Probability 41901: Assignment 5

Due Tuesday Week 10

1. (Random Walk). For a simple random walk $S_t$ with $S_0 = 0$ and $p < \frac{1}{2}$ show that $M = \max(S_t : t \geq 0)$ satisfies

$$P(M \geq x) = \left(\frac{p}{q}\right)^x$$

2. (Asymmetric Random Walk). Let $S_n$ be an asymmetric random walk with $p > \frac{1}{2}$. That is, $S_n = X_1 + X_2 + \ldots + X_n$ where $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$. Let $\sigma^2 = 1 - (p - q)^2$. Show that, $Z_n$, is a martingale where

$$Z_n = (S_n - (p - q)n)^2 - \sigma^2 n.$$  

Suppose $p > 1/2$ and let $\phi(x) = (q/p)^x$. Show that $\phi(S_n)$ is a martingale.

Let $T_x = \inf\{n : S_n = x\}$. Then for $a < 0 < b$ show that

$$P(T_a < T_b) = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)}$$

Also, show that

$$E(T_b) = \frac{b}{2p - 1}$$

3. (Martingales). Let $X_n$ be a sub-martingale and $u(x)$ be a convex non-decreasing function. Show that $u(X_n)$ is a submartingale.

4. (Brownian Motion). Let $B_t$ be a standard Brownian motion.
   - Which of the following are also a Brownian motion: $-B_t$, $\sqrt{t}B_1$ or $B_{2t} - B_t$?
   - Find the mean function and variance for the following: $X_t = |B_t|$ and $Y_t = e^{B_t}$.

5. (Ruin Probability). Let $Y_t$ be the amount of assets in an insurance company after $t$ years. Each year they receive premiums $P$ and payout claims $C_t$. Suppose that claims follow an i.i.d. sequence of normal distributions with mean $\mu$ and variance $\sigma^2$. Show that the probability of ultimate bankruptcy is

$$P(Y_t \leq 0 \text{ for some } t) \leq \exp\left(-\frac{2(P - \mu)Y_0}{\sigma^2}\right)$$