Option-Implied Currency Risk Premia

by Jakub W. Jurek and Zhikai Xu

Discussion

Pietro Veronesi

The University of Chicago Booth School of Business
What Does This Paper Do?
What Does This Paper Do?

\[ k_{s_{t+1}}^{j}[u] = \left( s_{t}^{ji} - \alpha_{t}^{j} + \alpha_{t}^{i} \right) \cdot u + k_{L}^{g_{t+1}} \left[ \left( \xi_{t}^{i} - \xi_{t}^{j} \right) \cdot u \right] + k_{L+1}^{i} [u] + k_{L+1}^{j} [-u] \quad (10) \]

\[ k_{L}^{i_{t+1}} [u] = \ln E_{t}^{i} \left[ \exp \left( u \cdot L_{t+1}^{g} \right) \right] = \ln E_{t}^{i} \left[ \exp \left( y_{t, t+1}^{i} + (m_{t+1}^{i} - m_{t}^{i}) + u \cdot L_{t+1}^{g} \right) \right] \]

\[ = \left( k_{L}^{g_{t+1}} [u - \xi_{t}^{i}] - k_{L}^{g_{t+1}} [-\xi_{t}^{i}] \right) \cdot Z_{t} \quad (11a) \]

\[ k_{L}^{i_{t+1}} [u] = \left( k_{L}^{i_{t+1}} [u - 1] - k_{L}^{i_{t+1}} [-1] \right) \cdot Y_{t}^{i} \quad (11b) \]

\[ k_{L}^{i_{t+1}} [u] = k_{L}^{i_{t+1}} [u] \cdot Y_{t}^{j} \quad (11c) \]

\[ k_{s_{t+1}}^{j}[u] = \left( s_{t}^{ji} - \alpha_{t}^{j} + \alpha_{t}^{i} \right) \cdot u + k_{L}^{g_{t+1}} \left[ \left( \xi_{t}^{i} - \xi_{t}^{j} \right) \cdot u \right] \cdot Z_{t} + k_{L+1}^{i} [u] \cdot Y_{t}^{i} + k_{L+1}^{j} [-u] \cdot Y_{t}^{j} \quad (12) \]

\[ k_{s_{t+1}}^{j}[u] = \left( s_{t}^{ji} - \alpha_{t}^{j} + \alpha_{t}^{i} \right) \cdot u + \left( k_{L}^{g_{t+1}} \left[ \left( \xi_{t}^{i} - \xi_{t}^{j} \right) \cdot u - \xi_{t}^{i} \right] - k_{L}^{g_{t+1}} [-\xi_{t}^{i}] \right) \cdot Z_{t} + \]

\[ + \left( k_{L}^{i_{t+1}} [u - 1] - k_{L}^{i_{t+1}} [-1] \right) \cdot Y_{t}^{i} + k_{L}^{j_{t+1}} [-u] \cdot Y_{t}^{j} \quad (13) \]

\[ \lambda_{HML,t}^{j} = \left( k_{L}^{g_{t+1}} \left[ \xi_{t}^{i} - \xi_{t}^{j} \right] + k_{L}^{g_{t}} [-\xi_{t}^{i}] - k_{L}^{g_{t+1}} [-\xi_{t}^{j}] \right) \cdot Z_{t} \quad (20) \]

\[ \lambda_{refFX,t}^{i} = \left( k_{L}^{i_{t+1}} [1] + k_{L}^{i_{t+1}} [-1] \right) \cdot Y_{t}^{i} \quad (21) \]

\[ \lambda_{t}^{j,k} = \left( k_{L}^{g_{t+1}} \left[ \xi_{t}^{i} - \xi_{t}^{j} \right] - k_{L}^{g_{t+1}} \left[ \xi_{t}^{i} - \xi_{t}^{k} \right] + k_{L}^{g_{t+1}} [-\xi_{t}^{k}] - k_{L}^{g_{t+1}} [-\xi_{t}^{j}] \right) \cdot Z_{t} \quad (22) \]
What Does This Paper Do?

Adjusted $R^2 : 85.97\%$
What Does This Paper Do?

• Proposes a relatively general model of option pricing in the FX market with non-Gaussian global and local shocks

• Proposes a methodology to use cross-sectional data on FX options to estimate options’ implied currency risk premia

• Implements such methodology using a panel of currency options.

• Finds evidence of global risk factor in currency risk premia, similar to the $HML_{FX}$ factor identified by Lustig, Roussanov, Verdelhan (2011, 2013).

• Uses ex-ante identification to carry out a number of empirical tests, variance decomposition etc.
Assessment and Outline of Discussion
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(1) Ambitious paper
  • Obtaining risk premia from cross-section of options is kind of the holy grail of this literature.
  • The methodology is relatively general (cumulants and the like), but it seems it can only be used in the case of currency options.

(2) Paper would benefit from a deeper discussion of the methodology to compute risk premia from options
  • The empirical results support the fact that the methodology indeed manages to estimate currency risk premia in the cross-section.
  • Why it is able to do so though is not transparent.
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- The empirical results support the fact that the methodology indeed manages to estimate currency risk premia in the cross-section.
- Why it is able to do so though is not transparent.

Outline of discussion:

(A) Review basic no arbitrage relations in currency markets.

(B) Use simple “baby models” to discuss why it is very hard to estimate risk premia from options.

(C) Go back to cross-section.
State Price Densities and Exchange Rates

- Consider 2 currencies: dollar $ and sterling pound £. Let $S = \frac{\$}{£}$. 
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- Two countries and two pricing kernels $M_t^\$ and $M_t^\£$. 
State Price Densities and Exchange Rates

• Consider 2 currencies: dollar $ and sterling pound £. Let $S = \frac{\$}{\£}$.

• Two countries and two pricing kernels $M^\$$ and $M^\£$.

• No arbitrage: Every £- security $V_t^\£$ with payoff $V_T^\£$ at $T$, must satisfy

$$M_t^\£ V_t^\£ = E_t \left[ M_T^\£ V_T^\£ \right]$$ (1)
State Price Densities and Exchange Rates

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$$M_t^\£ V_t^\£ = E_t \left[ M_T^\£ V_T^\£ \right]$$

(1)

• The same security expressed in dollars, $V_t^\$ = S_t V_t^\£$, must satisfy $\$" pricing relation:

$$M_t^\$ V_t^\$ = E_t \left[ M_T^\$ V_T^\$ \right]$$
State Price Densities and Exchange Rates

• Consider 2 currencies: dollar $ and sterling pound £. Let $S = \frac{\$}{\£}$.

• Two countries and two pricing kernels $M_t^\$\,$ and $M_t^\£$.

• No arbitrage: Every £- security $V_t^\£$ with payoff $V_T^\£$ at $T$, must satisfy

$$M_t^\£ V_t^\£ = E_t \left[ M_T^\£ V_T^\£ \right] \quad (1)$$

• The same security expressed in dollars, $V_t^\$ = $S_t V_t^\£$, must satisfy $\$ pricing relation:

$$M_t^\$ V_t^\$ = E_t \left[ M_T^\$ V_T^\$ \right]$$

or, substituting

$$M_t^\$ S_t V_t^\£ = E_t \left[ M_T^\$ S_T V_T^\£ \right] \quad (2)$$
State Price Densities and Exchange Rates

• Consider 2 currencies: dollar $ and sterling pound £. Let \( S = \frac{\$}{\£} \).

• Two countries and two pricing kernels \( M_t^\$ \) and \( M_t^\£ \).

• No arbitrage: Every £- security \( V_t^\£ \) with payoff \( V_T^\£ \) at \( T \), must satisfy

\[
M_t^\£ V_t^\£ = E_t \left[ M_T^\£ V_T^\£ \right] \tag{1}
\]

• The same security expressed in dollars, \( V_t^\$ = S_t V_t^\£ \), must satisfy $ pricing relation:

\[
M_t^\$ V_t^\$ = E_t \left[ M_T^\$ V_T^\$ \right]
\]

or, substituting

\[
M_t^\$ S_t V_t^\£ = E_t \left[ M_T^\$ S_T V_T^\£ \right] \tag{2}
\]

• Clearly, (1) and (2) are satisfied if (but not only if):

\[
M_t^\$ S_t = M_t^\£.
\]
State Price Densities and Exchange Rates

- Consider 2 currencies: dollar $ and sterling pound £. Let $S = \frac{\$}{£}$.
- Two countries and two pricing kernels $M^\$ t$ and $M^\£ t$.
- No arbitrage: Every £- security $V^\£ t$ with payoff $V^\£ T$ at $T$, must satisfy
  \[ M^\£ t V^\£ t = E^t [M^\£ T V^\£ T] \] (1)
- The same security expressed in dollars, $V^\$ t = S^t V^\£ t$, must satisfy $\$ pricing relation:
  \[ M^\$ t V^\$ t = E^t [M^\$ T V^\$ T] \]

  or, substituting
  \[ M^\$ t S^t V^\£ t = E^t [M^\$ T S^T T V^\£ T] \] (2)
- Clearly, (1) and (2) are satisfied if (but not only if):
  \[ M^\$ t S^t = M^\£ t. \]
- Other relations may also work.
Baby Model I

- Consider the following two SDFs for US and UK:

\[
\frac{dM^S}{M^S} = -r_S dt + \sigma_{S,g} dW_g + \sigma_{S,$\$} dW_\$ \\
\frac{dM^£}{M^£} = -r_£ dt + \sigma_{£,g} dW_g + \sigma_{£,$£} dW_£
\]
Baby Model I

- Consider the following two SDFs for US and UK:

\[
\begin{align*}
\frac{dM^S}{M^S} &= -r_S dt + \sigma_{S,g} dW_g + \sigma_{S,s} dW_s \\
\frac{dM^\£}{M^\£} &= -r_\£ dt + \sigma_{\£,g} dW_g + \sigma_{\£,\£} dW_\£
\end{align*}
\]
Baby Model I

- Consider the following two SDFs for US and UK:

\[
\begin{align*}
\frac{dM^S}{M^S} &= -r^S dt + \sigma^S g dW_g + \sigma^S dW^S \\
\frac{dM^E}{M^E} &= -r^E dt + \sigma^E g dW_g + \sigma^E dW^E
\end{align*}
\]
Baby Model I

• Consider the following two SDFs for US and UK:

\[
\frac{dM^S}{M^S} = -r_S dt + \sigma_{S,g} dW_g + \sigma_{S,$dW$_S}
\]

\[
\frac{dM^£}{M^£} = -r_£ dt + \sigma_{£,g} dW_g + \sigma_{£,$dW$_£}
\]

• Let the exchange rate follow the process

\[
\frac{dS_t}{S_t} = (\mu_S - r_£) dt + \sigma_{S,g} dW_g + \sigma_{S,$dW$_S} + \sigma_{S,£dW$_£}
\]
Baby Model I

- Consider the following two SDFs for US and UK:

\[
\begin{align*}
\frac{dM^\$$}{M^\$$} &= -r^\$$ dt + \sigma_{\$$,g} dW_g + \sigma_{\$$,\$$} dW_\$$ \\
\frac{dM^\£}{M^\£} &= -r^\£ dt + \sigma_{\£,g} dW_g + \sigma_{\£,\£} dW_\£
\end{align*}
\]

- Let the exchange rate follow the process

\[
\frac{dS_t}{S_t} = (\mu_S - r^\£) dt + \sigma_{S,g} dW_g + \sigma_{S,\$$} dW_\$$ + \sigma_{S,\£} dW_\£
\]

- All the shocks are justified by \( S_t = M_t^\£ / M_t^\$$ \).
Baby Model I

- Consider the following two SDFs for US and UK:

\[
\frac{dM^\$}{M^\$} = -r^\$ dt + \sigma^\$g dW_g + \sigma^\$,$ dW_$
\]

\[
\frac{dM^£}{M^£} = -r^£ dt + \sigma^£g dW_g + \sigma^£,$ dW_£
\]

- Let the exchange rate follow the process

\[
\frac{dS_t}{S_t} = (\mu_S - r^£) dt + \sigma_{S,g} dW_g + \sigma_{S,\$} dW_\$ + \sigma_{S,£} dW_£
\]

- All the shocks are justified by \( S_t = M_t^£ / M_t^\$ \).

- Let \( V^\$ \) be any dollar valued security. Then, no arbitrage implies

\[
E_t \left[ d \left( V^\$ M^\$ \right) \right] = 0
\]
Baby Model I

- Consider the following two SDFs for US and UK:

\[
\frac{dM^S}{M^S} = -r_S dt + \sigma_{S,g} dW_g + \sigma_{S,\$} dW_\$
\]
\[
\frac{dM^£}{M^£} = -r_£ dt + \sigma_{£,g} dW_g + \sigma_{£,\$} dW_\$
\]

- Let the exchange rate follow the process

\[
\frac{dS_t}{S_t} = (\mu_S - r_£) dt + \sigma_{S,g} dW_g + \sigma_{S,\$} dW_\$ + \sigma_{S,\£} dW_\£
\]

- All the shocks are justified by \( S_t = M^£_t / M^S_t \).

- Let \( V^\$ \) be any dollar valued security. Then, no arbitrage implies

\[
E_t \left[ d \left( V^\$ M^\$ \right) \right] = 0
\]

- Using Ito’s Lemma, this gives the PDE

\[
r_S V^\$ = \frac{\partial V^\$}{\partial t} + \frac{\partial V^\$}{\partial S} (\mu_S - r_£) S + \frac{1}{2} \frac{\partial^2 V^\$}{\partial S^2} S^2 \sigma_S \sigma' + \frac{\partial V^\$}{\partial S} S \sigma_S \sigma'_{M,\$}
\]
Baby Model I

- This PDE must hold also for $V = S$ (with the addition of the “dividend yield” $r_F$), which gives the risk premium for a US investor to purchase British pounds

$$
\mu_S - r_\$ = -\sigma_S \sigma'_M,\$ = -Cov \left( \frac{dS}{S}, \frac{dM^\$}{M^\$} \right)
$$
Baby Model I

• This PDE must hold also for $V = S$ (with the addition of the “dividend yield” $r_{\mathcal{£}}$), which gives the risk premium for a US investor to purchase British pounds

$$\mu_S - r_\$ = -\sigma_S \sigma'_{M,\$} = -\text{Cov}\left(\frac{dS}{S}, \frac{dM^\$}{M^\$}\right)$$

• Nothing new so far. We can substitute $\mu_S$ in PDE

$$0 = -r_\$V^\$ + \frac{\partial V^\$}{\partial t} + \frac{\partial V^\$}{\partial S} (r_\$ - r_{\mathcal{£}}) S + \frac{1}{2} \frac{\partial^2 V^\$}{\partial S^2} S^2 \sigma_S \sigma'_S$$
Baby Model I

- This PDE must hold also for $V = S$ (with the addition of the “dividend yield” $r_\£$), which gives the risk premium for a US investor to purchase British pounds

$$\mu_S - r_\$ = -\sigma_S \sigma'_M,\$ = -Cov \left( \frac{dS}{S}, \frac{dM\$}{M\$} \right)$$

- Nothing new so far. We can substitute $\mu_S$ in PDE

$$0 = -r_\$ V^\$ + \frac{\partial V^\$}{\partial t} + \frac{\partial V^\$}{\partial S} (r_\$ - r_\£) S + \frac{1}{2} \frac{\partial^2 V^\$}{\partial S^2} S^2 \sigma_S \sigma'_S$$

- This is the usual Black, Scholes, and Merton PDE
Baby Model I

- This PDE must hold also for \( V = S \) (with the addition of the “dividend yield” \( r_L \)), which gives the risk premium for a US investor to purchase British pounds

\[
\mu_S - r_S = -\sigma_S \sigma'_M, S = -Cov\left(\frac{dS}{S}, \frac{dM^S}{M^S}\right)
\]

- Nothing new so far. We can substitute \( \mu_S \) in PDE

\[
0 = -r_S V^S + \frac{\partial V^S}{\partial t} + \frac{\partial V^S}{\partial S} (r_S - r_L) S + \frac{1}{2} \frac{\partial^2 V^S}{\partial S^2} S \sigma_S \sigma'_S
\]

- This is the usual Black, Scholes, and Merton PDE

- No reference to expected return \( \mu_S \) nor to the pricing kernel parameters \( \sigma_{M, S} \).

\( \Rightarrow \) Options cannot tell us anything about expected returns.

\( \Rightarrow \) This is of course what Black, Scholes, Merton taught us long ago.
Baby Model I

- This PDE must hold also for $V = S$ (with the addition of the “dividend yield” $r_\mathcal{E}$), which gives the risk premium for a US investor to purchase British pounds

$$\mu_S - r_\$ = -\sigma_S \sigma'_M,\$ = -\text{Cov} \left( \frac{dS}{S}, \frac{dM\$}{M\$} \right)$$

- Nothing new so far. We can substitute $\mu_S$ in PDE

$$0 = -r_\$ V\$ + \frac{\partial V\$}{\partial t} + \frac{\partial V\$}{\partial S} (r_\$ - r_\mathcal{E}) S + \frac{1}{2} \frac{\partial^2 V\$}{\partial S^2} S^2 \sigma_S \sigma'_S$$

- This is the usual Black, Scholes, and Merton PDE

- No reference to expected return $\mu_S$ nor to the pricing kernel parameters $\sigma_{M,\$}$.

$\implies$ Options cannot tell us anything about expected returns.

$\implies$ This is of course what Black, Scholes, Merton taught us long ago.

- Note that all shocks are “Gaussian” $\implies$ related to Jurek and Xu finding that Gaussian shocks do not allow one to estimate risk premia from options.
Baby Model II
Baby Model II

• Let’s just add a BIG common shock to pricing kernels:

\[
\frac{dM^S}{M^S} = - \left[ r^S + \lambda g E_t \left( J^S_{M,g} - 1 \right) \right] dt + \sigma_{M^S} dW + \left( J^S_{M,g} - 1 \right) dQ_g
\]

\[
\frac{dM^L}{M^L} = - \left[ r^L + \lambda g E_t \left( J^L_{M,g} - 1 \right) \right] dt + \sigma_{M^L} dW + \left( J^L_{M,g} - 1 \right) dQ_g
\]

• with \( dQ_g \) being a Poisson increment with intensity \( \lambda g \).
Baby Model II

- Let’s just add a BIG common shock to pricing kernels:

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\frac{dM^S}{M^S} = - \left[ r^S + \lambda_g E_t \left( J^S_{M,g} - 1 \right) \right] dt + \sigma_{M,S} dW + \left( J^S_{M,g} - 1 \right) dQ_g
\]

\[
\frac{dM^E}{M^E} = - \left[ r^E + \lambda_g E_t \left( J^E_{M,g} - 1 \right) \right] dt + \sigma_{M,E} dW + \left( J^E_{M,g} - 1 \right) dQ_g
\]

- with \(dQ_g\) being a Poisson increment with intensity \(\lambda_g\).

- Consider again the same steps. Now \(S_t\) must depend on \(dQ_g\) as well (at least):

\[
\frac{dS_t}{S_t} = (\mu_S - r^E - \lambda_g E_t [J^S_{S,g} - 1]) dt + \sigma_S dW + (J^S_{S,g} - 1) dQ_g
\]
• Let’s just add a BIG common shock to pricing kernels:

\[
\frac{dM^S}{M^S} = - \left[ r^S + \lambda_g E_t \left( J_{M,g}^S - 1 \right) \right] dt + \sigma_{M,S} dW + \left( J_{M,g}^S - 1 \right) dQ_g
\]

\[
\frac{dM^L}{M^L} = - \left[ r^L + \lambda_g E_t \left( J_{M,g}^L - 1 \right) \right] dt + \sigma_{M,L} dW + \left( J_{M,g}^L - 1 \right) dQ_g
\]

• with \( dQ_g \) being a Poisson increment with intensity \( \lambda_g \).

• Consider again the same steps. Now \( S_t \) must depend on \( dQ_g \) as well (at least):

\[
\frac{dS_t}{S_t} = (\mu_S - r^L - \lambda_g E_t [J_{S,g} - 1]) dt + \sigma_S dW + (J_{S,g} - 1) dQ_g
\]

• Like before, a dollar security \( V^S \) must satisfy \( E_t [d (V^S M^S)] = 0 \).
Baby Model II

• Let’s just add a BIG common shock to pricing kernels:

\[
\frac{dM^S}{M^S} = - \left[ r^S + \lambda g E_t \left( J^S_{M,g} - 1 \right) \right] dt + \sigma_{M,S} dW + \left( J^S_{M,g} - 1 \right) dQ_g
\]

\[
\frac{dM^£}{M^£} = - \left[ r^£ + \lambda g E_t \left( J^£_{M,g} - 1 \right) \right] dt + \sigma_{M,£} dW + \left( J^£_{M,g} - 1 \right) dQ_g
\]

• with \( dQ_g \) being a Poisson increment with intensity \( \lambda_g \).

• Consider again the same steps. Now \( S_t \) must depend on \( dQ_g \) as well (at least):

\[
\frac{dS_t}{S_t} = (\mu_S - r^£ - \lambda g E_t \left[ J^S_{S,g} - 1 \right]) dt + \sigma_S dW + \left( J^S_{S,g} - 1 \right) dQ_g
\]

• Like before, a dollar security \( V^S \) must satisfy \( E_t \left[ d \left( V^S M^S \right) \right] = 0 \).

• Ito’s Lemma in this case gives

\[
r^S V^S = \frac{\partial V^S}{\partial t} + \frac{\partial V^S}{\partial S} (\mu_S - r^£) S + \frac{1}{2} \frac{\partial^2 V^S}{\partial S^2} S^2 \sigma_S \sigma'_S + \frac{\partial V^S}{\partial S} S \sigma_S \sigma'_{M,S}
\]

\[
- \frac{\partial V^S}{\partial S} S \lambda g E \left[ J^S_{S,g} - 1 \right] + \lambda g E \left[ \left( V^S (J^S_{S,g} S) - V^S (S) \right) J^S_{M,g} \right]
\]
Baby Model II

• Let’s just add a BIG common shock to pricing kernels:

\[
\begin{align*}
\frac{dM^S}{M^S} &= - \left[ r_S + \lambda g E_t \left( J_{M,g}^S - 1 \right) \right] dt + \sigma_{M,S} dW + \left( J_{M,g}^S - 1 \right) dQ_g \\
\frac{dM^\ell}{M^\ell} &= - \left[ r_\ell + \lambda g E_t \left( J_{M,g}^\ell - 1 \right) \right] dt + \sigma_{M,\ell} dW + \left( J_{M,g}^\ell - 1 \right) dQ_g
\end{align*}
\]

• with \( dQ_g \) being a Poisson increment with intensity \( \lambda_g \).

• Consider again the same steps. Now \( S_t \) must depend on \( dQ_g \) as well (at least):

\[
\frac{dS_t}{S_t} = (\mu_S - r_\ell - \lambda g E_t \left[ J_{S,g} - 1 \right]) dt + \sigma_S dW + (J_{S,g} - 1) dQ_g
\]

• Like before, a dollar security \( V^S \) must satisfy \( E_t \left[ d \left( V^S M^S \right) \right] = 0 \).

• Ito’s Lemma in this case gives

\[
\begin{align*}
\sigma_{M,S} &= \frac{\partial V^S}{\partial t} + \frac{\partial V^S}{\partial S} (\mu_S - r_\ell) S + \frac{1}{2} \frac{\partial^2 V^S}{\partial S^2} S^2 \sigma_{S} \sigma'_{S} + \frac{\partial V^S}{\partial S} S \sigma_{S} \sigma'_{M,S} \\
&- \frac{\partial V^S}{\partial S} S \lambda g E \left[ J_{S,g} - 1 \right] + \lambda g E \left[ \left( V^S (J_{S,g}, S) - V^S (S) \right) \right] J_{M,g}^S
\end{align*}
\]
Baby Model II

• Let’s just add a BIG common shock to pricing kernels:

\[
\frac{dM^S}{M^S} = - \left[ r_S + \lambda g E_t \left( J_{M,g}^S - 1 \right) \right] dt + \sigma_{M,S} dW + \left( J_{M,g}^S - 1 \right) dQ_g
\]

\[
\frac{dM^L}{M^L} = - \left[ r_L + \lambda g E_t \left( J_{M,g}^L - 1 \right) \right] dt + \sigma_{M,L} dW + \left( J_{M,g}^L - 1 \right) dQ_g
\]

• with \( dQ_g \) being a Poisson increment with intensity \( \lambda_g \).

• Consider again the same steps. Now \( S_t \) must depend on \( dQ_g \) as well (at least):

\[
\frac{dS_t}{S_t} = (\mu_S - r_L - \lambda g E_t [J_{S,g} - 1]) dt + \sigma_S dW + (J_{S,g} - 1) dQ_g
\]

• Like before, a dollar security \( V^S \) must satisfy \( E_t \left[ d \left( V^S M^S \right) \right] = 0 \).

• Ito’s Lemma in this case gives

\[
\begin{align*}
\frac{d}{dt} V^S &= \frac{\partial V^S}{\partial t} + \frac{\partial V^S}{\partial S} (\mu_S - r_L) S + \frac{1}{2} \frac{\partial^2 V^S}{\partial S^2} S^2 \sigma_S \sigma'_S + \frac{\partial V^S}{\partial S} S \sigma_S \sigma'_M,
\end{align*}
\]

\[
- \frac{\partial V^S}{\partial S} S \lambda g E \left[ J_{S,g} - 1 \right] + \lambda g E \left[ \left( V^S (J_{S,S'}) - V^S (S) \right) J_{M,g}^S \right]
\]

JUMP-RELATED COMPONENT OF PDE
Baby Model II

- This equation must hold for \( V^S = S \) (adding the “dividend yield” \( r_E \)), obtaining:

\[
\mu_S - r_S = \underbrace{-\sigma_S \sigma'_M}_{\text{Diffusive Cov} \left( \frac{dS}{S}, \frac{dM^S}{M^S} \right)} \quad \text{(Diffusive Risk Premium)}
\]

\[
- \frac{1}{\sigma'_M} \left[ \lambda_g E_t \left[ (J_{S,g} - 1) \left( J_{M,g}^S - 1 \right) \right] \right]
\]

\[
- \underbrace{\text{Jump Cov} \left( \frac{dS}{S}, \frac{dM^S}{M^S} \right)}_{\text{Jump Risk Premium}}
\]
Baby Model II

- This equation must hold for $V^S = S$ (adding the “dividend yield” $r_L$), obtaining:

\[
\mu_S - r_S = \underbrace{-\sigma_S \sigma_M^S}_{\text{(Diffusive Risk Premium)}} - \underbrace{\lambda_g E_t \left[ (J_{S,g} - 1) (J_{M,g}^S - 1) \right]}_{\text{(Jump Risk Premium)}} - \underbrace{\text{Diffusive Cov} \left( dS/S, dM^S/M^S \right)}_{\text{(Diffusive Risk Premium)}} - \underbrace{\text{Jump Cov} \left( dS/S, dM^S/M^S \right)}_{\text{(Jump Risk Premium)}}
\]

- Substitute $\mu_S$ again in PDE

\[
0 = \text{BSMPDE} - \frac{\partial V^S}{\partial S} S \lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^S \right] + \lambda_g E \left[ \left( V^S (J_{S,g} S) - V^S (S) \right) J_{M,g}^S \right]
\]
Baby Model II

- This equation must hold for \( V^S = S \) (adding the “dividend yield” \( r_L \)), obtaining:

\[
\mu_S - r_S = -\sigma_S \sigma'_{M,S} - \lambda g E_t \left[ (J_{S,g} - 1) \left( J^S_{M,g} - 1 \right) \right]
\]

(Diffusive Risk Premium)

\[
-\text{Diffusive Cov} \left( \frac{dS}{S}, \frac{dM^S}{M^S} \right)
\]

(Jump Risk Premium)

- Substitute \( \mu_S \) again in PDE

\[
0 = \text{BSMPDE} - \frac{\partial V^S}{\partial S} S \lambda g E \left[ (J_{S,g} - 1) J^S_{M,g} \right] + \lambda g E \left[ \left( V^S (J_{S,g}S) - V^S (S) \right) J^S_{M,g} \right]
\]

- Note:

1. Diffusive risk premium again disappears from pricing equation \( \implies \) options cannot inform about that component.
Baby Model II

- This equation must hold for \( V^\$ = S \) (adding the “dividend yield” \( r_L \)), obtaining:

\[
\mu_S - r_S = -\sigma_S \sigma_M^\$ \left( -\text{DiffusiveCov} \left( dS/S, dM^\$/M^\$ \right) \right) - \lambda g E_t \left[ (J_{S,g} - 1) (J_{M,g}^\$ - 1) \right] - \lambda g E \left[ (V^\$ (J_{S,g} S) - V^\$ (S)) J_{M,g}^\$ \right]
\]

- Substitute \( \mu_S \) again in PDE

\[
0 = BSMPDE - \frac{\partial V^\$}{\partial S} S \lambda g E \left[ (J_{S,g} - 1) J_{M,g}^\$ \right] + \lambda g E \left[ (V^\$ (J_{S,g} S) - V^\$ (S)) J_{M,g}^\$ \right]
\]

- Note:
  1. Diffusive risk premium again disappears from pricing equation \( \implies \) options cannot inform about that component.
  2. Options though depend on jump component of pricing kernel \( \implies \) options may reveal some information about it.
Baby Model II

- This equation must hold for $V^S = S$ (adding the "dividend yield" $r_E$), obtaining:

\[
\mu_S - r_S = \mu_S - r_S = -\sigma_S \sigma'_M, S - \lambda g E_t \left[ (J_{S,g} - 1) (J_{M,g}^S - 1) \right] - \text{Diffusive Cov} \left( \frac{dS}{S}, \frac{dM^S}{M^S} \right) - \text{Jump Cov} \left( \frac{dS}{S}, \frac{dM^S}{M^S} \right)
\]

(Diffusive Risk Premium)

- Substitute $\mu_S$ again in PDE

\[
0 = \text{BSMPDE} - \frac{\partial V^S}{\partial S} S \lambda g E \left[ (J_{S,g} - 1) J_{M,g}^S \right] + \lambda g E \left[ \left( V^S (J_{S,g} S) - V^S (S) \right) J_{M,g}^S \right]
\]

- Note:

1. Diffusive risk premium again disappears from pricing equation $\implies$ options cannot inform about that component.

2. Options though depend on jump component of pricing kernel $\implies$ options may reveal some information about it.

- But what exactly can be identified? Let’s expand the jump risk premium
Baby Model II

Jump Risk Premium = \(-\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^\$ \right]\) + \(\lambda_g E [J_{S,g} - 1]\)
Baby Model II

\[ \text{Jump Risk Premium} = -\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^s \right] + \lambda_g E [J_{S,g} - 1] \]

Identifiable from Options (Risk Neutral Dist.)

Not Identifiable from Options (Physical Dist.)
Jump Risk Premium = $-\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^S \right] + \lambda_g E [J_{S,g} - 1]$  

- Identifiable from Options (Risk Neutral Dist.)
- Not Identifiable from Options (Physical Dist.)

- But let’s do a bit more work. From $S_t = M_t^\mathcal{E} \left( M_t^S \right)^{-1} \implies J_{S,g} = J_{M,g}^\mathcal{E} \left( J_{M,g}^S \right)^{-1}$. 

Baby Model II
Jump Risk Premium  

\[
\text{Jump Risk Premium} = -\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^\$ \right] + \lambda_g E \left[ J_{S,g} - 1 \right]
\]

\begin{align*}
&\underbrace{-\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^\$ \right]}_{\text{Identifiable from Options}} + \underbrace{\lambda_g E \left[ J_{S,g} - 1 \right]}_{\text{Not Identifiable from Options}} \\
&\quad \text{(Risk Neutral Dist.)} \quad \text{(Physical Dist.)}
\end{align*}

• But let’s do a bit more work. From \( S_t = M_t^\ell (M_t^\$)^{-1} \implies J_{S,g} = J_{M,g}^\ell \left( J_{M,g}^\$ \right)^{-1}. \)

Jump Risk Premium  

\[
\text{Jump Risk Premium} = -\lambda_g \left[ E \left( J_{M,g}^\ell \right) - E \left( J_{M,g}^\$ \right) \right] + \lambda_g E \left[ J_{M,g}^\ell \left( J_{M,g}^\$ \right)^{-1} - 1 \right]
\]
Baby Model II

Jump Risk Premium = \(-\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^S \right] \) \(\underbrace{\text{Identifiable from Options}}_{\text{(Risk Neutral Dist.)}}\) \(+\) \(\lambda_g E [J_{S,g} - 1] \) \(\underbrace{\text{Not Identifiable from Options}}_{\text{(Physical Dist.)}}\)

• But let’s do a bit more work. From \(S_t = M_t^\ell (M_t^S)^{-1} \implies J_{S,g} = J_{M,g}^\ell \left( J_{M,g}^S \right)^{-1} \).

Jump Risk Premium = \(-\lambda_g \left[ E \left( J_{M,g}^\ell \right) - E \left( J_{M,g}^S \right) \right] \) + \(\lambda_g E \left[ J_{M,g}^\ell \left( J_{M,g}^S \right)^{-1} - 1 \right] \)

• Assume factor structure of jumps: \( J_{M,g}^S = \beta^S J_g; \ J_{M,g}^\ell = \beta^\ell J_g. \)
Baby Model II

Jump Risk Premium = \(-\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^S \right] + \lambda_g E [J_{S,g} - 1]\)

Identifiable from Options (Risk Neutral Dist.)

Not Identifiable from Options (Physical Dist.)

• But let’s do a bit more work. From \(S_t = M_t^\mathcal{E} \left( M_t^S \right)^{-1}\) \(\implies J_{S,g} = J_{M,g}^\mathcal{E} \left( J_{M,g}^S \right)^{-1}\).

Jump Risk Premium = \(-\lambda_g \left[ E \left( J_{M,g}^\mathcal{E} \right) - E \left( J_{M,g}^S \right) \right] + \lambda_g E \left[ J_{M,g}^\mathcal{E} \left( J_{M,g}^S \right)^{-1} - 1 \right]\)

• Assume factor structure of jumps: \(J_{M,g}^S = \beta^S J_g; \ J_{M,g}^\mathcal{E} = \beta^\mathcal{E} J_g\).

• Normalize \(E[J_g] = 1\) (wlog) and substitute:

Jump Risk Premium = \(-\lambda_g \left( \beta^\mathcal{E} - \beta^S \right) + \frac{\lambda_g}{\beta^S} \left( \beta^\mathcal{E} - \beta^S \right)\)
Baby Model II

Jump Risk Premium \( = -\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^s \right] \) (Identifiable from Options (Risk Neutral Dist.)) + \( \lambda_g E \left[ J_{S,g} - 1 \right] \) (Not Identifiable from Options (Physical Dist.))

- But let’s do a bit more work. From \( S_t = M_t^\mathcal{L} (M_t^s)^{-1} \) \( \implies J_{S,g} = J_{M,g}^\mathcal{L} (J_{M,g}^s)^{-1} \).

Jump Risk Premium \( = -\lambda_g \left[ E \left( J_{M,g}^\mathcal{L} \right) - E \left( J_{M,g}^s \right) \right] + \lambda_g E \left[ J_{M,g}^\mathcal{L} (J_{M,g}^s)^{-1} - 1 \right] \)

- Assume factor structure of jumps: \( J_{M,g}^s = \beta^s J_g; \quad J_{M,g}^\mathcal{L} = \beta^\mathcal{L} J_g \).

- Normalize \( E[J_g] = 1 \) (wlog) and substitute:

Jump Risk Premium \( = -\lambda_g \left( \beta^\mathcal{L} - \beta^s \right) + \frac{\lambda_g}{\beta^s} \left( \beta^\mathcal{L} - \beta^s \right) \)

\( = \lambda_g \left( \beta^\mathcal{L} - \beta^s \right) \left( 1 - \beta^s \right) (\beta^s)^{-1} \)
Baby Model II

Jump Risk Premium = $-\lambda_g E \left[ (J_{S,g} - 1) J_{M,g}^s \right]$ + $\lambda_g E \left[ J_{S,g} - 1 \right]$

Identifiable from Options (Risk Neutral Dist.)

Not Identifiable from Options (Physical Dist.)

• But let’s do a bit more work. From $S_t = M_t^\mathcal{L} (M_t^s)^{-1} \Rightarrow J_{S,g} = J_{M,g}^\mathcal{L} \left( J_{M,g}^s \right)^{-1}$.

Jump Risk Premium = $-\lambda_g \left[ E \left( J_{M,g}^\mathcal{L} \right) - E \left( J_{M,g}^s \right) \right] + \lambda_g E \left[ J_{M,g}^\mathcal{L} \left( J_{M,g}^s \right)^{-1} - 1 \right]$

• Assume factor structure of jumps: $J_{M,g}^s = \beta^s J_g$; $J_{M,g}^\mathcal{L} = \beta^\mathcal{L} J_g$.

• Normalize $E[J_g] = 1$ (wlog) and substitute:

Jump Risk Premium = $-\lambda_g \left( \beta^\mathcal{L} - \beta^s \right) + \frac{\lambda_g}{\beta^s} \left( \beta^\mathcal{L} - \beta^s \right)$

= $\lambda_g \left( 1 - \beta^s \right) \left( \beta^s \right)^{-1}$

• Not bad…. So: how do we identify $\lambda_g$, $\beta^\mathcal{L}$ and $\beta^s$?
• The PDE now becomes

\[ 0 = \text{BSMPDE} - \frac{\partial V^S}{\partial S} S \lambda g \beta^S \left( \frac{\beta^L}{\beta^S} - 1 \right) + \lambda g \beta^S \left( V^S \left( \frac{\beta^L}{\beta^S} S \right) - V^S (S) \right) \]
Baby Model II

• The PDE now becomes

\[ 0 = \text{BSMPDE} - \frac{\partial V^\$}{\partial S} S \lambda_g \beta^g \left( \frac{\beta^f}{\beta^g} - 1 \right) + \lambda_g \beta^g \left( V^\$ \left( \frac{\beta^f}{\beta^g} S \right) - V^\$ (S) \right) \]

• The PDE only features the parameter combinations \( \lambda_g \beta^g \) and \( \frac{\beta^f}{\beta^g} \)
Baby Model II

- The PDE now becomes

\[ 0 = \text{BSMPDE} - \frac{\partial V^S}{\partial S} S \lambda_g \beta^S \left( \frac{\beta^L}{\beta^S} - 1 \right) + \lambda_g \beta^S \left( V^S \left( \frac{\beta^L}{\beta^S} S \right) - V^S (S) \right) \]

- The PDE only features the parameter combinations \( \lambda_g \beta^S \) and \( \frac{\beta^L}{\beta^S} \)

\[ \implies \text{options on } \$/L \text{ can then identify them.} \]
Baby Model II

• The PDE now becomes

\[
0 = \text{BSMPDE} - \frac{\partial V^S}{\partial S} S \lambda_g \beta^S \left( \frac{\beta^S}{\beta^S} - 1 \right) + \lambda_g \beta^S \left( V^S \left( \frac{\beta^S}{\beta^S} S \right) - V^S (S) \right)
\]

• The PDE only features the parameter combinations \( \lambda_g \beta^S \) and \( \frac{\beta^L}{\beta^S} \)

\[\Rightarrow\] options on $/$£ can then identify them.

• Indeed, in this case, the option pricing formula is very simple (Merton (1976))

\[
V^S(S) = \sum_{n=0}^{\infty} \frac{e^{-(\lambda_g \beta^S)T}[(\lambda_g \beta^S)T]^n}{n!} BSM \left( S e^{b(n)T}, K, r_S, r_L, \sigma^2_S \sigma'^2_S, T \right)
\]

– where BSM is the Black, Scholes, Merton formula, and

\[
b(n) = - (\lambda_g \beta^S) \left[ \left( \frac{\beta^L}{\beta^S} \right) - 1 \right] + \frac{n}{T} \log \left( \frac{\beta^L}{\beta^S} \right)
\]
Baby Model II

• The PDE now becomes

\[ 0 = \text{BSMPDE} - \frac{\partial V^S}{\partial S} S \lambda_g \beta^S \left( \frac{\beta^L}{\beta^S} - 1 \right) + \lambda_g \beta^S \left( V^S \left( \frac{\beta^L}{\beta^S} S \right) - V^S (S) \right) \]

• The PDE only features the parameter combinations \( \lambda_g \beta^S \) and \( \frac{\beta^L}{\beta^S} \)

\[ \Rightarrow \] options on \$/\ell \text{ can then identify them.}

• Indeed, in this case, the option pricing formula is very simple (Merton (1976))

\[ V^S(S) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_g \beta^S T \left[ \left( \lambda_g \beta^S \right) T \right]^n}}{n!} \text{BSM} \left( S e^{b(n) T}, K, r_S, r_{\ell}, \sigma_S \sigma_S', T \right) \]

– where BSM is the Black, Scholes, Merton formula, and

\[ b(n) = -\lambda_g \beta^S \left[ \left( \frac{\beta^L}{\beta^S} \right) - 1 \right] + \frac{n}{T} \log \left( \frac{\beta^L}{\beta^S} \right) \]

• Given a bunch of options data, we can estimate \( (\lambda_g \beta^S) \) and \( \left( \frac{\beta^L}{\beta^S} \right) \).
Baby Model II

- The above shows that from $$/\mathcal{L}$$ options we can “observe:”

\[
\hat{d}_{1,\$}\mathcal{L} = \lambda_g \beta^\$; \quad \hat{d}_{2,\$}\mathcal{L} = \beta^\mathcal{L} / \beta^\$
\]

- 2 equations in 3 unknowns. Not enough.
Baby Model II

- The above shows that from $/£$ options we can “observe:”

\[
\hat{d}_{1,\$/£} = \lambda_g \beta^\$/ \quad \hat{d}_{2,\$/£} = \beta^\$/ / \beta^\$
\]

- 2 equations in 3 unknowns. Not enough.

- Consider now one more option on $/€$:

\[
\hat{d}_{1,\$/€} = \lambda_g \beta^\$/ \quad \hat{d}_{2,\$/€} = \beta^\$/ / \beta^\$
\]

- 3 equations in 4 unknowns. Still not enough.
Baby Model II

• The above shows that from \$/£ options we can “observe:”

\[ \hat{d}_{1,\$/$\£} = \lambda_g \beta^\$; \quad \hat{d}_{2,\$/$\£} = \beta^\£/\beta^\$

• 2 equations in 3 unknowns. Not enough.

• Consider now one more option on \$/€:

\[ \hat{d}_{1,\$/$\€} = \lambda_g \beta^\$; \quad \hat{d}_{2,\$/$\€} = \beta^\€/\beta^\$

• 3 equations in 4 unknowns. Still not enough.

• Consider the cross rate £/€:

\[ \hat{d}_{1,\£/$\€} = \lambda_g \beta^\£; \quad \hat{d}_{2,\£/$\€} = \beta^\€/\beta^\£

• 4 equations in 4 unknowns.

[middle equations do not add anything to the ones at the bottom and the top]
So, we have 4 equations in 4 unknowns. Can we actually identify the JRP?

\[
JRP = \lambda_g \left( \beta^L - \beta^S \right) \left( 1 - \beta^S \right) \left( \beta^S \right)^{-1}
\]
Baby Model II

• So, we have 4 equations in 4 unknowns. Can we actually identify the JRP?

$$JRP = \lambda g \left( \beta^\ell - \beta^s \right) \left( 1 - \beta^s \right) \left( \beta^s \right)^{-1}$$

• Unfortunately not.

• In fact, the equations are not independent. Recall they are:

$$\hat{d}_{1,\$/\ell} = \lambda g \beta^\$$;  $$\hat{d}_{2,\$/\ell} = \beta^\ell / \beta^$$;  $$\hat{d}_{1,\ell/\epsilon} = \lambda g \beta^\ell$$;  $$\hat{d}_{2,\ell/\epsilon} = \beta^\epsilon / \beta^\ell$$
Baby Model II

- So, we have 4 equations in 4 unknowns. Can we actually identify the JRP?

\[ JRP = \lambda_g \left( \beta^L - \beta^S \right) \left( 1 - \beta^S \right) \left( \beta^S \right)^{-1} \]

- Unfortunately not.

- In fact, the equations are not independent. Recall they are:

\[
\begin{align*}
\hat{d}_{1,/&/\ell} &= \lambda_g \beta^S; \\
\hat{d}_{2,/&/\ell} &= \beta^L / \beta^S; \\
\hat{d}_{1,/&/\iota} &= \lambda_g \beta^L; \\
\hat{d}_{2,/&/\iota} &= \beta^\iota / \beta^L
\end{align*}
\]

- But e.g. we also have

\[ \frac{\hat{d}_{2,/&/\ell}}{\beta^S} = \frac{\hat{d}_{1,/&/\ell}}{\hat{d}_{1,/&/\iota}} \implies \text{Not enough restrictions.} \]
Baby Model II

• So, we have 4 equations in 4 unknowns. Can we actually identify the JRP?

\[ \text{JRP} = \lambda_g \left( \beta^\ell - \beta^\$ \right) \left( 1 - \beta^\$ \right) \left( \beta^\$ \right)^{-1} \]

• Unfortunately not.

• In fact, the equations are not independent. Recall they are:

\[ \hat{d}_{1,\$ / \ell} = \lambda_g \beta^\$; \quad \hat{d}_{2,\$ / \ell} = \beta^\ell / \beta^\$; \quad \hat{d}_{1,\ell / \epsilon} = \lambda_g \beta^\ell; \quad \hat{d}_{2,\ell / \epsilon} = \beta^\epsilon / \beta^\ell \]

• But e.g. we also have \[ \frac{\hat{d}_{2,\$ / \ell}}{\hat{d}_{1,\ell / \epsilon}} = \frac{\beta^\ell / \beta^\$}{\hat{d}_{1,\$ / \ell}} \implies \text{Not enough restrictions} \]

• For instance, if \( \hat{\beta}^i = x \beta^i \), for \( i = \ell, \$, \epsilon \), and \( \hat{\lambda}_g = (1/x) \lambda_g \), then

\[ \hat{\lambda}_g \hat{\beta}^\$ = \lambda_g \beta^\$; \quad \hat{\beta}^\ell / \hat{\beta}^\$ = \beta^\ell / \beta^\$; \quad \hat{\lambda}_g \hat{\beta}^\ell = \lambda_g \beta^\ell; \quad \hat{\beta}^\epsilon / \hat{\beta}^\ell = \beta^\epsilon / \beta^\ell \]
Baby Model II

- So, we have 4 equations in 4 unknowns. Can we actually identify the JRP?

\[ JRP = \lambda_g \left( \beta^\ell - \beta^$ \right) \left( 1 - \beta^$ \right) \left( \beta^$ \right)^{-1} \]

- Unfortunately not.

- In fact, the equations are not independent. Recall they are:

\[ \hat{d}_{1,s/\ell} = \lambda_g \beta^$; \quad \hat{d}_{2,s/\ell} = \beta^\ell / \beta^$; \quad \hat{d}_{1,\ell/\epsilon} = \lambda_g \beta^\ell; \quad \hat{d}_{2,\ell/\epsilon} = \beta^\epsilon / \beta^\ell \]

- But e.g. we also have \[ \frac{\hat{d}_{2,s/\ell}}{\beta^s} = \frac{\hat{d}_{1,s/\ell}}{\hat{d}_{1,\ell/\epsilon}} \implies \text{Not enough restrictions.} \]

- For instance, if \( \hat{\beta}^i = x \beta^i \), for \( i = \ell, $, \epsilon \), and \( \hat{\lambda}_g = (1/x) \lambda_g \), then

\[ \hat{\lambda}_g \hat{\beta}^$ = \lambda_g \beta^$; \quad \hat{\beta}^\ell / \hat{\beta}^$ = \beta^\ell / \beta^$; \quad \hat{\lambda}_g \beta^\ell = \lambda_g \beta^\ell; \quad \hat{\beta}^\epsilon / \hat{\beta}^\ell = \beta^\epsilon / \beta^\ell \]

- But the JRP is different: If e.g. \( \beta^\ell > \beta^$ \), then

\[ JRP = \hat{\lambda}_g \left( \hat{\beta}^\ell - \hat{\beta}^$ \right) \left( 1 - \hat{\beta}^$ \right) \left( \hat{\beta}^$ \right)^{-1} = \lambda_g \left( \beta^\ell - \beta^$ \right) \left( 1 - x \beta^$ \right) \left( x \beta^$ \right)^{-1} \in \left[ -\lambda_g \left( \beta^\ell - \beta^$ \right), \infty \right] \]
Baby Model III

- Let’s try another “baby model”.
Baby Model III

• Let’s try another “baby model”.

• Consider a standard Heston’s stochastic volatility model:

\[
\frac{dM^S}{M^S} = -r_S dt + \beta_S \sqrt{v} dW_g + \sigma_S^d dW_S
\]

\[
\frac{dM^\mathcal{E}}{M^\mathcal{E}} = -r_\mathcal{E} dt + \beta_\mathcal{E} \sqrt{v} dW_g + \sigma_\mathcal{E} dW_\mathcal{E}
\]

• with

\[dv = k(\theta - v)dt + s_v \sqrt{v} dW_v.\]
Baby Model III

• Let’s try another “baby model”.

• Consider a standard Heston’s stochastic volatility model:

\[
\frac{dM^S}{M^S} = -r_S dt + \beta_S \sqrt{v} dW_g + \sigma_{S,S} dW_S,
\]

\[
\frac{dM^E}{M^E} = -r_E dt + \beta_E \sqrt{v} dW_g + \sigma_{E,E} dW_E,
\]

• with

\[d v = k(\theta - v) dt + s_v \sqrt{v} dW_v.\]

• From no arbitrage, this time we obtain

\[
\frac{dS}{S} = (\mu_S(v) - r_E) dt + (\beta_E - \beta_S) \sqrt{v} dW_g + \sigma_{E,E} dW_E - \sigma_{S,S} dW_S.
\]
Baby Model III

• Let’s try another “baby model”.

• Consider a standard Heston’s stochastic volatility model:

\[
\frac{dM^S}{M^S} = -r_S dt + \beta_s \sqrt{v} dW_g + \sigma_{S,S} dW_S
\]
\[
\frac{dM^F}{M^F} = -r_F dt + \beta_F \sqrt{v} dW_g + \sigma_{F,F} dW_F
\]

• with

\[dv = k(\theta - v)dt + s_v \sqrt{v} dW_v.\]

• From no arbitrage, this time we obtain

\[
\frac{dS}{S} = (\mu_S(v) - r_F) dt + (\beta_F - \beta_S) \sqrt{v} dW_g + \sigma_{F,F} dW_F - \sigma_{S,S} dW_S
\]

• The risk premium, to identify, is

\[
\mu_S(v) - r_S = -Cov\left(\frac{dM^S}{M^S}, \frac{dS}{S}\right) = - (\beta_F - \beta_S) \beta_S v_t + \sigma_{S,S}^2
\]
Baby Model III

- Let’s try another “baby model”.

- Consider a standard Heston’s stochastic volatility model:

  \[
  \frac{dM^S}{M^S} = -r_S dt + \beta_S \sqrt{v} dW_S + \sigma_S, dW_S \\
  \frac{dM^\mathcal{L}}{M^\mathcal{L}} = -r_\mathcal{L} dt + \beta_\mathcal{L} \sqrt{v} dW_S + \sigma_\mathcal{L}, dW_\mathcal{L}
  \]

- with \( dv = k(\theta - v)dt + s_v \sqrt{v} dW_v \).

- From no arbitrage, this time we obtain

  \[
  \frac{dS}{S} = (\mu_S(v) - r_\mathcal{L}) dt + (\beta_\mathcal{L} - \beta_S) \sqrt{v} dW_S + \sigma_\mathcal{L}, dW_\mathcal{L} - \sigma_S, dW_S
  \]

- The risk premium, to identify, is

  \[
  \mu_S(v) - r_S = -Cov\left( \frac{dM^S}{M^S}, \frac{dS}{S} \right) = - (\beta_\mathcal{L} - \beta_S) \beta_S v_t + \sigma^2_S, S
  \]

- Do options allow us to identify \( \beta_S, \beta_\mathcal{L}, v_t \) and \( \sigma^2_S, S \)?
Baby Model III

- The PDE this time is

\[ r_* V^* = \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial S} S (r_* - r_\ell) + \frac{1}{2} \frac{\partial^2 V^*}{\partial S^2} S^2 \left( (\beta \ell - \beta^*)^2 v + \sigma_{\ell,\ell}^2 + \sigma_{\^*,\^*}^2 \right) \]

\[ + \frac{\partial V}{\partial v} (k(\theta - v)) + \frac{1}{2} \frac{\partial^2 V}{\partial v^2} s^2 v \]
Baby Model III

- The PDE this time is

\[
\begin{align*}
    r_s V^s &= \frac{\partial V^s}{\partial t} + \frac{\partial V^s}{\partial S} S (r_s - r_L) + \frac{1}{2} \frac{\partial^2 V^s}{\partial S^2} S^2 \left( \beta_L - \beta_s \right)^2 v + \sigma^2_{L,L} + \sigma^2_{S,S} \\
    &\quad + \frac{\partial V}{\partial v} (k(\theta - v)) + \frac{1}{2} \frac{\partial^2 V}{\partial v^2} s^2 v
\end{align*}
\]

- Note that the $\beta$’s only enter as a difference $\beta_L - \beta_s$. 
Baby Model III

• The PDE this time is

\[ r_v V^v = \frac{\partial V^v}{\partial t} + \frac{\partial V^v}{\partial S} S (r_v - r_L) + \frac{1}{2} \frac{\partial^2 V^v}{\partial S^2} S^2 \left( (\beta_L - \beta_v)^2 v + \sigma_{L,L}^2 + \sigma_{v,v}^2 \right) \]

+ \frac{\partial V}{\partial v} (k(\theta - v)) + \frac{1}{2} \frac{\partial^2 V}{\partial v^2} s^2 v

• Note that the \( \beta \)'s only enter as a difference \( \beta_L - \beta_v \).

• \( \implies \) At most, from options, we can identify

\[ \$/L \text{ options } \implies \hat{d}_{v/L} = \beta_L - \beta_v \]

\[ \$/\€ \text{ options } \implies \hat{d}_{v/\€} = \beta_\€ - \beta_v; \]

\[ \€/L \text{ options } \implies \hat{d}_{\€/L} = \beta_L - \beta_\€; \]
Baby Model III

• The PDE this time is

\[ r_SV_S = \frac{\partial V_S}{\partial t} + \frac{\partial V_S}{\partial S}S (r_S - r_L) + \frac{1}{2} \frac{\partial^2 V_S}{\partial S^2} S^2 \left( (\beta_L - \beta_S)^2 v + \sigma_L^2 + \sigma_S^2 \right) \]

\[ + \frac{\partial V}{\partial v} (k(\theta - v)) + \frac{1}{2} \frac{\partial^2 V}{\partial v^2} s^2 v \]

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• \( \Longrightarrow \) At most, from options, we can identify

\( \$/L \) options \( \Longrightarrow \hat{d}_S/L = \beta_L - \beta_S \)

\( \$/\epsilon \) options \( \Longrightarrow \hat{d}_S/\epsilon = \beta_\epsilon - \beta_S \)

\( \epsilon/L \) options \( \Longrightarrow \hat{d}_{\epsilon/L} = \beta_L - \beta_\epsilon \)

• Although these are 3 equations in 3 unknowns, they are linearly dependent, as

\[ \hat{d}_{\epsilon/L} = \hat{d}_S/L - \hat{d}_S/\epsilon \]
Baby Model III

• The PDE this time is

\[ r_\$ V^\$ = \frac{\partial V^\$}{\partial t} + \frac{\partial V^\$}{\partial S} S (r_\$ - r_£) + \frac{1}{2} \frac{\partial^2 V^\$}{\partial S^2} S^2 \left( (\beta_£ - \beta_\$)^2 v + \sigma^2_£,£ + \sigma^2_\$,\$ \right) \]

\[ + \frac{\partial V}{\partial v} (k(\theta - v)) + \frac{1}{2} \frac{\partial^2 V}{\partial v^2} s^2 v \]

• Note that the \( \beta \)'s only enter as a difference \( \beta_£ - \beta_\$ \).

• \( \implies \) At most, from options, we can identify

\( \$/£ \) options \( \implies \hat{d}_{\$/£} = \beta_£ - \beta_\$ \)
\( \$/€ \) options \( \implies \hat{d}_{\$/€} = \beta_€ - \beta_\$ \);
\( €/£ \) options \( \implies \hat{d}_{€/£} = \beta_£ - \beta_€ \);

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\[ \hat{d}_{€/£} = \hat{d}_{\$/£} - \hat{d}_{\$/€} \]

• Again, the risk premium cannot be identified from options.
Baby Model III

• The PDE this time is

\[
\frac{\partial V^S}{\partial t} + \frac{\partial V^S}{\partial S} S (r^S - r_L) + \frac{1}{2} \frac{\partial^2 V^S}{\partial S^2} S^2 \left( (\beta_L - \beta_S)^2 v + \sigma^2_{L,L} + \sigma^2_{S,S} \right) \\
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• $\implies$ At most, from options, we can identify

$$/L \text{ options } \implies \hat{d}_{S/L} = \beta_L - \beta_S$$

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$$$/L \text{ options } \implies \hat{d}_{\epsilon/L} = \beta_L - \beta_{\epsilon};$$

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$$\hat{d}_{\epsilon/L} = \hat{d}_{S/L} - \hat{d}_{S/\epsilon}$$

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  – Note on discrete horizon $[t, t + \tau]$, returns are not Gaussian.
Summary from Baby Models

• Two simple “baby models” with non-Gaussian shocks do not allow for the identification of currency risk premium only from the cross-section of options.

• The main problem is that even though one can obtain a number of equations equal to the number of parameters under the physical measure, there are cross-currencies restrictions that decrease the number of independent equations.

• A similar problem seems to occur in Jurek and Xu.
  – Indeed, numerous identifying equations rely on differences in loadings.
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• A similar problem seems to occur in Jurek and Xu.
  – Indeed, numerous identifying equations rely on differences in loadings.

• But then, what is the source of the empirical success of the model?
Summary from Baby Models

- Ex-ante option-implied currency risk premia line up with *realized* excess returns.
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• Options predict currency risk premia better than interest rate differential.
Back to Baby Model II

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Indeed, note that for a US investor, the $ return against currency $j$ is always equal

$$JRP^{S,j} = \lambda_g \left( \beta^j - \beta^S \right) \left( 1 - \beta^S \right) \left( \beta^S \right)^{-1}$$
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- Therefore, the ratio of $-JRP over any two currencies are

$$\frac{JRP^{\$;j}}{JRP^{\$;i}} = \frac{\beta^j - \beta^\$}{\beta^i - \beta^\$} = \frac{\beta^j / \beta^\$ - 1}{\beta^i / \beta^\$ - 1}$$
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These ratios can indeed be identified from options, even if the levels cannot.
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Of course, this holds also for £-JPRs, or €-JPRs etc. (but not cross-currencies JPRs, such as $\frac{JR{P^\$}_i}{JR{P^{\£}}_i}$).
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- These ratios can indeed be identified from options, even if the levels cannot.
- Of course, this holds also for £-JPRs, or €-JPRs etc. (but not cross-currencies JPRs, such as \( \frac{JPR^{\$,i}}{JPR^{\$,i}} \)).
- This result is unfortunately not general, but it points at a potentially “easier” identification of the cross-section of risk premia from option.
Conclusions

• Ambitious paper.
• Getting at risk premia from options is tough.
• This paper seems to be getting quite interesting empirical results from the cross-section of options.
• Next step is to clearly understand how and why the estimation methodology works
  – More work there is clearly needed, but it is likely to bring about similar empirical results.
• It would be interesting to explore the difference between estimation of level of FX risk premia, and cross-section of FX risk premia, which may require different estimation methods, as shown in the simple example.