Technical Appendix to Accompany Was There a Nasdaq Bubble in the Late 1990s?

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June 7, 2005

Appendix

(A) The Stochastic Discount Factor

The properties of the SDF are described in detail in Pástor and Veronesi (2005; PV). This appendix contains a brief summary. The process in equation (10) implies a normal unconditional distribution for y_t with mean $\overline{y}$ and variance $\sigma_y^2/2k_y$. Let $y_D = \overline{y} - 4\sigma_y/\sqrt{2k_y}$ and $y_U = \overline{y} + 4\sigma_y/\sqrt{2k_y}$ be the boundaries between which y_t lies 99.9% of the time. To ensure that s_t (log surplus) conforms to the economic intuition of a habit formation model, PV impose the following parametric restrictions: $a_2 < 0$, $a_1 > -2a_2 y_U$ and $a_0 < 1/4\left(a_1^2/a_2\right)$. The resulting process for the stochastic discount factor $\pi_t = e^{-\eta t - \gamma(\varepsilon_t + s_t)}$ is given by

$$d\pi_t = -r_{f,t}\pi_t dt - \pi_t \sigma_{\pi,t} dW_{0,t},$$

where

$$r_{f,t} = R_0 + R_1 y_t + R_2 y_t^2,$$

with

$$\begin{aligned} R_0 &= \eta + \gamma\mu_\varepsilon + \gamma a_1 k_y \overline{y} - \frac{1}{2}\gamma^2\sigma_\varepsilon^2 + (\gamma a_2 - \frac{1}{2}\gamma^2 a_1^2)\sigma_y^2 - \gamma^2 a_1 \sigma_\varepsilon \sigma_y \\ R_1 &= \gamma\left(2a_2 k_y \overline{y} - a_1 k_y - 2a_2\gamma\left(\sigma_\varepsilon\sigma_y + a_1\sigma_y^2\right)\right) \\ R_2 &= 2a_2\gamma\left(-k_y - \gamma a_2 \sigma_y^2\right) \end{aligned}$$

and

$$\sigma_{\pi,t} = \gamma\left(\sigma_\varepsilon + \left(a_1 + 2a_2 y_t\right)\sigma_y\right). \tag{A1}$$

The parameter restrictions imposed earlier imply that $\sigma_{\pi,t}$ decreases as y_t increases. As a result, expected return and return volatility are low when y_t is high. See PV for more details.

(B) Proofs

Lemma 1: Let $\widetilde{b}_t$ follow the process

$$d\widetilde{b}_t = \left(\zeta_0\overline{\rho}_t + \zeta_1\rho_t^i - \zeta_2\right)dt,$$

where ρ_t^i and $\overline{\rho}_t$ follow the processes in equations (2) and (4), and ζ_i are constants. Define $\mathbf{Y}_t = \left(v\widetilde{b}_t - \gamma\varepsilon_t, y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i\right)'$ and $g\left(\mathbf{Y}_T\right) = e^{Y_{1,T} - \gamma a_1 Y_{2,T} - \gamma a_2 Y_{2,T}^2}$, where v is a constant, $Y_{i,t}$ denotes the i-th element of $\mathbf{Y}_t$, and γ, a_1, and a_2 are taken from equations (8) and (9). Then

$$E_t\left[e^{-\eta(T-t)} g\left(\mathbf{Y}_T\right) |\overline{\psi}_t^i\right] \equiv H\left(\mathbf{Y}_t, t; T\right) = e^{K_0(t;T) + \mathbf{K}(t;T)'\cdot\mathbf{Y}_t + K_6(t;T)Y_{2,t}^2} \tag{A2}$$

where $K_0(t;T)$, $\mathbf{K}(t;T) = \left(K_1(t;T), .., K_5(t;T)\right)'$, and $K_6(t;T)$ satisfy a system of ordinary differential equations (ODE)

$$\frac{dK_6(t;T)}{dt} = -2K_6^2(t;T)\sigma_y^2 + 2K_6(t;T)k_y \tag{A3}$$

$$\left(\frac{d\mathbf{K}(t;T)}{dt}\right)' = -\mathbf{K}(t;T)'\cdot\left[\mathbf{B}_Y + 2K_6(t;T)\left[\mathbf{\Sigma}_Y\mathbf{\Sigma}_Y'\right]_2 \mathbf{e}_2\right] - 2K_6(t;T)k_y\overline{y}\mathbf{e}_2 \tag{A4}$$

$$\frac{dK_0(t;T)}{dt} = \eta - \mathbf{K}(t;T)'\cdot\mathbf{A}_Y - \frac{1}{2}\mathbf{K}(t;T)'\mathbf{\Sigma}_Y\mathbf{\Sigma}_Y'\mathbf{K}(t;T) - K_6(t;T)\sigma_y^2 \tag{A5}$$

subject to the final condition $K_6(T;T) = -\gamma a_2$, $\mathbf{K}(T;T) = (1, -\gamma a_1, 0, 0, 0)$, and $K_0(T;T) = 0$. In the above, $\mathbf{e}_2 = (0, 1, 0, .., 0)$, and

$$\mathbf{A}_Y = \begin{pmatrix} -\gamma\mu_\epsilon - v\zeta_2 \\ k_y\overline{y} \\ k_L\overline{\rho}_L \\ 0 \\ 0 \end{pmatrix}; \mathbf{B}_Y = \begin{pmatrix} 0 & 0 & v\zeta_0 & v\zeta_1 & 0 \\ 0 & -k_y & 0 & 0 & 0 \\ 0 & 0 & -k_L & 0 & 0 \\ 0 & 0 & \phi^i & -\phi^i & \phi^i \\ 0 & 0 & 0 & 0 & -k_\psi \end{pmatrix}; \mathbf{\Sigma}_Y = \begin{pmatrix} -\gamma\sigma_\epsilon & 0 & 0 \\ \sigma_y & 0 & 0 \\ \sigma_{L,0} & \sigma_{L,L} & 0 \\ \sigma_{i,0} & 0 & \sigma_{i,i} \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof of Lemma 1: From the definition of the vector $\mathbf{Y}_t$, we have

$$d\mathbf{Y}_t = (\mathbf{A}_Y + \mathbf{B}_Y\mathbf{Y}_t)\,dt + \mathbf{\Sigma}_Y d\mathbf{W}_t.$$

The Feynman-Kac theorem implies that $H(\mathbf{Y}_t, t)$ from (A2) solves the partial differential equation

$$\frac{\partial H}{\partial t} + \sum_{i=1}^{5}\left(\frac{\partial H}{\partial Y_i}\right)[\mathbf{A}_Y + \mathbf{B}_Y\mathbf{Y}_t]_i + \frac{1}{2}\sum_{i=1}^{5}\sum_{j=1}^{5}\frac{\partial^2 H}{\partial Y_i \partial Y_j}\left[\mathbf{\Sigma}_Y\mathbf{\Sigma}_Y'\right]_{ij} = \eta H \tag{A6}$$

subject to the boundary condition

$$H(\mathbf{Y}_T, T; T) = g(\mathbf{Y}_T). \tag{A7}$$

It can be easily verified that the exponential quadratic function (A2) indeed satisfies (A6) subject to (A7), as long as $K_0(t;T)$, $\mathbf{K}(t;T)$, and $K_6(t;T)$ are the solutions to the system of ODEs in (A3) through (A5) under the final conditions presented in the claim of the Lemma.■

Lemma 2. If average excess profitability $\overline{\psi}_t^i$ is observable and T_i is known, the firm's ratio of market value to book value of equity is given by

$$\frac{M_t^i}{B_t^i} = c^i \int_0^{T_i - t} \widetilde{Z}^i\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s\right) ds + \widetilde{Z}^i\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, T_i - t\right),$$

where

$$\widetilde{Z}^i\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s\right) = e^{\widetilde{Q}_0(s) + \mathbf{Q}(s)'\cdot\mathbf{N}_t + Q_5(s)y_t^2},$$

and where $\mathbf{N}_t = \left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i\right)$ is the vector of state variables characterizing firm i, $\widetilde{Q}_0(s) = K_0(0;s)$, $Q_i(s) = K_{i+1}(0;s)$ for $i = 2, 3, 4$, $Q_1(s) = K_2(0;s) + \gamma a_1$, and $Q_5(s) = K_6(0;s) + \gamma a_2$, where $K_i(.;s)$ are in Lemma 1, all for the parameterization $\zeta_0 = 0$, $\zeta_1 = v = 1$, and $\zeta_2 = c^i$.

Proof of Lemma 2: Let $t = 0$, for notational simplicity. For given T_i, the pricing formula is

$$M_0^i = E_0\left[\int_0^{T_i}\frac{\pi_s}{\pi_0}D_s^i ds\right] + E_0\left[\frac{\pi_{T_i}}{\pi_0}B_{T_i}\right] = c^i\int_0^{T_i} E_0\left[\frac{\pi_s}{\pi_0}B_s^i\right]ds + E_0\left[\frac{\pi_{T_i}}{\pi_0}B_{T_i}\right].$$

We need to compute the folowing expectation:

$$E_0\left[\frac{\pi_s}{\pi_0}B_s^i\right] = e^{\gamma\varepsilon_0 + \gamma a_1 y_0 + \gamma a_2 y_0^2}E_0\left[e^{-\eta s}e^{b_s^i - \gamma\varepsilon_s - \gamma a_1 y_s - \gamma a_2 y_s^2}\right] = e^{\gamma\varepsilon_0 + \gamma a_1 y_0 + \gamma a_2 y_0^2}H(\mathbf{Y}_0, 0; s),$$

where the H function is given in equation (A2). Since $\mathbf{B}_Y$ has only zeros in its first column, we have $\left[\mathbf{K}(t;T_i)' \cdot \mathbf{B}_Y\right]_1 = 0$ in equation (A4). This implies $\frac{dK_1(t;T_i)}{dt} = 0$ and thus $K_1(t;T_i) = 1$ for $t \leq T_i$. By substituting in $H(\mathbf{Y}_0, 0; s)$, we obtain

$$E_0\left[\frac{\pi_s}{\pi_0}B_s^i\right] = e^{\gamma\varepsilon_0+\gamma a_1 y_0+\gamma a_2 y_0^2} \times H(\mathbf{Y}_0, 0; s) = B_0^i \times e^{\gamma a_1 y_0+\gamma a_2 y_0^2} \times e^{K_0(0;s)+\sum_{i=2}^5 K_i(0;s)Y_{i,0}+K_6(0;s)Y_{2,0}^2}$$

This expression leads immediately to the claim upon redefinition of the variables. ■

Proof of Proposition 1:

The density of the exponential distribution is $h(s,p) = pe^{-ps}$. We assume throughout that parameters are chosen such that $Q_0(s) = -ps + \widetilde{Q}_0(s) \to -\infty$, and all $Q_i(s)$ for $i \neq 0$ converge to finite numbers, where $\widetilde{Q}_0(s)$ and $Q_i(s)$ are defined in Lemma 2. Such parameters exist, because $\mathbf{B}_Y$ in Lemma 1 has negative eigevalues, and thus the convergence conditions are met for instance if $\mathbf{\Sigma}_Y = \mathbf{0}$. We now prove Proposition 1 under these conditions.

For given T, the expected discounted value of the future cash flow is given in Lemma 2:

$$E_t\left[\int_t^T \frac{\pi_\tau}{\pi_t} D_\tau^i d\tau | T\right] + E_t\left[\frac{\pi_T}{\pi_t} B_T^i | T\right] = B_t^i c^i \int_t^T \widetilde{Z}^i\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s-t\right) ds + B_t^i \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, T-t\right)$$

Integrating over all possible T's, the value of the stock today is given by

$$M_t^i = B_t^i c^i \int_t^\infty \left(pe^{-p(T-t)}\right) \int_t^T \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s-t\right) ds\, dT + B_t^i \int_t^\infty pe^{-p(T-t)} \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, T-t\right) dT. \tag{A8}$$

Using integration by parts and recalling that $\int pe^{-p(T-t)} dT = \frac{p}{-p} e^{-p(T-t)} = -e^{-p(T-t)}$, we find

$$\begin{aligned}\int_t^\infty \left(pe^{-p(T-t)}\right) \int_t^T \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s-t\right) ds\, dT &= \left[-e^{-p(T-t)} \int_t^T \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s-t\right) ds\right]_{T=t}^{T=\infty} \\ &\quad - \int_t^\infty -e^{-p(T-t)} \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, T-t\right) dT.\end{aligned}$$

Under the assumption stated earlier ($Q_0(s) \to -\infty$ and $Q_i(s)$'s converge to finite numbers), we have $e^{-p(T-t)} \int_t^T \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s-t\right) ds \to 0$ as $T \to \infty$. From equation (A5), the leading term in $\tilde{Q}_0(s)$ is linear in s, while the other terms converge to finite numbers. Thus, the properties of the integral $\int_t^T \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s-t\right) ds$ as $T \to \infty$ are determined by a term of the form $\int_t^T e^{m(s-t)} ds$ for some constant m determined as part of the solution of (A5). Under the assumptions stated earlier, $e^{-p(T-t)} \int_t^T e^{m(s-t)} ds = 1/m \left(e^{(-p+m)(T-t)} - e^{-p(T-t)}\right) \to 0$ as $T \to \infty$. Thus,

$$\int_t^\infty \left(pe^{-p(T-t)}\right) \int_t^T \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, s-t\right) ds dT = \int_t^\infty e^{-p(T-t)} \widetilde{Z}\left(y_t, \overline{\rho}_t, \rho_t^i, \overline{\psi}_t^i, T-t\right) dT.$$

Substituting this back into equation (A8), we find the relation (13) in Proposition 1. ■

Proof of Proposition 2: By the law of iterated expectations, the pricing function is

$$M_t^i = E_t\left[\int_t^{T_i} \frac{\pi_s}{\pi_t} D_s^i ds + \frac{\pi_{T_i}}{\pi_t} B_{T_i}^i\right] = E_t\left[E_t\left[\int_t^{T_i} \frac{\pi_s}{\pi_t} D_s^i ds + \frac{\pi_{T_i}}{\pi_t} B_{T_i}^i | \overline{\psi}_t^i\right]\right].$$

The inner expectation is computed in Proposition 1. Thus

$$M_t^i = B_t^i \left(c^i + p\right) \times E_t \left[\int_0^\infty e^{Q_0(s) + \mathbf{Q}(s)' \cdot \mathbf{N}_t + Q_5(s) y_t^2} ds\right].$$

Under the assumptions stated in the proof of Proposition 1, the integral exists. The only variable in $\mathbf{N}_t$ that is not known at t is $\overline{\psi}_t$. The claim of Proposition 2 then follows from the rules of the lognormal distribution, as

$$E_t \left[e^{Q_4(s)\overline{\psi}_t^i}\right] = e^{E\left[Q_4(s)\overline{\psi}_t^i\right] + \frac{1}{2} Var\left[Q_4(s)\overline{\psi}_t^i\right]} = e^{Q_4(s)\widehat{\psi}_t^i + \frac{1}{2} Q_4^2(s)\widehat{\sigma}_{i,t}^2}. \quad \blacksquare$$

Lemma 3 (Learning). Let $\overline{\psi}_t^i$ follow equation (5), $\mathbf{Z}_t = \left(\rho_t^i, \overline{\rho}_t, y_t\right)'$, and the prior distribution of $\overline{\psi}_t^i$ at $t = 0$ be normal, $N(\widehat{\psi}_0^i, \widehat{\sigma}_{i,0}^2)$. The posterior of $\overline{\psi}_t^i$ conditional on $\mathcal{F}_t = \{\mathbf{Z}_\tau : 0 \leq \tau \leq t\}$ is also normal, and the posterior moments $\widehat{\psi}_t^i = E_t \left[\overline{\psi}_t^i\right]$ and $\widehat{\sigma}_{i,t}^2 = E_t \left[\left(\overline{\psi}_t^i - \widehat{\psi}_t^i\right)^2\right]$ at $t > 0$ follow

$$d\widehat{\psi}_t^i = -k_\psi \widehat{\psi}_t^i dt + \widehat{\sigma}_{i,t}^2 \left(\frac{\phi^i}{\sigma_{ii}}\right) d\widehat{W}_{i,t} \tag{A9}$$

$$\frac{d\widehat{\sigma}_{i,t}^2}{dt} = -2k_\psi \widehat{\sigma}_{i,t}^2 - \left(\widehat{\sigma}_{i,t}^2\right)^2 \left(\frac{\phi^i}{\sigma_{ii}}\right)^2. \tag{A10}$$

Above, $\widehat{W}_{i,t}$ is the third entry in the vector of expectation errors, $\widehat{\mathbf{W}}_t = \left[\widehat{W}_{0,t}, \widehat{W}_{L,t}, \widehat{W}_{i,t}\right]$, which follows $d\widehat{\mathbf{W}}_t = \mathbf{\Sigma}_Z^{-1} \left[d\mathbf{Z}_t - E_t \left(d\mathbf{Z}_t\right)\right]$. To obtain the dynamics of $\mathbf{Z}_t$, we can define matrices $\mathbf{A}_Z$, $\mathbf{B}_Z$, $\mathbf{C}_Z$ and $\mathbf{\Sigma}_Z$ such that equations (2), (4), and (10) can be combined into one as

$$d\mathbf{Z}_t = \left(\mathbf{A}_Z + \mathbf{B}_Z \mathbf{Z}_t + \mathbf{C}_Z \overline{\psi}_t^i\right) dt + \mathbf{\Sigma}_Z d\mathbf{W}_t,$$

where $\mathbf{W}_t = [W_{0,t}, W_{L,t}, W_{i,t}]$. Proof of Lemma 3 follows from Liptser and Shiryayev (1977).

Expected Return and Volatility. Let $M_t^i / B_t^i \equiv \Phi^i \left(\rho_t^i, \overline{\rho}_t, y_t, \widehat{\psi}_t^i, \widehat{\sigma}_{i,t}^2\right)$, following Proposition 2. Ito's Lemma implies that firm i's return volatility is given by $\sqrt{\sigma_R^i \sigma_R^{i\prime}}$, where

$$\sigma_R^i = \frac{1}{\Phi^i} \left(\frac{\partial \Phi^i}{\partial y_t} \bar{\sigma}_y + \frac{\partial \Phi^i}{\partial \overline{\rho}_t} \sigma_L + \frac{\partial \Phi^i}{\partial \rho_t^i} \sigma_i + \frac{\partial \Phi^i}{\partial \widehat{\psi}_t^i} \sigma_{\widehat{\psi},t}\right), \tag{A11}$$

$\bar{\sigma}_y = (\sigma_y, 0, 0)$, $\sigma_L = (\sigma_{L,0}, \sigma_{L,L}, 0)$, $\sigma_i = (\sigma_{i,0}, 0, \sigma_{i,i})$, and $\sigma_{\widehat{\psi},t} = \left(0, 0, \frac{\phi^i}{\sigma_{i,i}} \widehat{\sigma}_{i,t}^2\right)$. We also have

$$E\left[dR_t^i\right] = \sigma_{R,1}^i \sigma_{\pi,t},$$

for expected excess return, where $\sigma_{R,1}^i$ is the first element in σ_R^i and $\sigma_{\pi,t}$ is given in equation (A1).

Proposition 3: The M/B value of the old economy is given by

$$M_t^O / B_t^O = \Phi\left(\overline{\rho}_t, y_t\right) = c^O \int_0^\infty Z\left(y_t, \overline{\rho}_t, s\right) ds, \tag{A12}$$

where

$$Z\left(y_t, \overline{\rho}_t, s\right) = e^{Q_0^O(s) + Q_1^O(s) y_t + Q_2^O(s) \overline{\rho}_t + Q_3^O(s) y_t^2},$$

and $Q_0^O(s) = K_0(0;s)$, $Q_1^O(s) = K_2(0;s) + \gamma a_1$, $Q_2^O(s) = K_3(0;s)$ and $Q_3^O(s) = K_6(0;s) + \gamma a_2$, where $K_i(.;s)$ are in Lemma 1, all for the parametrization $\zeta_0 = \zeta_2 = v = 1$, and $\zeta_1 = 0$.

Proof of Proposition 3: The claim follows from the same argument as in Proposition 1, but for the parameterization $\zeta_0 = \zeta_2 = v = 1$, and $\zeta_1 = 0$ in Lemma 1. The functions of time $Q_j^O(s)$, $j = 0, .., 3$, are computed as in Proposition 1. ■

The return volatility of the old economy is $\sqrt{\sigma_R^O \sigma_R^{O\,\prime}}$, where

$$\sigma_R^O = \frac{1}{\Phi}\left(\frac{\partial \Phi}{\partial y_t}\bar{\sigma}_y + \frac{\partial \Phi}{\partial \overline{\rho}_t}\sigma_L\right),$$

$\bar{\sigma}_y = (\sigma_y, 0)$, and $\sigma_L = (\sigma_{L,0}, \sigma_{L,L})$. The old economy's expected excess return is $E\left[dR_t^O\right] = \sigma_{R,1}^O \sigma_{\pi,t}$, where $\sigma_{R,1}^O$ is the first element in σ_R^O.

Lemma 4. (e.g., Duffie, 1996). For any linear vector process $\mathbf{z}_t$ that satisfies

$$d\mathbf{z}_t = (\mathbf{A}_z + \mathbf{B}_z \mathbf{z}_t)\, dt + \mathbf{\Sigma}_z d\mathbf{W}_t, \tag{A13}$$

we have

$$\mathbf{z}_{t+\tau}|_{\mathbf{z}_t} \sim N\left(\mu_{\mathbf{z}}(\mathbf{z}_t, \tau), \mathbf{S}_{\mathbf{z}}(\tau)\right),$$

where μ_z and $\mathbf{S}_z$ are given by

$$\begin{aligned}
\mu_{\mathbf{z}}(\mathbf{z}_t, \tau) &= \Psi(\tau)\, \mathbf{z}_t + \int_0^\tau \Psi(\tau - s)\, \mathbf{A}_z ds \\
\mathbf{S}_z(\tau) &= \int_0^\tau \Psi(\tau - s)\, \mathbf{\Sigma}_z \mathbf{\Sigma}_z' \Psi(\tau - s)\, ds
\end{aligned}$$

and $\Psi(\tau) = \mathbf{U} e^{\mathbf{\Lambda}\tau} \mathbf{U}^{-1}$, where $\mathbf{\Lambda}$ is the diagonal matrix with the eigenvalues of $\mathbf{B}_z$ on its principal diagonal, $\mathbf{U}$ is the matrix collecting the respective eigenvectors on each column, and $e^{\mathbf{\Lambda}\tau}$ is the diagonal matrix with $\mathrm{e}^{\lambda_{ii}\tau}$ on its principal diagonal.

(C) The Gordon growth model with an uncertain growth rate.

This section formalizes the discussion in the third paragraph of the introduction. Let D_t denote the dividend rate. The Gordon model assumes that the drift rate g of dividends is constant:

$$\frac{dD_t}{D_t} = g dt + \sigma_D dW_D. \tag{A14}$$

We consider two different specifications of the stochastic discount factor.

C.1. The Stochastic Discount Factor Independent of the Dividend Process.

Suppose that the SDF is governed by the following process with constant drift and volatility:

$$\frac{d\pi_t}{\pi_t} = -r_f dt - \sigma_\pi dW_\pi.$$

The price of the asset is then given by

$$P_t = E_t\left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds\right] = D_t\; E_t\left[\int_t^\infty \frac{\pi_s D_s}{\pi_t D_t} ds\right], \tag{A15}$$

assuming that the expectation exists. Let $x_t = \log(\pi_t D_t)$. Ito's lemma implies that

$$dx_t = \left(-r_f + g - \sigma_\pi \sigma_D \rho_{D,\pi} - \frac{1}{2}\left(\sigma_\pi^2 + \sigma_D^2 - 2\sigma_\pi \sigma_D \rho_{D,\pi}\right)\right) dt - \sigma_\pi dW_\pi + \sigma_D dW_D,$$

where $\rho_{D,\pi}$ is the correlation between dW_D and dW_π. Using the properties of the lognormal distribution,

$$E_t\left[\frac{\pi_s D_s}{\pi_t D_t}\right] = E_t\left[e^{(x_s - x_t)}\right] = e^{-\left(r_f + \sigma_\pi \sigma_D \rho_{D,\pi} - g\right)(s-t)}.$$

The price of the asset is then

$$\begin{aligned} P_t &= D_t \int_t^\infty E_t\left[\frac{\pi_s D_s}{\pi_t D_t}\right] ds = D_t \int_t^\infty e^{-\left(r_f + \sigma_\pi \sigma_D \rho_{D,\pi} - g\right)(s-t)} ds = \frac{D_t}{r_f + \sigma_\pi \sigma_D \rho_{D,\pi} - g} \\ &= \frac{D_t}{r-g}, \end{aligned}$$

where $r = r_f + \sigma_\pi \sigma_D \rho_{D,\pi}$ is the sum of the risk-free rate and the risk premium. This is the well-known Gordon growth formula in a continuous-time framework.

When g is unknown, it follows from the law of iterated expectations that

$$P_t = E_t\left[\int_t^\infty \frac{\pi_s D_s}{\pi_t} ds\right] = E_t\left[E_t\left[\int_t^\infty \frac{\pi_s D_s}{\pi_t} ds \mid g\right]\right] = D_t \; E_t\left[\frac{1}{r-g}\right].$$

That is, the P/D ratio is equal to the expectation of the P/D ratio in the case where g is known. (This expectation exists only under the assumption that the distribution of g assigns positive likelihood only to the values of g that satisfy a transversality condition.) Note that the risk premium is unchanged compared to the case of known g. By explicitly modeling the learning process, this fact can be proven directly by adapting the results in Veronesi (2000).

C.2. The Stochastic Discount Factor Dependent on the Dividend Process.

Following Campbell (1986) and Abel (1999), we assume the existence of a representative agent with a CRRA utility over aggregate consumption, which is given by

$$C_t = D_t^\lambda.$$

In a dynamic economy, the SDF is given by the marginal utility of consumption

$$\pi_t = e^{-\eta t} C_t^{-\gamma},$$

where γ is the coefficient of risk aversion. The SDF then follows the process

$$\frac{d\pi_t}{\pi_t} = -\left(\eta + \lambda\gamma g - \frac{1}{2}\lambda\gamma(1+\lambda\gamma)\sigma_D^2\right) dt - \lambda\gamma\sigma_D dW_D \tag{A16}$$

$$= -\left(\eta + \lambda\gamma E_t[g] - \frac{1}{2}\lambda\gamma(1+\lambda\gamma)\sigma_D^2\right) dt - \lambda\gamma\sigma_D d\widetilde{W}_D. \tag{A17}$$

The process (A16) is written with respect to the true Brownian motion W_D from equation (A14), whereas the (equivalent) process (A17) is written with respect to the Brownian motion $\widetilde{W}_D$ perceived by the agent with incomplete information about g. The equality between the processes in equations (A16) and (A17) follows from Girsanov's theorem. Importantly, equation (A17) shows that uncertainty about g has no impact on the volatility of the SDF.

The price of the asset is given by

$$P_t = E_t\left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds\right] = D_t\ E_t\left[\int_t^\infty e^{-\eta(s-t)}\left(\frac{D_s}{D_t}\right)^{1-\lambda\gamma} ds\right], \tag{A18}$$

assuming that the expectation exists.

If g is observable, the same calculation as in Section C.1 shows that

$$\frac{P_t}{D_t} = \int_t^\infty e^{-\left(\eta-(1-\lambda\gamma)g+\frac{1}{2}\lambda\gamma(1-\lambda\gamma)\sigma_D^2\right)(s-t)} ds = \frac{1}{\eta-(1-\lambda\gamma)g+\frac{1}{2}\lambda\gamma(1-\lambda\gamma)\sigma_D^2}. \tag{A19}$$

Note that as long as $\lambda\gamma \neq 1$, the P/D ratio is convex in g.

If g is unobservable, the law of iterated expectations implies that

$$\frac{P_t}{D_t} = E_t\left[\frac{1}{\eta-(1-\lambda\gamma)g+\frac{1}{2}\lambda\gamma(1-\lambda\gamma)\sigma_D^2}\right].$$

Due to the previously mentioned convexity, an increase in uncertainty about g (i.e., a mean-preserving spread on the density of g) leads to an increase in the P/D ratio.

By following the approach in Veronesi (2000), it is also possible to show that an increase in uncertainty decreases (increases) expected excess return if and only if $\gamma > 1/\lambda$ ($\gamma < 1/\lambda$).

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