alpha_\ell \). Consider \( \beta = \hat{\alpha}_h \). Recall that \( (1 - P)(1 - \hat{\alpha}_h) = 1/\rho \). Therefore, the LHS of equation (12) is equal to \( [\rho(1 - P/v_\ell)]^{\hat{\alpha}_h} (1 - \hat{\alpha}_h/\alpha_\ell)^{1 - \hat{\alpha}_h/\alpha_\ell} \). Now, \( (1 - P/v_\ell) < \rho(1 - P + (1 - q)/v_\ell) < 1 \), where the second inequality is assumed in the statement of the proposition. Also, \( 1 - \frac{\hat{\alpha}_h}{\alpha_\ell} < 1 \). Therefore, the LHS of (12) is less than 1, and the RHS is clearly weakly greater than 1.

Therefore, the LHS of (12) is greater than the RHS at \( \beta = 0 \) and lower than the RHS at \( \beta = \hat{\alpha}_h \). By continuity, there exists a \( \beta^* \in (0, \hat{\alpha}_h) \) at which the equation is satisfied.

Now, observe that \( \hat{\alpha}_h \leq \alpha_\ell \iff 1 - \frac{1}{\rho(1 - P)} \leq \frac{v_\ell + P}{1 - P} \iff \rho \leq \frac{1}{1 - v_\ell} \), which is condition (i) in the statement of the proposition.
Consider the LHS of equation (12) when $\beta = \alpha \ell$. Using $(1 - P)(1 - \alpha \ell) = 1 - v_\ell$, we obtain the LHS $\rho(1 - v_\ell)^{1 - \eta P} (1 - P/v_\ell)^{\eta - P}$. The RHS is $\exp\left(1 - \frac{1}{\rho(1 - P)}\right)$. Therefore, the LHS is strictly lower than the RHS at $\beta = \alpha \ell$ if

$$\rho(1 - v_\ell)^{1 - \eta P} (1 - P/v_\ell)^{\eta - P} < \exp\left(1 - \frac{1}{\rho(1 - P)}\right).$$

Notice that this is condition (b-ii) in the statement of the proposition. If it is satisfied, then there exists a $\beta^* \in (0, \alpha \ell)$ that satisfies equation (12).

**Step 3:** Uniqueness of $\beta^*$.

Observe that $\frac{G_{\ell}(\alpha)}{G_{h}(\alpha)} = \frac{\exp\left(\int_0^\alpha \ln \frac{\tilde{s}_h(a)}{s_h(a)} da\right)}{\exp\left(\int_0^\alpha \ln \frac{s_h(a)}{s_h(a)} da\right)} = \exp\left(\int_0^\alpha \ln \frac{\tilde{s}_h(a)}{s_h(a)} da\right)$. Further, $q > 0$ implies that $\tilde{s}_h(a) < s_h(a)$ for all $\alpha < 1$. Therefore, $\int_0^\alpha \ln \frac{s_h(a)}{\tilde{s}_h(a)} da$ is decreasing in $\alpha$, so $\frac{G_{\ell}(\alpha)}{G_{h}(\alpha)}$ is strictly decreasing in $\alpha$. Now, $\beta^*$ is defined by the equation $\frac{G_{\ell}(\beta^*)}{G_{h}(\beta^*)} = \frac{1}{G_{h}(\alpha \ell)}$. Because $\frac{G_{\ell}(\beta^*)}{G_{h}(\beta^*)}$ is strictly monotone, this equation can have at most one solution.

**Step 4:** $G_{\ell}(a)$ has a global maximum at $a = 0$.

Recall that $\beta^*$ satisfies the equation $G_{\ell}(\beta) = G_{h}(\alpha \ell)$, so that

$$G_{\ell}(\beta^*) \exp\left(\int_{\beta^*}^{\alpha \ell} \ln(\rho \tilde{s}_h(a)) da\right) = 1.$$  \hspace{1cm} (13)

Now, suppose that $G_{\ell}$ has a local maximum at some $\hat{\alpha}_\ell \in (\alpha \ell, 1)$. To find $\hat{\alpha}_\ell$, set $G_{\ell}' = 0$, which yields $\hat{\alpha}_\ell = 1 - \frac{1}{\rho(1 - q)(1 - P)} < \hat{\alpha}_h$. Observe that $G_{\ell}(\hat{\alpha}_\ell) = G_{\ell}(\beta^*) \exp\left(\int_{\beta^*}^{\alpha \ell} \ln(\rho \tilde{s}_h(a)) da\right)$. The second term in the latter expression is strictly smaller than the corresponding term in equation (13), whereas the first terms are equal. Therefore, $G_{\ell}(\hat{\alpha}_\ell) < 1$, so that $G_{\ell}(a)$ has a global maximum at $a = 0$.

**Step 5:** $G_{h}(a)$ has a global maximum at $a = \hat{\alpha}_h$.

The maximum value of $G_{h}(a)$ in the region $[0, \beta^*]$ is 1, attained at $a = 0$. The maximum value in the region $[\beta^*, 1]$ is $G_{h}(\hat{\alpha}_h)$. Therefore, if $G_{h}(\hat{\alpha}_h) > 1$, then $G_{h}(a)$ has a global maximum at $a = \hat{\alpha}_h$. Now, we can write $G_{h}(\hat{\alpha}_h) = G_{h}(\beta^*) \exp\left(\int_{\beta^*}^{\alpha \ell} \ln(\rho s_h(a)) da\right)$. Here, the first term is strictly larger than the first term in equation (13), whereas the second terms are equal. Therefore, $G_{h}(\hat{\alpha}_h) > 1$.

**Step 6:** The best response strategy of the consumers leads to the conjectured belief for uninformed consumers.
From Lemma 4, using $G_\ell(0) = 1$, it follows that $\lim_{c \to 0+} \left( \frac{\pi_\ell([aN])]}{\pi_h([aN])]} \right)^{\frac{1}{N}} = G_\ell(\alpha) G_h(\alpha)$. We have shown that $\frac{\partial G_\ell(\alpha)}{\partial \alpha}$ is strictly decreasing in $\alpha$. Therefore, $\lim_{c \to 0+} \left( \frac{\pi_\ell([aN])]}{\pi_h([aN])]} \right)^{\frac{1}{N}}$ is greater than 1 for $\alpha < \beta^*$ and less than 1 for $\alpha > \beta^*$. It follows that $\lim_{N \to \infty} \pi_\ell([aN])] / \pi_h([aN])]$ is infinite if $\alpha < \beta^*$ and 0 if $\alpha > \beta^*$. Then, $\gamma(P,n) = 0$ for $n < \beta^* N$ and $\gamma(P,n) = 1$ for $n > \beta^* N$. That is, in the limit as $c \to 0^+$, the consumer best response strategies indeed generate the conjectured belief. By continuity of the $G_\theta$ functions, the result holds approximately when $c$ is close to zero. That is, there exist $\bar{N}$ (alternatively $\bar{c}$), $\epsilon_c$ and $\epsilon_v$ such that if $N > \bar{N}$ (alternatively, $c < \bar{c}$) then $\gamma(n,P) < \epsilon_c$ if $\alpha < \beta^* - \epsilon_c$ and $\gamma(n,P) > 1 - \epsilon_c$ if $\alpha > \beta^* + \epsilon_v$. Further, $\epsilon_c, \epsilon_v \to 0^+$ as $N \to \infty$ (or $c \to 0^+$).

**Proof of Proposition 4**

Substituting $P = v_\ell/2$ in the statement of Proposition 3, after simplification we obtain conditions (i) through (iii). Therefore, Proposition 3 applies. In the rest of the proof, we show that (a) no firm has an incentive to deviate and (b) the off-equilibrium belief satisfies the Intuitive Criterion.

(a) Let the off-equilibrium belief be $\psi(P) = 0$ for any $P \neq P^*$. Then, firm $\ell$ clearly has no incentive to deviate, since its maximal deviation profit is also $\frac{\Lambda v_\ell}{2}$. From Lemma 2, given the equilibrium belief, the maximal profit of firm $h$ if it deviates is $\frac{\Lambda}{4} \max\{q, \frac{v_\ell}{qv_\ell + 1 - q}\}$. Observe that $q < \frac{v_\ell}{qv_\ell + 1 - q} \iff q(1 - q) < v_\ell(1 - q^2) \iff q < v_\ell(1 + q) \iff q < \frac{v_\ell}{1 - v_\ell}$. There are therefore two cases to consider: $q \leq \frac{v_\ell}{1 - v_\ell}$ and $q > \frac{v_\ell}{1 - v_\ell}$.

**Case 1:** Suppose $q \leq \frac{v_\ell}{1 - v_\ell}$. Then, if firm $h$ deviates, its maximal profit is $\frac{\Lambda}{4} \frac{v_\ell}{qv_\ell + 1 - q}$. For firm $h$ to continue to price at $P^*$, it must be that $\frac{\Lambda v_\ell}{2} \geq \frac{\Lambda}{4} \frac{v_\ell}{qv_\ell + 1 - q}$, or $q(1 - v_\ell) \leq 1 - \frac{\beta}{2}$. As $q \leq 1 - \frac{\beta}{2}$ and $v_\ell < 1$, the latter inequality is satisfied.

**Case 2:** Suppose $q \in \left[\frac{v_\ell}{1 - v_\ell}, \frac{2v_\ell}{\Lambda}\right]$. Then, the maximal profit of firm $h$ if it deviates on price is $\frac{\Lambda v_\ell}{4}$. The no-deviation condition is therefore $\frac{\Lambda v_\ell}{4} \leq \frac{\Lambda v_\ell}{2}$, or $q \leq \frac{2v_\ell}{\Lambda}$. Therefore, firm $h$ has no incentive to deviate. As before, firm $\ell$ also has no incentive to deviate.

(b) Suppose that, after a deviation, uninformed consumers believe the firm has high quality. Then, if firm $h$ strictly gains by deviating, we show that firm $\ell$ also strictly gains by deviating. That is, for the set of prices $\mathcal{P}$ such that $\Pi_h(P,1) > \Pi_h^*$, we also have $\Pi_\ell(P,1) > \Pi_\ell^*$. We proceed in two steps.

**Step 1:** Consider the set of prices $\mathcal{P}$ such that $\Pi_h(P,1) > \Pi_h^*$ has the form $(P^*, \bar{P})$. Suppose firm $h$ deviates and uninformed consumers believe the firm has high quality. The maximal throughput of firm $h$ at any price is $\mu$. Therefore, at any price $P < P^*$, the maximal profit...
it can earn is $\mu P < \frac{\mu v_\ell}{2}$. Hence, there is no price $P < P^*$ in $\mathcal{P}$. Consider prices above $P^*$. The throughput of firm $h$ at such prices is $\min\{\mu, \Lambda(1 - P)\}$. Therefore, for any price $P < 1 - \frac{1}{\rho}$, the throughput is $\mu$ and for any price $P > 1 - \frac{1}{\rho}$ the throughput is $\Lambda(1 - P)$. It is immediate that $(P^*, 1 - 1/\rho) \in \mathcal{P}$.

Consider prices $P > 1 - \frac{1}{\rho}$. The profit of firm $h$ after the deviation is $\Lambda P(1 - P)$. The profit-maximizing price in the region $[1/\rho, 1]$ is $\hat{P} = \frac{1}{2}$. As $\Pi_h(1/\rho, 1) > \Pi_h^*$, and $\Pi_h(P, 1)$ is increasing in the region $[1/\rho, 1/2]$, it follows that the region $[1/\rho, 1/2] \in \mathcal{P}$. In the region $[1/2, 1]$, the condition $\Lambda P(1 - P) \geq \frac{\mu v_\ell}{2}$ implies $P \leq \frac{1}{2} \left(1 + \sqrt{1 - \frac{2v_\ell}{\rho}}\right) \equiv \bar{P}$. Observe that $2(1 - 1/\rho) < \rho/2$, so $v_\ell \leq 2(1 - 1/\rho)$ implies that $2v_\ell/\rho < 1$. Therefore, $\bar{P}$ is well-defined and we have $\mathcal{P} = (P^*, \bar{P})$.

**Step 2:** We now need to show that $\Pi_\ell(P, 1) > \Pi_\ell^*$ for all $P \in \mathcal{P}$. The throughput of firm $\ell$ is $\min\{\mu, \Lambda(1 - P(q/v_\ell + 1 - q))\}$ for prices in the range $[P^*, v_\ell]$. The two terms in the “min” expression are equal if $P = \frac{1 - 1/\rho}{q/v_\ell + 1 - q} \equiv P_1$. For $P < P_1$, the throughput is $\mu$, and for $P > P_1$ it is $\Lambda(1 - P(q/v_\ell + 1 - q))$. Observe that the condition $P_1 > P^*$ reduces to $q + (1 - q)v_\ell < 2(1 - 1/\rho)$, which is satisfied as $q + (1 - q)v_\ell < v_\ell$ and $v_\ell \leq 2(1 - 1/\rho)$. Therefore, we have that $P_1 > P^*$. Then, the profit of firm $\ell$ for prices in $[P^*, P_1]$ is $\mu P$. From $\rho < 2$, it follows that $\mu P^* > \frac{1}{2} P^*$, so $\mu P^* > \Pi_\ell^*$. Further, the profit increases in $P$ for $P \in [P^*, P_1]$, so it follows that $\Pi_\ell(P, 1) > \Pi_\ell^*$ for all $P \leq P_1$.

That leaves the region $[P_1, \bar{P}]$. First, suppose $v_\ell < \bar{P}$. Then there are two possibilities:

1. $v_\ell \leq P_1$. In this case the profit of firm $\ell$ in this region is $\Lambda(1 - q)P(1 - P)$. This function has a unique maximum over $[P_1, 1]$. Therefore, if $\Pi_\ell(\bar{P}, 1) > \Pi_\ell^*$, it follows that $\Pi_\ell(P, 1) > \Pi_\ell^*$ for all $P \in [P_1, P_\ell]$.

2. $v_\ell \in (P_1, \bar{P})$. Then, the profit of firm $\ell$ in the region $[P_1, v_\ell]$ is $\Lambda(1 - P(q/v_\ell + 1 - q))P$. Again, this function has a unique maximum over $[P_1, v_\ell]$. Therefore, if $\Pi_\ell(v_\ell, 1) > \Pi_\ell^*$, it follows that $\Pi_\ell(P, 1) > \Pi_\ell^*$ for all $P \in [P_1, v_\ell]$. In the region $[v_\ell, \bar{P}]$, again the function $\Pi_\ell(P, 1)$ has a unique maximum, so it is sufficient to further check that $\Pi_\ell(P, 1) \geq \Pi_\ell^*$. Now, $v_\ell \leq \bar{P} \iff \sqrt{1 - \frac{2v_\ell}{\rho}} \geq 2v_\ell - 1$. Squaring both sides and simplifying, $2(v_\ell - 1) \leq 1/\rho$, or $v_\ell \leq 1 - \frac{1}{2\rho}$. Suppose the latter inequality holds, so we are either in case (a) or (b) from above. Now, $\Pi_\ell(\bar{P}, 1) = \Lambda(1 - q)\bar{P}(1 - \bar{P})$. Recall that $\bar{P} = \frac{1}{2} \left(1 + \sqrt{1 - \frac{2v_\ell}{\rho}}\right)$, so $1 - \bar{P} = \frac{1}{2} \left(1 - \sqrt{1 - \frac{2v_\ell}{\rho}}\right)$. Multiplying the two, we have $\bar{P}(1 - \bar{P}) = \frac{v_\ell}{2\rho}$. Therefore, the condition $\Pi_\ell(\bar{P}, 1) \geq \Pi_\ell^*$ reduces to $\Lambda(1 - q)\frac{v_\ell}{2\rho} \geq \Lambda\frac{v_\ell}{4}$, or $q \leq 1 - 1/\rho$, or $q \leq \bar{q}$ (since $v_\ell \leq 1 - 1/\rho/2$, the first case of equation (7) applies).

Now, ignoring whether case (1) or (2) applies, observe that $\Pi_\ell(v_\ell, 1) = \Lambda v_\ell(1 - q)(1 - v_\ell)$. Therefore, $\Pi_\ell(v_\ell, 1) > \Pi_\ell^* \iff (1 - q)(1 - v_\ell) > 1/4$. The conditions in (i) imply that
1 − q > ρ/2 and 1 − vℓ ≥ 1/(2p), so (1 − q)(1 − vℓ) > 1/4. Therefore, in both cases (1) and (2), the off-equilibrium beliefs satisfy the Intuitive Criterion.

Next, suppose vℓ > P. Here, Πℓ( P, 1 = Λ(1 − P(q/vℓ + 1 − q))P = Λ[ P(1 − P) − q1/vℓP2].

From above, P(1 − P) = 1/(4vℓ). Further, P2 = 1/2(1 − vℓ + (1 − vℓ2)/(1 − 2vℓ)). Making these substitutions, the condition Πℓ(P, 1 ≥ Πℓ∗ reduces to Λ[2vℓ − 2q1/vℓ(1 − vℓ + (1 − 2vℓ))]/4 > Λ4vℓ, or q ≤ q (here, we are in the second case of equation (7). Therefore, the belief ψ(P) satisfies the Intuitive Criterion.

Proof of Proposition 5
(a) Suppose the conditions in Proposition 4 are satisfied and vℓ > q(1 − q). For vℓ < 1 − µ/2λ, it follows that q(vℓ) = 1 − λ/2µ < 1/2 ≤ max{vℓ, 1 − vℓ}. Consider vℓ ∈ [1 − µ/2λ, 2(1 − λ)]). In this region, q is strictly increasing and convex in vℓ. As 1 − µ/2λ > 1/2, we have max{vℓ, 1 − vℓ} = vℓ.

Evaluating q(vℓ) at vℓ = 2(1 − µ/λ), we find q < vℓ. Therefore, q(vℓ) < max{vℓ, 1 − vℓ}, and from Proposition 2, it follows that there is a separating equilibrium with Pℓ∗ > 1/2.

In each of the separating and pooling equilibria considered, the profit of firm ℓ is Πℓ(Pℓ∗, 1) in the separating equilibrium and Πℓ(Pℓ, 1) in the pooling equilibrium. Suppose Πℓ(Pℓ∗, 1) > Πℓ(Pℓ, 1). In the pooling equilibrium, the off-equilibrium belief ψ(P) = 0 for any P ≠ vℓ/2 satisfies the Intuitive Criterion. Therefore, it must be that Πℓ(Pℓ∗, 1) ≥ Πℓ(Pℓ, 1) = Λvℓ4, so we have a contradiction. Hence, it must be that Πℓ(Pℓ∗, 1) ≤ Πℓ(Pℓ, 1).

(b) Suppose vℓ < q(1 − q). This implies that q < max{vℓ, 1 − vℓ}. From the proofs of Propositions 1 and 2, it follows that there is no separating equilibrium. ■
6.2 Numerical evaluation of conditions in Proposition 3

We set $P = v\ell/2$ and numerically evaluate conditions (b-i) and (b-ii) in Proposition 3 for values of $v\ell$ between 0 and 1 and $\frac{\mu}{\Lambda}$ between 1 and 2. At $P = v\ell/2$, condition (b-ii) becomes $\frac{\Lambda}{(1-v\ell)} \left( \frac{1}{2} - \frac{v\ell}{2} \right) < \exp \left( 1 - \frac{1}{(\Lambda/\mu)(1-v\ell/2)} \right)$. We exhibit our results in Figure 5. The dashed downward-sloping line is condition (b-i). To the left this line, condition (i) in the statement is satisfied. In between the two solid lines, condition (b-ii) is satisfied for $P = v\ell/2$. Numerically, therefore, the conditions in Proposition 3 are not particularly restrictive.

References


