Special Notes:

1. This is a closed-book exam. You may use an $8 \times 11$ piece of paper for the formulas.
2. Throughout this paper, $N(\mu, \sigma^2)$ will denote a normal distribution with mean $\mu$ and variance $\sigma^2$.
3. This is a 2 hour exam.

Problem A. True or False: Please Explain your answers in detail. Partial credit will be given (60 points)

1. Suppose that $X \sim N(2, 4)$ then $E(e^X) = e^2$.

   False. Use $E(e^{tX}) = e^{\mu t + \frac{1}{2} \sigma^2 t^2}$ and get $E(e^X) = e^4$.

2. For any random variable $X$, with finite mean and variance, there is a 75% then the realization of $X$ is within 2 standard deviations of the mean.

   True. Chebyshev’s inequality $P(|X - \mu| \geq t\sigma) \leq 1/t^2$.

3. For any two random variables $X$ and $Y$, $\text{Var}(X) - E(\text{Var}(X|Y))$ is always positive.

   True. For any random variables $\text{Var}(X) - E(\text{Var}(X|Y)) = \text{Var}(E(X|Y)) \geq 0$
4. If $X$ has a log-normal distribution then so does $\frac{1}{X}$.

*True.* If $X = e^Y$ where $Y \sim N(\mu, \sigma^2)$ then $\frac{1}{X} = e^{-Y}$ is also log-normal as $-Y \sim N(-\mu, \sigma^2)$.

5. Let $B_t$ be a standard Brownian motion then $\frac{1}{\sqrt{t}}B_t \sim N(0, 1)$ for all $t$.

*True.* $B_t \sim N(0, t)$ for all $t$, by definition. Hence $\frac{1}{\sqrt{t}}B_t \sim N(0, 1)$ for all $t$.

6. If $X$ and $Y$ are independent Gamma distributions, then the distribution of $X + Y$ is also Gamma.

*False.* If $X \sim \Gamma(\alpha, \beta)$ then $M_X(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha$. Similarly for $Y$. Moreover $M_{X+Y}(t) = M_X(t)M_Y(t)$ which is not necessarily of the same form as that of a Gamma distribution.

7. Let $X_1, \ldots, X_9$ be a random sample from an exponential distribution with mean 2, then $E(X_{\text{max}}) = 18$

*False.* $F_X(x) = 1 - e^{-\frac{1}{2}x}$. Moreover, $F_{X_{\text{max}}}(x) = (1 - e^{-\frac{1}{2}x})^9$ which doesn’t have a mean of 18.

8. Geometric Brownian motion, $X_t = e^{B_t}$, has independent ratios, that is $X_t/X_s$ is independent of $X_s/X_0$ for $0 < s < t$.

*True.* $X_t/X_s = e^{B_t-B_s}$ and results follow from properties of Brownian motion itself.
9. The stochastic process \( \frac{1}{4} B_{4t} \) is a standard Brownian motion.

\[
\text{False. Let } X_t = \frac{1}{4} B_{4t}. \text{ Now } B_{4t} - B_{4s} \sim N(0, 4(t-s)) \text{ and so } \text{Var}(X_t - X_s) = \frac{1}{4}(t-s). \text{ This isn’t } t - s \text{ and so } X_t \text{ is not a standard Brownian motion.}
\]

10. The equilibrium distribution \( \pi \) of a Markov chain with transition matrix \( P \), satisfies \( \pi P^k = \pi \) for any positive integer \( k \).

\[
\text{True. By definition, } \pi P = \pi \text{ and hence } \pi P^k = \pi \text{ for any positive integer } k.
\]

11. If \( P(A|C) \geq P(B|C) \) and \( P(A|\bar{C}) \geq P(B|\bar{C}) \), then \( P(A) \geq P(B) \).

\[
\text{True. The sure-thing principle. Use } P(A) = P(A|C)P(C) + P(A|\bar{C})P(\bar{C}).
\]

12. Consider the following bet: in rolling a pair of fair dice is it more likely that a double six will occur before two consecutive rolls of seven?

\[
\text{True. First, let } p \text{ be the probability that a double six will occur before two consecutive rolls of seven. Now use the law of total probability and argue as follows. If a double six occurs on the first roll you win; if a seven occurs on the first roll then there are three possibilities on the next roll: either a seven or twelve doesn’t occur on the next roll; or a twelve occurs and you win; or another seven occurs and you lose. In terms of probabilities this means that}
\]

\[
p = \frac{1}{36} + \frac{29}{36}p + \frac{6}{36} \frac{1}{36} + \frac{6}{36} \frac{29}{36}
\]

\[
\text{which implies that } p = 0.538.
\]

\[
\text{Secondly, you can use a waiting time argument and get } E(T_{66}) = 36 \text{ and } E(T_{77}) = 36 + 6 = 42.
\]
Problem B. (20 points)
Let $X$ and $Y$ be independent exponential random variables with mean $\frac{1}{\lambda}$ and densities: $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \lambda e^{-\lambda y}$

1. Find the joint distribution of $U = X + Y$ and $V = X/Y$.
2. Also find the marginal distributions for $U$ and $V$. Are $U$ and $V$ independent?

First, the joint density is

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}$$

The transformation $u = x + y$ and $v = x/y$ has inverse $x = uv/(1 + v)$ and $y = u/(1 + v)$.
The Jacobian is

$$|J| = \frac{u}{(1 + v)^2}$$

Hence

$$f_{U,V}(u,v) = \lambda^2 u e^{-\lambda u} \frac{1}{(1 + v)^2}$$

Therefore

$$f_U(u) = \lambda^2 u e^{-\lambda u} \sim \Gamma(2, \lambda)$$

$$f_V(v) = \frac{1}{(1 + v)^2}$$

and the random variables are independent.
**Problem C.** (20 points).

A manufacturer of DVD’s purchases a particular microchip called the CL2000 chips from three suppliers Matsushita Electric, Philips Electronics, and Hitachi.

From historical experience we know that 30% of CL2000 chips are purchased from Matsushita; 20% from Philips and the remaining 50% from Hitachi. The manufacturer has extensive histories on the reliability of the chips. We know that 3% of the Matsushita chips are defective; 5% of the Philips and 4% of the Hitachi chips are defective.

- A chip is later found to be defective; what is the probability it was manufactured by each of the manufacturers?

Let $A_1$ denotes the event that the chip is purchased from Matsushita. Similarly $A_2$ and $A_3$. Let $D$ be the event that the chip is defective and $\bar{D}$ is not. We know

\[
P(D|A_1) = 0.03 \quad P(D|A_2) = 0.05 \quad P(D|A_3) = 0.04
\]

and

\[
P(A_1) = 0.30 \quad P(A_2) = 0.20 \quad P(A_3) = 0.50
\]

The probability that it was manufactured by Philips given that its defective is given by Bayes rule which gives

\[
P(A_2|D) = \frac{P(D|A_2)P(A_2)}{P(D|A_1)P(A_1) + P(D|A_2)P(A_2) + P(D|A_3)P(A_3)}
\]

That is

\[
P(A_2|D) = \frac{0.20 \times 0.05}{0.30 \times 0.03 + 0.20 \times 0.05 + 0.50 \times 0.04} = \frac{10}{39}
\]

Similarly, $P(A_1|D) = 9/39$ and $P(A_3|D) = 20/39$