Technical Appendix to Accompany Technological Revolutions and Stock Prices

Ľuboš Pástor
University of Chicago, CEPR, and NBER

Pietro Veronesi
University of Chicago, CEPR, and NBER

October 28, 2008

Technical Appendix

Lemma A1: For later reference, we prove a more general version of Lemma A1. In particular, we cover three cases: (i) the new economy does not exist, and learning only occurs only by observing the old economy; (ii) the new economy exists, and learning occurs for $t \in [t^*, t^{**}]$; (iii) the new economy exists, adoption takes place at t^{**} and learning occurs for $t \geq t^{**}$. The learning dynamics for $t > t^{**}$ in the case of no adoption at t^{**} is identical to case (ii). For $t \geq t^*$ we then have

$$\begin{aligned} d\widehat{\psi}_t &= \widehat{\sigma}_t^2 c \frac{\phi}{\sigma} d\widetilde{Z}_{0,t} + c_N \widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}} \left(1 - c\frac{\sigma_{N,0}}{\sigma}\right) d\widetilde{Z}_{1,t} && \text{(C1)} \\ \frac{d\widehat{\sigma}_t^2}{dt} &= -\left(\widehat{\sigma}_t^2\right)^2 g && \text{(C2)} \end{aligned}$$

where g, c and c_N are constants given by

$$\begin{aligned} g &= \left(\left(\frac{c\phi}{\sigma}\right)^2 + c_N \left(\frac{\phi}{\sigma_{N,1}}\right)^2 \left(1 - c\frac{\sigma_{N,0}}{\sigma}\right)^2 \right) && \text{(C3)} \\ (c, c_N) &= \begin{cases} (1,0) & \text{if} \quad \text{only old economy exists} \\ (1,1) & \text{if} \quad t \geq t^{**} \text{ and adoption occurs at } t^{**} \\ (0,1) & \quad\quad \text{otherwise} \end{cases} && \text{(C4)} \end{aligned}$$

and the vector of expectation errors follows the process

$$\begin{pmatrix} d\tilde{Z}_{0,t} \\ d\tilde{Z}_{1,t} \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ \sigma_{N,0} & \sigma_{N,1} \end{pmatrix}^{-1} \left(\begin{matrix} d\rho_t \\ d\rho_t^N \end{matrix} - E_t \left[\begin{matrix} d\rho_t \\ d\rho_t^N \end{matrix} \right] \right). \quad \text{(C5)}$$

From equation (C2), we have

$$\widehat{\sigma}_t^2 = \begin{cases} \left(\widehat{\sigma}_{t^{**}}^{-2} + g\left(t - t^{**}\right)\right)^{-1} & \text{if} \quad t \geq t^{**} \text{ and the adoption occurs at } t^{**} \\ \left(\widehat{\sigma}_{t^{*}}^{-2} + g\left(t - t^{*}\right)\right)^{-1} & \quad\quad \text{otherwise} \end{cases} \quad \text{(C6)}$$

Proof: We consider only case (ii) and (iii). The simpler case (i) can be shown using similar steps. In these two cases, the new economy exists and thus the observation equations are

$$\begin{aligned} d\rho_t &= \phi\left(\overline{\rho} + c\psi - \rho_t\right) dt + \sigma dZ_{0,t} \\ d\rho_t^N &= \phi\left(\overline{\rho} + \psi - \rho_t^N\right) dt + \sigma_{N,0} dZ_{0,t} + \sigma_{N,1} dZ_{1,t} \end{aligned}$$

where c is given in (C4). Defining $\mathbf{s}_t = (\rho_t, \rho_t^N)'$, this can be written compactly as

$$d\mathbf{s}_t = \left(\mathbf{A} + \mathbf{B}\mathbf{s_t} + \mathbf{C}\psi\right) dt + \mathbf{\Sigma} d\mathbf{Z}$$

where $\mathbf{C} = (c\phi, \phi)'$ and

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma & 0 \\ \sigma_{N,0} & \sigma_{N,1} \end{pmatrix}$$

Liptser and Shiryaev (1977) show that the process for $\widehat{\psi}_t = E_t[\psi]$ is given by

$$d\widehat{\psi}_t = \widehat{\sigma}_t^2 \mathbf{C}' \left(\mathbf{\Sigma}'\right)^{-1} d\widetilde{\mathbf{Z}} \quad \text{(C7)}$$

where $\widetilde{\mathbf{Z}}_t = (\widetilde{Z}_{0,t}, \widetilde{Z}_{1,t})'$ follows the process in equation (C5) and

$$\frac{d\widehat{\sigma}_t^2}{dt} = -\left(\widehat{\sigma}_t^2\right)^2 \mathbf{C}' \left(\mathbf{\Sigma}\mathbf{\Sigma}'\right)^{-1} \mathbf{C}$$

Substituting $\mathbf{C}$ and $\mathbf{\Sigma}$, we find immediately

$$\mathbf{C}' \left(\mathbf{\Sigma}'\right)^{-1} = \left(c\frac{\phi}{\sigma}, -c\phi\frac{\sigma_{N,0}}{\sigma\sigma_{N,1}} + \frac{\phi}{\sigma_{N,1}}\right)$$

Substituting this expression in (C7) and defining $g = \mathbf{C}' \left(\mathbf{\Sigma}\mathbf{\Sigma}'\right)^{-1} \mathbf{C}$ we obtain (C1) and (C2) for $c_N = 1$. It is simple to verify that (C6) satisfies (C2), yielding the conclusion. Q.E.D.

Lemma A2: For $t \in [t^*, t^{**})$, the density of $\widehat{\psi}_{t^{**}}$ conditional on $\widehat{\psi}_t$ is normal, and given by

$$\widehat{\psi}_{t^{**}}|_{\widehat{\psi}_t} \sim N\left(\widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)$$

where

$$\sigma^2_{\widehat{\psi},t} = \widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2$$

and $\widehat{\sigma}_t^2$ is given in (C6) for the case $t \in [t^*, t^{**})$. In addition, the probability of adoption is

$$p_t \equiv p\left(\widehat{\psi}_t, t\right) = \Pr\left(\widehat{\psi}_{t^{**}} > \underline{\psi}|\widehat{\psi}_t\right) = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)$$

where $\mathcal{N}\left(.; a, s^2\right)$ the cdf of a normal distribution with mean a and variance s^2.

Proof: For $t \in [t^*, t^{**})$, the process for the posterior mean $\widehat{\psi}_t$ is a linear diffusion with deterministic volatility, as given in (C1) for $c = 0$ and $c_N = 1$. The integral representation is

$$\widehat{\psi}_{t^{**}} = \widehat{\psi}_t + \frac{\phi}{\sigma_{N,1}} \int_t^{t^{**}} \widehat{\sigma}_s^2 d\widetilde{Z}_{1,s}$$

which immediately implies that

$$\widehat{\psi}_{t^{**}}|\widehat{\psi}_t \sim N\left(\widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)$$

where

$$\sigma^2_{\widehat{\psi},t} = \left(\frac{\phi}{\sigma_{N,1}}\right)^2 \int_t^{t^{**}} \left(\widehat{\sigma}_s^2\right)^2 ds$$

Using (C6) for $t < t^{**}$ we can compute

$$\int_t^{t^{**}} \left(\widehat{\sigma}_s^2\right)^2 ds = \frac{1}{\left(\phi/\sigma_{N,1}\right)^2} \left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right]$$

Thus $\sigma^2_{\widehat{\psi},t} = \widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2$. In addition, it then immediately follows that

$$p_t \equiv p\left(\widehat{\psi}_t, t\right) = \Pr\left(\widehat{\psi}_{t^{**}} > \underline{\psi}|\widehat{\psi}_t\right) = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)$$

Q.E.D.

It is convenient to rewrite the original processes under the filtered measure. Let $b_t = \log(B_t)$ and $b_t^N = \log\left(B_t^N\right)$. For $t > t^*$ we have

$$\begin{aligned}
db_t &= \rho_t dt && \text{(C8)}\\
d\rho_t &= \phi\left(\overline{\rho} + c\widehat{\psi}_t - \rho_t\right)dt + \sigma d\widetilde{Z}_{0,t} && \text{(C9)}\\
d\widehat{\psi} &= \widehat{\sigma}_t^2 c\frac{\phi}{\sigma}d\widetilde{Z}_{0,t} + c_N\widehat{\sigma}_t^2\frac{\phi}{\sigma_{N,1}}\left(1 - c\frac{\sigma_{N,0}}{\sigma}\right)d\widetilde{Z}_{1,t} && \text{(C10)}\\
d\widehat{\sigma}_t^2 &= -\left(\widehat{\sigma}_t^2\right)^2\left(\left(\frac{c\phi}{\sigma}\right)^2 + c_N\left(\frac{\phi}{\sigma_{N,1}}\right)^2\left(1 - c\frac{\sigma_{N,0}}{\sigma}\right)^2\right)dt && \text{(C11)}\\
db_t^N &= \rho_t^N dt && \text{(C12)}\\
d\rho_t^N &= \phi\left(\overline{\rho} + \widehat{\psi}_t - \rho_t^N\right)dt + \sigma_{N,0}d\widetilde{Z}_{0,t} + \sigma_{N,1}d\widetilde{Z}_{1,t} && \text{(C13)}
\end{aligned}$$

Lemma A3: Let $\tau = T - t$. The expectation in equation (A6) is given by

$$V\left(B_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2, \tau\right) = E_t\left[\frac{B_T^{1-\gamma}}{1-\gamma}\right] = \frac{B_t^{1-\gamma}}{1-\gamma}e^{A_0(\tau)+(1-\gamma)A_1(\tau)\rho_t+(1-\gamma)A_2(\tau)\widehat{\psi}_t+\frac{1}{2}(1-\gamma)^2A_2(\tau)^2\widehat{\sigma}_t^2} \quad \text{(C14)}$$

where

$$\begin{aligned}
A_0(\tau) &= (1-\gamma)\overline{\rho}(\tau - A_1(\tau)) + \frac{\sigma^2}{2}\frac{(1-\gamma)^2}{\phi^2}\left\{\tau + \frac{1-e^{-2\phi\tau}}{2\phi} - 2\frac{1-e^{-\phi\tau}}{\phi}\right\}\\
A_1(\tau) &= \frac{1-e^{-\phi\tau}}{\phi} \text{ and } A_2(\tau) = \tau - A_1(\tau)
\end{aligned}$$

Proof: By definition

$$V\left(b_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2, t; T\right) = (1-\gamma)^{-1}E_t\left[e^{(1-\gamma)b_T}\right]$$

Denoting $\mathbf{x}_t = \left(b_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2\right)$, the Feynman-Kac theorem shows that V has to satisfy the PDE

$$0 = \frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial x_i}E_t[dx_i] + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 V}{\partial x_i \partial x_j}E_t[dx_i dx_j]$$

with the boundary condition $V(\mathbf{x}_T) = (1-\gamma)^{-1}e^{(1-\gamma)x_{1,T}}$. Using (C8) - (C11) with $c = 1$ and $c_N = 0$, it is simple to verify that (C14) satisfies this PDE with the boundary condition. Finally, $A_2(\tau) > 0$ is immediate. Rewrite $A_2(\tau) = f(\tau) = \tau - \frac{1-e^{-\phi\tau}}{\phi}$. Note that $f(0) = 0$. Since $f'(\tau) = 1 - e^{-\phi\tau} > 0$, we have $f(\tau) > 0$ for every $\tau > 0$. QED.

Proof of Proposition 1: Since $\gamma > 1$ we have that V in (C14) is decreasing in $\widehat{\sigma}_t^2$. It immediately follows that $V(B_{t^*}(1-\kappa), \rho_{t^*}, 0, \widehat{\sigma}_{t^*}^2, \tau^*) < V(B_{t^*}, \rho_{t^*}, 0, 0, \tau^*)$. Q.E.D.

Proof of Proposition 2: Using (C14) it is immediate to verify that equation (8) follows from the adoption condition $V\left(B_{t^{**}}(1-\kappa), \rho_{t^{**}}, \widehat{\psi}_{t^{**}}, \widehat{\sigma}_{t^{**}}^2, \tau^{**}\right) \geq V(B_{t^{**}}, \rho_{t^{**}}, 0, 0, \tau^{**})$.. Q.E.D.

To prove Proposition 3 we need the following lemma:

Lemma A4: The value function

$$\mathcal{V}_t = E_t\left[\max_{yes,no} E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\right]$$

at time $t^* \leq t < t^{**}$ is given by

$$\mathcal{V}\left(B_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2; \tau\right) = \frac{B_t^{1-\gamma}}{1-\gamma}\left\{(1-p_t)\, G_t^{no} + p_t G_t^{yes}\right\} \tag{C15}$$

where

$$\begin{aligned} G_t^{no} &\equiv E_t\left[\left(\frac{B_T}{B_t}\right)^{1-\gamma} |\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = e^{A_0(\tau)+(1-\gamma)A_1(\tau)\rho_t} \\ G_t^{yes} &\equiv E_t\left[\left(\frac{B_T}{B_t}\right)^{1-\gamma} |\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] = G_t^{no}\,(1-\kappa)^{1-\gamma}\, R_t e^{(1-\gamma)A_2(\tau^{**})\widehat{\psi}_t + \frac{1}{2}(1-\gamma)^2 A_2(\tau^{**})^2 \widehat{\sigma}_t^2} \end{aligned}$$

and

$$R_t = \frac{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + (1-\gamma)\, A_2\,(\tau^{**})\, \sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)}{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)} < 1 \tag{C16}$$

Proof: The value function is

$$\mathcal{V}_t = E_t\left[\max_{yes,no} E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\right] = (1-p_t)\, E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] + p_t E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right]$$

as the adoption at t^{**} occurs if and only if $\widehat{\psi}_{t^{**}} \geq \underline{\psi}$. Starting with the first expectation, we can use the law of iterated expectations

$$E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = E_t\left[E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right]|\widehat{\psi}_{t^{**}} < \underline{\psi}\right]$$

We can use again equation (C14) to compute the inner expectation. In fact, if $\widehat{\psi}_{t^{**}} < \underline{\psi}$, the technology does not change at t^{**}. Moreover, equations (C9) - (C10) show that ρ_t and $\widehat{\psi}_t$ are independent as $c = 0$ (see equation C4). Thus, Lemma A3 implies

$$E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = V\left(B_{t^{**}}, \rho_{t^{**}}, 0, 0, t^{**}; T\right) = \frac{B_{t^{**}}^{1-\gamma}}{1-\gamma} e^{A_0(t^{**};T)+A_1(t^{**};T)\rho_{t^{**}}}$$

Thus,

$$\begin{aligned} E_t\left[\frac{B_{t^{**}}^{1-\gamma}}{1-\gamma} e^{A_0(t^{**};T)+A_1(t^{**};T)\rho_{t^{**}}}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] &= E_t\left[\frac{B_{t^{**}}^{1-\gamma}}{1-\gamma} e^{A_0(t^{**};T)+A_1(t^{**};T)\rho_{t^{**}}}\right] \\ &= \frac{B_t^{1-\gamma}}{1-\gamma} e^{A_0(t;T)+A_1(t;T)\rho_t} \end{aligned}$$

where the first equality stems from the independence of ρ_t and $\widehat{\psi}_t$, and the second equality stems from an application of Feynman - Kac thorem, similar to the argument used in Lemma A3.

The second expectation is more involved, as until t^{**} capital employs the old technology, and only then it switches to the new technology. In addition, the switch occurs only if $\widehat{\psi}_{t^{**}}$ is high enough, and this must be taken into account in the computation. Using again the law of iterated expectations, we have

$$\begin{aligned} E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] &= E_t\left[E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right]|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ &= E_t\left[V\left(B_{t^{**}}\left(1-\kappa\right), \rho_{t^{**}}, \widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, t^{**}; T\right)|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \end{aligned}$$

where the second equality stems from Lemma A3 and the fact that if $\widehat{\psi}_{t^{**}} > \underline{\psi}$, the adoption occurs. We can use the explicit formula for $V(.)$ to compute this expectation. In particular, from (C8) - (C11), $\widehat{\psi}_t$ is independent of both ρ_t and b_t, and $\widehat{\sigma}^2_{t^{**}}$ is a known constant. Thus, we can write

$$\begin{aligned} & E_t\left[V\left(B_{t^{**}}\left(1-\kappa\right), \rho_{t^{**}}, \widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, t^{**}; T\right)|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ = & \frac{(1-\kappa)^{1-\gamma}}{1-\gamma}E_t\left[e^{(1-\gamma)b_{t**}+A_0(t^{**};T)+(1-\gamma)A_1(t^{**};T)\rho_{t**}+\frac{1}{2}(1-\gamma)^2A_2(t^{**};T)^2\widehat{\sigma}^2_{t^{**}}}\right] \\ & \times E_t\left[e^{(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ = & e^{(1-\gamma)b_t+A_0(t;T)+(1-\gamma)A_1(t;T)\rho_t+\frac{1}{2}(1-\gamma)^2A_2(t^{**};T)^2\widehat{\sigma}^2_{t^{**}}}E_t\left[e^{(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \end{aligned}$$

Since from Lemma A2, $\widehat{\psi}_{t^{**}} \sim N\left(\widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)$, we have that the conditional density required to compute the last expectation is given by

$$f\left(\widehat{\psi}_{t**}|\widehat{\psi}_{t**} > \underline{\psi}\right) = \frac{f\left(\widehat{\psi}_{t**}; \widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)1_{\left\{\widehat{\psi}_{t**} > \underline{\psi}\right\}}}{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)}$$

where $f\left(\widehat{\psi}_{t^{**}}; \widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)$ is the density of a normal distribution with mean $\widehat{\psi}_t$ and variance $\widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}$. We sometimes abbreviate this function simply as $f(\widehat{\psi}_{t^{**}})$. Using this density, we find

$$\begin{aligned} E\left[e^{(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] &= \frac{1}{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)}\int_{\underline{\psi}}^{\infty} e^{(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}}f\left(\widehat{\psi}_{t**}\right)d\widehat{\psi}_{t^{**}} \\ &= e^{\frac{1}{2}(1-\gamma)^2A_2^2(\tau^{**})\left(\widehat{\sigma}^2_t-\widehat{\sigma}^2_{t^{**}}\right)+(1-\gamma)A_2(\tau^{**})\widehat{\psi}_t}R\left(\widehat{\psi}_t\right) \end{aligned}$$

where $R\left(\widehat{\psi}_t\right) = R_t$ is given in (C16). Putting all these elements together, we obtain (C15).

Lemma A5: $G_t^{yes} < G_t^{no}$.

Proof: Consider the expression

$$J_t = E_t\left[e^{(1-\gamma)\log(1-\kappa)+(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}+\frac{1}{2}(1-\gamma)^2A_2(t^{**};T)^2\widehat{\sigma}^2_{t^{**}}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right]$$

Using the definition of $\underline{\psi}$ in equation (8) in the paper, this can be written as

$$\begin{aligned} J_t &= E_t\left[e^{-(1-\gamma)A_2(\tau^{**})\left[-\frac{\log(1-\kappa)}{A_2(\tau^*)}-\widehat{\psi}_{t**}-\frac{1}{2}(1-\gamma)A_2(t^{**};T)\widehat{\sigma}^2_{t^{**}}\right]}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ &= E_t\left[e^{(1-\gamma)A_2(\tau^{**})\left[\widehat{\psi}_{t**}-\underline{\psi}\right]}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \end{aligned}$$

Thus, $J_t < 1$, as it is the expectation of a random variable that is constrained to be less than 1. By using the same steps as in Lemma A4, we find

$$\begin{aligned} J_t &= E_t\left[e^{(1-\gamma)A_2(\tau^{**})\left[\widehat{\psi}_{t**}-\underline{\psi}\right]}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ &= e^{-(1-\gamma)A_2(\tau^{**})\underline{\psi}}E_t\left[e^{(1-\gamma)A_2(\tau^{**})\widehat{\psi}_{t**}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ &= e^{-(1-\gamma)A_2(\tau^{**})\underline{\psi}+(1-\gamma)A_2(\tau^*)\widehat{\psi}_t+\frac{1}{2}(1-\gamma)^2A_2(\tau^*)^2\left(\widehat{\sigma}^2_t-\widehat{\sigma}^2_{t^{**}}\right)} \times R_t \\ &= e^{(1-\gamma)\log(1-\kappa)+(1-\gamma)A_2(\tau^*)\widehat{\psi}_t+\frac{1}{2}(1-\gamma)^2A_2(\tau^*)^2\widehat{\sigma}^2_t} \times R_t \\ &= \frac{G^{yes}_t}{G^{no}_t} \end{aligned}$$

yielding the conclusion. Q.E.D.

Proposition 3: Experimenting is always optimal at time t^*, that is

$$\mathcal{V}\left(B_{t^*},\rho_{t^*},0,\widehat{\sigma}^2_{t^*};\tau^*\right) > V\left(B_{t^*},\rho_{t^*},0,0;\tau^*\right)$$

where $V\left(B_{t^*},\rho_{t^*},0,0;\tau^*\right)$ is defined in equation (C14).

Proof: Since from Lemma A5 $G^{yes}_t < G^{no}_t$, the result follows from the fact that we can rewrite $V\left(B_{t^*},\rho_{t^*},0,0;\tau^*\right) = \frac{B^{1-\gamma}_{t^*}}{1-\gamma}G^{no}_{t^*}$ and $\gamma > 1$. Q.E.D.

Derivation of State Price Density in Eq. (9). We assume a unit mass of identical investors. Given the equilibrium state price density π_t and market completeness, Cox and Huang (1989) imply each agent i maximizes intertemporal utility by solving the static optimization

$$\max_{W_{i,T}} E_0\left[\frac{W^{1-\gamma}_{i,T}}{1-\gamma}\right] \quad \text{subject to} \quad E_0\left[\pi_T W_{i,T}\right] \leq W_{0,i}$$

The first order condition of the Lagrangean $\mathcal{L} = E_0\left[\frac{W^{1-\gamma}_{i,T}}{1-\gamma} - \lambda_i(\pi_T W_{i,T} - W_{0,i})\right]$ imply

$$W_{i,T} = \lambda_i^{-\frac{1}{\gamma}}\pi_T^{-\frac{1}{\gamma}} \tag{C17}$$

where the Lagrange multiplier λ_i is determined from the budget constraint $E_0\left[\pi_T W_{i,T}\right] = W_{0,i}$. In equilibrium, market clearing imposes that $\int_0^1 W_{i,T}\,di = B_T$, which from (C17) implies $B_T = \left(\int_0^T \lambda_i^{-\frac{1}{\gamma}}di\right)\pi_T^{-\frac{1}{\gamma}}$. Since all agents have the same initial endowment, $\lambda_i = \lambda_j = \lambda$, which yields $B_T = \lambda^{-\frac{1}{\gamma}}\pi_T^{-\frac{1}{\gamma}}$. Solving for π_T, we obtain the general equilibrium restriction

$$\pi_T = \lambda^{-1}B_T^{-\gamma}$$

Finally, because we renormalize the interest rate to zero, the state price density must be a martingale conditional on the information set of investors, implying $\pi_t = E_t[\pi_T] = \lambda^{-1} E_t\left[B_T^{-\gamma}\right]$. Q.E.D.

Re Proposition 4: The functions $\widetilde{G}_t^{no}$ and $\widetilde{G}_t^{yes}$ are given by

$$\widetilde{G}_t^{no} \equiv E_t\left[\left(\frac{B_T}{B_t}\right)^{-\gamma} |\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = e^{\overline{A}_0(\tau)-\gamma A_1(\tau)\rho_t} \tag{C18}$$

$$\widetilde{G}_t^{yes} \equiv E_t\left[\left(\frac{B_T}{B_t}\right)^{-\gamma} |\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] = \widetilde{G}_t^{no}(1-\kappa)^{-\gamma} \widetilde{R}_t e^{-\gamma A_2(\tau^{**})\widehat{\psi}_t + \frac{1}{2}\gamma^2 A_2(\tau^{**})^2 \widehat{\sigma}_t^2}, \tag{C19}$$

where

$$\widetilde{R}_t = \frac{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t - \gamma A_2(\tau^{**})\sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)}{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)} < 1 \tag{C20}$$

$$\overline{A}_0(\tau) = -\gamma\overline{\rho}(\tau - A_1(\tau)) + \frac{\sigma^2}{2}\frac{\gamma^2}{\phi^2}\left\{\tau + \frac{1-e^{-2\phi\tau}}{2\phi} - 2\frac{1-e^{-\phi\tau}}{\phi}\right\}. \tag{C21}$$

Proof of Proposition 4: The proof is identical to the one of Lemma A4, where "$(1-\gamma)$" is substituted with "$-\gamma$". Using this fact, we immediately obtain

$$\pi_t = \lambda^{-1} B_t^{-\gamma}\left\{(1-p_t)\widetilde{G}_t^{no} + p_t \widetilde{G}_t^{yes}\right\} \tag{C22}$$

where $\widetilde{G}_t^{no}$ and $\widetilde{G}_t^{yes}$ are given by (C18) and (C19), respectively. Q.E.D.

Proof of Corollary 1: The corollary follows from an application of Ito's Lemma, so that

$$\frac{d\pi_t}{\pi_t} = -\sigma_{\pi,t} d\widetilde{\mathbf{Z}}_t$$

where

$$\sigma_{\pi,t} = \gamma A_1(\tau)\sigma_\rho + S_{\pi,t}\widetilde{\sigma}_{\psi,t}$$

and

$$S_{\pi,t} = \frac{\left(\gamma A_2(\tau^{**}) - \frac{1}{\widetilde{p}}\frac{\partial\widetilde{p}}{\partial\widehat{\psi}}\right) p_t \widetilde{G}_t^{yes} + \frac{\partial p}{\partial\widehat{\psi}}\widetilde{G}_t^{no}}{(1-p_t)\widetilde{G}_t^{no} + p_t\widetilde{G}_t^{yes}} \tag{C23}$$

where

$$\widetilde{p}_t \equiv \widetilde{p}\left(\widehat{\psi}_t, t\right) = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t - \gamma A_2(\tau^{**})\sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)$$

and $\sigma_\rho = (\sigma, 0)$,

$$\widetilde{\sigma}_{\psi,t} = \left(0, \widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}}\right). \tag{C24}$$

Q.E.D.

Proof of Proposition 5 (old economy): The result about the old economy is immediate from the pricing formula $M_t = E_t[\pi_T B_T]/\pi_t = E_t\left[B_T^{1-\gamma}\right]/\pi_t$, and the results in Lemma A4 and Proposition 4. Q.E.D.

For better referencing, it is convenient to restate Proposition 5 for the new economy:

Proposition 5 (new economy) Let $\tau = T - t$. For $t^* \leq t < t^{**}$, the market to book ratio of the new economy is given by

$$\frac{M_t^N}{B_t^N} = \frac{(1-p_t)\,K^{no} + p_t K^{yes}}{(1-p_t)\,\widetilde{G}_t^{no} + p_t \widetilde{G}_t^{yes}} \tag{C25}$$

where $\widetilde{G}_t^{no}$ and $\widetilde{G}_t^{yes}$ are given in Proposition 4, and

$$\begin{aligned} K^{no} &\equiv E_t\left[\left(\frac{B_T}{B_t}\right)^{-\gamma}\frac{B_T^N}{B_t^N}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = K_t R_{L,t}^N \\ K^{yes} &\equiv E_t\left[\left(\frac{B_T}{B_t}\right)^{-\gamma}\frac{B_T^N}{B_t^N}|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] = (1-\kappa)^{-\gamma} K_t^N R_{H,t}^N \end{aligned}$$

$$\begin{aligned} K_t &= e^{C_0(\tau) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho_t^N + A_2(\tau)\widehat{\psi}_t + \frac{1}{2}A_2^2(\tau)\widehat{\sigma}_t^2} \\ K_t^N &= K_t e^{-\gamma A_2(\tau^{**})\widehat{\psi}_t + \frac{1}{2}\gamma A_2(\tau^{**})(\gamma A_2(\tau^{**}) - 2A_2(\tau))\widehat{\sigma}_t^2} \end{aligned}$$

and

$$\begin{aligned} R_{L,t}^N &= \frac{\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma_{y\widehat{\psi}}^L, \sigma_{\widehat{\psi},t}^2\right)}{\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma_{\widehat{\psi},t}^2\right)} \text{ with } \sigma_{y\widehat{\psi}}^L = A_2(\tau)\,\widehat{\sigma}_t^2 - A_2(\tau^{**})\,\widehat{\sigma}_{t^{**}}^2 \\ R_{H,t}^N &= \frac{1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma_{y\widehat{\psi}}^H, \sigma_{\widehat{\psi},t}^2\right)}{1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma_{\widehat{\psi},t}^2\right)} \text{ with } \sigma_{y\psi}^H = \sigma_{y\psi}^L - \gamma A_2(\tau^{**})\,\sigma_{\widehat{\psi},t}^2 \end{aligned}$$

Above, $C_0(\tau)$ is given by

$$\begin{aligned} C_0(\tau) &= (1-\gamma)\,\overline{\rho}\,(\tau - A_1(\tau)) \\ &\quad + \frac{1}{2\phi^2}\left\{\tau + \frac{1-e^{-2\phi\tau}}{2\phi} - 2\frac{1-e^{-\phi\tau}}{\phi}\right\}\left(\gamma^2\sigma^2 - 2\gamma\sigma_{N,0}\sigma + \left(\sigma_{N,0}^2 + \sigma_{N,1}^2\right)\right) \end{aligned}$$

We start the proof with two lemmas:

Lemma A6: For $t \geq t^{**}$, let $\tau = T - t$. Then

$$V^N\left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2, \tau\right) \equiv E_t\left[e^{-\gamma b_T + b_T^N}\right]$$

is given by

$$V^N\left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2, \tau\right) = e^{-\gamma b_t + b_t^N + C_0(\tau) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho_t^N + (1-c\gamma)A_2(\tau)\widehat{\psi}_t + \frac{1}{2}(1-c\gamma)^2 A_2^2(\tau)\widehat{\sigma}_t^2} \tag{C26}$$

where $c = 1$ if the adoption occurred at time t^{**} , and 0 otherwise, $A_1(.)$ and $A_2(.)$ are as in Lemma A3, and

$$C_0(\tau) = (1-\gamma)\overline{\rho}(\tau - A_1(\tau)) + \frac{1}{\phi^2}\left(\tau + \frac{1-e^{-2\phi\tau}}{2\phi} - 2A_1(\tau)\right)\frac{1}{2}(\sigma^*)^2$$

and

$$(\sigma^*)^2 = \gamma^2\sigma^2 + \sigma_{N,0}^2 + \sigma_{N,1}^2 - 2\gamma\sigma_{N,0}\sigma$$

Proof: As in Lemma A3, denoting $\mathbf{x}_t = \left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2\right)$, the Feynman-Kac theorem shows that V^N has to satisfy the PDE

$$0 = \frac{\partial V^N}{\partial t} + \sum_i \frac{\partial V^N}{\partial x_i} E_t[dx_i] + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 V^N}{\partial x_i \partial x_j} E_t[dx_i dx_j]$$

with the boundary condition $V(\mathbf{x}_T) = e^{-\gamma x_{1,T} + x_{2,T}}$. Using (C8) - (C13) for the cases where $c = 1$ or $c = 0$ (with $c_N = 1$) in Lemma A1, it is simple to verify that (C26) satisfies this PDE with the boundary condition provided. Q.E.D.

Lemma A7: Define

$$y_{t^{**}} = -\gamma b_{t^{**}} + b_{t^{**}}^N - \gamma A_1(\tau^{**})\rho_{t^{**}} + A_1(\tau^{**})\rho_{t^{**}}^N + (1-c_1)A_2(\tau^{**})\widehat{\psi}_{t^{**}}$$

where $c_1 > 0$ is a constant. Then

$$\begin{pmatrix} y_{t^{**}} \\ \widehat{\psi}_{t^{**}} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{y,t} \\ \widehat{\psi}_t \end{pmatrix}, \begin{pmatrix} \sigma_y^2 & \sigma_{y\psi} \\ \sigma_{y\psi} & \sigma_{\widehat{\psi},t}^2 \end{pmatrix}\right)$$

where

$$\begin{aligned}
\mu_{y,t} &= -\gamma b_t + b_t^N + (1-\gamma)\overline{\rho}a(t) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho_t^N + (A_2(\tau) - c_1 A_2(\tau^{**}))\widehat{\psi}_t \\
\sigma_y^2 &= (1-c_1)^2 A_2(\tau^{**})^2\left(\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right) + a(t)^2\widehat{\sigma}_t^2 + 2A_2(\tau^{**})(1-c_1)a(t)\widehat{\sigma}_t^2 + (\sigma^*)^2 a_2(t) \\
\sigma_{\widehat{\psi},t}^2 &= \widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2 \\
\sigma_{y\psi} &= (1-c_1)A_2(\tau^{**})\left(\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right) + a(t)\widehat{\sigma}_t^2
\end{aligned}$$

and

$$a(t) = t^{**} - t - \frac{e^{-\phi(T-t^{**})} - e^{-\phi(T-t)}}{\phi} \quad \text{(C27)}$$

$$a_2(t) = \frac{1}{\phi^2}\left(t^{**} - t + \frac{e^{-2\phi(T-t^{**})} - e^{-2\phi(T-t)}}{2\phi} - 2\frac{e^{-\phi(T-t^{**})} - e^{-\phi(T-t)}}{\phi}\right) \quad \text{(C28)}$$

$$(\sigma^*)^2 = \gamma^2\sigma^2 + \sigma_{N,0}^2 + \sigma_{N,1}^2 - 2\gamma\sigma_{N,0}\sigma \quad \text{(C29)}$$

Proof: The proof of this lemma is rather lengthy, and so it is provided separately below.

Proof of Proposition 5 (new economy): The pricing formula is $M_t^N = E_t\left[\pi_T B_T^N\right]/\pi_t$. Thus, we need to compute

$$E_t\left[B_T^{-\gamma}B_T^N\right] = (1-p_t)\,E_t\left[B_T^{-\gamma}B_T^N|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] + p_t E_t\left[B_T^{-\gamma}B_T^N|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \tag{C30}$$

Starting with the first expectation, note that if $\widehat{\psi}_{t^{**}} < \underline{\psi}$, no adoption occurs at t^{**}. Thus,

$$\begin{aligned}
E_t\left[B_T^{-\gamma}B_T^N|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] &= E_t\left[E_{t^{**}}\left[B_T^{-\gamma}B_T^N|\widehat{\psi}_{t^{**}} < \underline{\psi}\right]|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] \\
&= E_t\left[V^N\left(b_{t^{**}}, b_{t^{**}}^N, \rho_{t^{**}}, \rho_{t^{**}}^N, \widehat{\psi}_{t^{**}}, \widehat{\sigma}_{t^{**}}^2, t^{**}; T\right)|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] \\
&= e^{C_0(\tau^{**})+\frac{1}{2}A_2^2(\tau^{**})\widehat{\sigma}_{t^{**}}^2} \\
&\quad \times E_t\left[e^{-\gamma b_{t^{**}}+b_{t^{**}}^N-\gamma A_1(\tau^{**})\rho_{t^{**}}+A_1(\tau^{**})\rho_{t^{**}}^N+A_2(\tau^{**})\widehat{\psi}_{t^{**}}}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right]
\end{aligned}$$

where the first equality stems from the law of iterated expectations, the second from the fact that $\widehat{\psi}_{t^{**}}$ is known at t^{**}, the third from Lemma A6, with $c = 0$ as the adoption does not occur at t^{**}. Note that the exponent in the expectation is simply $y_{t^{**}}$ in Lemma A7 with $c_1 = 0$. For notational convenience, let

$$a_0(t) = (1-\gamma)\,\overline{\rho}a(t)\,.$$

Using Lemma A7 with $c_1 = 0$ and denoting by L the corresponding quantities in Lemma A7 for this case, we can compute

$$E\left[e^{y_{t^{**}}}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = \frac{\int_{-\infty}^{\underline{\psi}} E\left[e^{y_{t^{**}}}|\widehat{\psi}_{t^{**}}\right] f\left(\widehat{\psi}_{t^{**}}; \widehat{\psi}_t, \sigma_{\widehat{\psi},t}^2\right) d\widehat{\psi}_{t^{**}}}{\Pr\left(\widehat{\psi}_{t^{**}} < \underline{\psi}\right)}$$

where $f\left(\widehat{\psi}_{t^{**}}; \widehat{\psi}_t, \sigma_{\widehat{\psi},t}^2\right)$ is the density of a normal with mean $\widehat{\psi}_t$ and variance $\sigma_{\widehat{\psi},t}^2$. The rules of the conditional normal distribution yield the following expression for this expectation:

$$E_t\left[e^{y_{t^{**}}}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = B_t^{-\gamma}B_t^N e^{a_0(t)-\gamma A_1(t;T)\rho_t+A_1(t;T)\rho_t^N+A_2(t;T)\widehat{\psi}_t+\frac{1}{2}\sigma_{Ly}^2} R_{L,t}^N$$

where $R_{L,t}^N$ is given in Proposition 5. So, finally, the first expectation is given by

$$\begin{aligned}
E_t\left[B_T^{-\gamma}B_T^N|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] &= B_t^{-\gamma}B_t^N e^{C_0(t^{**};T)+\frac{1}{2}A_2^2(t^{**};T)\widehat{\sigma}_{t^{**}}^2} e^{a_0(t)-\gamma A_1(t;T)\rho_t+A_1(t;T)\rho_t^N+A_2(t;T)\widehat{\psi}_t+\frac{1}{2}\sigma_{L,y}^2} R_{L,t}^N \\
&= B_t^{-\gamma}B_t^N e^{C_0(t;T)+\frac{1}{2}A_2^2(t;T)\widehat{\sigma}_t^2-\gamma A_1(t;T)\rho_t+A_1(t;T)\rho_t^N+A_2(t;T)\widehat{\psi}_t} R_{L,t}^N
\end{aligned}$$

where the second equality is obtained from the first after some tedious algebra.

We now turn to the second expectation in (C30). The methodology is the same as before, although now we must set $c = 1$ in (C26) and note that $B_{t^{**}} = (1-\kappa)\,B_{t_-^{**}}$, which implies $b_{t^{**}} = b_{t_-^{**}} + \log(1-\kappa)$. Specifically, we have that for $t \leq t^{**}$

$$\begin{aligned}
E_t\left[B_T^{-\gamma}B_T^N|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] &= E_t\left[V^N\left(\log(1-\kappa) + b_{t_-^{**}}, b_{t^{**}}^N, \rho_{t^{**}}, \rho_{t^{**}}^N, \widehat{\psi}_{t^{**}}, \widehat{\sigma}_{t^{**}}^2, \tau\right)|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] \\
&= (1-\kappa)^{-\gamma}\, e^{C_0(\tau^{**})+\frac{1}{2}(1-\gamma)^2A_2^2(\tau^{**})\widehat{\sigma}_{t^{**}}^2} \times \\
&\quad \times E_t\left[e^{-\gamma b_{t^{**}}+b_{t^{**}}^N-\gamma A_1(\tau^{**})\rho_{t^{**}}+A_1(\tau^{**})\rho_{t^{**}}^N+(1-\gamma)A_2(\tau^{**})\widehat{\psi}_{t^{**}}}|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right]
\end{aligned}$$

Comparing to the case with $\left\{\widehat{\psi}_{t^{**}} < \underline{\psi}\right\}$, we see that the term in the expectation is identical, but for the coefficient of $\widehat{\psi}_{t^{**}}$, which is multiplied by $(1-\gamma)$. The distribution of the exponent is given in Lemma A7 for $c_1 = \gamma$. In this case, defining

$$y_{H,t^{**}} = -\gamma b_{t^{**}} + b^N_{t^{**}} - \gamma A_1(\tau^{**})\rho_{t^{**}} + A_1(\tau^{**})\rho^N_{t^{**}} + (1-\gamma) A_2(\tau^{**})\widehat{\psi}_{t^{**}}$$

we have that

$$\mu_{H,y,t} = E\left[y_{H,t^{**}}\right] = -\gamma b_t + b^N_t + a_0(t) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho^N_t + \left(A_2(\tau) - \gamma A_2(\tau^{**})\right)\widehat{\psi}_t$$

The same steps then show

$$\begin{aligned} E_t\left[e^{y_{H,t^{**}}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] &= \frac{1}{1 - N\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi}}\right)} \int_{\underline{\psi}}^{\infty} E\left[e^{y_{H,t^{**}}}|\widehat{\psi}_{t^{**}}\right] f\left(\widehat{\psi}_{t^{**}}\right) d\widehat{\psi}_{t^{**}} \\ &= B_t^{-\gamma} B_t^N e^{a_0(t) - \gamma A_1(t;T)\rho_t + A_1(t;T)\rho^N_t + (A_2(\tau;T) - \gamma A_2(t^{**};T))\widehat{\psi}_t + \frac{1}{2}\sigma^2_{Hy}} R^N_{H,t} \end{aligned}$$

where $R^N_{H,t}$ is defined in Proposition 5.

So, we finally obtain

$$\begin{aligned} E_t\left[B_T^{-\gamma} B_T^N|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] &= B_t^{-\gamma} B_t^N (1-\kappa)^{-\gamma} e^{C_0(\tau^{**}) + \frac{1}{2}(1-\gamma)^2 A_2(\tau^{**})^2 \widehat{\sigma}^2_{t^{**}}} \\ &\quad \times e^{a_0(t) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho^N_t + (A_2(\tau) - \gamma A_2(\tau^{**}))\widehat{\psi}_t + \frac{1}{2}\sigma^2_{Hy}} R^N_{H,t} \\ &= B_t^{-\gamma} B_t^N (1-\kappa)^{-\gamma} e^{C_0(\tau) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho^N_t + (A_2(\tau) - \gamma A_2(\tau^{**}))\widehat{\psi}_t + \frac{1}{2}(A_2(\tau) - \gamma A_2(\tau^{**}))^2 \widehat{\sigma}^2_t} R^N_{H,t} \end{aligned}$$

where the second equality is obtained from the first after some tedious algebra. Putting all terms together, we obtain the expression in Proposition 5. Q.E.D.

Corollary A1: For any $t \in [t^*, t^{**})$, the stock return processes are given by

$$\frac{dM_t}{M_t} = \mu_{M,t}dt + \sigma^0_{M,t}d\widetilde{Z}^0_t + \sigma^1_{M,t}d\widetilde{Z}^1_t \quad \text{and} \quad \frac{dM^N_t}{M^N_t} = \mu^N_{M,t}dt + \sigma^{N,0}_{M,t}d\widetilde{Z}^0_t + \sigma^{N,1}_{M,t}d\widetilde{Z}^1_t,$$

where expected returns are equal to the return covariances with $d\pi_t/\pi_t$,

$$\mu_{M,t} = -\sigma^0_{M,t}\sigma^0_{\pi,t} - \sigma^1_{M,t}\sigma^1_{\pi,t}; \qquad \mu^N_{M,t} = -\sigma^{N,0}_{M,t}\sigma^0_{\pi,t} - \sigma^{N,1}_{M,t}\sigma^1_{\pi,t}, \tag{C31}$$

and the components of the return volatilities are

$$\sigma^0_{M,t} = A_1(\tau)\sigma; \qquad \sigma^1_{M,t} = (S_{M,t} + S_{\pi,t})\widehat{\sigma}^2_t \frac{\phi}{\sigma_{N,1}} \tag{C32}$$

$$\sigma^{N,0}_{M,t} = A_1(\tau)\sigma_{N,0}; \qquad \sigma^{N,1}_{M,t} = A_1(\tau)\sigma_{N,1} + \left(S^N_{M,t} + S_{\pi,t}\right)\widehat{\sigma}^2_t \frac{\phi}{\sigma_{N,1}}, \tag{C33}$$

with $S_{M,t}$ and $S^N_{M,t}$ given by

$$S_{M,t} = \frac{-\frac{\partial p_t}{\partial \widehat{\psi}_t} G^{no}_t + \left((1-\gamma) A_2(\tau^{**}) + \frac{1}{\overline{p}_t}\frac{\partial \overline{p}_t}{\partial \widehat{\psi}_t}\right) p_t G^{yes}_t}{(1-p_t) G^{no}_t + p_t G^{yes}_t} \tag{C34}$$

$$S^N_{M,t} = \frac{\left(A_2(\tau) + \frac{1}{p^N_{L,t}}\frac{\partial p^N_{L,t}}{\partial\widehat{\psi}}\right)(1-p_t)K^{no}_t + \left((A_2(\tau) - \gamma A_2(\tau^{**})) + \frac{1}{p^N_{H,t}}\frac{\partial p^N_{H,t}}{\partial\widehat{\psi}}\right)p_t K^{yes}_t}{(1-p_t)K^{no}_t + p_t K^{yes}_t} \quad \text{(C35)}$$

$$\overline{p}_t = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + (1-\gamma)A_2(\tau^{**})\sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right) \quad \text{(C36)}$$

$$p^N_{L,t} = \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma^L_{y\widehat{\psi}}, \sigma^2_{\widehat{\psi},t}\right) \quad \text{(C37)}$$

$$p^N_{H,t} = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma^H_{y\widehat{\psi}}, \sigma^2_{\widehat{\psi},t}\right). \quad \text{(C38)}$$

Proof of Corollary A1: The proof follows from an application of Ito's Lemma to the respective pricing functions. For the old economy, the diffusion is

$$\sigma_M = A_1(\tau)\sigma_\rho + (S_{M,t} + S_{\pi,t})\,\widetilde{\sigma}_\psi$$

where $\sigma_\rho = (\sigma, 0)$, $S_{M,t}$ is in (C34), $S_{\pi,t}$ is in (C23), and $\widetilde{\sigma}_\psi$ is in (C24). For the new economy,

$$\sigma^N_M = A_1(\tau)\sigma_N + \left(S^N_{M,t} + S_{\pi,t}\right)\widetilde{\sigma}_\psi$$

where $\sigma_N = (\sigma_{N,0}, \sigma_{N,1})$, and $S^N_{M,t}$ is in (C35). Q.E.D.

Proof of Proposition 6: Consider the old economy first. Rewrite the M/B of the old economy as

$$MB_t = \frac{G^{no}_t + p_t H_t}{\widetilde{G}^{no}_t + p_t\widetilde{H}_t}$$

where $H_t = G^{yes}_t - G^{no}_t$, and $\widetilde{H}_t = \widetilde{G}^{yes}_t - \widetilde{G}^{no}_t$. Given the closed form formulas for all the functions, we can compute the first derivative of MB_t with respect to the probability of adoption of the new technology p_t :

$$\frac{\partial MB_t}{\partial p_t} = \frac{H_t\widetilde{G}^{no}_t - G^{no}_t\widetilde{H}_t}{\left(\widetilde{G}^{no}_t + p_t\widetilde{H}_t\right)^2}$$

That is, the M/B increases in p_t if and only if $H_t\widetilde{G}^{no}_t > G^{no}_t\widetilde{H}_t$. Substituting the closed form expressions, we obtain the condition $h_{old} > 0$ where

$$h_{old} = -\widetilde{\kappa} + A_2(\tau^{**})\widehat{\psi}_t + \frac{1}{2}(1-2\gamma)A_2(\tau^{**})^2\widehat{\sigma}^2_t \quad \text{(C39)}$$

$$-\log\left(\frac{1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t - \gamma A_2(\tau^{**})\sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)}{1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + (1-\gamma)A_2(\tau^{**})\sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)}\right) \quad \text{(C40)}$$

Consider now the new economy

$$MB^N_t = \frac{K_t R^N_{L,t} + p_t\overline{J}_t}{\widetilde{G}^{no}_t + p_t\widetilde{H}_t}$$

where $\overline{J}_t = (1-\kappa)^{-\gamma} K^N_t R^N_H - K_t R^N_L$. The first derivative with respect to p_t is

$$\frac{\partial MB^N_t}{\partial p_t} = \frac{\overline{J}_t\widetilde{G}^{no}_t - K_t R^N_{L,t}\widetilde{H}_t}{\left(\widetilde{G}^{no}_t + p_t\widetilde{H}_t\right)^2}$$

Once again, the M/B of the new economy increases in p_t if and only if $\overline{J}_t\widetilde{G}_t^{no} - K_t\widetilde{H}_tR_{L,t}^N > 0$. Substituting, we obtain the condition $h_{new} > 0$

$$h_{new} = -\gamma A_2\left(\tau^{**}\right)A_2\left(\tau\right)\widehat{\sigma}_t^2 - \log\left(\frac{\mathcal{N}\left(\underline{\psi};\widehat{\psi}_t + \sigma_{y\widehat{\psi}}^L, \sigma_{\widehat{\psi},t}^2\right)}{\mathcal{N}\left(\underline{\psi};\widehat{\psi}_t, \sigma_{\widehat{\psi},t}^2\right)}\right) \tag{C41}$$

$$-\log\left(\frac{1-\mathcal{N}\left(\underline{\psi};\widehat{\psi}_t - \gamma A_2\left(\tau^{**}\right)\sigma_{\widehat{\psi},t}^2, \sigma_{\widehat{\psi},t}^2\right)}{1-\mathcal{N}\left(\underline{\psi};\widehat{\psi}_t - \gamma A_2\left(\tau^{**}\right)\sigma_{\widehat{\psi},t}^2 + \sigma_{y\psi}^L, \sigma_{\widehat{\psi},t}^2\right)}\right) \tag{C42}$$

Proof of Proposition 7: Consider $\frac{M^N}{B^N} = \frac{\Phi^N}{\widetilde{\pi}}$, where Φ^N and $\widetilde{\pi}$ are defined appropriately. Then,

$$\frac{\partial\left(\frac{M^N}{B^N}\right)}{\partial\widehat{\psi}_t} = \frac{\widetilde{\pi}\partial\Phi^N/\partial\widehat{\psi}_t - \Phi^N\partial\widetilde{\pi}/\partial\widehat{\psi}_t}{\widetilde{\pi}^2} > 0$$

if and only if $S_{M,t}^N + S_{\pi,t} > 0$ where $S_{M,t}^N$, and $S_{\pi,t}$ are defined above. The probability of adoption as of time t^* is given by

$$p_{t^*} = \int_{f\left(\kappa,\gamma,\widehat{\sigma}_{t^*}^2;\tau^*\right)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx$$

where

$$f\left(\kappa,\gamma,\widehat{\sigma}_{t^*}^2;\tau^*\right) = -\log\left(1-\kappa\right)/A_2(\tau^{**})\left(\frac{\left(\widehat{\sigma}_{t^*}^2\right)^{-1} + \left(\frac{\phi}{\sigma_{N,1}}\right)^2\left(t^{**}-t^*\right)}{\widehat{\sigma}_{t^*}^2\left(\frac{\phi}{\sigma_{N,1}}\right)^2\left(t^{**}-t^*\right)}\right)^{\frac{1}{2}}$$
$$+\frac{\frac{1}{2}\left(\gamma-1\right)A_2(\tau^{**})}{\left(\frac{\phi}{\sigma_{N,1}}\right)\left(t^{**}-t^*\right)^{\frac{1}{2}}\left(1+\widehat{\sigma}_{t^*}^2\left(\frac{\phi}{\sigma_{N,1}}\right)^2\left(t^{**}-t^*\right)\right)^{\frac{1}{2}}}$$

Thus, p_t is small whenever $f\left(\kappa,\gamma,\widehat{\sigma}_{t^*}^2;\tau^*\right)$ is large. We can see that $f\left(\kappa,\gamma,\widehat{\sigma}_{t^*}^2;\tau^*\right)$ is large when κ is high, γ is high and, finally, when $\widehat{\sigma}_{t^*}^2$ is small (if $\kappa > 0$). (In addition, we can see that f is large when $(t^{**} - t^*)$ is small and T is large, the latter due to the increase in $A_2(\tau^{**}) = (T - t^{**}) - \left(1 - e^{-\phi(T-t^{**})}\right)/\phi)$. In all of these cases, the formulas for the various quantities in $S_{M,t}^N + S_{\pi,t}$ imply that the latter becomes positive. Q.E.D.

Re Corollary 2. In the corollary, $\overline{C}_0(\tau^{**}) = C_0(\tau^{**}) - \overline{A}_0(\tau^{**})$. The proof is immediate from Proposition 5 for $p_{t^{**}} = 1$ and $p_{t^{**}} = 0$. Q.E.D.

Optimal Stopping Time.

The value function is

$$\mathcal{V}\left(B_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2, t; T\right) = \max_{t^{**}} E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \tag{C43}$$

where $W_T = B_T$ depends on time t^{**}, as at that time c in (C9) switches from $c = 0$ to $c = 1$.

Proposition 8: The value function in equation (C43) is given by

$$\mathcal{V}\left(B_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2, t; T\right) = B_t^{1-\gamma} e^{(1-\gamma)A_1(T-t)\rho_t} \mathcal{V}_2\left(\widehat{\psi}_t, t; T\right), \tag{C44}$$

where $\mathcal{V}_2\left(\widehat{\psi}_t, t; T\right)$ satisfies the PDE

$$0 = \frac{\partial \mathcal{V}_2}{\partial t} + \left((1-\gamma) A_1(T-t)\phi\overline{\rho} + \frac{1}{2}(1-\gamma)^2 A_1(T-t)^2\sigma^2\right)\mathcal{V}_2 + \frac{1}{2}\frac{\partial^2 \mathcal{V}_2}{\partial \widehat{\psi}^2}\left(\widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}}\right)^2, \tag{C45}$$

with the boundary conditions $\mathcal{V}_2\left(\widehat{\psi}_T, T\right) = \frac{1}{1-\gamma}$ if $t^{**} > T$ and

$$\mathcal{V}_2\left(\widehat{\psi}_t, t; T\right) \geq \frac{(1-\kappa)^{1-\gamma}}{1-\gamma} e^{A_0(\tau)+(1-\gamma)A_2(\tau)\widehat{\psi}_t + \frac{1}{2}(1-\gamma)^2 A_2(\tau)^2 \widehat{\sigma}_t^2},$$

where the equality holds at $t = t^{**}$.

Proof: Since $\widehat{\sigma}_t^2$ is a deterministic function of time, we write the value function simply as $\mathcal{V}(B_t, \rho_t, \widehat{\psi}_t, t; T)$. For $t \leq t^{**}$, $\mathcal{V}$ must satisfy the Bellman equation

$$0 = \frac{\partial \mathcal{V}}{\partial t} + \frac{\partial \mathcal{V}}{\partial B_t} E_t[dB_t] + \frac{\partial \mathcal{V}}{\partial \rho} E_t[d\rho_t] + \frac{\partial \mathcal{V}}{\partial \widehat{\psi}} E_t\left[d\widehat{\psi}_t\right] + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \rho^2} E_t\left[d\rho_t^2\right] + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \widehat{\psi}^2} E_t\left[d\widehat{\psi}_t^2\right] + \frac{\partial^2 \mathcal{V}}{\partial \rho \partial \widehat{\psi}} E_t\left[d\rho_t d\widehat{\psi}_t\right],$$

with the boundary conditions $\mathcal{V}(B_t, \rho_t, \widehat{\psi}_t, t; T) \geq V(B_t(1-\kappa), \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t, t; T)$ (and equality at t^{**}) and $\mathcal{V}(B_T, \rho_T, \widehat{\psi}_T, \widehat{\sigma}_T, T; T) = B_T^{1-\gamma}/(1-\gamma)$ if $T < t^{**}$. It is easy to verify that this Bellman equation is satisfied by the value function (C44) with $\mathcal{V}_2$ satisfying the PDE and the boundary conditions given in Proposition 8. Q.E.D.

Proposition 9: For $t^* \leq t < t^{**}$ the state price density is given by

$$\pi_t = \lambda^{-1} B_t^{-\gamma} e^{-\gamma A_1(T-t)\rho_t} F_2\left(\widehat{\psi}_t, t; T\right), \tag{C46}$$

where F_2 satisfies the partial differential equation

$$0 = \frac{\partial F_2}{\partial t} + \left(-\gamma A_1(T-t)\phi\overline{\rho} + \frac{1}{2}\gamma^2 A_1(T-t)^2\sigma^2\right)F_2 + \frac{1}{2}\frac{\partial^2 F_2}{\partial \widehat{\psi}^2}\left(\widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}}\right)^2$$

with boundary conditions $F_2\left(\widehat{\psi}_T, T; T\right) = 1$ and

$$F_2\left(\widehat{\psi}_{t^{**}}, t^{**}; T\right) = (1-\kappa)^{-\gamma} e^{\overline{A}_0(\tau^{**}) - \gamma A_2(\tau^{**})\widehat{\psi}_{t^{**}} + \frac{1}{2}\gamma^2 A_2(\tau^{**})^2 \widehat{\sigma}_{t^{**}}^2}$$

where time t^{**} is obtained endogenously in Proposition 8.

Proof: The state price density π_t must follow a martingale. Define $\pi_t = F\left(B_t, \rho_t, \widehat{\psi}_t, t; T\right)$, then the martingale condition and Ito's Lemma imply

$$0 = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial B_t} E_t[dB_t] + \frac{\partial F}{\partial \rho} E_t[d\rho_t] + \frac{\partial F}{\partial \widehat{\psi}_t} E_t\left[d\widehat{\psi}\right] + \frac{1}{2}\frac{\partial^2 F}{\partial \rho^2} E_t\left[d\rho_t^2\right] + \frac{1}{2}\frac{\partial^2 F}{\partial \widehat{\psi}^2} E_t\left[d\widehat{\psi}_t^2\right] + \frac{\partial^2 F}{\partial \rho \partial \widehat{\psi}} E_t\left[d\rho_t d\widehat{\psi}_t\right],$$

with boundary conditions $F\left(B_T, \rho_T, \widehat{\psi}_T, T; T\right) = \lambda^{-1} B_T^{-\gamma}$ and

$$\begin{aligned} F\left(B_{t^{**}}, \rho_{t^{**}}, \widehat{\psi}_{t^{**}}, t^{**}; T\right) &= \lambda^{-1} E_{t^{**}}\left[B_T^{-\gamma}\right] \\ &= \lambda^{-1}(1-\kappa)^{-\gamma} B_{t^{**}}^{-\gamma} e^{\overline{A}_0(\tau^{**}) - \gamma A_1(\tau^{**})\rho_{t^{**}} - \gamma A_2(\tau^{**})\widehat{\psi}_{t^{**}} + \frac{1}{2}\gamma^2 A_2(\tau^{**})^2 \widehat{\sigma}^2_{t^{**}}} \end{aligned}$$

It is easy to verify that this PDE is satisfied by function (C46), with the corresponding boundary conditions.

Proposition 10: For $t^* \le t < t^{**}$, the market-to-book ratios of the old and new economies are:

$$\frac{M_t}{B_t} = e^{A_1(T-t)\rho_t} \frac{(1-\gamma)\mathcal{V}_2\left(\widehat{\psi}_t, t; T\right)}{F_2\left(\widehat{\psi}_t, t; T\right)} \tag{C47}$$

$$\frac{M_t^N}{B_t^N} = e^{A_1(T-t)\rho_t^N} \frac{F_2^N\left(\widehat{\psi}_t, t; T\right)}{F_2\left(\widehat{\psi}_t, t; T\right)} \tag{C48}$$

where $\mathcal{V}_2$ and F_2 are given in Propositions 8 and 9, and F_2^N satisfies the PDE

$$\begin{aligned} 0 &= \frac{\partial F_2^N}{\partial t} + F_2^N\left(A_1(T-t)\phi\widehat{\psi}_t + (1-\gamma)A_1(T-t)\phi\overline{\rho} + \frac{1}{2}A_1(T-t)^2(\sigma^*)^2\right) \\ &\quad + \frac{1}{2}\frac{\partial^2 F_2^N}{\partial \widehat{\psi}_t^2}\left(\widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}}\right)^2 + A_1(T-t)\frac{\partial F_2^N}{\partial \widehat{\psi}_t}\widehat{\sigma}_t^2\phi \end{aligned} \tag{C49}$$

where the boundary conditions are $F_2^N\left(\widehat{\psi}_T, T\right) = 1$ and

$$F_2^N\left(\widehat{\psi}_{t^{**}}, t^{**}; T\right) = (1-\kappa)^{-\gamma} e^{C_0(\tau^{**}) + (1-\gamma)A_2(\tau^{**})\widehat{\psi}_t + \frac{1}{2}(1-\gamma)^2 A_2^2(\tau^{**})\widehat{\sigma}^2_{t^{**}}}$$

Proof. The old economy price is

$$M_t = \frac{E_t\left[\pi_T B_T\right]}{\pi_t} = \lambda^{-1}\frac{E_t\left[B_T^{1-\gamma}\right]}{\pi_t} = \frac{(1-\gamma)B_t^{1-\gamma}e^{(1-\gamma)A_1(T-t)\rho_t}\mathcal{V}_2\left(\widehat{\psi}_t, t; T\right)}{B_t^{-\gamma}e^{-\gamma A_1(T-t)\rho_t}F_2\left(\widehat{\psi}_t, t; T\right)}$$

The last equality stems from Proposition 8, as the value function $\mathcal{V} = \frac{1}{1-\gamma}E_t\left[B_T^{1-\gamma}\right]$ under the optimal stopping time policy t^{**}. The result then follows by combining Propositions 8 and 9.

For the new economy, we must compute the quantity

$$F^N\left(B_t, B_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, t; T\right) = E_t\left[\pi_T B_T^N\right] = \lambda^{-1}E_t\left[B_T^{-\gamma}B_T^N\right]$$

After accounting for the conversion cost κ, we have from Lemma A6 that at t^{**}

$$\begin{aligned} F^N\left(B_{t^{**}}, B^N_{t^{**}}, \rho_{t^{**}}, \rho^N_{t^{**}}, \widehat{\psi}_{t^{**}}, t^{**}; T\right) &= \lambda^{-1}(1-\kappa)^{-\gamma}B_{t^{**}}^{-\gamma}B_{t^{**}}^N \\ &\quad \times e^{C_0(\tau^{**}) - \gamma A_1(\tau^{**})\rho_{t^{**}} + A_1(\tau^{**})\rho^N_{t^{**}} + (1-\gamma)A_2(\tau^{**})\widehat{\psi}_{t^{**}} + \frac{1}{2}(1-\gamma)^2 A_2^2(\tau^{**})\widehat{\sigma}^2_{t^{**}}} \end{aligned}$$

Feynman Kac theorem shows that for $t \leq t^{**}$

$$\begin{aligned}
0 &= \frac{\partial F^N}{\partial t} + \frac{\partial F^N}{\partial B_t} E_t[dB_t] + \frac{\partial F^N}{\partial B_t^N} E_t[dB_t^N] + \frac{\partial F^N}{\partial \rho_t} E_t[d\rho_t] + \frac{\partial F^N}{\partial \rho_t^N} E_t[d\rho_t^N] + \frac{\partial F^N}{\partial \widehat{\psi}_t} E_t[d\widehat{\psi}_t] \\
&+ \frac{1}{2}\frac{\partial^2 F^N}{\partial \rho_t^2} E_t[d\rho^2] + \frac{1}{2}\frac{\partial^2 F^N}{\partial \left(\rho_t^N\right)^2} E_t[d\rho_N^2] + \frac{1}{2}\frac{\partial^2 F^N}{\partial \widehat{\psi}_t^2} E_t[d\widehat{\psi}^2] \\
&+ \frac{\partial^2 F^N}{\partial \rho_t \partial \rho_t^N} E_t[d\rho_t d\rho_t^N] + \frac{\partial^2 F^N}{\partial \rho_t \partial \widehat{\psi}_t} E_t[d\rho_t d\widehat{\psi}_t] + \frac{\partial^2 F^N}{\partial \rho_t^N \partial \widehat{\psi}_t} E_t[d\rho_t^N d\widehat{\psi}_t]
\end{aligned}$$

It is easy to verify that this PDE is satisfied by

$$F^N\left(B_t, B_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, t; T\right) = \lambda^{-1} B_t^{-\gamma} B_t^N e^{-\gamma A_1(T-t)\rho_t + A_1(T-t)\rho_t^N} F_2^N\left(\widehat{\psi}_t, t; T\right)$$

where F_2^N satisfies (C49). The result then follows from the pricing formula $M_t^N = E_t[\pi_T B_T]/\pi_t$. Q.E.D.

We solve the Partial Differential Equations by using the finite difference method.

Proof of Lemma A7: Let

$$y_{t^{**}} = -\gamma b_{t^{**}} + b_{t^{**}}^N - \gamma A_1\left(t^{**}; T\right)\rho_{t^{**}} + A_1\left(t^{**}; T\right)\rho_{t^{**}}^N + (1 - c_1) A_2\left(t^{**}; T\right)\widehat{\psi}_{t^{**}}$$

The fact that $y_{t^{**}}$ and $\widehat{\psi}_{t^{**}}$ are jointly normally distributed stems from the linearity of all of the processes. To compute the means, variances and covariances, we can compute the joint moment generating function. That is, let $\alpha_1, \alpha_2 > 0$, and define

$$N\left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2, t\right) = E_t\left[e^{\alpha_1 y_{t^{**}} + \alpha_2 \widehat{\psi}_{t^{**}}}\right]$$

where the processes of stochastic variables are given by (C8) - (C13) with $c = 0$. Let $\mathbf{x}_t = \left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2\right)$, the Feynman-Kac theorem shows that N must satisfy the PDE

$$0 = \frac{\partial N}{\partial t} + \sum_i \frac{\partial N}{\partial x_i} E_t\left[dx\right] + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 N}{\partial x_i \partial x_j} E_t\left[dx_i dx_j\right]$$

with the boundary condition $N\left(b_{t^{**}}, b_{t^{**}}^N, \rho_{t^{**}}, \rho_{t^{**}}^N, \widehat{\psi}_{t^{**}}, \widehat{\sigma}_{t^{**}}^2, t^{**}\right) = e^{\alpha_1 y_{t^{**}} + \alpha_2 \widehat{\psi}_{t^{**}}}$. It can be verified that the solution to the PDE is given by

$$N_t = e^{\alpha_1\left\{-\gamma b_t + b_t^N - \gamma C_1(t;T)\rho_t + C_1(t;T)\rho_t^N\right\} + \alpha_1 C_0(t;T) + \left\{(1-c_1)\alpha_1 C_2(t;T) + \alpha_2\right\}\widehat{\psi}_t + \alpha_1 C_3(t;T)\widehat{\sigma}_t^2}$$

where

$$\begin{aligned}
C_1\left(t; T\right) &= \frac{1 - e^{-\phi(T-t)}}{\phi} = A_1\left(t; T\right) \\
C_2\left(t; T\right) &= A_2\left(t^{**}; T\right) + \frac{1}{(1 - c_1)} a\left(t\right) \\
\alpha_1 C_3\left(t; T\right) &= \widetilde{C}_3\left(t; T\right) \\
&= \frac{1}{2}\left(\left(1 - c_1\right)\alpha_1 C_2 + \alpha_2\right)^2 - \frac{1}{2}\left(\left(1 - c_1\right)\alpha_1 A_2\left(t^{**}; T\right) + \alpha_2\right)^2
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 C_0\left(t;T\right) &= \widetilde{C}_0\left(t;T\right) \\
&= \alpha_1\left(1-\gamma\right)\overline{\rho}a\left(t\right)+\frac{1}{2}\left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right)+\alpha_2\right)^2\left[\widehat{\sigma}_t^2-\widehat{\sigma}_{t^{**}}^2\right] \\
&\quad +\alpha_1^2\frac{1}{2}\left(\sigma^*\right)^2 a_2\left(t\right)
\end{aligned}$$

Above, $a\left(t\right)$, $a_2\left(t\right)$ and σ^* are given by (C27) - (C29). Rewrite $N_t = e^{g(\alpha_1,\alpha_2)}$ where

$$g\left(\alpha_1,\alpha_2\right)=\alpha_1\left\{-\gamma b_t+b_t^N-\gamma C_1\left(t;T\right)\rho_t+C_1\left(t;T\right)\rho_t^N\right\}+\widetilde{C}_0\left(t;T\right)+\left\{\left(1-c_1\right)\alpha_1 C_2\left(t;T\right)+\alpha_2\right\}\widehat{\psi}_t+\widetilde{C}_3\left(t;T\right)\widehat{\sigma}_t^2$$

Thus

$$\frac{\partial N}{\partial\alpha_1}=e^g\left\{\left(-\gamma b_t+b_t^N-\gamma C_1\rho_t+C_1\rho_t^N\right)+\frac{\partial\widetilde{C}_0}{\partial\alpha_1}+\left(1-c_1\right)C_2\widehat{\psi}_t+\frac{\partial\widetilde{C}_3}{\partial\alpha_1}\widehat{\sigma}_t^2\right\}$$

We can use

$$\begin{aligned}
\frac{\partial\widetilde{C}_0}{\partial\alpha_1} &= \left(1-\gamma\right)\overline{\rho}a\left(t\right)+\left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right)+\alpha_2\right)\left(1-c_1\right)A_2\left(t^{**};T\right)\left[\widehat{\sigma}_t^2-\widehat{\sigma}_{t^{**}}^2\right] \\
&\quad +\alpha_1\left(\sigma^*\right)^2 a_2\left(t\right)
\end{aligned}$$

and

$$\frac{\partial\widetilde{C}_3}{\partial\alpha_1}=\left(\left(1-c_1\right)\alpha_1 C_2+\alpha_2\right)\left(1-c_1\right)C_2-\left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right)+\alpha_2\right)\left(1-c_1\right)A_2\left(t^{**};T\right)$$

Thus

$$\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial\alpha_1}=\mu_y=\left\{-\gamma b_t+b_t^N-\gamma C_1\rho_t+C_1\rho_t^N+\left(1-\gamma\right)\overline{\rho}a\left(t\right)+\left(1-c_1\right)C_2\widehat{\psi}_t\right\}$$

Similarly

$$\frac{\partial N}{\partial\alpha_2}=e^g\left\{\frac{\partial\widetilde{C}_0}{\partial\alpha_2}+\widehat{\psi}_t+\frac{\partial\widetilde{C}_3}{\partial\alpha_2}\widehat{\sigma}_t^2\right\}$$

Since

$$\begin{aligned}
\frac{\partial\widetilde{C}_0}{\partial\alpha_2} &= \left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right)+\alpha_2\right)\left[\widehat{\sigma}_t^2-\widehat{\sigma}_{t^{**}}^2\right] \\
\frac{\partial\widetilde{C}_3}{\partial\alpha_2} &= \left(\left(1-c_1\right)\alpha_1 C_2+\alpha_2\right)-\left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right)+\alpha_2\right)
\end{aligned}$$

we find

$$\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial\alpha_2}=\mu_\psi=\widehat{\psi}_t$$

Turning to the second moments

$$\begin{aligned}
\frac{\partial^2 N}{\partial\alpha_1^2} &= e^g\left\{\left(-\gamma b_t+b_t^N-\gamma C_1\rho_t+C_1\rho_t^N\right)+\frac{\partial\widetilde{C}_0}{\partial\alpha_1}+\left(1-c_1\right)C_2\widehat{\psi}_t+\frac{\partial\widetilde{C}_3}{\partial\alpha_1}\widehat{\sigma}_t^2\right\}^2 \\
&\quad +e^g\left\{\frac{\partial^2\widetilde{C}_0}{\partial\alpha_1^2}+\frac{\partial^2\widetilde{C}_3}{\partial\alpha_1^2}\widehat{\sigma}_t^2\right\}
\end{aligned}$$

Since

$$\frac{\partial^2 \widetilde{C}_0}{\partial \alpha_1^2} = (1-c_1)^2 A_2(t^{**};T)^2 \left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right] + (\sigma^*)^2 a_2(t)$$
$$\frac{\partial^2 \widetilde{C}_3}{\partial \alpha_1^2} = ((1-c_1)C_2)^2 - ((1-c_1)A_2(t^{**};T))^2$$

we obtain

$$\begin{aligned}\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial^2 N}{\partial \alpha_1^2} &= \left\{-\gamma b_t + b_t^N - \gamma C_1\rho_t + C_1\rho_t^N + (1-\gamma)\overline{\rho}a(t) + (1-c_1)C_2\widehat{\psi}_t\right\}^2 \\ &+ (1-c_1)^2 A_2(t^{**};T)^2\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right] + (\sigma^*)^2 a_2(t) \\ &+ \left(((1-c_1)C_2)^2 - ((1-c_1)A_2(t^{**};T))^2\right)\widehat{\sigma}_t^2\end{aligned}$$

Thus

$$\begin{aligned}\sigma_y^2 &= \lim_{\alpha_1,\alpha_2\to 0}\frac{\partial^2 N}{\partial \alpha_1^2} - \left(\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial \alpha_1}\right)^2 \\ &= (1-c_1)^2 A_2(t^{**};T)^2\left(\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right) + a(\tau)^2\widehat{\sigma}_t^2 + 2A_2(t^{**};T)(1-c_1)a(\tau)\widehat{\sigma}_t^2 + (\sigma^*)^2 a_2(t)\end{aligned}$$

Similarly,

$$\frac{\partial^2 N}{\partial \alpha_2^2} = e^g\left\{\frac{\partial \widetilde{C}_0}{\partial \alpha_2} + \widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_2}\widehat{\sigma}_t^2\right\}^2 + e^g\left\{\frac{\partial^2 \widetilde{C}_0}{\partial \alpha_2^2} + \frac{\partial^2 \widetilde{C}_3}{\partial \alpha_2^2}\widehat{\sigma}_t^2\right\}$$

Since

$$\frac{\partial^2 \widetilde{C}_0}{\partial \alpha_2^2} = \left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right]$$
$$\frac{\partial \widetilde{C}_3}{\partial \alpha_2} = 0$$

we have

$$\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial^2 N}{\partial \alpha_2^2} = \widehat{\psi}_t^2 + \left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right]$$

and thus

$$\sigma_\psi^2 = \lim_{\alpha_1,\alpha_2\to 0}\frac{\partial^2 N}{\partial \alpha_2^2} - \left(\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial \alpha_2}\right)^2 = \widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2$$

Finally

$$\begin{aligned}\frac{\partial^2 N}{\partial \alpha_2 \partial \alpha_1} &= e^g\left\{\left(-\gamma b_t + b_t^N - \gamma C_1\rho_t + C_1\rho_t^N\right) + \frac{\partial \widetilde{C}_0}{\partial \alpha_1} + (1-c_1)C_2\widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_1}\widehat{\sigma}_t^2\right\}\left\{\frac{\partial \widetilde{C}_0}{\partial \alpha_2} + \widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_2}\widehat{\sigma}_t^2\right\} \\ &+ e^g\left\{\frac{\partial^2 \widetilde{C}_0}{\partial \alpha_2 \partial \alpha_1} + \frac{\partial^2 \widetilde{C}_3}{\partial \alpha_2 \partial \alpha_1}\widehat{\sigma}_t^2\right\}\end{aligned}$$

Since

$$\frac{\partial^2 \widetilde{C}_0}{\partial \alpha_2 \partial \alpha_1} = (1-c_1)A_2(t^{**};T)\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right]$$
$$\frac{\partial^2 \widetilde{C}_3}{\partial \alpha_2 \partial \alpha_1} = (1-c_1)(C_2 - A_2(t^{**};T))$$

we have

$$\begin{aligned}
\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial^2 N}{\partial\alpha_2\partial\alpha_1} &= \left\{-\gamma b_t + b_t^N - \gamma C_1\rho_t + C_1\rho_t^N + (1-\gamma)\,\overline{\rho}a\,(t) + (1-c_1)\,C_2\widehat{\psi}_t\right\}\left\{\widehat{\psi}_t\right\} \\
&\quad + \left\{(1-c_1)\,A_2\,(t^{**};T)\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right] + (1-c_1)\,(C_2 - A_2\,(t^{**};T))\,\widehat{\sigma}_t^2\right\}
\end{aligned}$$

implying

$$\begin{aligned}
\sigma_{y,\psi} &= \left(\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial^2 N}{\partial\alpha_2\partial\alpha_1}\right) - \left(\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial\alpha_1}\right)\left(\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial\alpha_2}\right) \\
&= (1-c_1)\,A_2\,(t^{**};T)\left(\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right) + a\,(t)\,\widehat{\sigma}_t^2
\end{aligned}$$

Some additional algebra yields the formulas in Lemma A7. Q.E.D.

Alternative Decentralized Model.

In this section we prove Proposition B1 and B2 in the appendix of the paper. In addition, we also show two additional claims about the Nash equilibrium at time t^*, as discussed in footnote 21.

In order to prove Proposition B1 and B2, we need to compute the following four pricing functions:

$$M_{i,t^{**}}^{s^N=1,yes} = E\left[\left(\frac{\pi_T^{s^N=1}}{\pi_{t^{**}}^{s^N=1}}\right) B_{i,T}|\text{adopt}\right] \text{ and } M_{i,t^{**}}^{s^N=1,no} = E\left[\left(\frac{\pi_T^{s^N=1}}{\pi_{t^{**}}^{s^N=1}}\right) B_{i,T}|\text{do not adopt}\right]$$
$$M_{i,t^{**}}^{s^N=0,yes} = E\left[\left(\frac{\pi_T^{s^N=0}}{\pi_{t^{**}}^{s^N=0}}\right) B_{i,T}|\text{adopt}\right] \text{ and } M_{i,t^{**}}^{s^N=0,no} = E\left[\left(\frac{\pi_T^{s^N=0}}{\pi_{t^{**}}^{s^N=0}}\right) B_{i,T}|\text{do not adopt}\right]$$

We start with the following Lemmas:

Lemma B1: Let a common factor x follow the process

$$dx_t = -\phi x dt + \sigma dZ_{0,t}, \qquad t^{**} \leq t \leq T$$

Consider two old firms i and j. Let $\rho_{t^{**}}^i = \rho_{t^{**}}^j$, and let i invest in the old technology from t^{**} to t, and j in the new technology. Then

$$\rho_t^i = x_t + \varepsilon_t^O \text{ and } \rho_t^j = x_t + \varepsilon_t^N$$

with $x_{t^{**}} = \rho_{t^{**}}^i = \rho_{t^{**}}^j$ and where

$$d\varepsilon_t^O = \phi\left(g\left(s^O\right) - \varepsilon_t^O\right) dt \text{ and } d\varepsilon_t^N = \phi\left(g\left(s^N\right) + \psi - \varepsilon_t^N\right) dt$$

with $\varepsilon_{t^{**}}^O = \varepsilon_{t^{**}}^N = 0$. Solving the differential equation, we obtain

$$\varepsilon_t^O = g\left(s^O\right)\left(1 - e^{-\phi(t-t^{**})}\right) \text{ and } \varepsilon_t^N = \left(g\left(s^N\right) + \psi\right)\left(1 - e^{-\phi(t-t^{**})}\right)$$

Proof of Lemma B1: From Ito's Lemma, ρ_t^i and ρ_t^j satisfy (B1) and (B3) in the paper. Q.E.D.

Lemma B2: (a) Let some amount of capital B_t (resp $\widetilde{B}_t$) be invested in the old (new) economy at t^{**}. Then for $t > t^{**}$

$$B_t = B_{t^{**}} e^{\int_{t^{**}}^t x_s ds + \int_{t^{**}}^t \varepsilon_s^O ds} = B_{t^{**}} e^{\int_{t^{**}}^t x_s ds + g\left(s^O\right) A_2(\tau^{**})}$$
$$\widetilde{B}_t = \widetilde{B}_{t^{**}} e^{\int_{t^{**}}^t x_s ds + \int_{t^{**}}^t \varepsilon_s^N ds} = \widetilde{B}_{t^{**}} e^{\int_{t^{**}}^t x_s ds + \left(g\left(s^N\right) + \psi\right) A_2(\tau^{**})}$$

where $A_2(\tau)$ is as in Proposition 2.

(b) For any two constants α and β, conditional on ψ being observable:

$$E_{t^{**}}\left[B_T^\alpha \widetilde{B}_T^\beta | \psi\right] = B_{t^{**}}^\alpha \widetilde{B}_{t^{**}}^\beta e^{(\alpha+\beta)\rho_{t^{**}} A_1(\tau^{**}) + \frac{\sigma^2(\alpha+\beta)^2}{2\phi^2} A_3(\tau^{**}) + \alpha g\left(s^O\right) A_2(\tau^{**}) + \beta\left[g\left(s^N\right) + \psi\right] A_2(\tau^{**})}$$

where $A_1(\tau)$ is as in Corollary 1, and $A_3(\tau) = \tau + \frac{1-e^{-2\phi\tau}}{2\phi} - 2A_1(\tau)$

(c) If ψ is unknown and $\psi \sim N\left(\widehat{\psi}, \widehat{\sigma}^2_{t^{**}}\right)$, then

$$E_{t^{**}}\left[B_T^\alpha \widetilde{B}_T^\beta\right] = B_{t^{**}}^\alpha \widetilde{B}_{t^{**}}^\beta e^{(\alpha+\beta)\rho_{t^{**}}A_1(\tau^{**})+\frac{\sigma^2(\alpha+\beta)^2}{2\phi^2}A_3(\tau^{**})+\alpha g\left(s^O\right)A_2(\tau^{**})+\beta g\left(s^N\right)A_2(\tau^{**})+\beta\widehat{\psi}_{t^{**}}A_2(\tau^{**})+\frac{1}{2}\beta^2\widehat{\sigma}^2_{t^{**}}A_2(\tau^{**})^2}$$

Proof of Lemma B2: Part (a) follows immediately from the integral representation of stochastic differential equations. Part (b) follows from

$$\begin{aligned}
E_{t^{**}}\left[B_T^\alpha \widetilde{B}_T^\beta|\psi\right] &= E_{t^{**}}\left[\left(B_{t^{**}}^\alpha e^{\alpha\int_{t^{**}}^T x_s ds+\alpha g\left(s^O\right)A_2(\tau^{**})}\right)\left(\widetilde{B}_{t^{**}}^\alpha e^{\beta\int_{t^{**}}^T x_s ds+\beta\left[g\left(s^N\right)+\psi\right]A_2(\tau^{**})}\right)|\psi\right] \\
&= B_{t^{**}}^\alpha \widetilde{B}_{t^{**}}^\beta E_{t^{**}}\left[e^{(\alpha+\beta)\int_{t^{**}}^T x_s ds}\right] e^{\alpha g\left(s^O\right)A_2(\tau^{**})+\beta\left[g\left(s^N\right)+\psi\right]A_2(\tau^{**})} \\
&= B_{t^{**}}^\alpha \widetilde{B}_{t^{**}}^\beta e^{(\alpha+\beta)x_{t^{**}}A_1(\tau^{**})+\frac{\sigma^2(\alpha+\beta)^2}{2\phi^2}A_3(\tau^{**})+\alpha g\left(s^O\right)A_2(\tau^{**})+\beta\left[g\left(s^N\right)+\psi\right]A_2(\tau^{**})} \\
&= B_{t^{**}}^\alpha \widetilde{B}_{t^{**}}^\beta e^{(\alpha+\beta)\rho_{t^{**}}A_1(\tau^{**})+\frac{\sigma^2(\alpha+\beta)^2}{2\phi^2}A_3(\tau^{**})+\alpha g\left(s^O\right)A_2(\tau^{**})+\beta\left[g\left(s^N\right)+\psi\right]A_2(\tau^{**})}
\end{aligned}$$

where from the properties of mean reverting processes we have

$$E\left[e^{(\alpha+\beta)\int_{t^{**}}^T x_s ds}\right] = e^{(\alpha+\beta)x_{t^{**}}A_1(\tau^{**})+\frac{\sigma^2(\alpha+\beta)^2}{2\phi^2}A_3(\tau^{**})}$$

Part (c) follows from the properties of the normal distribution. Q.E.D.

Proof of Proposition B1: (a) We must compute

$$M_{i,t^{**}}^{s^N=1,yes} = E_{t^{**}}\left[\left(\frac{\pi_T^{s^N=1}}{\pi_{t^{**}}^{s^N=1}}\right)B_{i,T}|\text{adopt}\right] \quad\text{and}\quad M_{i,t^{**}}^{s^N=1,no} = E_{t^{**}}\left[\left(\frac{\pi_T^{s^N=1}}{\pi_{t^{**}}^{s^N=1}}\right)B_{i,T}|\text{do not adopt}\right]$$

If at t^{**} all firms switch, i.e. $s_{t^{**}}^N = 1$, their productivity switches to process (B3) in the paper with $g(s^N) = g(1) = \overline{\rho}$. Since all firms are identical up to t^{**}, for all firms i and j, $\rho_t^i = \rho_t^j = \rho_t$ and $B_{t^{**}}^i = B_{t^{**}}^j = B_{t^{**}}$. Thus, aggregate capital also follows the process $dB_t = \rho_t B_t dt$ where ρ_t follows the process (B3) in the paper.

As a consequence, if firm i adopts the new technology at t^{**}, then

$$\pi_T^{s^N=1} \times B_{i,T} = B_T^{-\gamma} \times B_{i,T} = B_T^{1-\gamma}.$$

(We choose $\lambda = 1$ without loss of generality.) It follows from Lemma B2, with $\alpha = 0$ and $\beta = 1-\gamma$:

$$\begin{aligned}
&E_{t^{**}}\left[\left(\pi_T^{s^N=1}\right)B_{i,T}|\text{adopt}\right] = E_{t^{**}}\left[B_T^{1-\gamma}\right] \\
&= B_{t^{**}}^{1-\gamma}(1-\kappa)^{1-\gamma} e^{(1-\gamma)\rho_{t^{**}}A_1(\tau^{**})+\frac{\sigma^2(1-\gamma)^2}{2\phi^2}A_3(\tau^{**})+(1-\gamma)\overline{\rho}A_2(\tau^{**})+(1-\gamma)\widehat{\psi}_{t^{**}}A_2(\tau^{**})+\frac{1}{2}(1-\gamma)^2\widehat{\sigma}^2_{t^{**}}A_2(\tau^{**})^2} \qquad (C50)
\end{aligned}$$

Similarly, if firm i does not adopt, then $\pi_T^{s^N=1}B_{i,T} = B_T\widetilde{B}_T^{-\gamma}$, where we use the notation of Lemma B2 to differentiate capital invested in the old and in the new technology. It follows from

Lemma B2 with $\alpha = 1$ and $\beta = -\gamma$, and the fact that at t^{**} we have $B_{t^{**}} = \widetilde{B}_{i,t^{**}}$ (before the conversion costs are paid)

$$E_{t^{**}}\left[\pi_T^{s^N=1} B_{i,T} | \text{do not adopt}\right] = E_{t^{**}}\left[B_T \widetilde{B}_T^{-\gamma}\right]$$
$$= \quad B_{t^{**}} B_{t^{**}}^{-\gamma} (1-\kappa)^{-\gamma} e^{(1-\gamma)\rho_{t^{**}} A_1(\tau^{**}) + \frac{\sigma^2(1-\gamma)^2}{2\phi^2} A_3(\tau^{**}) + \overline{\rho}_0 A_2(\tau^{**}) - \gamma\overline{\rho} A_2(\tau^{**}) - \gamma\widehat{\psi}_{t^{**}} A_2(\tau^{**}) + \frac{1}{2}\gamma^2 \widehat{\sigma}^2_{t^{**}} A_2(\tau^{**})^2} \quad \text{(C51)}$$

Since the denominators in the pricing formulas are the same, $M_{i,t^{**}}^{s^N=1,\text{yes}} > M_{i,t^{**}}^{s^N=1,\text{no}}$ if and only if (C50) is greater than (C51). The claim follow after some algebra.

(b) This part is similar. We must compute

$$M_{i,t^{**}}^{s^N=0,yes} \quad = \quad E\left[\left(\frac{\pi_T^{s^N=0}}{\pi_{t^{**}}^{s^N=0}}\right) B_{i,T} | \text{adopt}\right] \quad \text{and} \quad M_{i,t^{**}}^{s^N=0,no} = E\left[\left(\frac{\pi_T^{s^N=0}}{\pi_{t^{**}}^{s^N=0}}\right) B_{i,T} | \text{do not adopt}\right]$$

If at t^{**} all firms do not switch, i.e. $s_{t^{**}}^N = 0$, then their productivity process is given by (B1) in the paper with $g(s^0) = g(1) = \overline{\rho}$. Since all firms are identical up to t^{**}, for all firms i and j, $\rho_t^i = \rho_t^j = \rho_t$ and $B_{t^{**}}^i = B_{t^{**}}^j = B_{t^{**}}$. Thus, aggregate capital also follows the process $dB_t = \rho_t B_t dt$ where ρ_t follows the process (B1) in the paper.

As a consequence, if firm i does not adopt the new technology at t^{**}, then

$$\pi_T^{s^N=0} B_{i,T} = B_T^{1-\gamma}$$

where we use the convention of Lemma B2 to denote capital invested in the old technology. It follows from Lemma B2, with $\alpha = 1-\gamma$ and $\beta = 0$ that

$$E_{t^{**}}\left[\left(\pi_T^{s^N=0}\right) B_{i,T} | \text{do not adopt}\right] \quad = \quad E_{t^{**}}\left[B_T^{1-\gamma}\right]$$
$$= \quad B_{t^{**}}^{1-\gamma} e^{(1-\gamma)\rho_{t^{**}} A_1(\tau^{**}) + \frac{\sigma^2(1-\gamma)^2}{2\phi^2} A_3(\tau^{**}) + (1-\gamma)\overline{\rho} A_2(\tau^{**})} \quad \text{(C52)}$$

Similarly, if firm i adopts the new technology, then

$$\pi_T^{s^N=0} B_{i,T} = B_T^{-\gamma} \widetilde{B}_T$$

Then, it follows from Lemma B2 with $\alpha = -\gamma$ and $\beta = 1$, and the fact that at t^{**} $B_{t^{**}} = \widetilde{B}_{i,t^{**}}$ (before the conversion costs are paid)

$$E_{t^{**}}\left[\pi_T^{s^N=0} B_{i,T} | \text{adopt}\right] = E_{t^{**}}\left[B_T^{-\gamma} \widetilde{B}_T\right]$$
$$= \quad B_{t^{**}}^{-\gamma} B_{t^{**}} (1-\kappa) e^{(1-\gamma)\rho_{t^{**}} A_1(\tau^{**}) + \frac{\sigma^2(1-\gamma)^2}{2\phi^2} A_3(\tau^{**}) - \gamma\overline{\rho} A_2(T-t^{**}) + \overline{\rho}_0 A_2(T-t^{**}) + \widehat{\psi}_{t^{**}} A_2(\tau^{**}) + \frac{1}{2}\widehat{\sigma}^2_{t^{**}} A_2(\tau^{**})^2} \quad \text{(C53)}$$

Since the denominators of the pricing formulas are the same, $M_{i,t^{**}}^{s^N=0,yes} > M_{i,t^{**}}^{s^N=0,no}$ if and only if (C53) is greater than (C52). The claim follows after some algebra. Q.E.D.

Proof of Proposition B2: The claim follows from Proposition B1 and the definition of a Nash Equilibrium. In particular, if $\widehat{\psi}_{t^{**}} \geq \underline{\psi}$ and all other firms switch to a new technology ($s^N = 1$), then Proposition B1 (a) shows that it is optimal for an individual firm i to adopt the new technology, thereby supporting the equilibrium outcome. Similarly, if $\widehat{\psi}_{t^{**}} < \underline{\psi}$ and all firms do not switch to the new technology ($s^N = 0$), then it is optimal for each individual firm not to adopt the new technology, thereby supporting again the equilibrium outcome. Finally, if in equilibrium, either $s^N = 0$ or $s^N = 1$ at t^{**}, it follows that the pricing kernel π_t is indeed the same as in the paper and the one used in Proposition B1 to obtain the pricing formulas. Since the pricing formulas determine the optimal firm choice at t^{**} as in Proposition B1, the Nash equilibrium is established. Q.E.D.

We now prove that adopting the new technology is never an equilibrium at t^* and, under a plausible sufficient condition, that not adopting the technology at t^* is in fact a Nash Equilibrium.

Proposition B3: $s^N_{t^*} = 1$ is not an equilibrium at t^*.

Proof. The claim follows from the same argument as in Proposition B1 and B2. If $s^N = 1$ at t^*, then a firm i has the choice of adopting at t^*, adopting it at t^{**} or never. Clearly, since at t^{**} firm i will be able to choose whether to adopt it or not, the price in which the firm retains the option to adopt must be larger than the price in which the firm commits never to adopt the technology. Thus, if we show that $M^{\text{never}}_{t^*} > M^{\text{adopt}}_{t^*}$, the statement follows. The same derivations as in Proposition B1 but for the pricing formulas

$$M^{s^N_{t^*}=1,\text{adopt}}_{i,t^*} = E\left[\left(\frac{\pi_T^{s^N_{t^*}=1}}{\pi_{t^*}^{s^N_{t^*}=1}}\right) B_{i,T}|\text{adopt}\right] \quad \text{and} \quad M^{s^N_{t^*}=1,\text{never}}_{i,t^*} = E\left[\left(\frac{\pi_T^{s^N_{t^*}=1}}{\pi_{t^*}^{s^N_{t^*}=1}}\right) B_{i,T}|\text{never adopt}\right]$$

show that adopting at t^* when everybody else adopts at t^* is optimal if and only if

$$\widehat{\psi}_{t^*} > -\frac{\log(1-\kappa)}{A_2(T-t^*)} + (\overline{\rho}_0 - \overline{\rho}) + \frac{1}{2}(2\gamma - 1)\widehat{\sigma}^2_{t^*} A_2(T-t^*)$$

At time t^*, however, $\widehat{\psi}_{t^*} = 0$, while the network externality gain is given by

$$\overline{\rho} - \overline{\rho}_0 = \frac{1}{2}\gamma\widehat{\sigma}^2_{t^{**}} A_2(T-t^{**}) \tag{C54}$$

Thus, if everybody adopts at time t^* it is optimal to adopt for an individual firm if and only if

$$0 > -\frac{\log(1-\kappa)}{A_2(T-t^*)} - \frac{1}{2}\gamma\widehat{\sigma}^2_{t^{**}} A_2(T-t^{**}) + \frac{1}{2}(2\gamma-1)\widehat{\sigma}^2_{t^*} A_2(T-t^*) \tag{C55}$$

Note that if $\gamma \geq 1$, the right hand side is always greater than zero, as $\widehat{\sigma}^2_{t^*} A_2(T-t^*) > \widehat{\sigma}^2_{t^{**}} A_2(T-t^{**})$. This implies that condition (C55) is never realized. It follows that $s^N = 1$ cannot be a Nash Equilibrium: if everybody switches, it is value maximizing for an individual firm to deviate and not to switch. Q.E.D.

Showing that $s^N_{t^*} = 0$ is instead a Nash Equilibrium is more challenging, as the pricing formulas are substantially more complicated in this case. We can however prove the following limiting result.

Proposition B4: Let $s^N_{t^*} = 0$. Then, there exists $\underline{p}$ such that if $p_{t^*} < \underline{p}$, an old firm i does not switch to the new technology at t^* if

$$\gamma > \frac{\widehat{\sigma}^2_{t^*} A_2(T-t^*)}{\widehat{\sigma}^2_{t^{**}} A_2(T-t^{**})} + 2\frac{\log(1-\kappa)}{A_2(T-t^*)} \tag{C56}$$

It follows that under condition (C56), if $p_{t^*} < \underline{p}$, then $s_{t^*}^N = 0$ is a Nash Equlibrium.[1]

Proof of Proposition B4: Let $s_{t^*}^N = 0$. A firm can (a) adopt at t^*, or (b) do not adopt at t^* but retain the option to adopt at t^{**}. Suppose the firm could commit also not to ever adopt the new technology. The price of the stock in this case cannot be larger than under (b). That is, $M_{i,t^*}^{s_{t^*}^N=0,\text{ option at } t^{**}} \geq M_{i,t^*}^{s_{t^*}^N=0,\text{never}}$. Thus, if we show that $M_{i,t^*}^{s_{t^*}^N=0,\text{never}} > M_{i,t^*}^{s_{t^*}^N=0,\text{adopt}}$, then the claim follows.

Let $p_{t^*} = Pr\left(\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right)$ denote the probability of large scale adoption at t^{**}. The pricing function is continuous in this probability p_{t^*}, and this probability can be decreased to essentially zero by increasing κ. In the limiting case in which $p_{t^*} \to 0$, the pricing formulas converge to

$$M_{i,t^*}^{s_{t^*}^N=0,\text{adopt}} = E\left[\left(\frac{\pi_T^{s_{t^*}^N=1}}{\pi_{t^*}^{s_{t^*}^N=1}}\right) B_{i,T} | \text{adopt}\right] \quad \text{and} \quad M_{i,t^*}^{s_{t^*}^N=0,\text{never}} = E\left[\left(\frac{\pi_T^{s_{t^*}^N=1}}{\pi_{t^*}^{s_{t^*}^N=1}}\right) B_{i,T} | \text{never adopt}\right]$$

In this case, we know that the threshold is

$$\underline{\psi}^{s_{t^*}^N=0} = -\frac{\log(1-\kappa)}{A_2(T-t^*)} + (\overline{\rho} - \overline{\rho}_0) - \frac{1}{2}\widehat{\sigma}_{t^*}^2 A_2(T-t^*)$$

Since $\widehat{\psi}_{t^*} = 0$, no adoption is optimal if $\underline{\psi}^{s_{t^*}^N=0} > 0$. Substituting the expression for $\overline{\rho} - \overline{\rho}_0$ obtained earlier (see (C54)), we find

$$\underline{\psi}^{s_{t^*}^N=0} = -\frac{\log(1-\kappa)}{A_2(T-t^*)} + \frac{1}{2}\gamma\widehat{\sigma}_{t^{**}}^2 A_2(T-t^{**}) - \frac{1}{2}\widehat{\sigma}_{t^*}^2 A_2(T-t^*)$$

The right hand side is positive if and only if (C56) holds. It follows that under this condition, if $s_{t^*}^N = 0$ then it is not optimal for an individual firm i to adopt the new technology in the limiting case in which $p_{t^*} \to 0$. Since the price inequality is strict, by continuity of the pricing functions it follows that there is a $\underline{p}$ such that if $p_{t^*} < \underline{p}$, it is not optimal to adopt the new technology when $s_{t^*}^N = 0$. Thus, $s_{t^*}^N = 0$ is a Nash Equilibrium. Q.E.D.

[1] In our base case calibration, the right hand side of (C56) is 1.88, which is indeed smaller than $\gamma = 4$ that we use.

Separating Productivity from Profitability

In our simple model, all output represents firm profits, so productivity and profitability coincide. In reality, technological advances lead to permanent increases in productivity but only temporary increases in profitability. In the long run, new technology tends to benefit workers and consumers, not producers. Therefore, we now also analyze a richer model in which labor income drives a wedge between productivity and profitability. Specifically, we assume that while productivity gains from new technology last until time T, profitability gains last only until a random time $t^{***} \leq T$, after which all productivity gains go to labor. The key insight from this analysis is the following: while profitability affects cash flow to stocks, the permanent change in productivity affects the discount rate, because it affects the total wealth produced at time T, independently of whether it is compensation to capital or to labor. As a consequence, the adoption of the new technology results in an increase in the discount rate, and our predictions remain largely unaffected.

More in detail, as in the paper, aggregate capital evolves according to the process

$$dB_t = \rho_t B_t dt$$

Aggregate capital B_t has now two components: One component, $\widetilde{B}_t$, accrues to the firm's shareholders, while the second component, $B_t - \widetilde{B}_t$, accrues to the workers. As in the paper, there are no payouts before T. At time T, the firms' payout to shareholders is $\widetilde{B}_T$ while its payout to labor is $B_T - \widetilde{B}_T$. Since we do not modify any of the model's assumptions about the aggregate capital, the utility maximizing choice at time t^{**} is the same as in the paper, namely, the new technology will be adopted if and only if $\widehat{\psi}_{t^{**}} \geq \underline{\psi}$, as in Proposition 2. In particular, if adoption occurs at t^{**}, then productivity ρ_t follows the process (4) as in the paper. The important implication is that the stochastic discount factor used to discount the future cash flow of the firm is the one given in Proposition 4.

We identify the old economy return on equity capital as the rate of growth of $\widetilde{B}_t$, that is, the share of capital that will accrue to the firm's shareholders. We refer to $\widetilde{B}_t$ as physical capital. Denote the old economy return on capital, or profitability, by $\widetilde{\rho}_t$. Then,

$$d\widetilde{B}_t = \widetilde{\rho}_t \widetilde{B}_t dt$$

For simplicity, before t^{**}, return on capital follows the same process as productivity $\widetilde{\rho}_t = \rho_t$. So long this condition is satisfied, the ratio of physical capital to total capital remains constant over time $\widetilde{B}_t / B_t = s =$ constant.

At t^{**} the new technology is either adopted or not. If no adoption occurs at time t^{**}, then $\widetilde{\rho}_t = \rho_t$ for $t > t^{**}$ as well. In this case, we have $\widetilde{B}_T = sB_T$.

If adoption occurs at t^{**}, then $\widetilde{\rho}_t = \rho_t$ only until t^{***}, where $t^{**} \leq t^{***} \leq T$. After t^{***}, the productivity process ρ_t and the profitability process $\widetilde{\rho}_t$ diverge. In particular, while the productivity process is given by (4) in the paper, the profitability process instead reverts back to

$$d\widetilde{\rho}_t = \phi \left(\overline{\rho} - \widetilde{\rho}_t \right) dt + \sigma dZ_{0,t}$$

Note that the profitability process is continuous at t^{***}, but the average profitability declines back to $\overline{\rho}$ from $\overline{\rho} + \psi$.

We assume that t^{***} is random, with $t^{***} - t^{**}$ distributed according to an exponential distribution with parameter p. For simplicity, t^{***} is observable. Since we are interested in the price pattern *before* t^{**}, this assumption is innocuous, but it simplifies the derivations.

Proposition C1. (a) If no adoption occurs at t^{**}, then the market to book ratio of the old economy at t^{**} is identical to the one in the paper. (b) If adoption occurs at t^{**}, then

$$\frac{\widetilde{M}_{t^{**}}}{\widetilde{B}_{t^{**}}} = e^{A_0(\tau^{**})-\overline{A}_0(\tau^{**})+A_1(\tau^{**})\rho_{t^{**}}} \times \widetilde{\mathcal{V}}_2\left(\widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau^{**}\right)$$

where $A_0(\tau^{**})$ is defined in Lemma A3, and $\overline{A}_0(\tau^{**})$ in the Proof of Proposition 4, and where

$$\begin{aligned}\widetilde{\mathcal{V}}_2\left(\widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau^{**}\right) &= \int_{t^{**}}^{T} p e^{-p(t^{***}-t^{**})+\widehat{\psi}_t(A_2(\tau^{**})-A_2(T-t^{***}))+\frac{1}{2}\left[\gamma^2 A_2^2(\tau^{**})+((1-\gamma)A_2(\tau^{**})-A_2(T-t^{***}))^2\right]\widehat{\sigma}^2_{t^{**}}} dt^{***} \\ &\quad + e^{-p(T-t^{**})} e^{+\widehat{\psi}_t(A_2(\tau^{**}))+\frac{1}{2}\left[\gamma^2 A_2^2(\tau^{**})+((1-\gamma)A_2(\tau^{**}))^2\right]\widehat{\sigma}^2_{t^{**}}}\end{aligned}$$

(c) For $t < t^{**}$, the market to book ratio of the old economy is given by

$$\widetilde{M}_t/\widetilde{B}_t = e^{A_1(\tau)\rho_t} \times \widehat{\mathcal{V}}_2(t, \widehat{\psi}_t)/F_2(t, \widehat{\psi}_t),$$

where $F_2(t, \widehat{\psi}_t)$ is given in Proposition 9 (for the case in which t^{**} is exogenous), and $\widehat{\mathcal{V}}_2(t, \widehat{\psi}_t)$ satisfies the partial differential equation

$$0 = \frac{\partial \widehat{\mathcal{V}}_2}{\partial t} + \left((1-\gamma) A_1(T-t)\phi\overline{\rho} + \frac{1}{2}(1-\gamma)^2 A_1(T-t)^2\sigma^2\right)\widehat{\mathcal{V}}_2 + \frac{1}{2}\frac{\partial^2 \widehat{\mathcal{V}}_2}{\partial \widehat{\psi}^2}\left(\widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}}\right)^2,$$

with boundary condition

$$\widehat{\mathcal{V}}_2(t^{**}, \widehat{\psi}_{t^{**}}) = \begin{cases} (1-\kappa)^{1-\gamma}\widetilde{\mathcal{V}}_2\left(\widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau^{**}\right) & \text{if } \widehat{\psi}_{t^{**}} \geq \underline{\psi} \\ 1 & \text{if } \widehat{\psi}_{t^{**}} < \underline{\psi} \end{cases}$$

We now turn to the new economy. As for the old economy, we assume that at time t^{***} the excess profitability gain ψ moves back to zero. We assume that this occurs independently of the choice of the old economy at t^{**}. The profitability process for the new economy is then

$$\begin{aligned} d\widetilde{\rho}_t^N &= \phi\left(\overline{\rho} + \psi - \widetilde{\rho}_t^N\right)dt + \sigma_{N,0}dZ_{0,t} + \sigma_{N,1}dZ_{1,t} \quad \text{for } t < t^{***} \\ d\widetilde{\rho}_t^N &= \phi\left(\overline{\rho} - \widetilde{\rho}_t^N\right)dt + \sigma_{N,0}dZ_{0,t} + \sigma_{N,1}dZ_{1,t} \qquad \text{for } t > t^{***} \end{aligned}$$

Physical capital $\widetilde{B}_t^N$ for the new economy evolves according to

$$d\widetilde{B}_t^N = \widetilde{\rho}_t^N \widetilde{B}_t^N dt$$

(the new economy final wage $B_T^N - \widetilde{B}_T^N$ has no impact on the SDF, as it is infinitesimal, and thus its dynamics is irrelevant).

In this case, the computation of asset prices is substantially more complicated, except in the special case in which $\sigma_{N,0} = \sigma$. For simplicity, we focus on this special case. Note that this is the case that we consider in our calibration (see Table 1 in the paper).

Proposition C2. Let $\sigma_{N,0} = \sigma$. Then, (a) if the technology is adopted, the market to book ratio of the new economy is

$$\frac{\widetilde{M}^N_{t^{**}}}{\widetilde{B}^N_{t^{**}}} = e^{\left[C_0(\tau^{**}) - \overline{A}_0(\tau^{**})\right] + A_1(\tau^{**})\rho^N_{t^{**}}} \times \widetilde{F}^N_2\left(\widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau^{**}\right)$$

where $C_0(\tau)$ is given in Lemma A6 (with $\sigma = \sigma_{N,0}$), and

$$\begin{aligned}\widetilde{F}^N_2\left(\widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau^{**}\right) &= \int_{t^{**}}^{T} pe^{-p(t^{***}-t^{**})} e^{+(A_2(\tau^{**}) - A_2(T-t^{***}))\widehat{\psi}_{t^{**}} + \frac{1}{2}\left[\gamma^2 A_2^2(\tau^{**}) + ((1-\gamma)A_2(\tau^{**}) - A_2(T-t^{***}))^2\right]\widehat{\sigma}^2_{t^{**}}} dt^{***} \\ &\quad + e^{-p(T-t^{**})} e^{A_2(\tau^{**})\widehat{\psi}_{t^{**}} + \frac{1}{2}\left[\gamma^2 A_2^2(\tau^{**}) + ((1-\gamma)A_2(\tau^{**}))^2\right]\widehat{\sigma}^2_{t^{**}}}\end{aligned}$$

(b) if the technology is not adopted, the market to book ratio of the new economy is

$$\frac{\widetilde{M}^N_{t^{**}}}{\widetilde{B}^N_{t^{**}}} = e^{\left[C_0(\tau^{**}) - \overline{A}_0(\tau^{**})\right] + A_1(\tau^{**})\rho^N_{t^{**}}} \times \widehat{F}^N_2\left(\widehat{\psi}_{t^{**}}, \widetilde{\sigma}^2_{t^{**}}, \tau^{**}\right)$$

where

$$\begin{aligned}\widehat{F}^N_2\left(\widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau^{**}\right) &= \int_{t^{**}}^{T} pe^{-p(t^{***}-t^{**})} e^{+A_5(t^{**},t^{***},T)\widehat{\psi}_{t^{**}} + \frac{1}{2}A_5^2(t^{**},t^{***},T)\widehat{\sigma}^2_{t^{**}}} dt^{***} \\ &\quad + e^{-p(T-t^{**})} e^{A_2(\tau^{**})\widehat{\psi}_{t^{**}} + \frac{1}{2}A_5^2 A_2(\tau^{**})\widehat{\sigma}^2_{t^{**}}}\end{aligned}$$

(c) For $t < t^{**}$, the market to book ratio of the new economy is given by

$$\widetilde{M}^N_t / \widetilde{B}^N_t = e^{A_1(\tau)\rho^N_t} \times F^N_2(t, \widehat{\psi}_t)/F_2(t, \widehat{\psi}_t),$$

where $F^N_2(t, \widehat{\psi}_t)$ satisfies the partial differential equation (C49), with boundary condition

$$F^N_2(t^{**}, \widehat{\psi}_{t^{**}}) = \begin{cases} (1-\kappa)^{-\gamma}\widetilde{F}^N_2\left(\widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau^{**}\right) & \text{if } \widehat{\psi}_{t^{**}} \geq \underline{\psi} \\ \widehat{F}^N_2\left(\widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau^{**}\right) & \text{if } \widehat{\psi}_{t^{**}} < \underline{\psi} \end{cases}$$

Proof of Proposition C1: If no adoption occurs at time t^{**}, there is no difference between productivity and profitability. It follows that $\widetilde{B}_T/B_T = \widetilde{B}_{t^*}/B_{t^*} = s$. Since the SDF depends on total capital B_T as well, we find:

$$\widetilde{M}_{t^{**}} = = \frac{E_{t^{**}}\left[B_T^{-\gamma}\widetilde{B}_T|\text{no adoption}\right]}{E_{t^{**}}\left[B_T^{-\gamma}|\text{no adoption}\right]} = \frac{\widetilde{B}_{t^{**}}}{B_{t^{**}}}\frac{E_{t^{**}}\left[B_T^{-\gamma}B_T|\text{no adoption}\right]}{E_{t^{**}}\left[B_T^{-\gamma}|\text{no adoption}\right]} = \frac{\widetilde{B}_{t^{**}}}{B_{t^{**}}}M_{t^{**}}$$

where $M_{t^{**}}$ denotes the discounted value of aggregate capital, i.e., the price as in the paper.

Consider now the case in which the new technology is adopted. In this case, from t^{***} a wedge appears between productivity and profitability. In particular, define the difference between productivity and profitability $x_t = \rho_t - \widetilde{\rho}_t$. It easy to see that for $t > t^{***}$,

$$x_t = \psi\left(1 - e^{-\phi(t-t^{***})}\right)$$

Thus, since by definition $\widetilde{\rho}_t = \rho_t - x_t$, we can write the firm capital process from t^{***} onward as $d\widetilde{B}_t = (\rho_t - x_t)\widetilde{B}_t dt$. This implies that the physical capital at T is

$$\begin{aligned}\widetilde{B}_T &= \widetilde{B}_{t^{**}} e^{\int_{t^{**}}^T \rho_u du - \int_{t^{***}}^T x_u du} = \widetilde{B}_{t^{**}} e^{\int_{t^{**}}^T \rho_u du} e^{-\psi \int_{t^{***}}^T \left(1-e^{-\phi(u-t^{***})}\right) du} = \widetilde{B}_{t^{**}} e^{\int_{t^{**}}^T \rho_u du} e^{-\psi A_2(\tau^{***})} \\ &= \left(\frac{\widetilde{B}_{t^{**}}}{B_{t^{**}}}\right) B_T e^{-\psi A_2(\tau^{***})}\end{aligned}$$

where the last equality stems from $B_T = B_{t^{**}} e^{\int_{t^{**}}^T \rho_u du}$. For $t \geq t^{**}$, it is easy to see that we can also decompose productivity as $\rho_t = \varepsilon_t + (\overline{\rho} + \psi)\left(1 - e^{-\phi(t-t^{**})}\right)$, where $\varepsilon_{t^{**}} = \rho_{t^{**}}$ and $d\varepsilon_t = -\phi\varepsilon_t dt + \sigma dZ_{0,t}$. Thus, we can write

$$B_T = B_{t^{**}} e^{\int_{t^{**}}^T \rho_u du} = B_{t^{**}} e^{\int_{t^{**}}^T \varepsilon_u du + \int_{t^{**}}^T (\overline{\rho}+\psi)\left(1-e^{-\phi(u-t^{**})}\right) du} = B_{t^{**}} e^{\int_{t^{**}}^T \varepsilon_u du + \overline{\rho} A_2(\tau^{**}) + \psi A_2(\tau^{**})}$$

It follows that for the old economy, we obtain

$$\begin{aligned}E_{t^{**}}\left[\pi_T \widetilde{B}_T | \text{adopt}, t^{***}\right] &= \left(\frac{\widetilde{B}_{t^{**}}}{B_{t^{**}}}\right) E_{t^{**}}\left[B_T^{1-\gamma} e^{-\psi A_2(\tau^{***})} | \text{adopt}, t^{***}\right] \\ &= \left(\frac{\widetilde{B}_{t^{**}}}{B_{t^{**}}}\right) E_{t^{**}}\left[B_{t^{**}}^{1-\gamma} e^{(1-\gamma)\int_{t^{**}}^T \varepsilon_u du + (1-\gamma)\overline{\rho} A_2(\tau^{**}) + (1-\gamma)\psi A_2(\tau^{**}) - \psi A_2(\tau^{***})} | \text{adopt}, t^{***}\right] \\ &= \widetilde{B}_{t^{**}} B_{t^{**}}^{-\gamma} E_{t^{**}}\left[e^{(1-\gamma)\int_{t^{**}}^T \varepsilon_u du + (1-\gamma)\overline{\rho} A_2(\tau^{**}) + \psi((1-\gamma)A_2(\tau^{**}) - A_2(\tau^{***}))} | \text{adopt}, t^{***}\right] \\ &= \widetilde{B}_{t^{**}} B_{t^{**}}^{-\gamma} e^{A_0(\tau^{**}) + (1-\gamma)A_1(\tau^{**})\rho_{t^{**}} + \widehat{\psi}_t((1-\gamma)A_2(\tau^{**}) - A_2(\tau^{***})) + \frac{1}{2}((1-\gamma)A_2(\tau^{**}) - A_2(\tau^{***}))^2 \widehat{\sigma}_{t^{**}}^2}\end{aligned}$$

The claim follows from this equation after (i) we integrate the right hand side over all possible t^{***}, which is distributed according to an exponential distribution with probability p,[2] and (ii) we divide by the SDF π_t, which is the same as in the paper.

(c) The pricing value before t^{**} follows from the same argument as in Proposition 10, since before t^{**} there are no differences between this model and the one in the paper (note that the PDE in part (c) equals the PDE (C45) in Proposition 8, as the $\mathcal{V}_2$ and $\widetilde{\mathcal{V}}_2$ are proportional to each other). The only difference is given by the boundary condition at t^{**}, which are given explicitly in the proposition. Q.E.D.

Proof of Proposition C2: (a) In case of adoption, the SDF is

$$\pi_T = B_T^{-\gamma} = B_{t^{**}}^{-\gamma} e^{\int_{t^{**}}^T -\gamma\varepsilon_u du - \gamma\overline{\rho} A_2(\tau^{**}) - \gamma\psi A_2(\tau^{**})}.$$

Moreover, we can write $\rho_t^N = \rho_t - x_t + \widetilde{\varepsilon}_t$ where $\widetilde{\varepsilon}_{t^{**}} = \rho_{t^{**}}^N - \rho_{t^{**}}$, x_t is zero between t^{**} and t^{***} and then it grows according to $dx_t = \phi(\psi - x_t)dt$ and[3]

$$d\widetilde{\varepsilon}_t = -\phi\widetilde{\varepsilon}_t dt + \sigma_{N,1} dZ_{1,t} \quad \text{(C57)}$$

Thus, we can write

$$\widetilde{B}_T^N = \widetilde{B}_{t^{**}}^N e^{\int_{t^{**}}^T \rho_u^N du} = \widetilde{B}_{t^{**}}^N e^{\int_{t^{**}}^T \rho_u du - \int_{t^{***}}^T x_u du +} e^{\int_{t^{**}}^T \widetilde{\varepsilon}_u du} = \widetilde{B}_{t^{**}}^N e^{\int_{t^{**}}^T \varepsilon_u du + \overline{\rho} A_2(\tau^{**}) + \psi A_2(\tau^{**}) - \psi A_2(\tau^{***})} e^{\int_{t^{**}}^T \widetilde{\varepsilon}_u du}$$

[2] In particular, we must be careful to assign probability $Pr(t^{***} > T) = e^{-p(T-t^{**})}$ to the expression with $t^{***} = T$ (which reduces to the same formulas in the paper)

[3] This can be verified by an application of Ito's Lemma.

where the last equality follows from $\rho_t = \varepsilon_t + (\overline{\rho} + \psi)\left(1 - e^{-\phi(t - t^{**})}\right)$, where recall that $\varepsilon_{t^{**}} = \rho_{t^{**}}$. It follows that conditioning on t^{***}

$$
\begin{aligned}
& E\left[\pi_T \widetilde{B}_T^N | \text{adopt}, t^{***}\right] = E_{t^{**}}\left[B_T^{-\gamma} \widetilde{B}_T^N | \text{adopt}, t^{***}\right] \\
= \;& B_{t^{**}}^{-\gamma} \widetilde{B}_{t^{**}}^N E_{t^{**}}\left[e^{(1-\gamma)\int_{t^{**}}^T \varepsilon_u du + (1-\gamma)\overline{\rho} A_2(\tau^{**}) + (1-\gamma)\psi A_2(\tau^{**}) - \psi A_2(\tau^{***})} | \text{adopt}, t^{***}\right] \times \\
& \times E\left[e^{\int_{t^{**}}^T \widetilde{\varepsilon}_u du} | \text{adopt}, t^{***}\right] \\
= \;& B_{t^{**}}^{-\gamma} \widetilde{B}_{t^{**}}^N e^{C_0(\tau^{**}) + (1-\gamma)A_1(\tau^*)\varepsilon_{t^{**}} + A_1(\tau^{**})\widetilde{\varepsilon}_{t^{**}} + ((1-\gamma)A_2(\tau^{**}) - A_2(\tau^{***}))\widehat{\psi}_{t^{**}} + \frac{1}{2}((1-\gamma)A_2(\tau^{**}) - A_2(\tau^{***}))^2 \widehat{\sigma}_{t^{**}}^2} \\
= \;& B_{t^{**}}^{-\gamma} \widetilde{B}_{t^{**}}^N e^{C_0(\tau^{**}) - \gamma A_1(\tau^*)\rho_{t^{**}} + A_1(\tau^{**})\rho_{t^{**}}^N + ((1-\gamma)A_2(\tau^{**}) - A_2(\tau^{***}))\widehat{\psi}_{t^{**}} + \frac{1}{2}((1-\gamma)A_2(\tau^{**}) - A_2(\tau^{***}))^2 \widehat{\sigma}_{t^{**}}^2}
\end{aligned}
$$

where $C_0(\tau)$ is in Lemma A6, and the last equality stems from $\varepsilon_{t^{**}} = \rho_{t^{**}}$ and $\widetilde{\varepsilon}_{t^{**}} = \rho_{t^{**}}^N - \rho_{t^{**}}$. The claim follows from the following additional steps: (i) integrate the right hand side over all possible t^{***}, which is distributed according to an exponential distribution with probability p, and (ii) we divide by the SDF π_t, which is the same as in the paper.

Part (b) is obtained in a similar fashion. In particular, in this case, the SDF is $\pi_T = B_T^{-\gamma} = B_{t^{**}}^{-\gamma} e^{\int_{t^{**}}^T -\gamma \varepsilon_u du - \gamma \overline{\rho} A_2(\tau^{**})}$. In this case we can write $\rho_t^N = \rho_t + x_t + \widetilde{\varepsilon}_t$ where $\widetilde{\varepsilon}_{t^{**}} = \rho_{t^{**}}^N - \rho_{t^{**}}$, where[4]

$$
x_t = \begin{cases} \psi\left(1 - e^{-\phi(t - t^{**})}\right) & t \in [t^{**}, t^{***}) \\ \psi\left(1 - e^{-\phi(t^{***} - t^{**})}\right) e^{-\phi(t - t^{***})} & t \geq t^{***} \end{cases}
$$

and $\widetilde{\varepsilon}_t$ follows (C57). This implies that

$$
\widetilde{B}_T^N = \widetilde{B}_{t^{**}}^N e^{\int_{t^{**}}^T \rho_u^N du} = \widetilde{B}_{t^{**}}^N e^{\int_{t^{**}}^T \rho_u du + \int_{t^{**}}^T x_u du +} e^{\int_{t^{**}}^T \widetilde{\varepsilon}_u du} = \widetilde{B}_{t^{**}}^N e^{\int_{t^{**}}^T \varepsilon_u du + \overline{\rho} A_2(\tau^{**}) + \psi A_5(t^{**}, t^{***}, T)} e^{\int_{t^{**}}^T \widetilde{\varepsilon}_u du}
$$

where

$$
A_5(t^{**}, t^{***}, T) = \int_{t^{**}}^T x_u du = t^{***} - t^{**} - \frac{\left(1 - e^{-\phi(t^{***} - t^{**})}\right)}{\phi} e^{-\phi(T - t^{***})}
$$

It follows that

$$
\begin{aligned}
& E\left[\pi_T \widetilde{B}_T^N | \text{no adoption}, t^{***}\right] = E_{t^{**}}\left[B_T^{-\gamma} \widetilde{B}_T^N | \text{no adoption}, t^{***}\right] \\
= \;& B_{t^{**}}^{-\gamma} \widetilde{B}_{t^{**}}^N E_{t^{**}}\left[e^{(1-\gamma)\int_{t^{**}}^T \varepsilon_u du + (1-\gamma)\overline{\rho} A_2(\tau^{**}) + \psi A_5(t^{**}, t^{***}, T)} | \text{no adoption}, t^{***}\right] \times \\
& \times E\left[e^{\int_{t^{**}}^T \widetilde{\varepsilon}_u du} | \text{no adoption}, t^{***}\right] \\
= \;& B_{t^{**}}^{-\gamma} \widetilde{B}_{t^{**}}^N e^{(1-\gamma)C_0(\tau^{**}) + (1-\gamma)A_1(\tau^{**})\varepsilon_{t^{**}} + A_1(\tau^{**})\widetilde{\varepsilon}_{t^{**}} + A_5(t^{**}, t^{***}, T)\widehat{\psi}_{t^{**}} + \frac{1}{2}A_2^2(t^{**}, t^{***}, T)\widehat{\sigma}_{t^{**}}^2} \\
= \;& B_{t^{**}}^{-\gamma} \widetilde{B}_{t^{**}}^N e^{C_0(\tau^{**}) - \gamma A_1(\tau^{**})\rho_{t^{**}} + A_1(\tau^{**})\rho_{t^{**}}^N + A_5(t^{**}, t^{***}, T)\widehat{\psi}_{t^{**}} + \frac{1}{2}A_5^2(t^{**}, t^{***}, T)\widehat{\sigma}_{t^{**}}^2}
\end{aligned}
$$

where the last equality follows from $\varepsilon_{t^{**}} = \rho_{t^{**}}$ and $\widetilde{\varepsilon}_{t^{**}} = \rho_{t^{**}}^N - \rho_{t^{**}}$. The claim follows from (i) integrating the right hand side over all possible t^{***}, which is distributed according to an exponential distribution with intensity p, and (ii) dividing by the SDF π_t, which is the same as in the paper for the case of no adoption.

(c) The pricing value before t^{**} follows from the same argument as in Proposition 10, since before t^{**} there are no differences between this model and the one in the paper. The only difference is given by the boundary condition at t^{**}, which are given explicitly in the proposition. Q.E.D.

[4]This claim can be verified by Ito's Lemma.