Name:

University of Chicago
Graduate School of Business

Business 41901: Probability

Special Notes:

1. This is a closed-book exam. You may use an 8 × 11 piece of paper for the formulas.

2. Throughout this paper, \( N(\mu, \sigma^2) \) will denote a normal distribution with mean \( \mu \) and variance \( \sigma^2 \).

3. This is a 2 hr exam.

Honor Code: By signing my name below, I pledge my honor that I have not violated the Booth Honor Code during this examination.

Signature:
Problem A. True or False: Please Explain your answers in detail. Partial credit will be given (60 points)

1. (Normal) Let $X$ be normal with mean zero, then the distribution of $X^2$ is a Rayleigh distribution.

2. (Russian Roulette) I have a revolver (six chambers) loaded with one bullet. I pull the trigger once, click, and hand you the revolver. Do you want to spin it first, or shoot directly?

3. (Children and Apples) You have three children, but only one apple. You want to toss a fair coin to decide who gets the apple. You want to be fair. You flip a coin twice. Assign $HH$, $HT$, and $TT$ to the children, and ignore $TH$ if you throw it. This is a fair allocation.
4. (Raisins per Cookie) A baker put 500 raisins into his dough, and made 100 cookies. You take a random cookie. The probability of finding exactly 2 raisins is 0.8.

5. (Bayes) If $(y|\theta) \sim N(\theta, 1)$ and $\theta$ has a Jeffrey’s flat uniform prior $p(\theta) \equiv 1$, then the posterior distribution $(\theta|y) \sim N(y, 1)$.

6. The reciprocal of a Cauchy distribution has finite mean and variance
7. (Order Statistics) Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with mean one. Then the probability density of any order statistic is also exponential.

8. (Transformations) Let $Y$ have a Pareto distribution. Then $\sqrt{Y}$ is a log-normal distribution.

9. (Brownian Motion) Let $B_t$ be a standard Brownian motion. Then we can calculate the mean $E\left(\sqrt{B_t - B_s}\right) = \sqrt{t-s}$ for $t > s$. 
10. (Moments) The function $1/(1 - t)^2$ cannot be a moment generating function, $M_X(t)$, of any random variable, $X$.

11. (Martingales) A sub-martingale has the property that $E(X_t) > X_0$ for any $t$.

12. (Odds) Consider betting $x$ against Secretariat and $y$ against Zenyatta. Crazy Eddie takes in $x + y$ in bets and given his beliefs in the following payout table

<table>
<thead>
<tr>
<th>Horse</th>
<th>Eddie’s Beliefs</th>
<th>Fair odds</th>
<th>If horse loses Eddie pays</th>
</tr>
</thead>
<tbody>
<tr>
<td>Secretariat</td>
<td>$\frac{2}{3}$</td>
<td>1 : 2</td>
<td>$x + 2x = 3x$</td>
</tr>
<tr>
<td>Zenyatta</td>
<td>$\frac{1}{2}$</td>
<td>1 : 1</td>
<td>$y + y = 2y$</td>
</tr>
</tbody>
</table>

Table 1: Eddies’ Payout

Then Eddie will always lose money if we bet $x = 2, y = 3$ against him.
Problem B. (20 points)

Let $X$ and $Y$ be independent with a joint distribution given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi} \sqrt{\frac{1}{xy}} \exp \left( \frac{-x^2 - y^2}{2} \right)$$

where $x, y > 0$.

Identify the following distributions

- The marginal distribution of $X$
- Compute the joint distribution of $U = X$ and $V = X + Y$
- Compute the marginal distribution of $V$. 

Problem C. (20 points)

1. (Hit and Run Taxi)
A certain town has two taxi companies: Blue Birds, whose cabs are blue, and Night Owls, whose cabs are black. Blue Birds has 15 taxis in its fleet, and Night Owls has 75. Late one night, there is a hit-and-run accident involving a taxi.

The town’s taxis were all on the streets at the time of the accident. A witness saw the accident and claims that a blue taxi was involved. At the request of the police, the witness undergoes a vision test under conditions similar to those on the night in question. Presented repeatedly with a blue taxi and a black taxi, in random order, he shows he can successfully identify the colour of the taxi 4 times out of 5.

- Which company is more likely to have been involved in the accident?

2. (Children) I tell you that I have two children, and that one of them is a girl (I say nothing about the other). What is the probability that I have two girls?

You come to visit, and a girl (who is my daughter) opens your door. What is the probability that I have two girls?