Technical Appendix

to accompany

Entrepreneurial Learning, the IPO Decision, and the Post-IPO Drop in Firm Profitability

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Learning: The learning dynamics in this paper are the same as in Pástor and Veronesi (2003, Lemma 1). We report here only the most relevant equations. Let $\mathbf{N}_t = (\log(\pi_t), \rho_t)'$, which follows the process

$$d\mathbf{N}_t = (\mathbf{A}_N + \mathbf{B}_N \overline{\rho} + \mathbf{C}_N \mathbf{N}_t)\, dt + \mathbf{\Sigma}_N d\mathbf{X}_t$$

where

$$\mathbf{A}_N = \begin{pmatrix} r - \frac{1}{2}\sigma_{\pi,1}^2 \\ 0 \end{pmatrix}; \quad \mathbf{B}_N = \begin{pmatrix} 0 \\ \phi \end{pmatrix}; \quad \mathbf{C}_N = \begin{pmatrix} 0 & 0 \\ 0 & -\phi \end{pmatrix}; \quad \mathbf{\Sigma}_N = \begin{pmatrix} \sigma_{\pi,1} & 0 \\ \sigma_{\rho,1} & \sigma_{\rho,2} \end{pmatrix};$$

Define the vector of orthogonalized expectation errors:

$$d\widehat{\mathbf{X}}_t = \mathbf{\Sigma}^{-1} \left[d\mathbf{N}_t - E_t \left(d\mathbf{N}_t \right) \right]$$

The Kalman-Bucy filter (e.g. Liptser and Shiryayev (1977)) implies that $\widehat{\rho}_t = E_t[\overline{\rho}]$ and $\widehat{\sigma}_t^2 = E_t\left[(\overline{\rho} - \widehat{\rho}_t)^2\right]$ follow the processes in equation (27). Conditional on the agents' information, the processes for π_t and ρ_t are given by

$$\begin{aligned} \frac{d\pi_t}{\pi_t} &= -r dt - \sigma_{\pi,1} d\widehat{X}_{1,t} \\ d\rho_t &= \phi\left(\widehat{\rho}_t - \rho_t\right) dt + \sigma_{\rho,1} d\widehat{X}_{1,t} + \sigma_{\rho,2} d\widehat{X}_{2,t} \end{aligned}$$

Market value: Let $\sigma_\pi = (\sigma_{\pi,1}, 0)$ and $\sigma_\rho = (\sigma_{\rho,1}, \sigma_{\rho,2})$. The firm's market value is given by

$$M_t = B_t Z\left(\rho_t, \widehat{\rho}_t, \widehat{\sigma}_t, T - t\right) \tag{A1}$$

where

$$Z\left(\rho_t, \widehat{\rho}_t, \widehat{\sigma}_t, T - t\right) = e^{Q_0(T-t) + Q_1(T-t)\rho_t + Q_2(T-t)\widehat{\rho}_t + \frac{1}{2}Q_2(T-t)^2\widehat{\sigma}_t^2} \tag{A2}$$

and

$$\begin{aligned} Q_0(s) &= -rs + \frac{\sigma_\rho \sigma_\rho'}{2\phi^2} Q_3(s) - \frac{\sigma_\pi \sigma_\rho'}{\phi} Q_2(s); \quad Q_1(s) = \frac{1}{\phi}\left(1 - e^{-\phi s}\right); \\ Q_2(s) &= s - Q_1(s); \quad Q_3(s) = s + \frac{1 - e^{-2\phi s}}{2\phi} - 2Q_1(s). \end{aligned}$$

Utility from selling the firm: After the IPO, the entrepreneur can invest in bonds, in a stock market index (which is perfectly correlated with the state price density (11)), or in his own firm, thereby retaining a stake in the firm. Given market completeness, we can use the Cox and Huang (1989) results to obtain the intertemporal utility from the IPO independently of the portfolio allocation. Specifically, from Cox and Huang (1989), the optimal investment strategy is the one that supports the solution to the static maximization problem

$$\max E_t \left[\int_t^T e^{-\beta(u-t)} \frac{c_u^{1-\gamma}}{1-\gamma} du + \eta e^{-\beta(T-t)} \frac{W_T^{1-\gamma}}{1-\gamma} \right] \tag{A3}$$

subject to

$$W_t = E_t \left[\int_t^T \frac{\pi_u}{\pi_t} c_u du + \frac{\pi_T}{\pi_t} W_T \right] \tag{A4}$$

The first-order conditions are

$$e^{-\beta(u-t)}c_u^{-\gamma} = \frac{\pi_u}{\pi_t}\lambda \quad \text{and} \quad e^{-\beta(T-t)}\eta W_T^{-\gamma} = \frac{\pi_T}{\pi_t}\lambda$$

where λ is a Lagrange multiplier determined by the budget constraint. Thus,

$$c_u = \left(\frac{\pi_u}{\pi_t}\right)^{-\frac{1}{\gamma}} \lambda^{-\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(u-t)}; \quad \text{and} \quad W_T = \left(\frac{\pi_T}{\pi_t}\right)^{-\frac{1}{\gamma}} \lambda^{-\frac{1}{\gamma}} \eta^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(T-t)} \tag{A5}$$

Substitute c_u and W_T in the budget constraint (A4) to obtain

$$W_t = \lambda^{-\frac{1}{\gamma}} E_t \left[\int_t^T \left(\frac{\pi_u}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} e^{-\frac{\beta}{\gamma}(u-t)} du + \left(\frac{\pi_T}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} \eta^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(T-t)} \right]$$

This yields λ :

$$\lambda^{-\frac{1}{\gamma}} = W_t \frac{1}{E_t \left[\int_t^T \left(\frac{\pi_u}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} e^{-\frac{\beta}{\gamma}(u-t)} du + \left(\frac{\pi_T}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} \eta^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(T-t)} \right]}$$

Thus, substituting back λ in the optimal consumption c_u and wealth W_T in (A5), and the resulting formulas in the utility function, we obtain

$$V(W_t, t) = E_t \left[\int_t^T e^{-\beta(u-t)} \frac{c_u^{1-\gamma}}{1-\gamma} du + \eta e^{-\beta(T-t)} \frac{W_T^{1-\gamma}}{1-\gamma} \right] \tag{A6}$$

$$= \frac{W_t^{1-\gamma}}{1-\gamma} \left\{ E_t \left[\int_t^T \left(\frac{\pi_u}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} e^{-\frac{\beta}{\gamma}(u-t)} du + \left(\frac{\pi_T}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} \eta^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(T-t)} \right] \right\}^{\gamma} \tag{A7}$$

Given the stochastic discount factor in (11), for every $u > t$

$$E_t \left[\left(\frac{\pi_u}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} \right] = e^{\frac{1-\gamma}{\gamma}\left(r + \frac{1}{2}\frac{1}{\gamma}\sigma_{\pi,1}^2\right)(u-t)} \tag{A8}$$

The value function after the IPO, $V(W_t, t)$ in (31), then follows, as we use the Fubini theorem to invert the order of integration in (A7), use result (A8), and solve for the resulting the integral, finding that

$$g(T-t) = \left\{ E_t \left[\int_t^T \left(\frac{\pi_u}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} e^{-\frac{\beta}{\gamma}(u-t)} du + \left(\frac{\pi_T}{\pi_t}\right)^{\frac{\gamma-1}{\gamma}} \eta^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(T-t)} \right] \right\}^{\gamma}$$

$$= \left(\frac{\left(1 + \eta^{\frac{1}{\gamma}} \frac{1-\gamma}{\gamma} \left(r - \frac{\beta}{1-\gamma} + \frac{1}{2}\frac{1}{\gamma}\sigma_{\pi,1}^2\right)\right) e^{\frac{1-\gamma}{\gamma}\left(r - \frac{\beta}{1-\gamma} + \frac{1}{2}\frac{1}{\gamma}\sigma_{\pi,1}^2\right)(T-t)} - 1}{\frac{1-\gamma}{\gamma}\left(r - \frac{\beta}{1-\gamma} + \frac{1}{2}\frac{1}{\gamma}\sigma_{\pi,1}^2\right)} \right)^{\gamma} \tag{A9}$$

Q.E.D.

Selling the whole firm in an IPO is optimal: We provide two proofs.

Proof #1. From (A5), the optimal consumption depends only on the systematic shocks $dX_{1,t}$, which are perfectly correlated with the returns on publicly-traded stocks (the stock market). Investing any amount in the entrepreneur's (tiny) firm would make the entrepreneur's consumption driven also by the firm's idiosyncratic shocks $dX_{2,t}$, which would make the consumption path suboptimal.

Proof #2. Consider the setting on page 42 of the paper, Proof 2. In this subsection we fill in the details of the proof. First, given the pricing formula for the firm's stock (equation (28) in the paper), an application of Ito's Lemma shows that the return process after the IPO is (see Pastor and Veronesi (2003, Proposition 2)):

$$\frac{dM_t}{M_t} = \left(r + \mu^f_{R,t}\right) dt + \sigma^f_{R,1,t} dX_{1,t} + \sigma^f_{R,2,t} dX_{2,t}$$

where

$$\begin{aligned} \mu^f_{R,t} &= \sigma^f_{R,1,t}\sigma_\pi = \left(1 - e^{-\phi(T-t)}\right) \frac{\sigma_{\rho,1}\sigma_\pi}{\phi} && \text{(A10)} \\ \sigma^f_{R,1,t} &= \left(1 - e^{-\phi(T-t)}\right) \frac{\sigma_{\rho,1}}{\phi} \\ \sigma^f_{R,2,t} &= \left(1 - e^{-\phi(T-t)}\right) \frac{\sigma_{\rho,2}}{\phi} + \left(\phi\,(T-t) - 1 + e^{-\phi(T-t)}\right) \widehat{\sigma}^2_t \frac{\phi}{\sigma_{\rho,2}} && \text{(A11)} \end{aligned}$$

Substitute the return processes in the budget equation (see paper, page 42, after eq. (29)):

$$dW_t = \left(W_t \left(\theta_t \mu_R + \theta^f_t \mu^f_{R,t} + r_t\right) - c_t\right) dt + W_t \left(\theta_t \sigma_R + \theta^f_t \sigma^f_{R,1,t}\right) dX_{1,t} + W_t \theta^f_t \sigma^f_{R,2,t} dX_{2,t} \tag{A12}$$

The Hamilton, Jacobi, Bellman (HJB) equation corresponding to problem in equation (29) in the paper is then:

$$0 = \max_{c_t, \theta_t, \theta^f_t} \frac{c_t^{1-\gamma}}{1-\gamma} - \beta V + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial W} E_t\left[dW_t\right] + \frac{1}{2}\frac{\partial^2 V}{\partial W^2} E_t\left[dW_t^2\right]$$

We now solve the HJB equation. First, substitute (A12) to find

$$\begin{aligned} 0 &= \max_{c_t, \theta_t, \theta^f_t} \frac{c_t^{1-\gamma}}{1-\gamma} - \beta V + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial W}\left(W_t \left(\theta_t \mu_R + \theta^f_t \mu^f_{R,t} + r\right) - c_t\right) \\ &\quad + \frac{1}{2}\frac{\partial^2 V}{\partial W^2} W^2 \left(\left(\theta_t \sigma_R + \theta^f_t \sigma^f_{R,1,t}\right)^2 + \left(\theta^f_t \sigma^f_{R,2,t}\right)^2\right) \end{aligned}$$

The first order condition (FOC) with respect to consumption is

$$c_t^{-\gamma} = \frac{\partial V}{\partial W} \tag{A13}$$

The FOC with respect to θ_t

$$\frac{\partial V}{\partial W}\mu_R + \frac{\partial^2 V}{\partial W^2} W_t \left(\theta_t \sigma_R + \theta_t^f \sigma_{R,1,t}^f\right) \sigma_R = 0$$

The FOC with respect to θ_t^f

$$\frac{\partial V}{\partial W}\mu_{R,t}^f + \frac{\partial^2 V}{\partial W^2} W_t \left(\left(\theta_t \sigma_R + \theta_t^f \sigma_{R,1,t}^f\right) \sigma_{R,1,t}^f + \left(\theta_t^f \sigma_{R,2,t}^f\right) \sigma_{R,2,t}^f\right) = 0$$

Substitute the equilibrium expected return $\mu_R = \sigma_\pi \sigma_R$ and $\mu_{R,t}^f = \sigma_\pi \sigma_{R,1,t}^f$ to find

$$\begin{aligned} \frac{\partial V}{\partial W}\sigma_\pi \sigma_R + \frac{\partial^2 V}{\partial W^2} W_t \left(\theta_t \sigma_R + \theta_t^f \sigma_{R,1,t}^f\right) \sigma_R &= 0 \qquad \text{(A14)} \\ \frac{\partial V}{\partial W}\sigma_\pi \sigma_{R,1,t}^f + W_t \frac{\partial^2 V}{\partial W^2} \left(\left(\theta_t \sigma_R + \theta_t^f \sigma_{R,1,t}^f\right) \sigma_{R,1,t}^f + \left(\theta_t^f \sigma_{R,2,t}^f\right) \sigma_{R,2,t}^f\right) &= 0 \end{aligned}$$

We now see that these two equations are satisfied by $\theta_t^f = 0$ and $\theta_t = \frac{\mu_R}{\gamma \sigma_R^2}$. Indeed, by setting $\theta_t^f = 0$ the two FOC equations become

$$\begin{aligned} \frac{\partial V}{\partial W}\sigma_\pi + \frac{\partial^2 V}{\partial W^2} W_t \theta_t \sigma_R &= 0 \\ \frac{\partial V}{\partial W}\sigma_\pi + W_t \frac{\partial^2 V}{\partial W^2} \theta_t \sigma_R &= 0 \end{aligned}$$

That is, they become the *same equation* which is in fact satisfied by the usual solution

$$\theta_t = -\frac{\sigma_\pi \frac{\partial V}{\partial W}}{W \frac{\partial^2 V}{\partial W^2} \sigma_R}$$

To conclude the proof, we conjecture (and verify later) the value function is given by

$$V(W,t) = \frac{W^{1-\gamma}}{1-\gamma} J(t)^\gamma \qquad \text{(A15)}$$

Taking the first derivatives with respect to wealth and time

$$\frac{\partial V}{\partial t} = \frac{W^{1-\gamma}}{1-\gamma} \gamma J(t)^{\gamma-1} \frac{\partial J}{\partial t}; \qquad \frac{\partial V}{\partial W} = W^{-\gamma} J(t)^\gamma; \qquad \frac{\partial^2 V}{\partial W^2} = -\gamma W^{-\gamma-1} J(t)^\gamma$$

Using also the fact that the risk premium is $\mu_R = \sigma_\pi \sigma_R$ we have $\sigma_\pi = \frac{\mu_R}{\sigma_R}$, which leads to the standard result

$$\theta_t = \frac{\mu_R}{\gamma \sigma_R^2}$$

Thus, to conclude, the optimal portfolio allocation in the two stocks is

$$\left(\theta_t, \theta_t^f\right) = \left(\frac{\mu_R}{\gamma \sigma_R^2}, 0\right)$$

As a final step, we can substitute everything back into the HJB equation, and after some tedious algebra, we find the ordinary differential equation

$$0 = 1 + \frac{\partial J}{\partial t} + aJ \tag{A16}$$

where

$$a = \frac{(1-\gamma)}{\gamma}\left(r - \frac{\beta}{1-\gamma} + \frac{\sigma_\pi^2}{2\gamma}\right)$$

The solution of (A16) subject to the final condition $J(T) = \eta^{1/\gamma}$ is

$$J(t) = \frac{\left(1 + \eta^{\frac{1}{\gamma}} a\right) e^{a(T-t)} - 1}{a}$$

obtaining the value function in (31) in the paper. Q.E.D.

Lemma A.1: Given the process (10) and $dB_t = \rho_t B_t dt$, for every $u > t$ we have

$$E_t\left[e^{-\beta(u-t)}\frac{B_u^{1-\gamma}}{1-\gamma}|B_t\right] = \frac{B_t^{1-\gamma}}{1-\gamma} Z^O\left(\rho_t, \widehat{\rho}_t, \widehat{\sigma}_t; u-t\right)$$

where

$$Z^O\left(\rho_t, \widehat{\rho}_t, \widehat{\sigma}_t; s\right) = e^{\overline{Q}_0(s) + (1-\gamma)Q_1(s)\rho_t + (1-\gamma)Q_2(s)\widehat{\rho}_t + \frac{1}{2}(1-\gamma)^2 Q_2(s)^2 \widehat{\sigma}_t^2}; \tag{A17}$$

$Q_i(.)$ are given above, and

$$\overline{Q}_0(s) = -\beta s + (1-\gamma)^2 \frac{\sigma_\rho \sigma_\rho'}{2\phi^2} Q_3(s) \tag{A18}$$

Proof of Lemma A.1: First, we compute

$$E_t\left[e^{-\beta(u-t)} B_u^{1-\gamma}|B_t, \overline{\rho}\right] = E\left[e^{-\beta(u-t)+p_u}|p_t, \overline{\rho}\right]$$

where we define $p_t = (1-\gamma)\log(B_t)$. p_t follows the process $dp_t = (1-\gamma)\rho_t dt$. Define the vector $\mathbf{Z}_t = [p_t, \rho_t]$, which follows the process

$$d\mathbf{Z}_t = (\mathbf{A} + \mathbf{B}\mathbf{Z}_t)\, dt + \mathbf{\Sigma} d\mathbf{X}_t$$

where

$$\mathbf{A} = \begin{pmatrix} 0 \\ \phi\overline{\rho} \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 0 & 1-\gamma \\ 0 & -\phi \end{pmatrix}; \mathbf{\Sigma} = \begin{pmatrix} 0 & 0 \\ \sigma_{\rho 1} & \sigma_{\rho 2} \end{pmatrix}$$

Standard results imply

$$\mathbf{Z}_u|\mathbf{Z}_t \sim N\left(\mu_Z\left(Z_t, u-t\right), \mathbf{\Sigma}_Z\left(u-t\right)\right)$$

where

$$\begin{aligned} \mu_Z(s) &= \mathbf{\Psi}(s)\mathbf{Z}_0 + \int_0^s \mathbf{\Psi}(s-t)\mathbf{A}dt \\ \mathbf{\Sigma}_Z(s) &= \int_0^s \mathbf{\Psi}(s-t)\mathbf{\Sigma}\mathbf{\Sigma}'\mathbf{\Psi}(s-t)'\, dt \end{aligned}$$

In our setting

$$\mathbf{\Psi}(s) = \begin{pmatrix} 1 & \frac{1-\gamma}{\phi}\left(1-e^{-\phi s}\right) \\ 0 & e^{-\phi s} \end{pmatrix}$$

so that we obtain the explicit formulas

$$\mu_{Z,1}(s) = p_t + (1-\gamma)Q_1(s)\rho_t + \overline{\rho}(1-\gamma)Q_2(s)$$

As for the variance, we only need to find the variance of p_u. Thus, since the (1,1) element in the integrand is

$$e_1\mathbf{\Psi}(s-t)\mathbf{\Sigma}\mathbf{\Sigma}'\mathbf{\Psi}(s-t)'e_1' = \frac{(1-\gamma)^2}{\phi^2}(1-e^{-\phi s})^2\sigma_\rho\sigma_\rho'$$

we have

$$\begin{aligned}\sigma_p(u) = e_1\Sigma_Z(s)e_1' &= \frac{(1-\gamma)^2\sigma_\rho\sigma_\rho'}{\phi^2}\int_0^s(1-e^{-\phi(s-t)})^2dt \\ &= \frac{(1-\gamma)^2\sigma_\rho\sigma_\rho'}{\phi^2}\left(s+\frac{1-e^{-2\phi s}}{2\phi}-2\frac{1-e^{-\phi s}}{\phi}\right)\end{aligned}$$

This implies

$$E\left[e^{-\beta(u-t)+p_u}|p_t,\overline{\rho}\right] = e^{p_t-\beta(u-t)+(1-\gamma)Q_1(u-t)\rho_t+\overline{\rho}(1-\gamma)Q_2(u-t)+\frac{1}{2}\frac{(1-\gamma)^2\sigma_\rho\sigma_\rho'}{\phi^2}Q_3(u-t)}$$

Finally, given $\overline{\rho}\sim N(\widehat{\rho}_t,\widehat{\sigma}_t^2)$, we obtain

$$E\left[e^{-\beta(u-t)}B_u^{1-\gamma}|B_t\right] = E\left[e^{\beta(u-t)+p_u}|p_t\right] = B_t^{1-\gamma}e^{\overline{Q}_0(u-t)+(1-\gamma)Q_1(u-t)\rho_t+(1-\gamma)Q_2(u-t)\widehat{\rho}_t+\frac{1}{2}(1-\gamma)^2Q_2(u-t)^2\widehat{\sigma}_t^2}$$

where

$$\overline{Q}_0(s) = -\beta s + \frac{1}{2}\frac{(1-\gamma)^2\sigma_\rho\sigma_\rho'}{\phi^2}Q_3(s)$$

Q.E.D.

Utility from Keeping the Firm: From Lemma A.1, the utility from owning the firm from τ to T is given by

$$\begin{aligned}V^O(B_\tau,\tau) &= E_\tau\left[\alpha^{1-\gamma}\int_\tau^T e^{-\beta(u-\tau)}\frac{B_u^{1-\gamma}}{1-\gamma}du + \eta e^{-\beta(T-\tau)}\frac{B_T^{1-\gamma}}{1-\gamma}\right] \\ &= \frac{B_\tau^{1-\gamma}}{1-\gamma}\left\{\alpha^{1-\gamma}\int_\tau^T Z^O\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau^2;u-\tau\right)du + \eta Z^O\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau^2;T-\tau\right)\right\} \quad \text{(A19)}\end{aligned}$$

Q.E.D.

IPO Condition: We restate the condition, for better referencing. An IPO takes place at time τ if and only if

$$f(T-\tau,\widehat{\sigma}_\tau,\sigma_\rho) < \alpha^{1-\gamma}\int_\tau^T \widehat{Z}(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau,\sigma_\rho;u-\tau;T)\,du, \quad \text{(A20)}$$

where

$$f\left(T-\tau,\widehat{\sigma}_\tau,\sigma_\rho\right) = e^{(1-\gamma)\left(-\left(r-\frac{\beta}{1-\gamma}\right)(T-\tau)+\gamma\frac{\sigma_\rho\sigma_\rho'}{2\phi^2}Q_3(T-\tau)-\frac{\sigma_\pi\sigma_\rho'}{\phi}Q_2(T-\tau)\right)+\frac{1}{2}\gamma(1-\gamma)Q_2(T-\tau)^2\widehat{\sigma}_\tau^2} g\left(T-\tau\right)-\eta \tag{A21}$$

$$\widehat{Z}\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau,\sigma_\rho;u-\tau;T\right) = e^{\widehat{Q}_0(u-\tau;T)+(1-\gamma)\widehat{Q}_1(u-\tau;T)\rho_\tau+(1-\gamma)\widehat{Q}_2(u-\tau;T)\widehat{\rho}_\tau+\frac{1}{2}(1-\gamma)^2\widehat{Q}_3(u-\tau;T)\widehat{\sigma}_\tau^2} \tag{A22}$$

Above, the g function is given in equation (A9) and

$$\begin{aligned}
\widehat{Q}_0\left(u-\tau;T\right) &= \overline{Q}_0\left(u-\tau\right)-\overline{Q}_0\left(T-\tau\right)\\
\widehat{Q}_1\left(u-\tau;T\right) &= Q_1\left(u-\tau\right)-Q_1\left(T-\tau\right)<0\\
\widehat{Q}_2\left(u-\tau;T\right) &= Q_2\left(u-\tau\right)-Q_2\left(T-\tau\right)<0\\
\widehat{Q}_3\left(u-\tau;T\right) &= Q_2\left(u-\tau\right)^2-Q_2\left(T-\tau\right)^2<0
\end{aligned}$$

Proof. An IPO occurs at τ if and only if

$$V\left(M_\tau,\tau\right)>V^O\left(B_\tau,\tau\right). \tag{A23}$$

Using the closed-form expressions for V and V^O, we obtain

$$\frac{M_\tau^{1-\gamma}}{1-\gamma}g\left(T-\tau\right)>\frac{B_\tau^{1-\gamma}}{1-\gamma}\left\{\eta Z^O\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau;T-\tau\right)+\alpha^{1-\gamma}\int_\tau^T Z^O\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau;u-\tau\right)du\right\}$$

Substitute M_τ from equation (A1) and delete common terms (remembering that $\gamma>1$) to obtain the condition

$$Z\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau;T-\tau\right)^{1-\gamma}g\left(T-\tau\right)-\eta Z^O\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau;T-\tau\right)<\alpha^{1-\gamma}\int_\tau^T Z^O\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau;u-\tau\right)du \tag{A24}$$

From the definition of $Z(.)$ and $Z^O(.)$ in (A2) and (A17) we can write

$$\begin{aligned}
Z\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau;T-\tau\right)^{1-\gamma} &= Z^O\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau;T-\tau\right)\times\\
&\quad\times e^{(1-\gamma)\left(-\left(r-\frac{\beta}{1-\gamma}\right)(T-\tau)+\gamma\frac{\sigma_\rho\sigma_\rho'}{2\phi^2}Q_3(T-\tau)-\frac{\sigma_\pi\sigma_\rho'}{\phi}Q_2(T-\tau)\right)+\frac{1}{2}\gamma(1-\gamma)Q_2(T-\tau)^2\widehat{\sigma}_\tau^2}
\end{aligned}$$

Thus, substituting for $Z\left(\right)^{1-\gamma}$ we have that (A24) becomes

$$Z^O\left(\rho_t,\widehat{\rho}_t,\widehat{\sigma}_t;T-\tau\right)f\left(T-\tau,\widehat{\sigma}_\tau,\sigma_\rho\right)<\alpha^{1-\gamma}\int_\tau^T Z^O\left(\rho_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau^2;u-\tau\right)du$$

where $f\left(T-\tau,\widehat{\sigma}_\tau,\sigma_\rho\right)$ is defined in (A21). Dividing through by Z^O we obtain the claim. The definitions of $\widehat{Q}_i$ stem immediately from the computation of the ratio

$Z^O\left(\rho_\tau, \widehat{\rho}_\tau, \widehat{\sigma}_t^2; u-\tau\right)/Z^O\left(\rho_\tau, \widehat{\rho}_\tau, \widehat{\sigma}_\tau^2; T-\tau\right)$. The inequality signs stem from the fact that $Q_1(s)$ and $Q_2(s)$ are increasing functions of s. Thus it immediately follows that for $\gamma > 1$

$$\frac{\partial \widehat{Z}}{\partial \rho_\tau} > 0 \text{ and } \frac{\partial \widehat{Z}}{\partial \widehat{\rho}_\tau} > 0$$

Q.E.D.

Proof of Proposition 1: Part (a) stems from the fact that the Left Hand Side (LHS) of (A20) is independent of α, while the Right Hand Side (RHS) of (A20) is decreasing in α (for $\gamma > 1$). Part (b) and (c) stem directly from the original condition (A23). From the formulas (A1) and (31), it is immediate to see that the LHS of (A23) is increasing in both $\widehat{\sigma}_\tau$ and $\sigma_{\rho,2}$, because market value M_τ in (A1) is. Similarly, from (32) and (A17) the RHS of (A23) is instead decreasing in both $\widehat{\sigma}_\tau$ and $\sigma_{\rho,2}$. Part (d) stems from the fact that the function $\widehat{Z}$ on the RHS of (A20) is increasing in both ρ_τ and $\widehat{\rho}_\tau$, while the LHS of (A20) does not depend on them. Q.E.D.

Lemma A.2: Define $x_t = \rho_t - \widehat{\rho}_t$. This variable follows the process

$$dx_t = -\phi x_t dt + \sigma_{\rho,1} d\widehat{X}_1 + \sigma_{x,t} d\widehat{X}_2$$

with

$$\sigma_{x,t} = \sigma_{\rho,2} - \widehat{\sigma}_t^2 \frac{\phi}{\sigma_{\rho,2}}$$

Define the function

$$\begin{aligned} h\left(x_\tau, \widehat{\rho}_\tau\right) &= \alpha^{1-\gamma} \int_\tau^T \widehat{Z}\left(\rho_\tau, \widehat{\rho}_\tau, \widehat{\sigma}_\tau, u-\tau, T\right) du \\ &= \alpha^{1-\gamma} \int_\tau^T \overline{Z}\left(x_\tau, \widehat{\rho}_\tau, \widehat{\sigma}_\tau, u-\tau, T\right) du \end{aligned}$$

where

$$\overline{Z}\left(x_\tau, \widehat{\rho}_\tau, \widehat{\sigma}_\tau, u-\tau, T\right) = e^{\widehat{Q}_0(u-\tau,T)+(1-\gamma)\left(\widehat{Q}_1(u-\tau,T)x_\tau+(u-T)\widehat{\rho}_\tau\right)+\frac{1}{2}(1-\gamma)^2\widehat{Q}_3(u-\tau,T)\widehat{\sigma}_\tau^2}$$

Then, the function $h\left(x_\tau, \widehat{\rho}_\tau\right)$ is monotonically increasing in both arguments.

Proof of Lemma A.2: First, from the definition of $\widehat{Q}_2\left(u-\tau, T\right)$ in Proposition 3, we can rewrite

$$\widehat{Q}_2\left(u-\tau, T\right) = (u-T) - \widehat{Q}_1\left(u-\tau, T\right)$$

Simple substitution then shows

$$\widehat{Z}\left(\rho_\tau, \widehat{\rho}_\tau, \widehat{\sigma}_\tau, u-\tau, T\right) = \overline{Z}\left(x_\tau, \widehat{\rho}_\tau, \widehat{\sigma}_\tau, u-\tau, T\right)$$

From the definition of h, we have

$$\begin{aligned}\frac{\partial h}{\partial x_\tau} &= \alpha^{1-\gamma}\int_\tau^T (1-\gamma)\,\widehat{Q}_1(u-\tau)\,\overline{Z}\left(x_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau^2,u-\tau,T\right)du>0\\ \frac{\partial h}{\partial \widehat{\rho}_\tau} &= \alpha^{1-\gamma}\int_\tau^T (1-\gamma)\,(u-T)\,\overline{Z}\left(x_\tau,\widehat{\rho}_\tau,\widehat{\sigma}_\tau^2,u-\tau,T\right)du>0\end{aligned}$$

The inequality signs follow from the fact that $\gamma > 1$, $\widehat{Q}_1 < 0$ (see Proposition 3) and $(u-T)<0$. Q.E.D.

Proposition 2 (Endogenous IPO Cutoff Rule): Define the cutoff $\underline{\rho}(x_\tau)$ by

$$\underline{\rho}(x_\tau) \text{ such that } h\left(x_\tau,\underline{\rho}(x_\tau)\right) = f\left(T-\tau,\widehat{\sigma}_\tau,\sigma_\rho\right)$$

Then, an IPO takes place at τ if and only if

$$\widehat{\rho}_\tau > \underline{\rho}(x_\tau) \tag{A25}$$

Proof of Proposition 2: We can restate the IPO condition as

$$f\left(T-\tau,\widehat{\sigma}_\tau,\sigma_\rho\right) < h\left(x_\tau,\widehat{\rho}_\tau\right)$$

Since $h\left(x_\tau,\widehat{\rho}_\tau\right)$ is increasing in both arguments, we have the equivalence

$$h\left(x_\tau,\widehat{\rho}_\tau\right) > f\left(T-\tau,\widehat{\sigma}_\tau,\sigma_\rho\right) \Longleftrightarrow \widehat{\rho}_\tau > \underline{\rho}(x_\tau)$$

The statement of the proposition follows. Q.E.D.

Lemma A.3: Let $\mathbf{Z}_t = (b_t, x_t, \widehat{\rho}_t)'$, Then

$$\mathbf{Z}_\tau|\mathbf{Z}_t \sim N\left(\mu_Z\left(\mathbf{Z}_t,\tau-t\right),\mathbf{S}\left(t,\tau\right)\right)$$

where

$$\begin{aligned}\mu_{Z,1} &= E_t[b_\tau] = b_t + Q_1(\tau-t)\,x_t + (\tau-t)\,\widehat{\rho}_t &\text{(A26)}\\ \mu_{Z,2} &= E_t[x_\tau] = x_t e^{-\phi(\tau-t)} &\text{(A27)}\\ \mu_{Z,3} &= E_t[\widehat{\rho}_\tau] = \widehat{\rho}_t &\text{(A28)}\end{aligned}$$

and

$$\mathbf{S}(t,\tau) = \int_t^\tau \mathbf{\Psi}(\tau-u)\,\mathbf{\Sigma}_{Z,u}\mathbf{\Sigma}_{Z,u}'\mathbf{\Psi}(\tau-u)'\,du \tag{A29}$$

where

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 & \frac{1-e^{-\phi t}}{\phi} & t\\ 0 & e^{-\phi t} & 0\\ 0 & 0 & 1\end{pmatrix} \tag{A30}$$

In particular,

$$\begin{aligned} S_{22}(t,\tau) &= \frac{1-e^{2\phi(\tau-t)}}{2\phi}\left(\sigma_{\rho,1}^2+\sigma_{\rho,2}^2\right)+\left(e^{-2\phi(\tau-t)}\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right) \\ S_{23}(t,\tau)=S_{32}(t,\tau) &= \widehat{\sigma}_\tau^2-e^{-\phi(\tau-t)}\widehat{\sigma}_t^2; \\ S_{33}(t,\tau) &= \widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2 \end{aligned}$$

Proof of Lemma A.3: By definition, $\mathbf{Z}_t$ follows the process

$$d\mathbf{Z}_t = \mathbf{B}\mathbf{Z}_t dt + \mathbf{\Sigma}_{Z,t} d\widehat{\mathbf{X}}$$

where

$$\mathbf{B}=\begin{pmatrix} 0 & 1 & 1 \\ 0 & -\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \text{and} \quad \mathbf{\Sigma}_{Z,t}=\begin{pmatrix} 0 & 0 \\ \sigma_{\rho,1} & \sigma_{\rho,2}-\widehat{\sigma}_t^2\frac{\phi}{\sigma_{\rho,2}} \\ 0 & \widehat{\sigma}_t^2\frac{\phi}{\sigma_{\rho,2}} \end{pmatrix}$$

Without loss of generality, let $t=0$. The vector $\mu_Z(\tau)$ and the matrix $\mathbf{S}(0,\tau)$ are then the solutions to (see e.g. Duffie (2001, page 293))

$$\frac{d\mu_Z(\tau)}{d\tau} = \mathbf{B}\mu_Z(\tau)$$

with $\mu(0)=\mathbf{Z}_0$ and to

$$\frac{d\mathbf{S}(0,\tau)}{d\tau} = \mathbf{B}\mathbf{S}(0,\tau)+\mathbf{S}(0,\tau)\mathbf{B}'+\mathbf{\Sigma}_{Z,\tau}\mathbf{\Sigma}_{Z,\tau}' \tag{A31}$$

with $\mathbf{S}(0,0)=0$. Defining the matrix $\mathbf{\Psi}(\tau)$ as the solution to the ODE

$$\frac{d\mathbf{\Psi}(\tau)}{d\tau} = \mathbf{B}\mathbf{\Psi}(\tau) \tag{A32}$$

with $\mathbf{\Psi}(0)=\mathbf{I}$, then it is simple to verify that the solution for μ_Z is given by

$$\mu_Z(\tau) = \mathbf{\Psi}(\tau)\mathbf{Z}_0$$

which leads to (A26) -(A28). Similarly, it is simple to verify that the solution $\mathbf{S}(0,\tau)$ of (A31) is given by (A29). The last step is then to show that the solution to (A32) is (A30). This can be easily verified by direct computation.

Finally, we can compute the integrals in (A29) explicitly, yielding the formulas for S_{ij}, $i=2,3$. Q.E.D.

Proposition 3 (Expected drop in profitability): The expected drop in profitability after the IPO is given by

$$E\left[\rho_\tau-\widehat{\rho}_\tau|IPO\right] = \frac{e^{-\phi(\tau-t)}x_t-\int x_\tau \mathcal{N}\left(k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t,\widehat{\sigma}_t^2\right)\right)\Phi\left(x_\tau;\mu_x(x),\sigma_x^2(t,\tau)\right)dx_\tau}{1-\int \mathcal{N}\left(k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t,\widehat{\sigma}_t^2\right)\right)\Phi\left(x_\tau;\mu_x(x),\sigma_x^2(t,\tau)\right)dx_\tau} \tag{A33}$$

where $\mathcal{N}(.)$ is the standard normal cdf,

$$k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t,\widehat{\sigma}_t^2\right)=\frac{\underline{\rho}\left(x_\tau\right)-\widehat{\rho}_t-a\left(t,\tau;\widehat{\sigma}_t^2\right)\left(x_\tau-e^{-\phi(\tau-t)}x_t\right)}{\sqrt{\left(\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)\left(1-b\left(t,\tau;\widehat{\sigma}_t^2\right)^2\right)}} \tag{A34}$$

the density $\Phi\left(x_\tau;\mu_x\left(x\right),\sigma_x^2\left(t,\tau\right)\right)$ is normal, with

$$\mu_x = e^{-\phi(\tau-t)}x_t \tag{A35}$$

$$\sigma_x^2 = \frac{1-e^{2\phi(\tau-t)}}{2\phi}\left(\sigma_{\rho,1}^2+\sigma_{\rho,2}^2\right)+\left(e^{-2\phi(\tau-t)}\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right) \tag{A36}$$

and finally,

$$\begin{aligned} a\left(t,\tau;\widehat{\sigma}_t^2\right) &= \frac{\widehat{\sigma}_\tau^2-e^{-\phi(\tau-t)}\widehat{\sigma}_t^2}{\frac{1-e^{2\phi(\tau-t)}}{2\phi}\left(\sigma_{\rho,1}^2+\sigma_{\rho,2}^2\right)+\left(e^{-2\phi(\tau-t)}\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)} \\ b\left(t,\tau;\widehat{\sigma}_t^2\right) &= \frac{\widehat{\sigma}_\tau^2-e^{-\phi(\tau-t)}\widehat{\sigma}_t^2}{\sqrt{\frac{1-e^{2\phi(\tau-t)}}{2\phi}\left(\sigma_{\rho,1}^2+\sigma_{\rho,2}^2\right)+\left(e^{-2\phi(\tau-t)}\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)}\sqrt{\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2}} \end{aligned}$$

Proof of Proposition 3: From Proposition 2, an IPO takes place at τ if and only if

$$\widehat{\rho}_\tau > \underline{\rho}\left(x_\tau\right)$$

We then have

$$\begin{aligned} E\left[\rho_\tau-\widehat{\rho}_\tau|IPO\right] &= E\left[x_\tau|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right] \\ &= \int\int x_\tau\Phi\left(x_\tau,\widehat{\rho}_\tau|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)d\widehat{\rho}_\tau dx_\tau \\ &= \frac{1}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)}\int\int x_\tau 1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right\}}\Phi\left(x_\tau,\widehat{\rho}_\tau\right)d\widehat{\rho}_\tau dx_\tau \\ &= \frac{1}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)}\int\int x_\tau 1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right\}}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)\Phi\left(x_\tau\right)d\widehat{\rho}_\tau dx_\tau \\ &= \frac{1}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)}\int x_\tau\left[\int 1_{\left\{\widehat{\rho}:\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right\}}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau\right]\Phi\left(x_\tau\right)dx_\tau \\ &= \frac{1}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)}\int x_\tau\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)|x_\tau\right)\Phi\left(x_\tau\right)dx_\tau \end{aligned}$$

From Lemma A.3. $\widehat{\rho}_\tau$ conditional on x_τ is normally distributed. Specifically

$$\widehat{\rho}_\tau|x_\tau\sim N\left(\widehat{\rho}_t+a\left(t,\tau\right)\left(x_\tau-e^{-\phi(\tau-t)}x_t\right),\left(\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)\left(1-b\left(t,\tau\right)^2\right)\right)$$

where $a(t,\tau)$ and $b(t,\tau)$ are given by

$$\begin{aligned} a(t,\tau) &= \frac{S_{23}(t,\tau)}{S_{22}(t,\tau)} = \frac{\widehat{\sigma}_\tau^2 - e^{-\phi(\tau-t)}\widehat{\sigma}_t^2}{\frac{1-e^{2\phi(\tau-t)}}{2\phi}\left(\sigma_{\rho,1}^2+\sigma_{\rho,2}^2\right)+\left(e^{-2\phi(\tau-t)}\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)} \\ b(t,\tau) &= \frac{S_{23}(t,\tau)}{\sqrt{S_{22}(t,\tau)S_{33}(t,\tau)}} = \frac{\widehat{\sigma}_\tau^2 - e^{-\phi(\tau-t)}\widehat{\sigma}_t^2}{\sqrt{\frac{1-e^{2\phi(\tau-t)}}{2\phi}\left(\sigma_{\rho,1}^2+\sigma_{\rho,2}^2\right)+\left(e^{-2\phi(\tau-t)}\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2\right)}\sqrt{\widehat{\sigma}_t^2-\widehat{\sigma}_\tau^2}} \end{aligned}$$

It is then immediate to find

$$\Pr\left(\widehat{\rho}_\tau > \underline{\rho}(x_\tau)\,|x_\tau\right) = 1-\Pr\left(\widehat{\rho}_\tau < \underline{\rho}(x_\tau)\,|x_\tau\right) = 1-\mathcal{N}\left(k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t\right)\right)$$

where $k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t\right)$ is given in (A34). It then follows that

$$\begin{aligned} \Pr\left(\widehat{\rho}_\tau > \underline{\rho}(x_\tau)\right) &= \int \Pr\left(\widehat{\rho}_\tau > \underline{\rho}(x_\tau)\,|x_\tau\right)\Phi(x_\tau)\,dx_\tau \\ &= \int \left(1-\mathcal{N}\left(k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t\right)\right)\right)\Phi(x_\tau)\,dx_\tau \\ &= 1-\int \mathcal{N}\left(k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t\right)\right)\Phi(x_\tau)\,dx_\tau \end{aligned}$$

To conclude, we find

$$\begin{aligned} E\left[\rho_\tau - \widehat{\rho}_\tau | IPO\right] &= \frac{\int x_\tau \Pr\left(\widehat{\rho}_\tau > \underline{\rho}(x_\tau)\,|x_\tau\right)\Phi(x_\tau)\,dx_\tau}{\Pr\left(\widehat{\rho}_\tau > \underline{\rho}(x_\tau)\right)} \\ &= \frac{\int x_\tau \Phi(x_\tau)\,dx_\tau - \int x_\tau \mathcal{N}\left(k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t\right)\right)\Phi(x_\tau)\,dx_\tau}{1-\int \mathcal{N}\left(k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t\right)\right)\Phi(x_\tau)\,dx_\tau} \\ &= \frac{e^{-\phi(\tau-t)}x_t - \int x_\tau \mathcal{N}\left(k\left(x_\tau,t,x_t,\widehat{\rho}_t\right)\right)\Phi(x_\tau)\,dx_\tau}{1-\int \mathcal{N}\left(k\left(x_\tau,\tau;t,x_t,\widehat{\rho}_t\right)\right)\Phi(x_\tau)\,dx_\tau} \end{aligned}$$

Q.E.D.

Corollary A.1 Let

$$y_\tau = b_\tau + Q_1(u-\tau)\,x_\tau + (u-\tau)\,\widehat{\rho}_\tau \tag{A37}$$

Then, the joint distribution of $(y_\tau, x_\tau, \widehat{\rho}_\tau)$ is given by

$$\begin{pmatrix} y_\tau \\ x_\tau \\ \widehat{\rho}_\tau \end{pmatrix} | \begin{pmatrix} y_0 \\ x_0 \\ \widehat{\rho}_0 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_y(0,\tau,u) \\ \mu_x(0,\tau,u) \\ \mu_{\widehat{\rho}}(0,\tau,u) \end{pmatrix}, \mathbf{\Sigma}(0,\tau,u)\right)$$

where $\mu_x(0,\tau,u)$ and $\mu_{\widehat{\rho}}(0,\tau,u)$ are given by $\mu_{\mathrm{Z},2}$ and $\mu_{\mathrm{Z},3}$ in Lemma A.3, respectively, $\Sigma_{i,j}(0,\tau,u) = S_{ij}(0,\tau)$, for $i,j=2,3$, $\mathbf{S}(0,\tau)$ is given in Lemma A.3, and finally

$$\begin{aligned} \Sigma_{11}(0,\tau,u) &= S_{11}(0,\tau) + Q_1(u-\tau)^2 S_{22}(0,\tau) + (u-\tau)^2 S_{33}(0,\tau) \\ &\quad +2Q_1(u-\tau)S_{12}(0,\tau) + 2(u-\tau)S_{13}(0,\tau) \\ &\quad +2Q_1(u-\tau)(T-\tau)S_{23}(0,\tau) \\ \Sigma_{12}(0,\tau,u) &= S_{12}(0,\tau) + Q_1(u-\tau)S_{22}(0,\tau) + (u-\tau)S_{23}(0,\tau) \\ \Sigma_{13}(0,\tau,u) &= S_{13}(0,\tau) + Q_1(u-\tau)S_{23}(0,\tau) + (u-\tau)S_{33}(0,\tau) \end{aligned}$$

Proof of Corollary A.1: Immediate from Lemma A.3. Q.E.D.

Utility from Starting a Private Firm at 0: The value function at time 0 is given by

$$\begin{aligned} V_0^O(B_0,0) &= \frac{B_0^{1-\gamma}}{1-\gamma}\times\left\{\alpha^{1-\gamma}\int_0^\tau Z^O\left(\rho_0,\widehat{\rho}_0,\widehat{\sigma}_\tau^2.u\right)du \right. && \text{(A38)}\\ &+e^{-\beta\tau}\left[g(T-\tau)\,e^{G_0(\tau,T)+G_1(\tau,T)x_0+G_2(\tau,T)\widehat{\rho}_0}H^y\left(x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2,\tau,T\right)\right. && \text{(A39)}\\ &+\int_\tau^T \alpha^{1-\gamma}e^{\overline{G}_0(\tau,u)+G_1(\tau,u)x_0+G_2(\tau,u)\widehat{\rho}_0}H^n\left(x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2,\tau,u\right)du && \text{(A40)}\\ &\left.\left.+\eta e^{\overline{G}_0(\tau,T)+G_1(\tau,T)x_0+G_2(\tau,T)\widehat{\rho}_0}H^n\left(x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2,\tau,T\right)\right]\right\} && \text{(A41)} \end{aligned}$$

where

$$H^y\left(x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2,\tau,u\right) = \int e^{G_3(\tau,u)x_\tau}\left(1-\mathcal{N}\left(k_2\left(x_\tau,\tau,u;0,x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right)\right)\right)\Phi\left(x_\tau;\mu_x(x),\sigma_x^2(t,\tau)\right)dx \quad \text{(A42)}$$

$$H^n\left(x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2,\tau,u\right) = \int e^{G_3(\tau,u)x_\tau}\mathcal{N}\left(k_2\left(x_\tau,\tau,u;0,x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right)\right)\Phi\left(x_\tau;\mu_x(x),\sigma_x^2(t,\tau)\right)dx_\tau \quad \text{(A43)}$$

with

$$k_2\left(x_\tau,\tau,u;x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right) = k\left(x_\tau,\tau;x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right)-(1-\gamma)\,a_2(\tau,u)\sqrt{\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(0,\tau)^2\right)}$$

In the formulas above,

$$\begin{aligned} G_0(\tau,u) &= (1-\gamma)\left(Q_0(u-\tau)+\frac{1}{2}Q_2(u-\tau)^2\widehat{\sigma}_\tau^2\right)\\ &\quad+\frac{1}{2}(1-\gamma)^2\left(\sigma_y^2(\tau,u)+a_2^2(\tau,u)\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(\tau)^2\right)\right)\\ \overline{G}_0(\tau,u) &= \overline{Q}_0(u-\tau)+\frac{1}{2}(1-\gamma)^2\left(Q_2(u-\tau)^2\widehat{\sigma}_\tau^2+\sigma_y^2(\tau,u)+a_2^2(\tau,u)\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(\tau)^2\right)\right)\\ G_1(\tau,u) &= (1-\gamma)\left(Q_1(u)-a_1(\tau,u)\,e^{-\phi\tau}-a(0,\tau)\,a_2(\tau,u)\,e^{-\phi\tau}\right)\\ G_2(\tau,u) &= (1-\gamma)\,u\\ G_3(\tau,u) &= (1-\gamma)\left(a_1(\tau,u)+a_2(\tau,u)\,a(\tau)\right)\\ \sigma_y^2(\tau,u) &= \mathbf{\Sigma}_{11}(\tau,u)-\mathbf{\Sigma}_{[1,2:3]}(\tau,u)\,\mathbf{S}^{-1}_{[2:3,2:3]}(0,\tau)\,\mathbf{\Sigma}_{[2:3,1]}(\tau,u) && \text{(A44)}\\ a_1(\tau,u) &= \mathbf{\Sigma}_{[1,2:3]}(\tau,u)\,\mathbf{S}^{-1}_{[2:3,2:3]}(0,\tau)\,[1,0]' && \text{(A45)}\\ a_2(\tau,u) &= \mathbf{\Sigma}_{[1,2:3]}(\tau,u)\,\mathbf{S}^{-1}_{[2:3,2:3]}(0,\tau)\,[0,1]' && \text{(A46)} \end{aligned}$$

Proof: The value function is given by

$$V_0^O(B_0,0) = E_0\left[\int_0^T e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt+\eta e^{-\beta T}\frac{W_T^{1-\gamma}}{1-\gamma}\right] \quad \text{(A47)}$$

$$
\begin{aligned}
&= E_0\left[\int_0^\tau e^{-\beta t}\frac{(\alpha B_t)^{1-\gamma}}{1-\gamma}dt\right] && \text{(A48)}\\
&\quad +e^{-\beta\tau}E_0\left[V\left(M_\tau,\tau\right)|\widehat{\rho}_\tau>\underline{\rho}\right]\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\right) && \text{(A49)}\\
&\quad +\ e^{-\beta\tau}E_0\left[V^O\left(B_\tau,\tau\right)|\widehat{\rho}_\tau<\underline{\rho}\right]\Pr\left(\widehat{\rho}_\tau<\underline{\rho}\right) && \text{(A50)}
\end{aligned}
$$

First, Lemma A.1 immediately implies that (A48) equals (A38). Next, we compute

$$
\begin{aligned}
&E_0\left[V\left(M_\tau\right),\tau\right)|\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\big]\\
&= E_0\left[\frac{B_\tau^{1-\gamma}}{1-\gamma}g(T-\tau)e^{(1-\gamma)Q_0(T-\tau)+(1-\gamma)Q_1(T-\tau)x_\tau+(1-\gamma)(T-\tau)\widehat{\rho}_\tau+\frac{1-\gamma}{2}Q_2(T-\tau)^2\widehat{\sigma}_\tau^2}|\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right]\\
&= \frac{g(T-\tau)}{1-\gamma}e^{(1-\gamma)\left\{Q_0(T-\tau)+\frac{1}{2}Q_2(T-\tau)^2\widehat{\sigma}_\tau^2\right\}}E_0\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right] && \text{(A51)}
\end{aligned}
$$

where y_τ is defined in (A37). The second equality stems from the fact that conditional on an IPO, the agent will receive M_τ and will optimally invest in stocks and bonds. The remaining equalities stem from the earlier results about the form of $V(M_\tau,\tau)$ and the market value itself. We then need to compute

$$
E\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right]=\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)dy_\tau
$$

The computation of this conditional expectation integral is tedious, but it follows directly from the application of Bayes rule, and the joint normality of all of the variables. The following derivations are given for convenience. First, we start from

$$
\begin{aligned}
\Phi\left(y_\tau|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right) &= \int\int\Phi\left(y_\tau|\widehat{\rho}_\tau,x_\tau\right)\Phi\left(\widehat{\rho}_\tau,x_\tau|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)d\widehat{\rho}_\tau dx_\tau\\
&= \int\int\Phi\left(y_\tau|\widehat{\rho}_\tau,x_\tau\right)\frac{1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right\}}\Phi\left(x_\tau,\widehat{\rho}_\tau\right)}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)}d\widehat{\rho}_\tau dx_\tau
\end{aligned}
$$

Substitute, to find

$$
\begin{aligned}
&E\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right]=\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)dy_\tau\\
&= \int\int\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau,x_\tau\right)\frac{1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right\}}\Phi\left(x_\tau,\widehat{\rho}_\tau\right)}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)}dx_\tau d\widehat{\rho}_\tau dy_\tau\\
&= \frac{1}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)}\int\int\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau,x_\tau\right)1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right\}}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)\Phi\left(x_\tau\right)dx_\tau d\widehat{\rho}_\tau dy_\tau\\
&= \frac{1}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right)}\int\int\left[\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau,x_\tau\right)dy_\tau\right]1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right\}}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)\Phi\left(x_\tau\right)dx_\tau d\widehat{\rho}_\tau
\end{aligned}
$$

The joint normality of $(y_\tau,x_\tau,\widehat{\rho}_\tau)$ and the rules of the conditional normal distribution imply

$$
y_\tau|\left(\widehat{\rho}_\tau,x_\tau\right)\sim N\left(\mu_y\left(\widehat{\rho}_\tau,x_\tau\right),\sigma_y^2\left(\tau\right)\right)
$$

where $\sigma_y^2(\tau,u)$ is given by (A44), and

$$\mu_y\left(\widehat{\rho}_\tau, x_\tau\right) = a_0\left(0,\tau,u\right) + a_1\left(0,\tau,u\right) x_\tau + a_2\left(0,\tau,u\right)\widehat{\rho}_\tau$$

where a_i, $i=1,2$ are given by (A45) and (A46), and

$$a_0\left(0,\tau,u\right) = b_0 + \left(Q_1\left(u\right) - a_1\left(0,\tau,u\right)e^{-\phi\tau}\right)x_0 + \left(u - a_2\left(0,\tau,u\right)\right)\widehat{\rho}_0$$

From now on, we suppress the time indices in a_i, and other variables, unless it is necessary for clarity. Since

$$\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau, x_\tau\right)dy_\tau = E\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau, x_\tau\right] = e^{(1-\gamma)\mu_y(\widehat{\rho}_\tau,x_\tau)+\frac{1}{2}(1-\gamma)^2\sigma_y^2(\tau,T)} \tag{A52}$$

we obtain

$$\begin{aligned}
&\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau > \underline{\rho}\left(x_\tau\right)\right)dy \\
&= \frac{1}{\Pr\left(\widehat{\rho}_\tau > \underline{\rho}\left(x_\tau\right)\right)}\int\left[\int_{\underline{\rho}(x_\tau)}^{\infty} e^{(1-\gamma)\mu_y(\widehat{\rho}_\tau,x_\tau)+\frac{1}{2}(1-\gamma)^2\sigma_y^2(\tau,T)}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau\right]\Phi\left(x_\tau\right)dx_\tau \\
&= \frac{e^{(1-\gamma)a_0+\frac{1}{2}(1-\gamma)^2\sigma_y^2(\tau,T)}}{\Pr\left(\widehat{\rho}_\tau > \underline{\rho}\left(x_\tau\right)\right)}\int e^{(1-\gamma)a_1x_\tau}\left[\int_{\underline{\rho}(x_\tau)}^{\infty} e^{(1-\gamma)a_2\widehat{\rho}_\tau}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau\right]\Phi\left(x_\tau\right)dx_\tau
\end{aligned} \tag{A53}$$

Recall now that

$$\widehat{\rho}_\tau|x_\tau \sim N\left(\widehat{\rho}_0 + a\left(0,\tau\right)\left(x_\tau - e^{-\phi\tau}x_0\right), \left(\widehat{\sigma}_0^2 - \widehat{\sigma}_\tau^2\right)\left(1 - b\left(0,\tau\right)^2\right)\right)$$

We now use the rule

$$\begin{aligned}
\int_a^{\infty} e^{kx}\Phi\left(x;b,s^2\right)dx &= e^{\frac{1}{2}k^2s^2+kb}\left(1-\mathcal{N}\left(a, b+ks^2, s^2\right)\right) \\
&= e^{\frac{1}{2}k^2s^2+kb}\left(1-\mathcal{N}\left(\frac{a-b}{s} - ks\right)\right)
\end{aligned} \tag{A54}$$

which implies

$$\begin{aligned}
&\int_{\underline{\rho}(x_\tau)}^{\infty} e^{(1-\gamma)a_2\widehat{\rho}_\tau}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau \\
&= e^{\frac{1}{2}(1-\gamma)^2a_2^2\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(0,\tau)^2\right)+(1-\gamma)a_2E[\widehat{\rho}_\tau|x_\tau]}\times\left(1-\mathcal{N}\left(k_2\left(x_\tau,\tau;0,x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right)\right)\right) \\
&= e^{\frac{1}{2}(1-\gamma)^2a_2^2\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(0,\tau)^2\right)+(1-\gamma)a_2\left(\widehat{\rho}_0-a(0,\tau)e^{-\phi\tau}x_0\right)+(1-\gamma)a_2a(0,\tau)x_\tau} \\
&\quad\times\left(1-\mathcal{N}\left(k_2\left(x_\tau,\tau;0,x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right)\right)\right)
\end{aligned}$$

where

$$\begin{aligned}
k_2\left(x_\tau,\tau;0,x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right) &= \frac{\underline{\rho}\left(x_\tau\right) - \widehat{\rho}_0 - a\left(0,\tau\right)\left(x_\tau - e^{-\phi\tau}x_0\right)}{\sqrt{\left(\widehat{\sigma}_0^2 - \widehat{\sigma}_\tau^2\right)\left(1-b\left(0,\tau\right)^2\right)}} \\
&\quad - \left(1-\gamma\right)a_2\sqrt{\left(\widehat{\sigma}_0^2 - \widehat{\sigma}_\tau^2\right)\left(1-b\left(0,\tau\right)^2\right)} \\
&= k\left(x_\tau,\tau;0,x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right) - \left(1-\gamma\right)a_2\sqrt{\left(\widehat{\sigma}_0^2 - \widehat{\sigma}_\tau^2\right)\left(1-b\left(0,\tau\right)^2\right)}
\end{aligned}$$

We can finally compute the integral in (A53) as

$$\begin{aligned}
& \int e^{(1-\gamma)a_1 x_\tau}\left[\int_{\underline{\rho}(x_\tau)}^{\infty} e^{(1-\gamma)a_2\widehat{\rho}_\tau}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau\right]\Phi\left(x_\tau\right)dx_\tau \\
= \; & e^{\frac{1}{2}(1-\gamma)^2 a_2^2\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(0,\tau)^2\right)+(1-\gamma)a_2\left(\widehat{\rho}_0-a(0,\tau)e^{-\phi\tau}x_0\right)} \\
& \times\int e^{(1-\gamma)(a_1+a_2a(0,\tau))x_\tau}\left(1-\mathcal{N}\left(k_2\left(x_\tau,\tau;0,x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right)\right)\right)\Phi\left(x_\tau\right)dx_\tau
\end{aligned}$$

Thus, putting all together, we obtain

$$\begin{aligned}
E\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau>\underline{\rho}\left(x_\tau\right)\right] \; = \; & \frac{e^{(1-\gamma)\left(a_0+a_2\left(\widehat{\rho}_0-a(0,\tau)e^{-\phi\tau}x_0\right)\right)+\frac{1}{2}(1-\gamma)^2\left(\sigma_y^2+a_2^2\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(0,\tau)^2\right)\right)}}{\Pr\left(\widehat{\rho}_\tau>\underline{\rho}(x_\tau)\right)}\times \\
& \times H^y\left(x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2,\tau,u\right)
\end{aligned}$$

where $H^y(.)$ is given in (A42). Substituting this expression in (A51) and using the identity

$$a_0+a_2\left(\widehat{\rho}_0-a\left(0,\tau\right)e^{-\phi\tau}x_0\right) \; = \; b_0+\left(Q_1\left(T\right)-a_1e^{-\phi\tau}-a\left(0,\tau\right)a_2e^{-\phi\tau}\right)x_0+T\widehat{\rho}_0 \tag{A55}$$

we finally find that (A49) equals (A39).

We now move to compute the expected utility conditional on no IPO at τ. We have

$$\begin{aligned}
& E_0\left[V^O\left(B_\tau,\tau\right)|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right] \\
= \; & \frac{1}{1-\gamma}\alpha^{1-\gamma}\int_\tau^T e^{\overline{Q}_0(u-\tau)+\frac{1}{2}(1-\gamma)^2Q_2(u-\tau)^2\widehat{\sigma}_\tau^2}E_0\left[e^{(1-\gamma)(b_\tau+Q_1(u-\tau)x_\tau+(u-\tau)\widehat{\rho}_\tau)}|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right]du \\
& +\frac{\eta}{1-\gamma}e^{\overline{Q}_0(T-\tau)+\frac{1}{2}(1-\gamma)^2Q_2(T-\tau)^2\widehat{\sigma}_\tau^2}E_0\left[e^{(1-\gamma)(b_\tau+Q_1(T-\tau)x_\tau+(T-\tau)\widehat{\rho}_\tau)}|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right]
\end{aligned}$$

and then

$$\begin{aligned}
& E_0\left[e^{(1-\gamma)(b_\tau+Q_1(u-\tau)x_\tau+(u-\tau)\widehat{\rho}_\tau)}|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right] \\
= \; & E\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right] \\
= \; & \int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right)dy_\tau \\
= \; & \int\int\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau,x_\tau\right)\frac{1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau<\underline{\rho}(x_\tau)\right\}}\Phi\left(x_\tau,\widehat{\rho}_\tau\right)}{\Pr\left(\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right)}dx_\tau d\widehat{\rho}_\tau dy_\tau \\
= \; & \frac{1}{\Pr\left(\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right)}\int\int\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau,x_\tau\right)1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau<\underline{\rho}(x_\tau)\right\}}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)\Phi\left(x_\tau\right)dx_\tau d\widehat{\rho}_\tau dy_\tau \\
= \; & \frac{1}{\Pr\left(\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right)}\int\int\left[\int e^{(1-\gamma)y_\tau}\Phi\left(y_\tau|\widehat{\rho}_\tau,x_\tau\right)dy_\tau\right]1_{\left\{x_\tau,\widehat{\rho}:\widehat{\rho}_\tau<\underline{\rho}(x_\tau)\right\}}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)\Phi\left(x_\tau\right)dx_\tau d\widehat{\rho}_\tau
\end{aligned}$$

Using (A52), we obtain

$$\begin{aligned} & E_0\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right] \\ = & \frac{1}{\Pr\left(\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right)}\int\left[\int_{-\infty}^{\underline{\rho}(x_\tau)}e^{(1-\gamma)\mu_y(\widehat{\rho}_\tau,x_\tau)+\frac{1}{2}(1-\gamma)^2\sigma_y^2}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau\right]\Phi\left(x_\tau\right)dx_\tau \\ = & \frac{e^{(1-\gamma)a_0+\frac{1}{2}(1-\gamma)^2\sigma_y^2}}{\Pr\left(\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right)}\int e^{(1-\gamma)a_1\widehat{x}_\tau}\left[\int_{-\infty}^{\underline{\rho}(x_\tau)}e^{(1-\gamma)a_2\widehat{\rho}_\tau}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau\right]\Phi\left(x_\tau\right)dx_\tau \end{aligned}$$

By using the same rule as in (A54), we obtain

$$\begin{aligned} \int_{-\infty}^{\underline{\rho}(x_\tau)}e^{(1-\gamma)a_2\widehat{\rho}_\tau}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau &= \int_{-\infty}^{\infty}e^{(1-\gamma)a_2\widehat{\rho}_\tau}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau-\int_{\underline{\rho}(x_\tau)}^{\infty}e^{(1-\gamma)a_2\widehat{\rho}_\tau}\Phi\left(\widehat{\rho}_\tau|x_\tau\right)d\widehat{\rho}_\tau \\ &= e^{\frac{1}{2}(1-\gamma)^2a_2^2\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(0,\tau)^2\right)+(1-\gamma)a_2\left(\widehat{\rho}_0-a(0,\tau)e^{-\phi\tau}x_0\right)+(1-\gamma)a_2a(0,\tau)x_\tau}\times \\ &\quad\times\mathcal{N}\left(k_2\left(x_\tau,\tau;0,x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2\right)\right) \end{aligned}$$

This yields

$$\begin{aligned} E_0\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right] &= \frac{e^{(1-\gamma)\left(a_0(\tau,u)+a_2(\tau,u)\left(\widehat{\rho}_0-a(0,\tau)e^{-\phi\tau}x_0\right)\right)+\frac{1}{2}(1-\gamma)^2\left(\sigma_y^2(\tau,u)+a_2^2(\tau,u)\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(0,\tau)^2\right)\right)}}{\Pr\left(\widehat{\rho}_\tau<\underline{\rho}(x_\tau)\right)}\times \\ &\quad\times H^n\left(x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2,\tau,u\right) \end{aligned}$$

Using the identity (A55), we finally find

$$\begin{aligned} & E_0\left[e^{(1-\gamma)y_\tau}|\widehat{\rho}_\tau<\underline{\rho}\left(x_\tau\right)\right] \\ = & \frac{B_0^{1-\gamma}e^{(1-\gamma)\left(\left(Q_1(u)-a_1(\tau,u)e^{-\phi\tau}-a(0,\tau)a_2(\tau,u)e^{-\phi\tau}\right)x_0+u\widehat{\rho}_0\right)+\frac{1}{2}(1-\gamma)^2\left(\sigma_y^2(\tau,u)+a_2^2(\tau,u)\left(\widehat{\sigma}_0^2-\widehat{\sigma}_\tau^2\right)\left(1-b(0,\tau)^2\right)\right)}}{\Pr\left(\widehat{\rho}_\tau<\underline{\rho}(x_\tau)\right)}\times \\ & \times H^n\left(x_0,\widehat{\rho}_0,\widehat{\sigma}_0^2,\tau,u\right) \end{aligned}$$

Using this result, and simple substitutions, we show that (A50) indeed equals the sum of (A40) and (A41), proving the claim. Q.E.D.

Optimal IPO Timing

The entrepreneur chooses the time τ^* of the IPO to maximize his utility. At this time, the entrepreneur sells the firm for its market value and obtains M_{τ^*} in equation (28). Using (A7) and (A9), the entrepreneur's utility from selling the firm at τ^* is

$$V\left(B_{\tau^*},\rho_{\tau^*},\widehat{\rho}_{\tau^*},\tau^*\right) = \frac{B_{\tau^*}^{1-\gamma}}{1-\gamma}g\left(T-\tau^*\right)e^{(1-\gamma)\left[Q_0(T-\tau^*)+Q_1(T-\tau^*)\rho_{\tau^*}+Q_2(T-\tau^*)\widehat{\rho}_{\tau^*}+\frac{1}{2}Q_2(T-\tau^*)^2\widehat{\sigma}_{\tau^*}^2\right]}$$

Note that there is no need to make explicit the dependence of value functions on $\widehat{\sigma}_t^2$, as this is a known function of time. For $t<\tau^*$, the entrepreneur's value function is

$$\mathcal{V}\left(B_t,\rho_t,\widehat{\rho}_t,t\right)=\max_{\tau^*}E_t\left[\int_t^{\tau^*}e^{-\beta(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds+e^{-\beta(\tau^*-t)}V\left(B_{\tau^*},\rho_{\tau^*},\widehat{\rho}_{\tau^*},\tau^*\right)\right]$$

where recall that $c_s = \alpha B_s$ for $s < \tau^*$. For $t < \tau^*$, the Bellman Equation is

$$\begin{aligned}\beta\mathcal{V} &= \frac{c_t^{1-\gamma}}{1-\gamma} + \frac{\partial\mathcal{V}}{\partial t} + \frac{\partial\mathcal{V}}{\partial B_t}E\left[dB_t\right] + \frac{\partial\mathcal{V}}{\partial\rho_t}E\left[d\rho_t\right] + \frac{\partial\mathcal{V}}{\partial\widehat{\rho}_t}E\left[d\widehat{\rho}_t\right] \\ &\quad + \frac{1}{2}\frac{\partial^2\mathcal{V}}{\partial\rho_t^2}E\left[d\rho_t^2\right] + \frac{1}{2}\frac{\partial^2\mathcal{V}}{\partial\widehat{\rho}_t^2}E\left[d\widehat{\rho}_t^2\right] + \frac{\partial^2\mathcal{V}}{\partial\rho_t\partial\widehat{\rho}_t}E\left[d\rho_t d\widehat{\rho}_t\right]\end{aligned}$$

with boundary condition $\mathcal{V}\left(B_T, \rho_T, \widehat{\rho}_T, T\right) = \frac{B_T^{1-\gamma}}{1-\gamma}$ and optimality conditions $\mathcal{V}\left(B_t, \rho_t, \widehat{\rho}_t, t\right) > V\left(B_t, \rho_t, \widehat{\rho}_t, t\right)$ for $t < \tau^*$, and $\mathcal{V}\left(B_{\tau^*}, \rho_{\tau^*}, \widehat{\rho}_{\tau^*}, \tau^*\right) = V\left(B_{\tau^*}, \rho_{\tau^*}, \widehat{\rho}_{\tau^*}, \tau^*\right)$.

Proposition A.1: The value function is $\mathcal{V}\left(B_t, \rho_t, \widehat{\rho}_t, t\right) = B_t^{1-\gamma}\Phi\left(\rho_t, \widehat{\rho}_t, t\right)$ where $\Phi(.)$ solves

$$\begin{aligned}0 &= \frac{\alpha^{1-\gamma}}{1-\gamma} + \frac{\partial\Phi}{\partial t} + \left(\left(1-\gamma\right)\rho_t - \beta\right)\Phi + \frac{\partial\Phi}{\partial\rho_t}\phi\left(\widehat{\rho}_t - \rho_t\right) \\ &\quad + \frac{1}{2}\frac{\partial^2\Phi}{\partial\rho_t^2}\left(\sigma_{\rho,1}^2 + \sigma_{\rho,2}^2\right) + \frac{1}{2}\frac{\partial^2\Phi}{\partial\widehat{\rho}_t^2}\left(\frac{\widehat{\sigma}_t^2\phi}{\sigma_{\rho,2}}\right)^2 + \frac{\partial^2\Phi}{\partial\rho_t\partial\widehat{\rho}_t}\left(\widehat{\sigma}_t^2\phi\right) \qquad \text{(A56)}\end{aligned}$$

with boundary condition $\Phi\left(\rho_T, \widehat{\rho}_T, T\right) = \left(1-\gamma\right)^{-1}$ and optimality conditions

$$\Phi\left(\rho_t, \widehat{\rho}_t, t\right) > \frac{1}{1-\gamma}e^{\left(1-\gamma\right)\left[Q_0(T-t) + Q_1(T-t)\rho_t + Q_2(T-t)\widehat{\rho}_t + \frac{1}{2}Q_2(T-t)^2\widehat{\sigma}_t^2\right]} \qquad \text{for} \qquad t < \tau^*$$

and equality at $t = \tau^*$.

Proof: It is simple to verify that $\mathcal{V}\left(B_t, \rho_t, \widehat{\rho}_t, t\right) = B_t^{1-\gamma}\Phi\left(\rho_t, \widehat{\rho}_t, t\right)$ satisfies the Bellman equation and all of the boundary conditions. Q.E.D.

Finally, we obtain $\Phi(.)$ and the optimal IPO time τ^* by numerically solving backward the Partial Differential Equation (A56) with a finite difference method.