41000+418000: Business Statistics

Fall 2018

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The University of Chicago Booth School of Business

October 1, 2018
Getting Started

http://faculty.chicagobooth.edu/nicholas.polson/teaching/41000/

► General Expectations

1. Read the notes/Practice
2. Be on schedule
3. Add R to your friend list!
Course Expectations

**Midterm: 40%**  First Saturday of November

T/F and three long questions. Cheat Sheet allowed.

**Final Project: 40%**  Individual Project

50% on Writing/Presentation skills

50% on Modeling.

**Homework: 20%**  Bi-weekly Assignments. Handed in at Class

Groups of 3-4. Otherwise it’s no fun!

Grading is ✓ +, ✓ , ✓ −.

**Fixed Grading Curve:** 3.33.
Course Overview

Weeks 1&2  Section 1: Introduction, Probability & Bayes
          OpenIntro Statistics, Chapters 2&3

Weeks 3&4  Section 2: Learning from Data
          Chapters 4,5&6

Week 5  Section 3: Modeling and Linear Regression
          Chapters 6&7

Week 6  Midterm

Weeks 7 & 8  Section 4: Multiple Regression and Classification
          Chapters 7&8

Week 9 & 10  Section 5: AI and Deep Learning
Business Statistics: 41000+41800

Section 1: Introduction
Probability and Bayes

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Suggested Reading
OpenIntro Statistics, Chapters 2&3
Review of Basic Probability Concepts

Probability lets us talk efficiently about things that we are uncertain about.

- What will Amazon’s sales be next quarter?
- What will the return be on my stocks next year?
- How often will users click on a particular Google ad?

All these involve estimating or predicting unknowns!!
Random Variables are numbers that we are not sure about.

There’s a list of potential outcomes.

We assign probabilities to each outcome.

Example: Suppose that we are about to toss two coins.

Let $X$ denote the number of heads.

We call $X$ the random variable that stands for the potential outcome.
**Probability** is a language designed to help us communicate about uncertainty.

We assign a number between 0 and 1 measuring how likely that event is to occur.

It’s immensely useful, and there’s only a few basic rules.

1. If an event $A$ is certain to occur, it has probability 1, denoted $P(A) = 1$

2. Either an event $A$ occurs or it does not.

\[ P(\text{not } A) = 1 - P(A) \]

3. If two events are mutually exclusive (both cannot occur simultaneously) then $P(A \text{ or } B) = P(A) + P(B)$

4. Conditional probability $P(A \text{ and } B) = P(A|B)P(B)$
### U.S. Presidential Election 2020

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Probability $P = \frac{1}{1 + O}$. 
We describe the behavior of random variables with a Probability Distribution.

**Example:** Suppose we are about to toss two coins.

Let $X$ denote the number of heads.

\[
X = \begin{cases} 
0 \text{ with prob. } 1/4 \\
1 \text{ with prob. } 1/2 \\
2 \text{ with prob. } 1/4 
\end{cases}
\]

$X$ is called a Discrete Random Variable.

**Question:** What is $Pr(X = 0)$? How about $Pr(X \geq 1)$?
Pete Rose of the Cincinnati Reds set a National League record of hitting safely in 44 consecutive games ...

- Rose was a 300 hitter.
  
  Assume he comes to bat 4 times each game.

- Each at bat is assumed to be independent, i.e., the current at bat doesn’t affect the outcome of the next.

What probability might reasonably be associated with that hitting streak?

Joe DiMaggio’s record is 56! His batting average was .325
Pete Rose Hitting Streak

Let $A_i$ denote the event that “Rose hits safely in the $i$th game”

$Pr$(Rose Hits Safely in 44 consecutive games) =
$P$(A$_1$ and A$_2$ . . . and A$_{44}$) = $P$(A$_1$)$P$(A$_2$) . . . $P$(A$_{44}$)

We now need to find $P$(A$_i$) . . . where $P$(A$_i$) = 1 − $P$(not A$_i$)

$P$(A$_1$) = 1 − $P$(not A$_1$)

= 1 − $P$(Rose makes 4 outs)

= 1 − (0.7)$^4$ = 0.76

So for the winning streak we have (0.76)$^{44}$ = 0.0000057 !!!
Pete Rose Hitting Streak

There are three basic inferences

▶ This means that the odds for a particular player as good as Pete Rose starting a hitting streak today are 175,470 to 1

▶ Doesn’t mean that the run of 44 won’t be beaten by some player at some time: the Law of Very Large Numbers

▶ Joe DiMaggio’s record is 56!!!!

It’s going to be hard to beat.

We have $(0.792)^{56} = 2.13 \times 10^{-6}$ or 2.13 million to 1
Patriots won 19 out of 25 coin tosses in 2014-15 season!

What is the probability of that happening?

Let $X$ be a random variable equal to 1 if the Patriots win and 0 otherwise.

It's reasonable to assume $Pr(X = 1) = \frac{1}{2}$

There are $25 \choose 19$ or 177,100 different sequences of 25 games where the Patriots win 19.

Each potential sequence has probability $0.5^{25}$ why?

$$Pr\left(\text{Patriots win 19 out 25 tosses}\right) = 177,100 \times 0.5^{25} = 0.005$$
Conditional, Joint and Marginal Distributions

Use probability to describe outcomes involving more than one variable at a time. Need to be able to measure what we think will happen to one variable relative to another.

In general the notation is...

- $\Pr(X = x, Y = y)$ is the joint probability that $Y$ equals $y$ and $X = x$
- $\Pr(X = x | Y = y)$ is the conditional probability that $Y$ equals $y$ given $X = x$
- $\Pr(X = x)$ is the marginal probability of $X = x$
Conditional, Joint and Marginal Distributions

Relationship between the joint and conditional ...

\[ pr(x, y) = p(x)p(y|x) \]

\[ = p(y)p(x|y) \]

Relationship between the joint and marginal ...

\[ pr(x) = \sum_y p(x, y) \]

\[ pr(y) = \sum_x p(x, y) \]
Independence

Two random variable $X$ and $Y$ are independent if

$$Pr(Y = y | X = x) = pr(Y = y)$$

for all possible $x$ and $y$ values.

Knowing $X = x$ tells you nothing about $Y$!

Example: Tossing a coin twice. What’s the probability of getting $H$ in the second toss given we saw a $T$ in the first one?
Example:

“happiness index” as a function of salary.

<table>
<thead>
<tr>
<th>Salary (X)</th>
<th>Happiness (Y)</th>
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<tbody>
<tr>
<td></td>
<td>0 (low)</td>
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<tr>
<td>low 0</td>
<td>0.03</td>
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<tr>
<td>medium 1</td>
<td>0.02</td>
</tr>
<tr>
<td>high 2</td>
<td>0.01</td>
</tr>
<tr>
<td>very high 3</td>
<td>0.01</td>
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</tbody>
</table>

Is $P(Y = 2|X = 3) > P(Y = 2)$?
Bayes Rule

Key fact: $Pr(x|y)$ is generally different from $Pr(y|x)$!

**Example:** Most people would agree

\[
Pr(\text{Practice hard}|\text{Play in NBA}) \approx 1
\]
\[
Pr(\text{Play in NBA}|\text{Practice hard}) \approx 0
\]

The main reason for the difference is that $Pr(\text{Play in NBA}) \approx 0$. 
Bayes Rule

The computation of $Pr(x|y)$ from $Pr(x)$ and $Pr(y|x)$ is called Bayes theorem ...

$$pr(x|y) = \frac{p(y, x)}{pr(y)} = \frac{p(y, x)}{\sum_x pr(y, x)} = \frac{pr(y|x)pr(x)}{\sum_x pr(y|x)pr(x)}$$

This shows now the conditional distribution is related to the joint and marginal distributions. You’ll be given all the quantities on the r.h.s.
Bayes Rule

Disease Testing example ....

Let $D = 1$ indicate you have a disease

Let $T = 1$ indicate that you test positive for it

If you get a positive result, you are really interested in the question: Given that you tested positive, what is the chance you have the disease?
Bayes Rule

We have a joint probability table

<table>
<thead>
<tr>
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<th>D</th>
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<tbody>
<tr>
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<td>0</td>
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<tr>
<td>T</td>
<td>0</td>
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<td>1</td>
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</table>

Bayes Probability

\[
p(D = 1 | T = 1) = \frac{0.019}{0.019 + 0.0098} = 0.66
\]
Let’s think about this intuitively ... imagine you are about to test 100,000 people.

- We assume that 2,000 of those have the disease.
- We also expect that 1% of the disease-free people to test positive, i.e., 980
  95% of the sick people to test positive. i.e. 1900.

In total, we expect 2,880 positive tests.

- Now choose one of those people at random ... what is the probability that he/she has the disease?

\[ p(D = 1|T = 1) = \frac{1,900}{2,880} = 0.66 \]

We get the same answer!!
Random Variables: Expectation $E(X)$

Example

- The **expected value** of a random variable is simply a weighted average of the possible values $X$ can assume.

- The weights are the probabilities of occurrence of those values.

\[
E(X) = \sum_x xP(X = x)
\]

We can also calculate $E(X^2) = \sum_x x^2 P(X = x)$
Standard Deviation $sd(X)$ and Variance $Var(X)$

The **variance** is calculated as

$$Var(X) = E \left( (X - E(X))^2 \right)$$

A simpler calculation is $Var(X) = E(X^2) - E(X)^2$.

The **standard deviation** is the square-root of variance.

$$sd(X) = \sqrt{Var(X)}$$
Example: $E(X)$ and $Var(X)$

Tortoise and Hare are selling cars.

Probability distributions, means and variances for $X$ the number of cars sold

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<thead>
<tr>
<th></th>
<th>X</th>
<th>Mean</th>
<th>Variance</th>
<th>sd</th>
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</thead>
<tbody>
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<td></td>
<td>0</td>
<td>1.5</td>
<td>0.25</td>
<td>0.5</td>
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<tr>
<td>Tortoise</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
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<tr>
<td>Hare</td>
<td>0.5</td>
<td>1.5</td>
<td>2.25</td>
<td>1.5</td>
</tr>
</tbody>
</table>
Let’s do Tortoise expectations and variances

- **The Tortoise**

  \[
  E(T) = \frac{1}{2}(1) + \frac{1}{2}(2) = 1.5
  \]
  \[
  Var(T) = E(T^2) - E(T)^2
  \]
  \[
  = (\frac{1}{2})(1)^2 + (\frac{1}{2})(2)^2 - (1.5)^2 = 0.25
  \]

- **Now the Hare’s**

  \[
  E(H) = \frac{1}{2}(0) + \frac{1}{2}(3) = 1.5
  \]
  \[
  Var(H) = (\frac{1}{2})(0)^2 + (\frac{1}{2})(3)^2 - (1.5)^2 = 2.25
  \]
Expectation and Variance

What do these tell us above the long run behavior?

- Tortoise and Hare have the same expected number of cars sold.
- Tortoise is more predictable than Hare. He has a smaller variance.
  - The standard deviations \( \sqrt{Var(X)} \) are 0.5 and 1.5, respectively.
- Given two equal means, you always want to pick the lower variance.
Covariance

Suppose that we have two random variables $X$ and $Y$

We need to measure whether they move together or in opposite directions

The Covariance is defined by

$$ Cov(X, Y) = E ((X - E(X))(Y - E(Y))) $$

In terms of probability distributions, we need to calculate

$$ Cov(X, Y) = \sum_{x,y} (x - E(X))(y - E(Y))pr(x, y) $$
The Correlation is defined by

\[ \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\text{sd}(X) \text{sd}(Y)} \]

- What are the units of \( \text{Corr}(X, Y) \)?
  
  They don’t depend on the units of \( X \) or \( Y \)!

- \(-1 \leq \text{Corr}(X, Y) \leq 1\)

If \( \text{Cov}(\text{Apple Sales}, \text{Economy}) = 5 \), \( \text{sd}(\text{Apple Sales}) = 2 \) and \( \text{sd}(\text{Economy}) = 3.5 \), then there’s a 71.4% correlation

\[ \text{Corr}(\text{Apple Sales}, \text{Economy}) = \frac{5}{2 \times 3.5} = \frac{5}{7} = 0.714 \]
Linear Combinations of Random Variables

Two key properties:

Let $a, b$ be given constants

**Example**

- Expectations and Variances

$$E(aX + bY) = aE(X) + bE(Y)$$

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y)$$

where $\text{Cov}(X, Y)$ is the covariance between random variables.
Tortoise and Hare

What about Tortoise and Hare?

We need to know $Cov(\text{Tortoise}, \text{Hare})$.

Let’s take $Cov(T, H) = -1$ and see what happens

Suppose $a = \frac{1}{2}, b = \frac{1}{2}$

Expectation and Variance

$$E \left( \frac{1}{2} T + \frac{1}{2} H \right) = \frac{1}{2} E(T) + \frac{1}{2} E(H) = \frac{1}{2} \times 1.5 + \frac{1}{2} \times 1.5 = 1.5$$

$$Var \left( \frac{1}{2} T + \frac{1}{2} H \right) = \frac{1}{4} 0.25 + \frac{1}{4} 2.25 - 2 \frac{1}{2} \frac{1}{2} = 0.625 - 0.5 = 0.125$$

Much lower!
Building a Portfolio of ETFs

What’s the appropriate investment decision for you?

ETF=Exchange Traded Fund. There’s many funds to choose from!

You have to decide which ones? and how much?

We’ll see how the expected return (mean) and risk (volatility) math works for you...

There’s no free lunch! You’ll have to take some risk ...
Vanguard has a suite of ETFs. Here’s a couple of combinations to choose from

1. Stocks and Bonds? SPY and TLT
2. Growth and Value? VUG and VTV
3. European or China? VGK and FXI

Let \( P = aX + bY \) be your portfolio. What \( a, b \) do you choose?
Binomial Distribution

Bernoulli Trials:

A sequence of repeated experiments are Bernoulli trials if:

1. The result of each trial is either a success or failure.
2. The probability $p$ of a success is the same for all trials.
3. The trials are independent.

If $X$ is the number of successes it is a Binomial Random Variable.
Binomial Distribution

We calculate probabilities using:

\[ P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \]

\( \binom{n}{x} \) counts the number of ways of getting \( x \) successes in \( n \) trials.

▶ The formula for \( \binom{n}{x} \) is

\[ \binom{n}{x} = \frac{n!}{x!(n-x)!} \]

where \( n! = n \times (n - 1) \times (n - 2) \times ... \times 2 \times 1 \).

In R: \( pbinom(z, n, p) \) and \( rbinom(1000, n, p) \)
The Mean and Variance of the Binomial are:

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<tr>
<th>Binomial Distribution</th>
<th>Parameters</th>
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<tr>
<td>Expected value</td>
<td>$\mu = E(X) = np$</td>
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<tr>
<td>Variance</td>
<td>$\sigma^2 = Var(X) = np(1 - p)$</td>
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- Used for discrete probability modeling
- Finds properties of proportions
Bernoulli Process

Let $X$ represent either \textit{success} ($X = 1$) or \textit{failure} ($X = 0$)

- A Bernoulli process is a binary outcome with

\[
p(X = 1) = p \quad \text{and} \quad p(X = 0) = 1 - p
\]

- We assume that trials are \textit{independent}.

The probability of two successes in a row is

\[
p(X_1 = 1, X_2 = 1) = p(X_1 = 1)p(X_2 = 1)
\]

\[
= p \cdot p
\]

\[
= p^2
\]

\textbf{Binomial distribution} counts the \textit{successes} of a Bernoulli process.
Continuous Random Variables

Suppose we are trying to predict tomorrow’s return on the S&P500...

There’s a number of questions that come to mind

▶ What is the random variable of interest?
▶ How can we describe our uncertainty about tomorrow’s outcome?
▶ Instead of listing all possible values we’ll work with intervals instead.

The probability of an interval is defined by the area under the probability density function.

They are continuous (as opposed to discrete) random variables.
Normal Distribution

$Z$ is a standard normal random variable

- The standard Normal has mean 0 and has a variance 1

$$Z \sim N(0,1)$$

- We have the probability statements

$$P(-1 < Z < 1) = 0.68$$

$$P(-1.96 < Z < 1.96) = 0.95$$

qnorm and pnorm

We can simulate 1000 draws using $\text{rnorm}(1000,0,1)$
pnorm and qnorm

We can find probabilities and quantiles in R. Here are the important values

>pnorm(2.58)
[1] 0.9950

>pnorm(1.96)
[1] 0.9750

>pnorm(1.64)
[1] 0.9499

qnorm is the inverse of pnorm.

Simulation rnorm. \( N = 1000, x = rnorm(N, 0, 1), p = \text{sum}(x < 1.96) / N \)
Here are two useful facts: If $X \sim N(\mu, \sigma^2)$, then

$$P(\mu - 2.58\sigma < X < \mu + 2.58\sigma) = 0.99$$

$$P(\mu - 1.96\sigma < X < \mu + 1.96\sigma) = 0.95.$$ 

The chance that $X$ will be within $2.58\sigma$ of its mean is 99%, and the chance that it will be within $2\sigma$ of its mean is about 95%.
The Normal Distribution

Our probability model is written $X \sim N(\mu, \sigma^2)$

$\mu$ is the mean, $\sigma^2$ is the variance

- **Standardization** if $X \sim N(\mu, \sigma^2)$ then

  $$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

- $\mu$: the center of the distribution

- $\sigma$: how spread out the data are

95% probability $X \in \mu \pm 1.96\sigma$. 
In R and Excel

In R: For a lower tail area, use `pnorm(z, mean, sd)`

Here $z$ is where you want to compute the tail area, while “mean” and “sd” are the mean and standard deviation of the normal distribution, respectively.

To compute an upper tail area, use `pnorm(z, mean, sd, lower.tail=F)`

In Excel: To compute a lower tail area, first click on a cell. In the formula bar, type `NORMDIST(z,mean,sd,TRUE)`. To find the $z$ value corresponding to a specified lower tail area $p$, use the expression `NORMINV(p,mean,sd)` in the formula bar, where $p$ is the lower tail area.
Pictorially

Figure: Examples of upper and lower tail areas. The lower tail area of 0.1 is at $z = -1.28$. The upper tail area of 0.05 is at $z = 1.64$. 

Lower Tail Area = 0.1
Upper Tail Area = 0.05
Example: The Crash

How extreme was the 1987 crash of −21.76%?

1. Prior to the October, 1987 crash SP500 monthly returns were 1.2% with a risk/volatility of 4.3%

$$X \sim N \left( 0.012, 0.043^2 \right)$$

Standardize:

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 0.012}{0.043} \sim N(0, 1)$$

2. Calculate the observed $Z$:

$$Z = \frac{-0.2176 - 0.012}{0.043} = -5.27$$

That's a 5-sigma event!
The Secretary Problem: also called the matching or marriage problem

▶ You will see items (spouses) from a distribution of types $F(x)$.

You clearly would like to pick the maximum.

You see these chronologically.

After you decide no, you can’t go back and select it.

▶ **Strategy**: wait for the length of time

$$\frac{1}{e} = \frac{1}{2.718281828} = 0.3678$$

Select after you observe an item greater than the current best.
What's your **best strategy**?

- Turns out its insensitive to the choice of distribution.
- Although there is the random sample i.i.d. assumption lurking.
- You’ll not doubt get married between 18 and 60.

  Waiting $\frac{1}{e}$ along this sequence gets you to the age 32!

Then, pick the next best person!
Business Statistics: 41000

Week 2: Bayes

Nick Polson

The University of Chicago Booth School of Business

http://faculty.chicagobooth.edu/nicholas.polson/teaching/41000/
This lecture will cover

- Bayes Rule
- Probability and Decision Trees

We’ll consider applications to:

1. Business Analytics
2. Artificial Intelligence (AI)
Monte Hall *Let’s make a Deal.*

You pick a door.

Monty then opens one of the other two doors, revealing a goat.

Monty can’t open your door or show you a car

You have the choice of switching doors.

Is it advantageous to switch?

Assume you pick door $A$ at random. Then $Pr(A) = (1/3)$.

You need to figure out $pr(A|MB)$ after Monte reveals door $B$ is a goat.

21: Monte Hall
Bayes and AI

Many applications in social media and business ...  

1. Google Translate: 3 billion words a day  
2. Amazon Alexa: Speech Recognition  
3. Driverless Car: Waymo  

Shannon’s autonomous mouse: Theseus  

Bayes solves these problems!!
AIQ: People & Robots Smarter Together

Read the book!!
Seven AI-IQ Stories

- Abraham Wald (October 31, 1902 – December 13, 1950)
- Henrietta Leavitt (July 4, 1868 – December 12, 1921)
- John Craven (August 16, 1940 –)
- Grace Hopper (December 9, 1906 – January 1, 1992)
- Isaac Newton (January 4, 1643 – March 31, 1727)
- Florence Nightingale (May 12, 1820 – August 13, 1910)
- Joe DiMaggio (November 25, 1914 – March 8, 1999)
Introduction

What Does “AI" Really Mean?

Think of an algorithm.

Two distinguishing features of AI algorithms:

1. Algorithms typically deal with probabilities rather than certainties.
2. There's the question of how these algorithms “know" what instructions to follow.
Abraham Wald

How Abraham Wald improved aircraft survivability

Table: Raw Reports from the Field

<table>
<thead>
<tr>
<th>Type of damage suffered</th>
<th>Returned (316 total)</th>
<th>Shot down (60 total)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engine</td>
<td>29</td>
<td>?</td>
</tr>
<tr>
<td>Cockpit</td>
<td>36</td>
<td>?</td>
</tr>
<tr>
<td>Fuselage</td>
<td>105</td>
<td>?</td>
</tr>
<tr>
<td>None</td>
<td>146</td>
<td>0</td>
</tr>
</tbody>
</table>

This fact would allow Wald to estimate:

\[ P(\text{damage on fuselage} | \text{returns safely}) = \frac{105}{316} \approx 32\% \]

You need the inverse probability:

\[ P(\text{returns safely} | \text{damage on fuselage}) \]
Abraham Wald

Wald invented a method to implement the missing data, which is called by data scientist as “imputation”.

Table: Wald Invented A Method for Reconstructing the Full Table

<table>
<thead>
<tr>
<th>Type of damage suffered</th>
<th>Returned (316 total)</th>
<th>Shot down (60 total)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engine</td>
<td>29</td>
<td>31</td>
</tr>
<tr>
<td>Cockpit</td>
<td>36</td>
<td>21</td>
</tr>
<tr>
<td>Fuselage</td>
<td>105</td>
<td>8</td>
</tr>
<tr>
<td>None</td>
<td>146</td>
<td>0</td>
</tr>
</tbody>
</table>

Then Wald got:

\[
\Pr(\text{return safely} | \text{damage on fuselage}) = \frac{105}{105 + 8} \approx 93\% \\
\Pr(\text{return safely} | \text{damage on engine}) = \frac{29}{29 + 31} \approx 48\%
\]
"Personalization" = "Conditional Probability"

Conditional probability is how AI systems express judgments in a way that reflects their partial knowledge.

Personalization runs on conditional probabilities, all of which must be estimated from massive data sets in which you are the conditioning event.
How does Netflix Give Recommendations?

Consider an easy case: assessing how probable it is that a subscriber will like the film Saving Private Ryan, given that he or she liked the HBO series Band of Brothers.

This seems like a good bet: both are epic dramas about the Normandy invasion and its aftermath.

The key insight is to frame the problem in terms of conditional probability.
Let’s say there are 100 people in your database, and every one of them has seen both films. Their viewing histories come in the form of a big “ratings matrix”.

Cross-tabulate the data from the ratings matrix by counting how many subscribers had a specific combination of preferences for two film, based on 100 subscribers:

<table>
<thead>
<tr>
<th></th>
<th>Liked Band of Brothers</th>
<th>Didn’t like it</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liked Saving Private Ryan</td>
<td>56 subscribers</td>
<td>6 subscribers</td>
</tr>
<tr>
<td>Didn’t like it</td>
<td>14 subscribers</td>
<td>24 subscribers</td>
</tr>
</tbody>
</table>

\[
\Pr(\text{likes Saving Private Ryan}|\text{likes Band of Brothers}) = \frac{56}{56 + 14} = 80\%
\]
How does Netflix Give Recommendations?

But real problem is much more complicated:

1. **Scale.** It has 100 million subscribers and ratings data on more than 10,000 shows. The ratings matrix has more than a trillion possible entries.

2. "**Missingness**". Most subscribers haven’t watched most films. Moreover, missingness pattern is informative.

3. **Combinatorial explosion.** In a database with 10,000 films, no one else’s history is exactly the same as yours.

The solution to all three issues is careful modeling.
How does Netflix Give Recommendations?

The fundamental equation is:

\[ \text{Predicted Rating} = \text{Overall Average} + \text{Film Offset} + \text{User Offset} + \text{User-Film Interaction} \]

These three terms provide a baseline for a given user/film pair:

- The overall average rating across all films is 3.7.
- Every film has its own offset. Popular movies have positive offsets.
- Every user has an offset. Some users are more or less critical than average.
The User-Film Interaction is calculated based on a person’s ratings of similar films exhibit patterns because those ratings are all associated with a latent feature of that person.

There’s not just one latent feature to describe Netflix subscribers, but dozens or even hundreds. There’s a “British murder mystery" feature, a “gritty character-driven crime drama" feature, a “cooking show" feature, a “hipster comedy films" feature, ...
These latent features are the magic elixir of the digital economy—a special brew of data, algorithms, and human insight that represents the most perfect tool ever conceived for targeted marketing.

Your precise combination of latent features—your tiny little corner of a giant multidimensional Euclidean space—makes you a demographic of one.

Netflix spent $130 million for 10 episodes on The Crown. Other network television: $400 million commissioning 113 pilots, of which 13 shows made it to a second season.
The Reverend and the Submarine (Thomas Bayes & John Craven)

How did John Craven found a lost submarine from 140 square miles of ocean floor?

How do self-driving car find themselves and dodge bicycle, snow, and kangaroo?

How large is the probability of actually having breast cancer with a positive mammogram result?

– Bayes’s Rule \( P(H|D) = \frac{P(H) \cdot P(D|H)}{P(D)} \)
Bayes’s Rule in Medical Diagnostics

Alice is a 40-year-old women, what is the chance that she really has breast cancer when she gets positive mammogram result, given the conditions:

1. The prevalence of breast cancer among people like Alice is 1%.
2. The test has an 80% detection rate.
3. The test has a 10% false-positive rate.

The posterior probability $P(\text{cancer|positive mammogram})$?
Of 1000 cases:

- 108 positive mammograms. 8 are true positives. The remaining 100 are false positives.
- 892 negative mammograms. 2 are false negatives. The other 890 are true negatives.
Calculation of Posterior Probability

\[ P(\text{cancer} | \text{positive mammogram}) = \frac{8}{108} \approx 7.4\% \]

Most women who test positive on a mammogram are healthy, because the vast majority of women who receive mammograms in the first place are healthy.
Bayes’s rule and Investing

What are the chances that your search can locate one of the rare exceptions, in a vast ocean of fund managers who cannot consistently beat the market?

Bayes’s rule gives a pretty clear answer: the chances are terrible.

Most mutual fund managers are just flipping coins.
If we were to compare the 10-year performance of Warren Buffett with that of five regular stock pickers, we might see something like the table below.

<table>
<thead>
<tr>
<th></th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
<th>Year 6</th>
<th>Year 7</th>
<th>Year 8</th>
<th>Year 9</th>
<th>Year 10</th>
<th>Total H’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jane Doe</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>T</td>
<td>H</td>
<td>T</td>
<td>T</td>
<td>H</td>
<td>T</td>
<td>T</td>
<td>5</td>
</tr>
<tr>
<td>John Bull</td>
<td>T</td>
<td>T</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>H</td>
<td>5</td>
</tr>
<tr>
<td>Jean Dupont</td>
<td>H</td>
<td>H</td>
<td>T</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>T</td>
<td>T</td>
<td>H</td>
<td>7</td>
</tr>
<tr>
<td>Jan Jansen</td>
<td>T</td>
<td>T</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>H</td>
<td>T</td>
<td>4</td>
</tr>
<tr>
<td>Max Mustermann</td>
<td>T</td>
<td>T</td>
<td>H</td>
<td>H</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>H</td>
<td>3</td>
</tr>
<tr>
<td>Warren Buffett</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>10</td>
</tr>
</tbody>
</table>

For our five average investors, their performance is just random. But for Buffett, his performance is driven by his superior stock-picking ability?like a special two-headed coin.
Imagine a jar containing 1,024 normal coins. Into this jar, a friend places a single two-headed coin.

Suppose the coin comes up heads on all 10 flips. In light of the evidence, are you holding the two-headed coin or one of the 1,024 ordinary coins?

▶ There are 1,025 coins in the jar: 1,024 are normal, and 1 has two heads.
▶ The single two-headed coin is guaranteed to come up heads 10 times in a row.
▶ Any randomly chosen normal coin has probability $1/1024$ of coming up heads 10 times in a row. (This is calculated by multiplying $1/2$ by itself 10 times.)
This matrix tells us that of the 1,025 coins in the jar, we’d expect two of them to come up heads 10 times in a row. Only one of them is actually the two-headed coin. There’s just a 50% chance that you’re holding it, even after 10 flips.

<table>
<thead>
<tr>
<th></th>
<th>At least 1 tail</th>
<th>10 heads in a row</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal coins</td>
<td>1,023 (true negatives)</td>
<td>1 (false positives)</td>
</tr>
<tr>
<td>Two-headed coin</td>
<td>0 (false negatives)</td>
<td>1 (true positives)</td>
</tr>
</tbody>
</table>

This matrix tells us that of the 1,025 coins in the jar, we’d expect two of them to come up heads 10 times in a row. Only one of them is actually the two-headed coin. There’s just a 50% chance that you’re holding it, even after 10 flips.
Suppose you believe that 1% of all stock pickers are true market beaters, and that the other 99% are just flipping coins. Then the posterior probability $P(\text{market beater} \mid \text{10-year winning streak})$ is $100/110$, or about 91%.

However, if you believed that excellence were much rarer, like $P(\text{true market beater}) = 1/10,000$? Then we have $P(\text{market beater} \mid \text{10-year winning streak}) = 1/11$, or about 9%.

Bayes’s rule implies that the right reaction to an investor’s track record depends strongly on the prior probability: whether excellent fund managers are common or rare.
Bayes’s Rule

We use the letter H to represent a hypothesis that might be true or false, and the letter D to represent some relevant data.

\[ p(H|D) = \frac{p(H) \times P(D|H)}{p(D)} \]
Test Marketing a New Product

Basic Problem:

- Your company is developing a new product and will be test marketing to better gauge the sales of the new product.
- Based on positive, neutral or negative reactions, what are the probability of high and low sales?

NetFlix Bayes and AI
Test Marketing

Suppose you are given the following information

- New products introduced in the marketplace have high sales 8% of the time and low sales 92% of the time.

- A marketing test has the following accuracies:
  
  If sales are high, then consumer test reaction is positive 70%, neutral 25% and negative 5%.
  
  If sales are low, then consumer test reaction is positive 15%, neutral 35% and negative 50%.
Test Marketing

Step 1: Set-up your notation. Let

\[ H = \text{high sales} \quad L = \text{low sales} \]

\[ \text{Pos} = \text{positive} \quad \text{Neu} = \text{Neutral} \quad \text{Neg} = \text{Negative} \]

Step 2: List the known conditional probabilities

For the marketing test we have

\[
P(\text{Pos}|H) = 0.70, P(\text{Neu}|H) = 0.25, P(\text{Neg}|H) = 0.05
\]

\[
P(\text{Pos}|L) = 0.15, P(\text{Neu}|L) = 0.35, P(\text{Neg}|L) = 0.50
\]

Finally, the base rates are \( P(H) = 0.08 \) and \( P(L) = 0.92 \)
Step 3: Describe the posterior probabilities that are required:

\[ P(H|Pos) \]

The probability of high sales given a positive marketing test.

Compute the probability of a positive test

\[
P(Pos) = P(Pos|H)P(H) + P(Pos|L)P(L)
\]

\[= 0.70 \times 0.08 + 0.15 \times 0.92 = 0.194\]
Now use Bayes Rule

\[ P(H|Pos) = \frac{P(Pos|H)P(H)}{P(Pos)} \]

\[ = \frac{0.70 \times 0.08}{0.194} = 0.288 \]

Hence 28.8% you’ll have high sales in the market.

We should interpret this relative to our initial probability of only 8%.
Two Headed Coin

Large jar containing 1024 fair coins and one two-headed coin.

- You pick one at random and flip it 10 times and get all heads.
- What’s the probability that the coin is the two-headed coin?

\[
\frac{1}{1025} \text{ probability of initially picking the two headed coin.}
\]

\[
\frac{1}{1024} \text{ chance of getting 10 heads in a row from a fair coin}
\]

Therefore, it’s a 50/50 bet.
Let $E$ be the event that you get 10 Heads in a row

$$P(\text{two headed}|E) = \frac{P(E|\text{two headed}) P(\text{two headed})}{P(E|\text{fair}) P(\text{fair}) + P(E|\text{two headed}) P(\text{two headed})}$$

Therefore, the posterior probability

$$P(\text{two headed}|E) = \frac{\frac{1}{1024} \times \frac{1}{1025}}{\frac{1}{1024} \times \frac{1}{1025} + \frac{1}{1025}} = 0.50$$
Next Toss?

What’s the probability that the next toss is a head?

- Using the law of total probability gives

\[
P(H|E) = P(H|\text{two headed})P(\text{two headed}|E) + P(H|\text{fair})P(\text{fair}|E)
\]
\[
= 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}
\]
\[
= \frac{3}{4}
\]
Sensitivity and Specificity

Two errors: An infected person may test negative, a well person tests positive.

**Sensitivity** = true positive rate (or recall)

% sick people who are correctly identified \( P(T|D) \).

In a perfect world, we’d like \( P(\bar{T}|D) \approx 0 \)

**Specificity** = true negative rate

% of negatives correctly identified as such \( P(\bar{T}|\bar{D}) \).

- Power = sensitivity = 1 – \( \beta \) where \( \beta \) is the type II error
- False positive rate, \( \alpha = \) type I error = 1 – specificity

We want the probability: \( p(D|T) \)
Bayes Classifier

Output $Y$ and input $X$:

$Y$: image of a cat

$X$: identify “cat”

Silicon Valley: Season 4: Not Hotdog

What’s our best decision?

Pick $\hat{Y}$ as the *most likely* category given that $X = x$, namely

$$\hat{Y} = \text{argmax}_Y p(Y = y | X = x)$$

I need the probability table $p(X = x, Y = y)$ and marginal $p(X = x)$. 
Decision Trees

\[
\begin{align*}
P(A) + P(B) &= 1.0 \\
P(D|A) + P(E|A) + P(F|A) &= 1.0 \\
P(G|B) + P(H|B) &= 1.0
\end{align*}
\]
Catastrophe Modeling

You live in a house that is somewhat prone to mud slides.

- Each rainy season there is a 1% chance of a mud slide occurring.
- You estimate that a mud slide would do $1 million in damage.
- You have the option of building a retaining wall that would help reduce the chance of a devastating mud slide.

The wall costs $40,000 to build, and if the slide occurs, the wall will hold with a 95% probability.

- You also have the option of a Geologist’s opinion.

Should you build the wall?

Should you use the Geologist’s Test and Bayes Rule?
Decision Tree

Let’s formally solve this as follows:

- Build a decision tree.

- The tree will list the probabilities at each node. It will also list any costs there are you going down a particular branch.

- Finally, it will list the expected cost of going down each branch, so we can see which one has the better risk/reward characteristics.

There’s also the possibility of a further test to see if the wall will hold.
Testing

Let’s include the testing option

- You also have the option of having a test done to determine whether or not a slide will occur in your location.
- The test costs $3000 and has the following accuracies.

\[ P(T|\text{Slide}) = 0.90 \quad \text{and} \quad P(\bar{T}|\text{No Slide}) = 0.85 \]

If you choose the test, then should you build the wall?
Bayes Rule is as follows:

- The initial prior probabilities are

\[ P(\text{Slide}) = 0.01 \quad \text{and} \quad P(\text{No Slide}) = 0.99 \]

- Therefore

\[ P(T) = P(T|\text{Slide})P(\text{Slide}) + P(T|\text{No Slide})P(\text{No Slide}) \]

\[ P(T) = 0.90 \times 0.01 + 0.15 \times 0.99 = 0.1575 \]

We’ll use this to find our optimal course of action.
Bayes Probabilities

\[ P(Slide|T) \]

The Bayes probability given a positive test is

\[
P(Slide|T) = \frac{P(T|Slide)P(Slide)}{P(T)} \\
= \frac{0.90 \times 0.01}{0.1575} = 0.0571
\]
Bayes Probabilities

\[ P(\text{Slide}|\text{Not } T) \]

The Bayes probability given a negative test is

\[
P(\text{Slide}|\bar{T}) = \frac{P(\bar{T}|\text{Slide}) P(\text{Slide})}{P(\bar{T})}
\]

\[
= \frac{0.1 \times 0.01}{0.8425}
\]

\[
= 0.001187
\]

Compare this to the initial base rate of a 1% chance of having a mud slide.
You build the wall without testing, what’s the probability that you lose everything?

With the given situation, there is one path (or sequence of events and decisions) that leads to losing everything:

1. Build without testing (given)
2. Slide (0.01)
3. Doesn’t hold (0.05)

\[ P(\text{losing everything} | \text{build w/o testing}) = 0.01 \times 0.05 = 0.0005 \]
Probability Lose Everything

You choose the test, what’s the probability that you’ll lose everything?

There are two paths that lead to losing everything:

1. First Path:

   There are three things that have to happen to lose everything

   Test +ve ($P = 0.1575$), Build, Slide ($P = 0.0571$), Doesn’t Hold ($P = 0.05$)

2. Second Path:

   Now you lose everything if

   Test -ve ($P = 0.8425$), Don’t Build, Slide given negative ($P = 0.001187$)
Conditional Probabilities

- For the first path

\[ P(\text{first path}) = 0.1575 \times 0.0571 \times 0.05 = 0.00045 \]

- For the second path

\[ P(\text{second path}) = 0.8425 \times 0.001187 = 0.00101 \]

Hence putting it all together

\[ P(\text{losing everything|testing}) = 0.00045 + 0.00101 = 0.00146 \]
## Risk and Reward

### Risk-Return Trade-off

<table>
<thead>
<tr>
<th>Choice</th>
<th>Expected Cost</th>
<th>Risk</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Don’t Build</td>
<td>$10,000</td>
<td>0.01</td>
<td>1 in 100</td>
</tr>
<tr>
<td>Build w/o testing</td>
<td>$40,500</td>
<td>0.0005</td>
<td>1 in 2000</td>
</tr>
<tr>
<td>Test</td>
<td>$10,760</td>
<td>0.00146</td>
<td>1 in 700</td>
</tr>
</tbody>
</table>

**Expected Cost: Fee + Build + Loss**

Expected cost: $3 + 40 \times 0.1575 + 1000 \times 0.00146 \text{ or } $10,760

**What do you choose?**
Summary

Probability and Random variables

- Joint Probability tables
- Bayes rule
- Decision Trees