

# Stock Market Overreaction to Bad News in Good Times: A Rational Expectations Equilibrium Model

Pietro Veronesi  
University of Chicago

This article presents a dynamic, rational expectations equilibrium model of asset prices where the drift of fundamentals (dividends) shifts between two unobservable states at random times. I show that in equilibrium, investors' willingness to hedge against changes in their own "uncertainty" on the true state makes stock prices overreact to bad news in good times and underreact to good news in bad times. I then show that this model is better able than conventional models with no regime shifts to explain features of stock returns, including volatility clustering, "leverage effects," excess volatility, and time-varying expected returns.

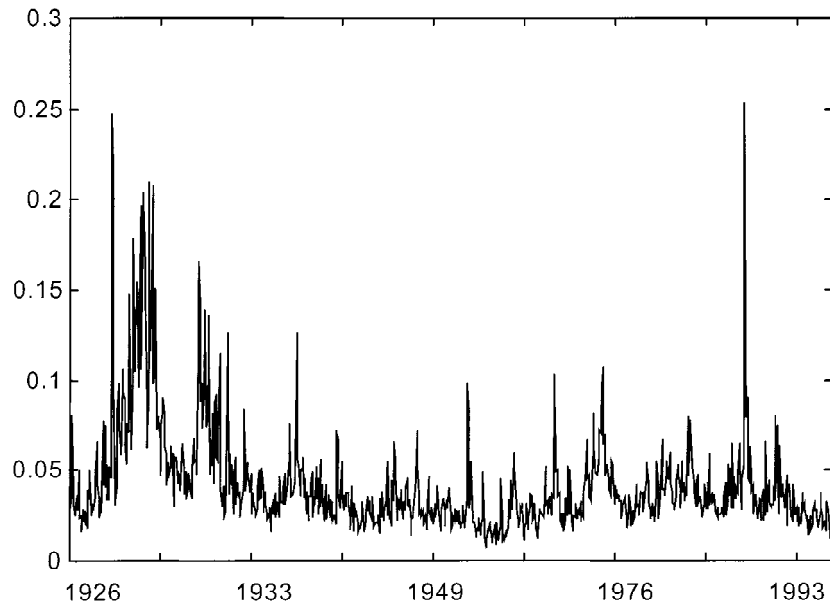
One of the most interesting aspects of financial data series is that stock return volatility changes widely across time. Historically the monthly volatility of stock returns has been as high as 20% in the early 1930s and as low as 2% in the early 1960s (see Figure 1). Moreover, changes in return volatility tend to be persistent, giving rise to the well-documented volatility clustering and "GARCH-type" behavior of returns [see, e.g., Bollerslev, Chou, and Kroner (1992) for an excellent survey of the literature].

Even though the statistical properties of return volatility have been deeply studied and uncovered by financial economists, several questions remain regarding their economic explanation. For example, why does return volatility tend to be higher in recessions? One explanation offered in the literature is that during recessions firms' riskiness is higher because they have higher debt-equity ratios [see Black (1976)]. However, evidence reported by Schwert (1989) shows that this cannot be the whole story.

Indeed, recent research has shown that investors' *uncertainty* over some important factors affecting the economy may greatly impact the volatility of stock returns. Bittlingmayer (1998) for instance argues that return volatility is related to political uncertainty, which also affects the level of future

---

I thank Nick Barberis, Gadi Barlevy, Domenico Cuoco, Chris Good, Ravi Jagannathan (the editor), Andrew Metrick, Luis Viceira, Jonathan Wright, and an anonymous referee for their comments on previous versions of this article. I also thank the seminar participants at Harvard, Chicago GSB, Northwestern, Columbia, Duke, Berkeley, London Business School, UCLA, Rochester, UBC, and Michigan. I am especially grateful to John Y. Campbell for his insightful suggestions while I was writing this article. Previous versions of this article circulated under the title "Regime Shifts and Changing Volatility in Stock Returns." Errors are my own. Address correspondence to Pietro Veronesi, Graduate School of Business, University of Chicago, 1101 E. 58th St., Chicago, IL 60637, or email: [pietro.veronesi@gsb.uchicago.edu](mailto:pietro.veronesi@gsb.uchicago.edu).



**Figure 1**  
Monthly volatility of stock returns: 1926:01–1995:12

output. Similarly, referring to a comment by Robert C. Merton about the high volatility during the 1930s, Schwert (1989) writes:

... the Depression was an example of the so called “Peso problem,” in the sense that there was legitimate uncertainty about whether the economic system would survive... Uncertainty about whether the “regime” had changed adds to the fundamental uncertainty reflected in past and future volatility of macroeconomic data.

More generally, there is evidence that investors tend to be more uncertain about the future growth rate of the economy during recessions, thereby partly justifying a higher volatility of stock returns. For example, Table 1 shows that economists’ forecasts about future real output are more dispersed — that is, they have a greater cross-sectional standard deviation — when the economy is contracting.

Even though it is intuitive that investors’ uncertainty may be related to return volatility, several questions still remain about its formal link to the business cycle, stock prices, and the intertemporal behavior of returns. This article develops an intertemporal, rational expectations equilibrium model of asset pricing to investigate these issues. Specifically, I assume that stock dividends are generated by realizations of a Gaussian diffusion process

**Table 1**  
**Uncertainty and Economic Contractions**

	Recession dummy	( <i>t</i> -statistic <sup>a</sup> )	Past GDP growth	( <i>t</i> -statistic <sup>a</sup> )
$\alpha$	2.6141	(32.36)	2.8151	(29.06)
$\beta$	0.4027	(2.95)	-8.0543	(-2.21)
$R^2$	0.09		0.06	

<sup>a</sup> *t*-statistics are corrected for heteroskedasticity and autocorrelation.

Results of the time-series regressions  $y_t = \alpha + \beta x_t + \varepsilon_t$ , where  $y_t$  is a measure of the dispersion of economists' forecasts at  $t$  about future real output growth and  $x_t$  is either a recession dummy or the  $t - 1/2$  real GDP growth (time unit is 6 months). Data on economists' forecasts are from the Livingston survey dataset (available at the Federal Reserve Bank of Philadelphia web site [www.phil.frb.org/econ/liv/index.html](http://www.phil.frb.org/econ/liv/index.html)), the sample period is 1951-1997 and survey dates were June and December.  $y_t$  is computed as follows: let  $GDP_{i,t}^F$  and  $CPI_{i,t}^F$  be the 12-month ahead forecast of individual  $i$  at time  $t$  for GNP (GDP since 1971) and CPI, and let  $GNP_t$  and  $CPI_t$  be base values economists had at their disposal at  $t$  (available from the data set). Then, the individual forecast of real GNP growth is computed by

$$F_t = \left( GDP_{i,t}^F / CPI_{i,t}^F \right) / \left( GNP_t / CPI_t \right).$$

The quantity  $y_t$  is the cross-sectional standard deviation  $STD(F_t)$  where cross-sectional outliers have been deleted (forecasts away from the mean by more than five standard deviations) and then a decreasing time-trend is eliminated. Instead, the recession dummy is one whenever at least (no more than) 5 months elapsed from the last peak (trough) date as determined by the NBER.

whose drift rate shifts between a high and a low state at random times. Of importance, I also assume that identical investors cannot observe the drift rate of the dividend process, but they have to infer it from the observation of past dividends. Depending on the actual path of past dividends, investors' uncertainty on the current state changes over time, being at its maximum when they assign probability .5 to each of the two states.

The main result of the article is that the equilibrium price of the asset is an increasing and convex function of investors' posterior probability of the high state. This result is rather intuitive: let  $\pi(t)$  denote investors' posterior probability that the state is high at  $t$ . Suppose now that investors believe times are good so that  $\pi(t) \approx 1$ . A bad piece of news decreases  $\pi$  and therefore decreases future expected dividends and increases investors' uncertainty about the true drift rate of the dividend process. In fact,  $\pi$  is now closer to .5. Since risk-averse investors want to be compensated for bearing more risk, they will require an additional discount on the price of the asset. As a consequence, the price drops by more than it would in a present-value model. By contrast, suppose that investors believe times are bad and hence  $\pi(t) \approx 0$ . A good piece of news increases their expectation of future dividends but also raises their uncertainty. Hence the equilibrium price of the asset increases, but not as much as it would in a present-value model. Overall the price function is increasing but convex in  $\pi$ . In other words, investors' willingness to "hedge" against changes in their level of uncertainty makes them overreact to bad news in good times and underreact

to good news in bad times, making the price of the asset more sensitive to news in good times than in bad times. Hence the form of the price function.

Given the results on the price function, I then turn my attention to the implications of the model for stock return volatility and the required equilibrium expected return. I first decompose the volatility of stock price changes, which is a measure of price sensitivity to news, into two components: the first is an “uncertainty component” that describes the sensitivity of prices to news due only to investors’ uncertainty, that is, as if they were risk neutral. I show that this component is a symmetric function of  $\pi$ , maximized at  $\pi = 0.5$ . Moreover, this component is decreasing in the level of interest rate and the frequency of shifts, while it is increasing in the magnitude of the dividend drift change when a switch occurs. The second additional component is a “risk-aversion component” which stems mainly from investors’ degree of risk aversion, that is, investors’ hedging behavior. Of importance, the risk aversion component is positive when  $\pi$  is high and negative when  $\pi$  is low. Hence risk aversion yields an asymmetric effect on the price sensitivity to news. Finally, I show by a numerical example that the *percentage* return volatility — the volatility of price changes divided by the asset price — is hump-shaped with respect to  $\pi$ . Specifically, its maximum is around  $\pi = 0.5$ , the point of maximum uncertainty, and its minimum is  $\pi = 1$ . Since recessions tend to be characterized by higher uncertainty, these results suggest that indeed we should observe higher volatility during recessions. Notice that these results also give theoretical support to Merton’s comment reported above: if investors are uncertain about whether a shift in regime has occurred, return volatility should be high. In addition, even if during some recessions uncertainty is not very high, that is,  $\pi$  is closer to 0 than to 0.5, it is still the case that return volatility should be higher than in booms. This is due to the properties of the equilibrium price function: since it is increasing and convex in  $\pi$ , percentage volatility decreases quickly when  $\pi$  approaches 1 so that this is the point of minimum return volatility.

These results also imply a persistence in return volatility changes, because investors’ beliefs tend to change slowly over time. Finally, as a result of changing volatility and investors’ risk aversion, the required expected return also varies over time. As a consequence, this model gives theoretical support to the assumption that expected returns should be proportional to (expected) stock volatility, as postulated by Merton (1980). French, Schwert, and Stambaugh (1987) and Campbell and Hentschel (1992) find some evidence of this relationship.

### **Review of the Literature**

The finance literature has offered several explanations for the high and/or changing stock volatility of stock returns. Following Shiller (1984), Campbell and Kyle (1993) introduce an exogenous source of noise and look at the equilibrium asset pricing function in a rational expectations model.

They find that the investors' willingness to hedge against "noise" produces higher levels of volatility and time-varying expected returns. Wang (1993) and Grundy and Kim (1995) motivate the high volatility of stock returns using information asymmetry among investors. Here informed traders hedge against the behavior of uninformed traders and cause high levels of volatility. Campbell and Cochrane (1999) introduce a "habit-formation" argument to explain the behavior of the stock market. They show that in their model, as the consumption level gets closer to the "habit" level, investors' risk aversion increases and hence so does the equilibrium conditional variance of stock returns. In contrast to these articles, my model assumes investors have standard preferences for consumption, they have homogeneous information, and there is no noise in the economy.

This article is closer in methodology to a second strand of literature, in which homogeneous investors have incomplete information about a relevant state variable but random variables are not assumed to have a Gaussian distribution [Detemple (1991) and David (1997)]. In fact, in continuous-time models the assumption of Gaussian-type diffusion processes for the unobservable variables has the undesirable outcome that, if investors' priors are also assumed to be normally distributed, the conditional variance of investors' expectations is deterministic. Therefore the assumption of Gaussian distributions is ill-suited to study issues related to changing volatility [see, e.g., Detemple (1986), Gennotte (1986), Brennan and Xia (1998)]. Departing from the Gaussian framework, Detemple (1991) investigates a Cox, Ingersoll, and Ross (1985) production economy, where the state variables are assumed to follow a Gaussian diffusion process but investors' priors over the state variables are not normally distributed. He shows the equilibrium *interest rate* follows a diffusion process with stochastic variance. Thus Detemple can model stochastic variance by assuming that beliefs are not Gaussian even though the state variables are. Alternatively, one could assume that the state variable does not follow a Gaussian process. This is explored by David (1997), who studies a Cox–Ingersoll–Ross production economy, where, as in the present article, the (unobservable) state variable follows a continuous-time, Markov switching model with only two states. He also assumes that the production side of the economy is formed by only two firms, which generate physical returns according to two diffusion processes. His key assumption is that the drifts of these processes depend on the state variable and, moreover, that they change inversely with one another, so that when one is high, the other is low, and vice versa. He then studies investors' optimal portfolio allocation and shows that equilibrium marketwide excess returns display changing volatility, negative skewness, and negative correlation with future volatility. The model presented below uses the underlying framework of David (1997), but differs from it in its emphasis. While David investigates the effects of uncertainty on investors' portfolio composition and its implications for marketwide excess returns, this article

investigates the effects of uncertainty and risk aversion on the form of the stock price function and how this relates to stock market volatility.

The article is organized as follows. In Section 1, I describe the model. Proof of existence of a rational expectations equilibrium is given in Section 2. Sections 3 and 4 discuss the implications of the results for stochastic volatility and time-varying expected returns, respectively. Section 5 gives an equivalence theorem with a full information economy. Section 6 concludes.

## 1. The Model

I consider an economy with a single physical consumption good, which can be allocated to investment or consumption, and a continuum of identical consumers/investors. There are only two investment assets: a risky asset and a risk-free asset. I make the following assumptions:

**Assumption 1. Dividend process.** *The dividend process of the risky asset is given by the stochastic differential equation*

$$dD = \theta dt + \sigma d\xi, \quad (1)$$

where  $\theta(t)$  is an unobservable state variable, described in assumption 2, and  $\xi(t)$  denotes a Wiener process.

**Assumption 2. Regime shifts.** *The dynamics of the state variable  $\theta(t)$  follow a two-state, continuous-time Markov process with transition probability matrix between time  $t$  and time  $t + \Delta$  given by:*

$$\mathbf{P}(\Delta) = \begin{pmatrix} 1 - \lambda\Delta & \lambda\Delta \\ \mu\Delta & 1 - \mu\Delta \end{pmatrix}. \quad (2)$$

I label the two states  $\{\bar{\theta}, \underline{\theta}\}$  with  $\bar{\theta} > \underline{\theta}$ .

Assumption 2 captures the idea that the dividend growth rate  $\theta(t)$  may change over time at random dates. It is assumed that  $\theta(t)$  is not observable by investors, who therefore face a “signal extraction” problem. The signal is provided by the observed level of dividend  $D$ . The accurate estimation of  $\theta(t)$  is important for investors in order to compute the expected future dividend payouts and thereby set the equilibrium price. More specifically, assumption 2 implies that during an infinitesimal interval  $\Delta$ , there is probability  $\lambda\Delta$  that dividend growth shifts from the high state  $\bar{\theta}$  to the low state  $\underline{\theta}$  and probability  $\mu\Delta$  that the dividend growth shifts from the low state  $\underline{\theta}$  to the high state  $\bar{\theta}$ . It is worth pointing out that assumption 2 implies that the drift rate of dividends follows a mean reverting process, as in Wang (1993). Other assumptions are

**Assmption 3. Preferences.** Investors are endowed with a CARA utility function over consumption  $U(c, t) = -e^{-\rho t - \gamma c}$ , where  $\rho$  is the parameter of time preference and  $\gamma$  is the coefficient of absolute risk aversion.

**Assmption 4. Asset supply.** The supply of the risky asset is fixed and normalized to 1. The risk-free asset is elastically supplied and has an instantaneous rate of return equal to  $r$ .

Assumption 4 postulates the existence of a storage technology whose rate of return is fixed and equal to  $r$ . Besides simplifying the analysis, this assumption also matches the empirical finding that the volatility of the risk-free rate is much lower than the volatility of market returns.<sup>1</sup>

Note that I do not assume any noise in this economy. In fact, under the assumption of symmetric information, the level of noise would be fully revealed by prices in equilibrium [see Wang (1993)]. Therefore, noise would only imply a higher level of stock volatility and hence a somewhat less risky position of risk-averse investors and a higher required expected return of the risky asset.

Since I assume investors possess common information (derived from observing the dividend stream), observing equilibrium prices does not refine their knowledge about the state of the world. Let  $\{\mathcal{F}(t)\}$  be the filtration generated by the dividend stream  $(D(\tau))_{\tau=0}^t$  and define the posterior probability of the good state  $\bar{\theta}$  by

$$\pi(t) = \Pr(\theta(t) = \bar{\theta} | \mathcal{F}(t)).$$

Lemma 1 characterizes the law of motion of  $\pi(t)$ :

**Lemma 1.**

$$d\pi = (\lambda + \mu)(\pi^s - \pi)dt + h(\pi)dv \quad (3)$$

where

$$dv = \frac{1}{\sigma} \{dD - E[dD | \mathcal{F}(t)]\} \quad (4)$$

$$h(\pi) = \left( \frac{\bar{\theta} - \theta}{\sigma} \right) \pi(1 - \pi) \quad (5)$$

$$\pi^s = \frac{\mu}{\mu + \lambda}.$$

Moreover,  $v(t)$  is a Wiener process with respect to  $\mathcal{F}(t)$ .

---

<sup>1</sup> Veronesi (1997) explores the properties of the model where the interest rate is endogenously determined by an equilibrium condition. In line with the findings of related research [see Campbell (1997) for a survey on the topic], the results make it very difficult to reconcile the low volatility of interest rates with the high volatility of the stock market. See also Veronesi (1999).

*Proof.* See Liptser and Shiryaev (1977: 348). See also Theorem 1 in David (1997). ■

Notice that  $\pi^s$  is the probability of  $\bar{\theta}$  under the Markov chain stationary distribution. Other properties of Equation (3) are discussed in Liptser and Shiryaev (1977) and David (1997). One important lemma is the following:

**Lemma 2.** *The values 0 and 1 are entrance boundaries for  $\pi$ , that is, there is probability zero that  $\pi$  equals either of these two values in any finite time.*

*Proof.* See Liptser and Shiryaev (1977) and David (1997). ■

Lemma 2 implies that the relevant domain for  $\pi$  is (0, 1). It is also worth noting that even though the state variable  $\theta(t)$  follows a “jump” process, the assumption that nobody can observe it, together with the result that the posterior probability  $\pi$  follows a continuous diffusion process, implies that equilibrium price paths are continuous. This will greatly simplify the analysis.

## 2. Equilibrium

**Definition 1.** *A rational expectations equilibrium (REE) is given by  $(P(D, \pi), X(W, P, D, \pi), c(W, P, D, \pi))$ , where  $P(D, \pi)$  is the price level for given dividend level  $D$  and belief  $\pi$ ,  $X(W, P, D, \pi)$  and  $c(W, P, D, \pi)$  are the demand for the risky asset and the consumption level for given level of wealth  $W$ , price  $P$ , dividend, and belief, respectively, such that*

(i) **Utility maximization.**  *$(c(\cdot), X(\cdot))$  maximizes investors’ expected intertemporal utility:*

$$J(W, \pi, t) \equiv \max_{c, X} E_t \left[ \int_t^\infty U(c, s) ds \right]$$

*subject to the dynamic budget constraint and transversality condition:*

$$dW = (rW - c)dt + XdQ, \quad \lim_{\tau \rightarrow \infty} E_t[J(W(t+\tau), \pi(t+\tau), t+\tau)] = 0,$$

*where  $dQ = (D - rP)dt + dP$  is the return to a zero-investment portfolio long one share of the risky asset.*

(ii) **Market clearing.**  *$P(\cdot, \cdot)$  adjusts so that  $X(W, P(D, \pi), \pi) = 1$  for every  $W$  and every pair  $(D, \pi)$ .*

The expectation  $E_t[\cdot] \equiv E[\cdot | \mathcal{F}(t)]$  denotes investors’ expectation conditional on their information  $\mathcal{F}(t)$ .

The assumption of CARA utility function has the convenient property that the demand for risky asset  $X(W, P, D, \pi)$  is independent of wealth level  $W$ . Therefore I will denote it as  $X(P, D, \pi)$  only. Similarly, consumption won’t depend on  $P$  and  $D$ .

### 2.1 The Benchmark Case: No Shifts in Parameters

Before analyzing the existence of an equilibrium with changes in trend, I look at the simpler case where the state  $\theta(t)$  is known and constant. In this case  $\lambda = \mu = 0$ . Let  $D$  be the level of dividends at time  $t$  and let  $P_b^*(D; \theta)$  denote the discounted expected dividend stream, discounted at the risk-free rate. We then have

**Lemma 3.**

$$P_b^*(D; \theta) \equiv E \left[ \int_0^\infty e^{-rs} D(t+s) ds \mid D(t) = D, \theta \right] = \frac{1}{r} D + \frac{1}{r^2} \theta.$$

*Proof.* This is a special case of Lemma 4. ■

The following proposition characterizes the price of the risky asset:

**Proposition 1.** *There exists an REE, with the following features:*

(i) *The price function is given by*

$$P_b(D; \theta) = p_0 + P_b^*(D; \theta) = p_0 + p_D D + p_\theta \theta, \quad (6)$$

where  $p_0 = -\frac{\gamma\sigma^2}{r^2}$ ,  $p_D = \frac{1}{r}$ , and  $p_\theta = \frac{1}{r^2}$ .

(ii) *The value function is*

$$J(W, t) = -e^{-\rho t - r\gamma W - \beta},$$

where  $\beta = \frac{\rho}{r} + \frac{\gamma^2\sigma^2}{2r} - 1 + \log(r)$ .

(iii) *The equilibrium consumption and investment in the risky asset are*

$$c(W; \theta) = rW + \frac{1}{\gamma}(\beta - \log(r)); \quad X(P, D; \theta) = \frac{\alpha_Q}{\gamma r \sigma_Q^2},$$

where  $\alpha_Q = E_t[dQ]$  and  $\sigma_Q^2 = E_t[dQ^2]$ .

*Proof.* Special case of Propositions 2 and 3. See also Campbell and Kyle (1993) and Wang (1993). ■

Proposition 1 shows that the equilibrium price is simply given by a fixed discount ( $p_0 < 0$ ) on the value of discounted expected future dividends. In particular, the discount  $p_0$  equals  $-\frac{\gamma\sigma^2}{r^2}$ , which, as we see below, is the negative of the instantaneous variance of dollar returns multiplied by the coefficient of risk aversion.

The particular linear form of the equilibrium price function also implies  $E_t[dP^2] = \frac{\sigma^2}{r^2}$ , which is constant. This result is rather at odds with the empirical evidence, which shows that return volatility is highly heteroskedastic. Even though a constant  $E_t[dP^2]$  entails that the *percentage* returns variance is higher in troughs than in booms, we will see in the results of Section 4 that the order of magnitude of the volatility implied by this assumption is

too small. Introducing unobservable shifts in the conditional mean of the dividend stream has the effect of increasing return volatility in periods of higher uncertainty.<sup>2</sup>

The *equilibrium* expected return is given by  $E_t[dQ] = \frac{\gamma\sigma^2}{r} > 0$ , which is constant. This is due to the assumption that the supply of the asset is constant. Campbell and Kyle (1993) and Wang (1993) show that assuming an exogenous source of noise in the economy yields a time-varying expected return  $E_t[dQ]$ . My model achieves this same result relying only on changes in investors' uncertainty.

### 2.2 The Case of Shifts in Parameters

I now generalize proposition 1 to allow for regime shifts in the dividend stream. As in the previous case, let  $P^*(D, \pi)$  be the discounted expected dividends when current dividend is  $D$  and belief is  $\pi$ .

**Lemma 4.**

$$P^*(D, \pi) \equiv E_t \left[ \int_0^\infty e^{-rs} D(t+s) ds \right] \quad (7)$$

$$= \frac{1}{r} D + \frac{1}{r^2} \theta^s + \left( \frac{\bar{\theta} - \theta}{r(\lambda + \mu + r)} \right) (\pi - \pi^s), \quad (8)$$

where  $\theta^s = \pi^s \bar{\theta} + (1 - \pi^s) \underline{\theta}$  is the unconditional expected value of  $\theta$ .

*Proof.* See Appendix A. ■

The expectation of discounted dividends is a linear function of the posterior probability  $\pi$  and of the dividend  $D$ . I will generally refer to  $P^*(D, \pi)$  as the *risk-neutral price*, since it is the equilibrium price when investors are risk neutral. The following two propositions prove and characterize the rational expectations equilibrium when shifts are allowed:

**Proposition 2.** *There exists an REE where the price function  $P(D, \pi)$  is given by*

$$P(D, \pi) = p_0 + S(\pi) + P^*(D, \pi) \quad (9)$$

$$= p_0 + S(\pi) + p_D D + p_1 + p_\pi \pi, \quad (10)$$

where  $p_0 = -\frac{\gamma\sigma^2}{r^2}$ ,  $p_D = \frac{1}{r}$ ,  $p_1 = \frac{\theta}{r^2} + \left( \frac{\bar{\theta} - \theta}{r^2(\lambda + \mu + r)} \right) \mu$ ,  $p_\pi = \frac{(\bar{\theta} - \theta)}{r(\lambda + \mu + r)}$ , and  $S(\cdot)$  is a negative, convex, and U-shaped function of  $\pi \in [0, 1]$ , which

---

<sup>2</sup> As Wang (1993) shows, the result that  $E_t[dP^2]$  is constant holds under more general assumptions. Specifically, he shows that it holds true even if (i) the total supply of the asset, here normalized to one, moves stochastically according to a continuous  $It\hat{o}$  process (such as an Ornstein–Uhlenbeck process, for instance), (ii) the state variable  $\theta(t)$  follows a continuous  $It\hat{o}$  process as well, and (iii) there is asymmetric information among investors, so that some observe  $\theta(t)$  and others do not.

satisfies the differential equation

$$S''(\pi)P_3(\pi) = S'(\pi)P_2(\pi) + rS(\pi) + P_0(\pi). \quad (11)$$

The coefficients  $P_i(\pi)$  are given in Appendix A. The boundary conditions are discussed below.

*Proof.* See Appendix A. ■

**Proposition 3.** (a) Given the price function [Equation (9)], investors' value function is given by

$$J(W, \pi, t) = -e^{-\rho t - r\gamma W - g(\pi) - \beta},$$

where  $\beta$  is as in Proposition 1 and  $g(\pi) = f(\pi) - r\gamma S(\pi)$  and  $f(\pi)$  is a negative, convex, and U-shaped function satisfying

$$f''(\pi)Q_3(\pi) = f'(\pi)^2Q_3(\pi) + f'(\pi)Q_2(\pi) + f(\pi)r + Q_0(\pi). \quad (12)$$

The coefficients  $Q_i(\pi)$  are given in Appendix A.

(b) Investors optimal consumption and demand for the risky asset are given by

$$c(W, \pi) = rW + \frac{1}{\gamma}(\beta + g(\pi) - \log(r))$$

and

$$X(D, P, \pi) = \frac{\alpha_Q(\pi)}{\gamma r \sigma_Q^2(\pi)} - \frac{g'(\pi)}{\gamma r \sigma_Q(\pi)} h(\pi), \quad (13)$$

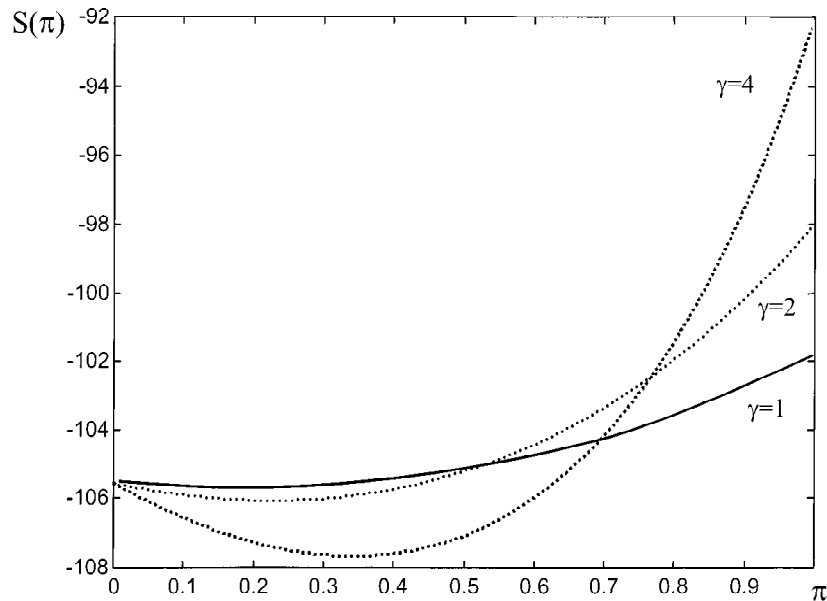
where  $\alpha_Q(\pi) = E_t[dQ]$ ,  $\sigma_Q^2(\pi) = E_t[dQ^2]$ , and  $h(\pi) = \left(\frac{\bar{\theta} - \theta}{\sigma}\right) \pi(1 - \pi)$ .

*Proof.* See Appendix A. ■

**2.2.1 The function  $S(\pi)$ .** The function  $S(\pi)$  is shown to be negative, convex, and U-shaped when we impose as the sole boundary condition for the differential equation which defines it [Equation (11)], the fact that it must not diverge to either  $+\infty$  or  $-\infty$ . Figure 2 plots the function for different values of the coefficient of absolute risk aversion. To make comparisons easier, I normalized the functions to have a common starting value.<sup>3</sup>

The fact that  $S(\pi)$  is negative implies that the equilibrium price function  $P(D, \pi)$  in Equation (9) is given by a discount  $p_0 + S(\pi) < 0$  over dis-

<sup>3</sup> All figures use the following parameters:  $\bar{\theta} = .0025$ ,  $\theta = -.015$ ,  $\sigma = .024$ ,  $r = .0041$ ,  $\lambda = .0167$ , and  $\mu = .1$ . In a previous version of the article, I estimated these parameters by applying the Hamilton (1989) algorithm to exponentially detrended data [see Campbell and Kyle (1993)]. The unit period is 1 month. All plots are generated by Matlab. Since the differential equations [Equations (12) and (11)] have singular points at 0 and 1, the range has been restricted to [.001, .999] or [.001, .995] depending on the ability of the numerical solver to get close to the boundaries. The proofs of Propositions 2 and 3 shows that the solution is convergent for  $\pi \rightarrow 1$  and  $\pi \rightarrow 0$ .



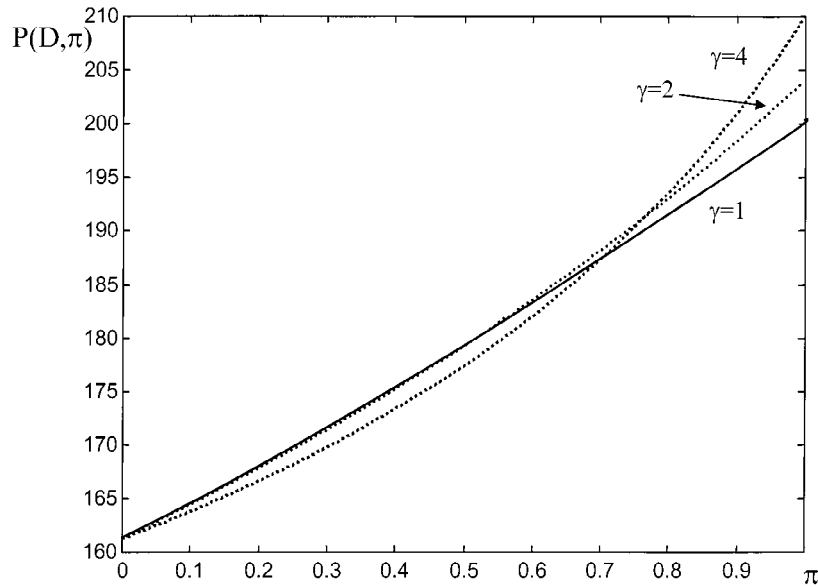
**Figure 2**  
 The discount functions  $S(\pi)$  for  $\gamma = 1, 2, 4$   
 All functions normalized to have a common starting value.

counted expected dividends  $P^*(D, \pi)$ , as was true in the benchmark case of Proposition 1. But in contrast to the case with no breaks, the discount changes with  $\pi$ . In particular, since  $S(\pi)$  is U-shaped, the discount is relatively smaller for extreme values of  $\pi$  (i.e., for  $\pi$  close to 0 and 1). This implies that for intermediate values of  $\pi$ , the price function will be further away from the discounted expected future dividends.

The shape of the  $S(\pi)$  function also implies that for  $\pi$  close to 1, the equilibrium price  $P(D, \pi)$  is more sensitive to changes in beliefs than the expected discounted dividends  $P^*(D, \pi)$ , because the former is steeper than the latter, whereas for  $\pi$  close to 0, it will be less reactive. Since the quantity  $P^*(D, \pi)$  is the fair price in a risk-neutral world, adding risk aversion both adds a discount to the risk-neutral price and changes the price sensitivity to news. This issue will be discussed at length in the next section. Figure 3 show the price functions  $P(D, \pi)$  for different values of the risk aversion coefficient. Again, to make comparisons easier, I normalized the functions to have a common starting value.

### 3. Stochastic Volatility

It is intuitive that during periods of high uncertainty, every new piece of information receives a large weight in the updating process of an investor's



**Figure 3**  
 The price functions  $P(D, \pi)$  for  $\gamma = 1, 2, 4$   
 All functions normalized to have a common starting value.

posterior distribution. This is quite apparent in the functional form of Equation (3), where at the point 0.5, the variance of  $d\pi$  is greatest. Since investors know that their expectations of future dividends will vary by a great deal when they are uncertain about the growth rate of the economy, they will try to “hedge” against changes in their own level of uncertainty [David (1997)]. The hedging behavior of investors is apparent in Equation (13), where there is an additional component to the standard demand for the asset determined by the expected return  $\alpha_Q(\pi)$ , volatility level  $\sigma_Q(\pi)$ , risk aversion, and risk free rate, as it was in the benchmark case. This behavior of course affects equilibrium asset prices. The next proposition characterizes the asset price sensitivity to news as a function of investors beliefs  $\pi$ . Let me define  $\sigma_P^2(\pi) = E_t[(dP)^2]/dt$ , so that  $\sigma_P(\pi)$  is the “price change” volatility.

**Proposition 4.** (a) The price change volatility takes the form

$$\sigma_P(\pi) = \frac{\sigma}{r} + p_\pi h(\pi) + S'(\pi)h(\pi), \quad (14)$$

where  $h(\pi) = \left(\frac{\bar{\theta} - \theta}{\sigma}\right) \pi(1 - \pi)$  and  $p_\pi = \frac{\bar{\theta} - \theta}{r(r + \lambda + \mu)}$ .

(b) Define  $\sigma_{P^*}^2(\pi) = E_t[(dP^*)^2]/dt$ , where  $P^*(D, \pi)$  is the risk-neutral price. Then

$$\sigma_{P^*}(\pi) = \frac{\sigma}{r} + p_\pi h(\pi). \quad (15)$$

*Proof.* (a) Immediate from *Itô's lemma* applied to the price function  $P(D, \pi)$ . (b) Immediate from *Itô's lemma* applied to the price function  $P^*(D, \pi)$ . ■

Proposition 4 shows that the “price change” volatility  $\sigma_P(\pi)$  can be decomposed into two components: the first is  $\sigma_{P^*}(\pi)$ , which I call “the uncertainty component,” and the second is  $S'(\pi)h(\pi)$ , which I call “the risk-aversion component.” The uncertainty component gives the amount of volatility stemming from investors’ uncertainty only, that is, the volatility that we would observe in a risk-neutral world. From part (b), the magnitude of the uncertainty component depends crucially on the parameter  $p_\pi = \frac{(\bar{\theta} - \theta)}{r(r + \lambda + \mu)}$ . This implies that uncertainty alone tends to increase volatility when changes are infrequent [i.e.,  $(\lambda + \mu)$  small], or more dramatic [i.e.,  $(\bar{\theta} - \theta)$  big], or when expected dividends in the far future are discounted at a low rate ( $r$  small).

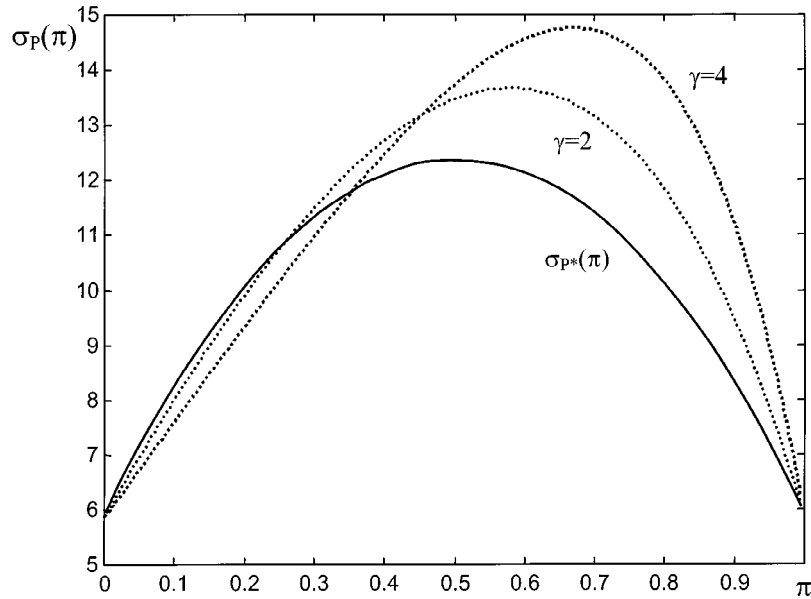
In addition to the uncertainty component, volatility depends on the “risk aversion component” given by  $S'(\pi)h(\pi)$ . This stems from investors’ attempts to “hedge” against changes in their own level of uncertainty. Before discussing this effect further, it is useful to state the following corollary:

**Corollary 1.** Let  $\hat{\pi}$  be such that  $S'(\hat{\pi}) = 0$ . Then

$$\sigma_P(\pi) > \sigma_{P^*}(\pi) \Leftrightarrow \pi > \hat{\pi} \quad (16)$$

*Proof.* Immediate from Equation (14). ■

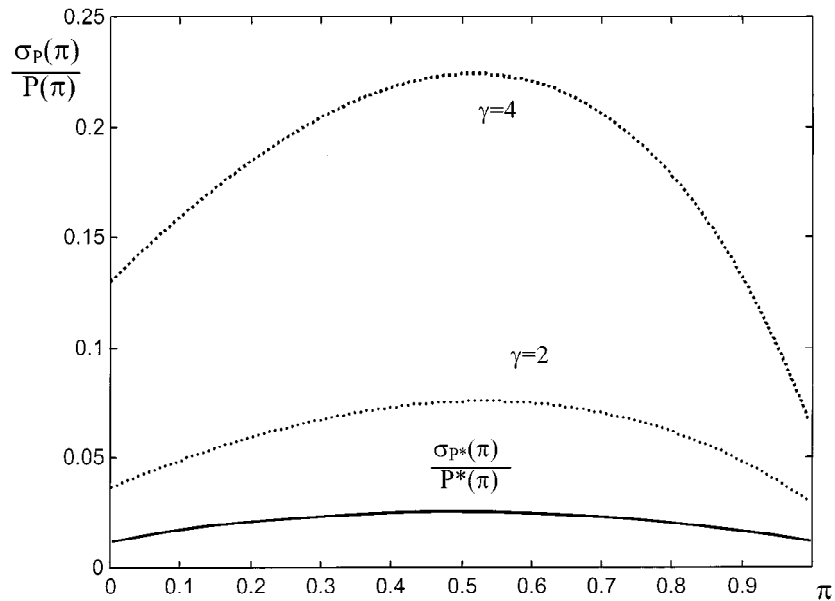
When investors assign high probability to the good state (i.e.,  $\pi > \hat{\pi}$ ), risk aversion increases stock price sensitivity to news, while the opposite is true when they assign high probability to the low state. The intuition for this result is as follows: consider the case where  $\pi$  is close to 1 and suppose investors receive a bad piece of news, that is, a realization of dividend that is below their expectations. This has the effect of reducing their posterior probability  $\pi$ , which in turn has two effects: the first is to decrease the expectation of future dividends and the second is to increase investors’ “degree of uncertainty,” because  $\pi$  gets closer to 0.5. We saw above that even under risk neutrality, higher uncertainty increases return volatility  $\sigma_{P^*}(\pi)$ , and moreover, this higher volatility is predictable. While risk-neutral investors would not be affected by this increase in predictable volatility, risk-averse investors are and require a higher discount than before. This implies that when  $\pi$  is high, in a rational expectations equilibrium, the decrease in the asset price due to a bad piece of news is greater under risk aversion than



**Figure 4**  
Price-change volatilities  $\sigma_{P^*}(\pi)$  and  $\sigma_P(\pi)$  for  $\gamma = 2, 4$

under risk neutrality. This higher sensitivity of prices to changes in beliefs implies that returns volatility should be higher when investors are risk averse than when they are risk neutral. This is shown in Figure 4. When  $\pi$  is close to zero, the converse holds: a good piece of news implies an increase in  $\pi$ , which increases the expected future dividends and the expected return volatility (higher uncertainty). However, since the second effect dampens the increase in price because of risk aversion, the price impact of news is not as great as in the case of risk neutrality.

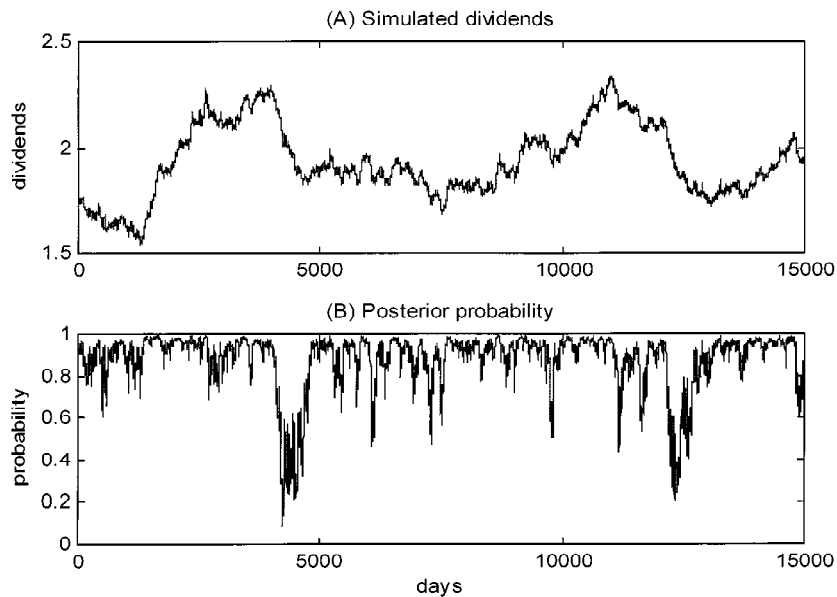
Even though price responsiveness to news is higher when  $\pi$  is high, the volatility of *percentage* returns need not be. In fact, the latter is given by  $\frac{\sigma_P(\pi)}{P(D,\pi)}$ . Figure 5 plots this quantity for different coefficients of risk aversion. Several comments are in order: First, the volatility of percentage returns is maximized for  $\pi \approx 0.5$ , that is, the point of maximum uncertainty. Moreover, because the function  $P(D, \pi)$  is increasing and convex in  $\pi$ , we can also see that the volatility of percentage returns  $\frac{\sigma_P(\pi)}{P(D,\pi)}$  is higher for  $\pi \approx 0$  than for  $\pi \approx 1$ . These findings lend theoretical support to the observation that return volatility is higher during recessions. In fact, on the one hand, empirical evidence suggests that recessions and to a greater extent depressions are characterized by higher uncertainty about the future growth rate of the economy (see discussion in the introduction and Table 1), which



**Figure 5**  
 Percentage return volatility  $\frac{\sigma_{P^*}(\pi)}{P^*(\pi)}$  and  $\frac{\sigma_P(\pi)}{P(\pi)}$  for  $\gamma = 2, 4$

in this model corresponds to the value of  $\pi = 0.5$ . On the other hand, even for those recessions where investors have low uncertainty, so that  $\pi$  is closer to 0 than to 0.5, it is still the case that return volatility is high because of the greater discount required on stock prices for very low  $\pi$ 's. Overall, we should then expect higher volatility during recessions.

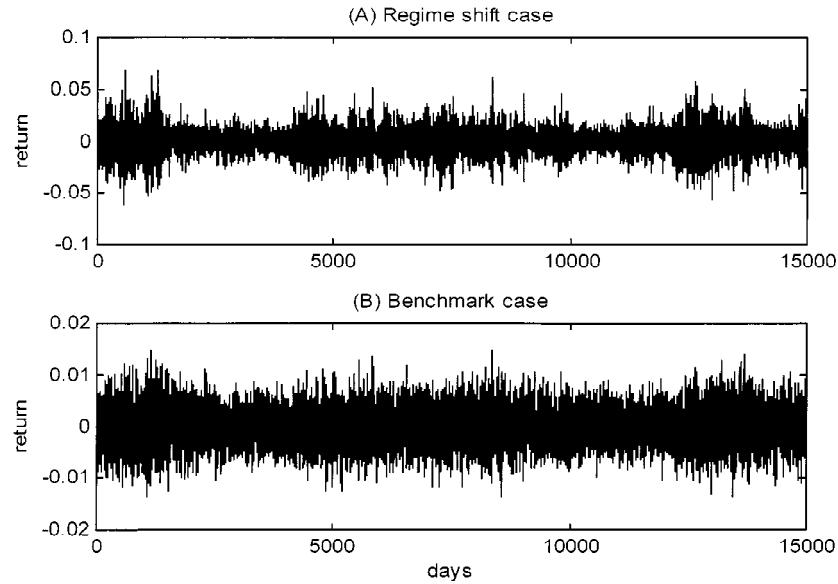
The effects of fluctuating uncertainty on volatility can be made clearer by looking at Figures 6–8. Figure 6A plots a time series of daily dividends simulated using the model outlined in Equations (1) and (2). Figure 6B instead plots the time series of the posterior probability  $\pi(t)$ , computed using Equation (3). It is apparent from the plot that after a few negative innovations in dividends, the probability  $\pi(t)$  dramatically decreases and becomes very volatile when it is around 0.5. From the simulated series of dividends and probabilities I then computed the series of daily returns implied by the prices  $P(D, \pi)$  in Equation (10). This is plotted in Figure 7A. For comparison, Figure 7B also plots the analogous return series implied by the prices  $P_b(D; \theta^s)$  in Equation (6), that is, the price series in the benchmark case where there are no shifts in regimes. It is apparent from Figure 7A that when there is a sequence of negative dividend innovations, returns in the regime shifts case become more volatile. In contrast, this effect is very small in the benchmark case. To emphasize this point further, Figure 8



**Figure 6**  
**Simulated dividends and probabilities**  
 Graph A plots a simulated series of daily dividend realizations, according to the regime shift model of section 1. Graph B plots the posterior probability distribution implied by Bayesian updating.

plots the computed monthly volatility (i.e., sum of squared returns over 25 days) for the two cases. We can see that in the regime shift case, return volatility is much higher and changes more dramatically over time than in the benchmark case. The reason for this is the following: First, in the regime shift case, prices are affected by both dividend changes and probability changes, whose innovations are perfectly correlated. Hence prices have higher volatility. Second, the form of the discount function in the regime shift case makes prices drop more substantially after negative innovations in dividends than in the benchmark case. Hence the increase percentage return volatility is higher in the former case.

An additional fact is apparent from Figures 6–8: negative dividend innovations tend to increase volatility, while positive innovations tend to decrease it. Since in this model there is only one random element conditional on investors' information, that is,  $dv$  in Equation (3), we have that standardized dividend innovations are equal to the standardized return innovations. Hence this model entails an asymmetric response of volatility to innovations in returns, as was found in Glosten, Jagannathan, and Runkle (1993). This point can be made more precise by studying the process of the uncertainty component of stock price volatility  $\sigma_{p^*}$  for the simpler case where  $\lambda = \mu$ .



**Figure 7**  
**Simulated stock returns**  
 Graph A plots the simulated daily stock return series in the case of regime shifts [using the price function in Equation (10)]. Graph B plots the simulated daily stock return series in the benchmark case [using the price function in Equation (6)].

Let  $\underline{\sigma}_{P^*} = \frac{\sigma}{r}$  and  $\overline{\sigma}_{P^*} = \frac{\sigma}{r} + p_{\pi} \frac{(\bar{\theta} - \theta)}{4\sigma}$  be the minimum and the maximum value that  $\sigma_{P^*}$  can take on, respectively.

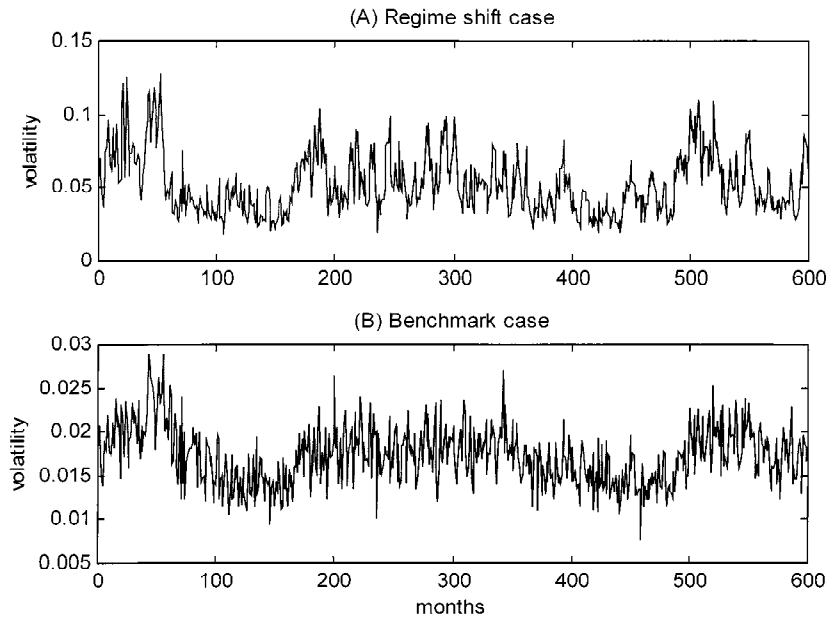
**Corollary 2.** If  $\lambda = \mu$ , the diffusion process for  $\sigma_{P^*}$  is

$$d\sigma_{P^*} = \left[ 4\lambda (\overline{\sigma}_{P^*} - \sigma_{P^*}) - \left( \frac{\bar{\theta} - \theta}{\sigma p_{\pi}} \right) (\sigma_{P^*} - \underline{\sigma}_{P^*})^2 \right] dt \pm \left( \frac{\bar{\theta} - \theta}{\sigma} \right) (\sigma_{P^*} - \underline{\sigma}_{P^*}) \sqrt{1 - \frac{(\sigma_{P^*} - \underline{\sigma}_{P^*})}{(\overline{\sigma}_{P^*} - \underline{\sigma}_{P^*})}} dv, \quad (17)$$

where the diffusion term has a positive sign when  $\pi < 0.5$  (the diffusion is zero for  $\pi = 0.5$ ).

*Proof.* See Appendix C. ■

This expression shows that  $\sigma_{P^*}$  is attracted by both its maximum value  $\overline{\sigma}_{P^*}$  and its minimum value  $\underline{\sigma}_{P^*}$ , thereby determining a sort of autoregressive behavior of volatility: when volatility is close to its maximum, the second component in the bracket parenthesis dominates, driving the volatility down,

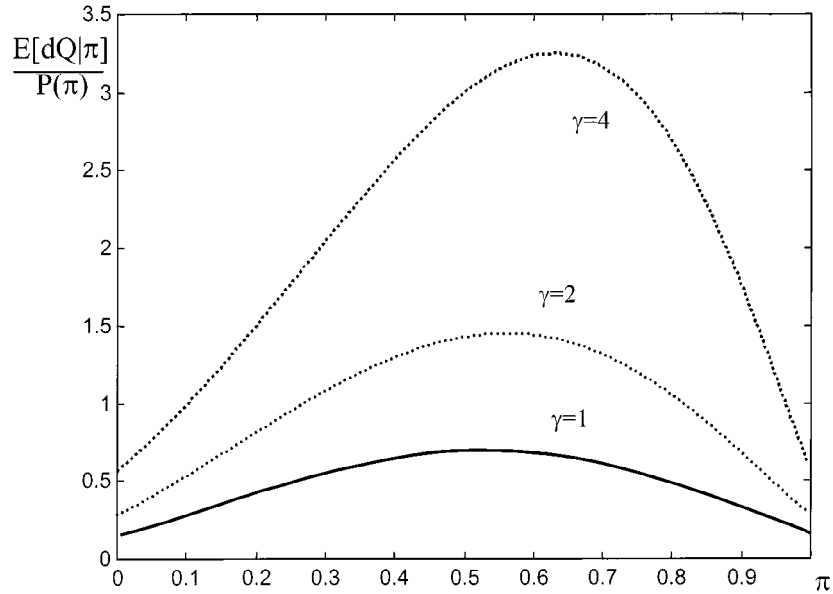


**Figure 8**  
**Simulated monthly volatility**  
 Graph A plots the monthly return volatility implied by the return series in Figure 7A. Graph B plots the monthly volatility implied by the return series in Figure 7B.

whereas the opposite is true when volatility is at its minimum. We can also see that the diffusion term gets closer to zero any time the volatility approaches its maximum or its minimum, leaving the drift term to dominate the changes in volatility. Finally, since booms last longer than recessions, we can expect that most of the time  $\pi(t) > 0.5$  (as in Figure 6B), which implies that most of the time the diffusion term in Equation (17) has a negative sign. Hence from Corollary 2, on average we should find that negative return innovations ( $dv < 0$ ) increase volatility, while positive innovations ( $dv > 0$ ) decrease it.

#### 4. Time-Varying Expected Returns

Given the discussion of the previous section, one could guess that in periods of high uncertainty, the required ex ante expected return increases to compensate risk-averse investors for the higher volatility of returns. The following proposition confirms this intuition. Recall that  $dQ = (D - rP)dt + dP$  is the return on a portfolio long one share of the risky asset financed by borrowing at the risk-free rate  $r$ .



**Figure 9**  
Expected excess return  $E_t[dQ]/P$  for  $\gamma = 1, 2, 4$

**Proposition 5.**

$$E_t[dQ] = [\gamma\sigma + (\gamma r p_\pi + f'(\pi))h(\pi)]\sigma_P(\pi)dt, \quad (18)$$

where  $h(\pi) = \left(\frac{\bar{\theta}-\theta}{\sigma}\right)\pi(1-\pi)$  and  $f(\pi)$  is defined in Proposition 3. Since  $\gamma r p_\pi + f'(\pi) > 0$  for all  $\pi$ ,  $E_t[dQ]$  increases with  $\sigma_P(\pi)$ .

*Proof.* See Appendix C. ■

This proposition shows that risk-averse investors want to be compensated for bearing higher risk. In fact, Equation (18) shows that there should be a positive albeit nonlinear relationship between expected returns and return volatility. Empirical studies [Merton (1980), French, Schwert, and Stambaugh (1987), Campbell and Hentschel (1992)] partially confirm that the observed movement in expected returns is due to a “volatility feedback,” that is, to increases in the premium due to an increase in the expected future return volatility. Figure 9 plots the conditional expected excess return  $E_t[dQ]/P$  as a function of  $\pi$ .

## 5. An Equivalent Economy with Complete Information

In the previous sections, I analyzed an economy where investors cannot observe the true instantaneous growth rate of dividends  $\theta(t)$ , which is assumed to switch between two values,  $\bar{\theta}$  and  $\underline{\theta}$ , over time. Since the conditional probability  $\pi(t)$  follows a diffusion process with “changing volatility,” which is due to a changing degree of uncertainty over time, equilibrium returns are shown to display changing volatility as well. Of course, in general there is no need to assume incomplete information in order to generate changing volatility in stock returns. Consider, for example, an economy described by the dividend process

$$dD = \theta^e dt + \sigma dB_1. \quad (19)$$

Assume the instantaneous dividend growth rate  $\theta^e$  is observable but stochastic. Specifically, assume that  $\theta^e$  follows the  $It\hat{o}$  process:

$$d\theta^e = a(\theta^e)dt + s(\theta^e)dB_2, \quad (20)$$

where  $a(\theta^e)$  and  $s(\theta^e)$  are two functions of  $\theta^e$ . The investor now faces an economy with fully observable state variables and the equilibrium price of the asset will be a function of the states,  $P^e(D, \theta^e)$ . Provided that the function  $s(\theta^e)$  is not constant,  $It\hat{o}$ 's lemma yields a process for returns  $dP^e$  with changing volatility. In fact, the economy described in Equations (19) and (20) can be obtained by the model presented in Section 1, by suitably defining the state variable  $\theta^e$ , the functions  $a(\theta^e)$  and  $s(\theta^e)$ , and imposing a restriction on the Wiener processes  $B_1(t)$  and  $B_2(t)$ .

**Proposition 6.** *The model described in Equations (1) and (2) is equivalent to the full information model,*

$$dD = \theta^e dt + \sigma dB_1 \quad (21)$$

$$d\theta^e = k(\theta^s - \theta^e)dt + \frac{(\bar{\theta} - \theta^e)(\theta^e - \underline{\theta})}{\sigma} dB_2, \quad (22)$$

where  $E[dB_1 dB_2] = 1$ ,  $k = \lambda + \mu$ , and  $\theta^s = \pi^s \bar{\theta} + (1 - \pi^s) \underline{\theta}$ .

*Proof.* Consider the model described by Equations (1)–(4). Define  $\theta^e(t) = \pi(t)\bar{\theta} + (1 - \pi(t))\underline{\theta}$ . Then  $It\hat{o}$ 's lemma and simple algebraic manipulation immediately give Equation (22), where  $B_2(t) = v(t)$ . Finally, since from Lemma 3,  $v(t)$  is a Wiener process with respect to  $\mathcal{F}_t$ , we can invert the relation [Equation (4)], to obtain  $dD = \theta^e dt + \sigma dv$ . Hence setting  $B_1(t) = v(t)$  yields the result. ■

Proposition 6 implies that a regime switching model with unobservable state variables is equivalent to a full information model where dividends follow an  $It\hat{o}$  process and where the instantaneous dividend growth rate

changes according to a Ornstein–Uhlenbeck process *with stochastic variance*. The regime shift assumption has therefore several implications for the full information model [Equations (19) and (20)]. First, it gives an intuitive explanation why  $\theta^e$  should follow a process with stochastic variance rather than assuming it exogenously. In fact,  $\theta^e$  is just the expected rate of dividend growth under the regime shift model, conditional on investor's information  $\mathcal{F}_t$ . Since we have seen that in periods of high uncertainty, the posterior probability reacts more to news, the expected dividend growth rate would also tend to be more volatile. Hence there are periods of time where  $\theta^e(t)$  should display higher volatility, as shown in Equation (22). Second, it provides restrictions on the functions  $a(\theta^e)$  and  $s(\theta^e)$  and hence offers an alternative way to estimate the parameters of the model in Equations (19) and (20). In fact, given that the econometrician can only observe realizations of the dividend process  $D(t)$ , we know from Merton (1980) that high-frequency data are not sufficient to estimate the drift parameter of a diffusion process, but that a sufficiently long sample period is necessary. Over this sample period, the drift parameter should be assumed constant. However, in order to generate changing volatility of stock returns, we need  $\theta^e(t)$  sufficiently variable over time, which increases the difficulty of estimating the parameters of the process [Equation (22)]. We can use Proposition 6 to map the model of Equations (21) and (22) into a model with unobservable regime shifts as given by Equations (1) and (2), whose parameter can be estimated by applying the Hamilton (1989) technique to the discrete approximation of the model.

## 6. Conclusion

This article explores the role of uncertainty and investors' degree of risk aversion to explain several of the empirical regularities of stock returns, including volatility clustering, "leverage effects," excess volatility, and time-varying risk premia. The key assumption is that *economic fundamentals*, such as the drift of dividend process, follow a process with unobservable regime shifts, which has been formalized by a two-state, continuous-time hidden Markov chain model. Investors formulate posterior probabilities on the two states, which depend on their observation of past dividends. The article shows that investors rationally anticipate that during periods of high uncertainty their expectations of future cash flows tend to react more swiftly to news. This predictable higher sensitivity to news tends to increase the asset price volatility, against which risk-averse investors are willing to hedge.

The main result is that the equilibrium price function is increasing and convex in investors' posterior probability of the high state. This is due to the extra discount investors require in anticipation of the higher volatility

of returns that occur when they are more uncertain about the true state of the world. In fact, when times are good — that is, when investors assign high probability to the good state — a bad piece of news makes investors increase the discount over expected future dividends in order to bear the risk of higher uncertainty. As a consequence of this “hedging behavior,” the price reduction due to a bad piece of news in good times is greater than the reduction in expected future dividends. Similarly, a good piece of news in bad times tends to increase the expected future dividends, but it also increases the discount investors require to hold the asset. Hence the increase in the price is lower than the increase in expected future dividends. Overall, the price function is increasing and convex in the posterior probability of the good state. I also show by numerical examples that this nonlinearity increases with the investors’ degree of risk aversion, thereby giving a precise role of risk aversion in determining asset price volatility. In addition, the characterization of the price function has several other implications: First, the reaction of prices to news tends to be high in good times and low in bad times. Second, the volatility of percentage returns tends to be higher in bad times than in good times and it is maximized during periods of highest uncertainty. Third, expected returns change over time as the investors’ level of uncertainty changes, as does return volatility.

## Appendix A

*Proof of Lemma 4.* We want to show that

$$E_t \left[ \int_0^\infty e^{-rs} D(t+s) ds \right] = \frac{D(t)}{r} + \frac{\theta^s}{r^2} - \frac{\bar{\theta} - \underline{\theta}}{r(\lambda + \mu + r)} (\pi^s - \pi(t)) \quad (23)$$

$$= \frac{D(t)}{r} + \frac{\underline{\theta}}{r^2} + \frac{(\bar{\theta} - \underline{\theta}) \mu}{r^2(\lambda + \mu + r)} + \frac{\bar{\theta} - \underline{\theta}}{r(\lambda + \mu + r)} \pi(t), \quad (24)$$

where  $\theta^s = \frac{\mu}{\lambda + \mu} \bar{\theta} + \frac{\lambda}{\lambda + \mu} \underline{\theta}$  is the unconditional expectation of  $\theta$ .

We first determine  $E_t[\theta(t+u)]$  for  $u > 0$ , where  $\theta(t)$  follows the continuous-time Markov chain in Equation (2). The infinitesimal matrix [see Karlin and Taylor (1975, p. 150)] associated with these probabilities is

$$\mathbf{M} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

The eigenvalues of  $\mathbf{M}$  are 0 and  $(\lambda + \mu)$  and let  $\mathbf{U}$  be the matrix whose columns are their associated eigenvectors. Then, if we define

$$\Lambda(s) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-(\lambda + \mu)s} \end{pmatrix},$$

by Karlin and Taylor (1975, p. 152), we can write the transition probability matrix between time  $t$  and time  $t + s$  as

$$\mathbf{P}(s) = \mathbf{U}\Lambda(s)\mathbf{U}^{-1} = \frac{1}{\lambda + \mu} \begin{pmatrix} \mu + \lambda e^{-(\lambda+\mu)s} & \lambda - \lambda e^{-(\lambda+\mu)s} \\ \mu - \mu e^{-(\lambda+\mu)s} & \lambda + \mu e^{-(\lambda+\mu)s} \end{pmatrix}.$$

Hence, if  $\Pi(t) = (\pi(t), 1 - \pi(t))$  and  $\Theta = (\bar{\theta}, \underline{\theta})'$ , we have

$$E_t[\theta(t + u)] = \Pi(t)\mathbf{P}(u)\Theta = \theta^s - (\bar{\theta} - \underline{\theta})(\pi^s - \pi(t))e^{-(\lambda+\mu)u}.$$

Therefore,

$$\begin{aligned} E_t[D(t + s)] &= D(t) + E_t \left[ \int_0^s \theta(t + u) du \right] + \sigma E_t[\xi(t + s) - \xi(t)] \\ &= D(t) + \int_0^s E_t[\theta(t + u)] du \\ &= D(t) + \int_0^s \left\{ \theta^s - (\bar{\theta} - \underline{\theta})(\pi^s - \pi(t))e^{-(\lambda+\mu)u} \right\} du \\ &= D(t) + \theta^s s + (\bar{\theta} - \underline{\theta})(\pi^s - \pi(t)) \left( \frac{e^{-(\lambda+\mu)s} - 1}{\lambda + \mu} \right). \end{aligned}$$

Hence, finally, we have

$$\begin{aligned} E_t \left[ \int_0^\infty e^{-rs} D(t + s) ds \right] &= \int_0^\infty \left\{ e^{-rs} D(t) + \theta^s s + (\bar{\theta} - \underline{\theta})(\pi^s - \pi(t)) \left( \frac{e^{-(\lambda+\mu)s} - 1}{\lambda + \mu} \right) \right\} ds \\ &= \frac{D(t)}{r} + \frac{\theta^s}{r^2} - \frac{\bar{\theta} - \underline{\theta}}{r(\lambda + \mu + r)}(\pi^s - \pi(t)), \end{aligned}$$

which shows Equation (23). Substituting for  $\theta^s$ ,  $\pi^s$ , and  $\frac{\lambda}{\lambda+\mu} = 1 - \frac{\mu}{\lambda+\mu}$  shows Equation (24). ■

*Proof of Propositions 2 and 3.* As is commonly done in the literature [see, e.g., Wang (1993)], I prove Propositions 2 and 3 by first postulating that the price function is Equation (10) and then checking that the Bellman equation

$$0 = \max_{c, X} U(c) + \frac{E_t[dJ]}{dt}, \tag{25}$$

the market clearing conditions, and the transversality condition are all satisfied by the solution proposed in Propositions 2 and 3. From the proof of Proposition 6, the dividend process can equivalently be written as  $dD = (\pi\bar{\theta} + (1 - \pi)\underline{\theta})dt + \sigma dv$ , where  $dv = \frac{1}{\sigma}(dD - E(dD | \mathcal{F}(t)))$  is the normalized expectation error introduced in Section 1. Suppose now that

$$P(D, \pi) = p_0 + S(\pi) + p_D D + p_1 + p_\pi \pi,$$

where  $p_D$ ,  $p_1$ , and  $p_\pi$  are given in Equation (24). From *Itô's lemma* we get

$$dP = p_D dD + (p_\pi + S'(\pi))d\pi + \frac{1}{2}S''(\pi)(d\pi)^2.$$

Let  $h(\pi) = \frac{\bar{\theta} - \theta}{\sigma} \pi(1 - \pi)$ , so that

$$E_t[d\pi] = (\pi^s - \pi)(\lambda + \mu)dt \quad E_t[d\pi^2] = h(\pi)^2 dt.$$

Hence, we can write

$$dP = \alpha_P(\pi)dt + \sigma_P(\pi)dv$$

where

$$\begin{aligned} \alpha_P(\pi)dt &= \left( p_D(\pi\bar{\theta} + (1 - \pi)\underline{\theta}) \right. \\ &\quad \left. + (p_\pi + S'(\pi))(\pi^s - \pi)(\lambda + \mu) + \frac{1}{2}S''(\pi)h(\pi)^2 \right) dt \\ \sigma_P(\pi) &= (p_D\sigma + (p_\pi + S'(\pi))h(\pi)). \end{aligned}$$

Let us solve the Bellman equation, given by Equation (25). We can write  $E_t[dJ]$  as follows:

$$\begin{aligned} E_t[dJ] &= J_t + J_W E_t[dW] + J_\pi E_t[d\pi] + \frac{1}{2}J_{WW} E_t[dW^2] \\ &\quad + \frac{1}{2}J_{\pi\pi} E_t[d\pi^2] + J_{W\pi} E_t[dWd\pi]. \end{aligned}$$

Hence we have

$$\begin{aligned} E_t[dW] &= [rW - c + X(D - rP)]dt + X E_t[dP] \\ &= [rW - c + X(D - rP + \alpha_P(\pi))]dt \\ E_t[dW^2] &= X^2 E_t[dP^2] = X^2 \sigma_P^2(\pi)dt \\ E_t[dWd\pi] &= X E_t[dPd\pi] = X E_t[(p_D dD + (p_\pi + S'(\pi))d\pi)d\pi] \\ &= X(p_D\sigma h(\pi) + (p_\pi + S'(\pi))h(\pi)^2) = Xh(\pi)\sigma_P(\pi)dt. \end{aligned}$$

We can now substitute in the Bellman equation to get

$$\begin{aligned} 0 &= \max_{c, X} U(c) + J_t + J_W[rW - c + X(D - rP + \alpha_P(\pi))] \\ &\quad + J_\pi(\pi^s - \pi)(\lambda + \mu) + \frac{1}{2}J_{WW}X^2\sigma_P^2(\pi) + \frac{1}{2}J_{\pi\pi}h(\pi)^2 + J_{W\pi}Xh(\pi)\sigma_P(\pi). \end{aligned}$$

The first-order conditions are

$$U'(c) = J_W \quad \text{and} \quad J_W(D - rP + \alpha_P(\pi)) + XJ_{WW}\sigma_P^2(\pi) + J_{W\pi}h(\pi)\sigma_P(\pi) = 0.$$

Solving for  $X$ ,

$$X = -\frac{J_W}{J_{WW}\sigma_P^2(\pi)}(D - rP + \alpha_P(\pi)) - \frac{J_{W\pi}}{J_{WW}\sigma_P(\pi)}h(\pi).$$

The utility function is  $U(c) = -e^{-\rho t - \gamma c}$ , and I guess a form of the value function as

$$J(W, \pi, t) = -e^{-\rho t - r\gamma W - g(\pi) - \beta},$$

where  $g(\pi)$  is some function of  $\pi$  to be determined in the final solution. We then have  $J_t = -\rho J$ ,  $J_W = -r\gamma J$ ,  $J_\pi = -g'(\pi)J$ ,  $J_{WW} = (r\gamma)^2 J$ ,  $J_{\pi\pi} = (g'(\pi)^2 - g''(\pi))J$ ,  $J_{W\pi} = r\gamma g'(\pi)J$ . By substituting in the FOC for  $c$  and  $X$  we get

$$c(W, \pi) = rW + \frac{1}{\gamma}(g(\pi) + \beta - \log(r))$$

$$X(D, P, \pi) = \frac{D - rP + \alpha_P(\pi)}{r\gamma\sigma_P^2(\pi)} - \frac{g'(\pi)}{r\gamma\sigma_P(\pi)}h(\pi).$$

Notice that  $\alpha_Q(\pi) = E_t[dQ] = D - rP + E_t[dP] = D - rP + \alpha_P(\pi)$  and  $\sigma_Q^2(\pi) = E_t[dQ^2] = E_t[dP^2] = \sigma_P^2(\pi)$ , confirming Equation (13). Now the market clearing condition requires  $X = 1$ , which implies:

$$D - rP + \alpha_P(\pi) - g'(\pi)\sigma_P(\pi)h(\pi) = r\gamma\sigma_P^2(\pi). \quad (26)$$

We know that, after substituting for  $P$ ,  $p_D$ ,  $p_1$ , and  $p_\pi$ , we have the identity

$$D - rP + \alpha_P(\pi) = -rp_0 - rS(\pi) + S'(\pi)(\pi^s - \pi)(\lambda + \mu) + \frac{1}{2}S''(\pi)h(\pi)^2. \quad (27)$$

Substituting for  $D - rP + \alpha_P(\pi)$  in Equation (26), we obtain

$$-rp_0 - rS(\pi) + S'(\pi)(\pi^s - \pi)(\lambda + \mu) + \frac{1}{2}S''(\pi)h(\pi)^2 - g'(\pi)\sigma_P(\pi)h(\pi) = r\gamma\sigma_P^2(\pi), \quad (28)$$

which is a differential equation for  $S$  [in terms of  $g(\pi)$ ].

The market clearing condition imposes  $X = 1$ , which has to be satisfied by the equilibrium equations. I therefore set  $X = 1$  in the Bellman equation and substitute for  $c$  to obtain:

$$\begin{aligned} 0 &\equiv \frac{1}{J}U(c) + \frac{1}{J} \frac{E_t[dJ]}{dt} = r + \frac{1}{J} \frac{E_t[dJ]}{dt} \\ &= r - \rho - r\gamma \left[ \frac{1}{\gamma}(\log(r) - g(\pi) - \beta) + (D - rP + \alpha_P(\pi)) \right] \\ &\quad - g'(\pi)(\pi^s - \pi)(\lambda + \mu) + \frac{(r\gamma)^2}{2}\sigma_P^2(\pi) + \frac{1}{2}(g'(\pi))^2 \\ &\quad - g''(\pi)h(\pi)^2 + r\gamma g'(\pi)h(\pi)\sigma_P(\pi). \end{aligned} \quad (29)$$

By using Equation (27) again, we can substitute for  $D - rP + \alpha_P(\pi)$  and  $\sigma_P(\pi)$  into the Bellman equation [Equation (29)] and then for  $p_0 = -\frac{\gamma\sigma^2}{r}$ ,  $\beta = \frac{\gamma^2\sigma^2}{2r} + \frac{\rho}{r} + \log(r) - 1$ , and  $g(\pi) = -r\gamma S(\pi) + f(\pi)$  for some function  $f(\pi)$ . After straightforward although tedious algebraic manipulation, one obtains a differential equation in  $f(\pi)$ :

$$-f''(\pi)Q_3(\pi) + f'(\pi)^2Q_3(\pi) + f'(\pi)Q_2(\pi) + f(\pi)r + Q_0(\pi) = 0, \quad (30)$$

where  $Q_3 = \frac{h(\pi)^2}{2}$ ,  $Q_2 = h(\pi)\sigma\gamma - (\pi^s - \pi)(\lambda + \mu) + \gamma r p_\pi h(\pi)^2$ , and  $Q_0 = \frac{(r\gamma)^2}{2} p_\pi^2 h(\pi)^2 + r\gamma^2 \sigma p_\pi h(\pi)$ .

Appendix B shows that there is a solution  $f(\pi)$  to this differential equation that is bounded on  $(0, 1)$ , with bounded first derivative. Moreover,  $f(\pi)$  is negative, convex, and U-shaped.

Substituting this solution into Equation (28) and rearranging,

$$S''(\pi)P_3(\pi) = S'(\pi)P_2(\pi) + rS(\pi) + P_0(\pi), \quad (31)$$

where  $P_3(\pi) = \frac{h(\pi)^2}{2}$ , and

$$P_2(\pi) = \gamma\sigma h(\pi) - (\pi^s - \pi)(\lambda + \mu) + \gamma r p_\pi h(\pi)^2 + f'(\pi)h(\pi)^2$$

$$P_0(\pi) = \gamma r p_\pi^2 h(\pi)^2 + 2\gamma\sigma p_\pi h(\pi) + f'(\pi)\frac{\sigma}{r}h(\pi) + f'(\pi)p_\pi h(\pi)^2.$$

Appendix B shows that, given  $f(\pi)$ , there is a solution  $S(\pi)$  to this differential equation that is bounded on  $(0, 1)$ , with bounded first derivative. Moreover,  $S(\pi)$  is negative, convex, and U-shaped. The function  $g(\pi)$  is obtained from the definition of  $f(\pi)$ :

$$g(\pi) = f(\pi) - r\gamma S(\pi). \quad (32)$$

Finally, the transversality condition

$$\lim_{\tau \rightarrow \infty} E[J(W(t + \tau), \pi(t + \tau), t + \tau) | \mathcal{F}(t)] = 0 \quad (33)$$

holds. In fact, from Equation (29) we can see that

$$\frac{1}{J}E_t[dJ] = -rdt,$$

so that we can write

$$\frac{dJ}{J} = -rdt + \sigma_J(\pi)dv,$$

with  $\sigma_J(\pi) = -(\gamma\sigma + (f'(\pi) + \gamma r p_\pi)h(\pi))$ . Appendix B shows that  $f'(\pi)$  is bounded and so is  $\sigma_J(\pi)$ . Hence, Equation (33) immediately follows.

Notice that we can recover the (standard) result of Proposition 1 by letting  $\bar{\theta} \rightarrow \underline{\theta}$ . By inspection, we see that  $h(\pi) \rightarrow 0$  and hence  $\pi \rightarrow \pi^s$ . Therefore, all coefficients  $P_0$ ,  $P_2$ ,  $P_3$ ,  $Q_0$ ,  $Q_2$ , and  $Q_3$  vanish, implying that  $S(\pi) = f(\pi) = g(\pi) = 0$ . The results of proposition 1 follow.

This completes the proof. ■

## Appendix B

In this appendix I show that the differential equations [Equations (12) and (11)] have solutions  $f(\pi)$  and  $S(\pi)$  on  $(0,1)$  which are bounded with bounded first derivative. We need to be careful in studying these differential equations because they have singular points at 0 and 1. As it is well known, singular points may imply the nonexistence of a solution on them. Even though this would be a minor concern, because the relevant domain for  $\pi$  is  $(0,1)$  because both 0 and 1 are entrance boundaries for the process  $\pi$ , still it is important to show that  $f(\pi)$  and  $S(\pi)$  are bounded on this interval.

We will make use of the following standard results: Let a system of the first-order differential equation be denoted by

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}, t), \tag{34}$$

where  $\mathbf{X}(\mathbf{x}, t)$  is defined on some open region  $\mathcal{R}$ .

**Theorem B1** [Theorem 11, Chap. 6, Birkoff and Rota (1989)]. *Let  $\mathbf{X}(\mathbf{x}, t)$  be defined and of class  $C^1$  in an open region  $\mathcal{R}$  of the  $(\mathbf{x}, t)$ -space. For any point  $(\mathbf{c}, a) \in \mathcal{R}$ , the DE [Equation (34)] has a unique solution  $\mathbf{x}(t)$  satisfying the initial condition  $\mathbf{x}(a) = \mathbf{c}$  and defined for an interval  $a \leq t < b$  with  $b \leq \infty$  such that, if  $b < \infty$ , either  $\mathbf{x}(t)$  approaches the boundary of the region, or  $\mathbf{x}(t)$  is unbounded as  $t \rightarrow b$ .*

**Theorem B2** [Theorem 12, Chap. 6, Birkoff and Rota (1989)]. *Let the vector function  $\mathbf{X}$  be of class  $C^1$ , and let  $\mathbf{x}(t, \mathbf{c})$  be the solution to the DE [Equation (34)], taking the initial value  $\mathbf{c}$  at  $t = a$ . Then  $\mathbf{x}(t, \mathbf{c})$  is a continuously differentiable function of each of the components  $c_j$  of  $\mathbf{c}$ .*

**Lemma B1.** *There exists a solution  $f(\pi)$  on  $(0, 1)$  to Equation (12) that is bounded with bounded first derivative, convex, and U-shaped.*

*Proof.* Let us rewrite Equation (12) as a system of first-order differential equations. Let  $x_1(\pi) = f(\pi)$  and  $x_2(\pi) = f'(\pi)$  so that we have

$$x_1' = x_2 \tag{35}$$

$$x_2' = x_2^2 + \frac{Q_2}{Q_3}x_2 + \frac{r}{Q_3}x_1 + \frac{Q_0}{Q_3}, \tag{36}$$

where again  $Q_3 = \frac{h(\pi)^2}{2}$ ,  $Q_2 = h(\pi)\sigma\gamma - (\pi^s - \pi)(\lambda + \mu) + \gamma r p_\pi h(\pi)^2$ , and  $Q_0 = \frac{(r\gamma)^2}{2} p_\pi^2 h(\pi)^2 + r\gamma^2 \sigma p_\pi h(\pi)$ .

Notice that the coefficients in Equation (36) depend on the underlying variable  $\pi$ . We now study the phase portrait of this system of differential equations. Let  $x_1$  be on the vertical axis and  $x_2$  on the horizontal axis (see Figure 10). The locus  $x_1' = 0$  is the vertical axis, so that to the right of the vertical axis the system moves north and to its left it moves south. Consider now the locus  $x_2' = 0$ . From Equation (36), for a given  $\pi$ ,  $x_2' = 0$  if and only if

$$x_1 = -\frac{Q_3}{r}x_2^2 - \frac{Q_2}{r}x_2 - \frac{Q_0}{r} \equiv \Psi(\pi, x_2), \tag{37}$$

which is a parabola.

Notice that as  $\pi \rightarrow 0$ , Equation (37) tends to the straight line  $x_1 = \frac{\mu}{r}x_2$ . Symmetrically, as  $\pi \rightarrow 1$ , Equation (37) tends to the straight line  $x_1 = -\frac{\lambda}{r}x_2$ . These lines are plotted in Figure 10 along with the parabola obtained for  $\pi = 0.5$ ,  $x_1 = \Psi(0.5, x_2)$ , and the parabola obtained for  $\pi = \varepsilon$ ,  $x_1 = \Psi(\varepsilon, x_2)$ . As  $\pi$  ranges from  $\varepsilon$  to 1, the locus  $x_2' = 0$  moves from line I to line II to line III in the graph.

I now show that there is a point  $(\hat{x}_2, \hat{x}_1)$  that lies on the line  $x_1 = x_2 \frac{\mu}{r}$  (i.e., the line which is the limit as  $\pi \rightarrow 0$  of the parabolas [Equation (37)] that does not move when  $\pi$  changes from  $\pi = 0$  to  $\pi = d\pi$ ).

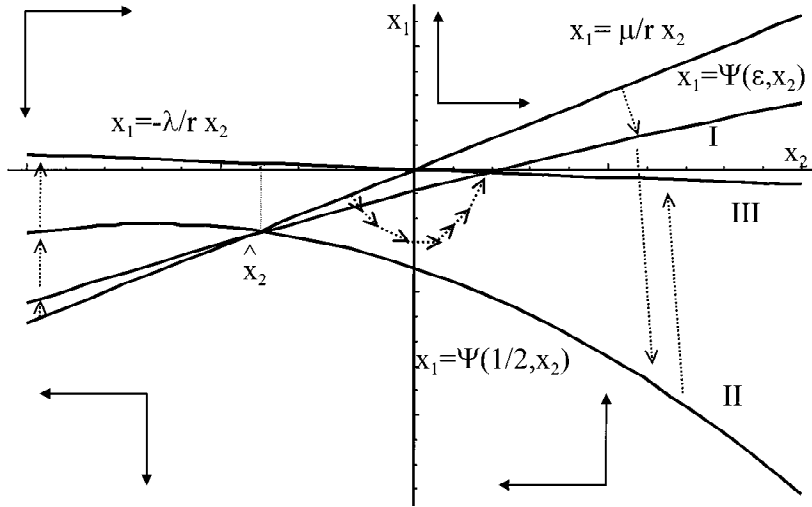


Figure 10  
Phase diagram of Lemma B1

Consider

$$\frac{dx_1}{d\pi} \Big|_{\pi=0} = -\frac{1}{r} \left( \left( \frac{dQ_3}{d\pi} \Big|_{\pi=0} \right) x_2^2 + \left( \frac{dQ_2}{d\pi} \Big|_{\pi=0} \right) x_2 + \frac{dQ_0}{d\pi} \Big|_{\pi=0} \right).$$

Since  $\frac{dQ_3}{d\pi} \Big|_{\pi=0} = 0$ ,  $\frac{dQ_2}{d\pi} \Big|_{\pi=0} = \gamma(\bar{\theta} - \underline{\theta}) + \lambda + \mu$  and  $\frac{dQ_0}{d\pi} \Big|_{\pi=0} = \gamma^2 r p_\pi (\bar{\theta} - \underline{\theta})$ , we have

$$\frac{dx_1}{d\pi} \Big|_{\pi=0} = -\frac{1}{r} \left( (\bar{\theta} - \underline{\theta}) \gamma + \lambda + \mu \right) x_2 + \gamma^2 r p_\pi (\bar{\theta} - \underline{\theta}).$$

We find  $\hat{x}_2$  which makes this derivative zero. This is given by

$$\hat{x}_2 = -\frac{\gamma^2 r p_\pi (\bar{\theta} - \underline{\theta})}{\gamma (\bar{\theta} - \underline{\theta}) + \lambda + \mu} < 0. \quad (38)$$

Hence

$$\frac{dx_1}{d\pi} \Big|_{\pi=0} \begin{cases} > 0 & \text{if } x_2 < \hat{x}_2 \\ = 0 & \text{if } x_2 = \hat{x}_2 \\ < 0 & \text{if } x_2 > \hat{x}_2. \end{cases}$$

This implies that to the left of  $\hat{x}_2$  the arm of the parabola moves upward and to the right of  $\hat{x}_2$  it moves downward.

I now show the existence of a converging solution as follows. We first show that for every  $\varepsilon > 0$ , there is a solution to the system [Equations (35) and (36)] on the interval  $[\varepsilon, 1)$  which is convergent to a finite number. Let  $x_2(\varepsilon) = x_2^\varepsilon$  be the initial condition for  $x_2$  and let  $x_1(\varepsilon) = x_2^\varepsilon \frac{\mu}{r}$  be the initial condition for  $x_1$ . These initial conditions lie on the line  $x_1 = x_2 \frac{\mu}{r}$ , therefore for every  $\varepsilon > 0$  (small enough) the point is above the parabola [Equation (37)] if  $x_2^\varepsilon > \hat{x}_2$  and below the parabola if  $x_2^\varepsilon < \hat{x}_2$ . If the system starts below

the parabola, by the phase diagram in Figure 10 it will start moving southwest, diverging to  $(-\infty, -\infty)$ . On the opposite side, if the system starts above the parabola, but with  $x_2^\varepsilon > 0$ , the system will move northeast, diverging to  $(\infty, \infty)$ . Hence we found some starting values for  $x_2^\varepsilon$  such that by Theorem B1 the system will diverge to  $-\infty$  and some other values such that the system will diverge to  $+\infty$ . Since by Theorem B2, the solution to a differential equation must be continuous in its initial conditions, there must exist an initial condition  $x_2^{\varepsilon,*} \in [\hat{x}_2, 0]$  such that the system converges to a finite value as  $\pi \rightarrow 1$ . Moreover, we see from the phase diagram that the converging path must be decreasing to start with and increasing after intersecting the vertical axis.

Finally, notice that for  $\varepsilon > 0$ , we have found a value  $x_2^{\varepsilon,*} \in [\hat{x}_2, 0]$  with a converging path. Choosing a sequence of  $\varepsilon_i$ 's going to zero, we have that  $x_2^{\varepsilon_i,*}$  is a bounded sequence in  $[\hat{x}_2, 0]$ . Since every bounded sequence must have a convergent subsequence, we have that  $x_2^{\varepsilon_i,*}$  has a convergent subsequence as  $\varepsilon_i \rightarrow 0$ , the limit  $x_2^*(0)$  of which gives the initial condition in the interval  $[\hat{x}_2, 0]$ . This construction implies the assert of this lemma. ■

**Lemma B2.** *There exists a solution  $S(\pi)$  on  $(0,1)$  to Equation (11) which is bounded with bounded first derivative, convex, and U-shaped. Moreover,  $P(D, \pi) \in [P(D; \underline{\theta}), P(D; \bar{\theta})]$  for all  $\pi \in [0, 1]$ , where  $P(D; \theta)$  is defined in Proposition 1.*

*Proof.* The proof is analogous to the one of Lemma B1. Let  $y_1(\pi) = S(\pi)$  and  $y_2(\pi) = S'(\pi)$ , so that we can write

$$y_1' = y_2 \tag{39}$$

$$y_2' = \frac{P_2}{P_3}y_2 + \frac{r}{P_3}y_1 + \frac{P_0}{P_3}, \tag{40}$$

where  $P_3(\pi) = \frac{h(\pi)^2}{2}$ ,  $P_2(\pi) = \gamma\sigma h(\pi) - (\pi^s - \pi)(\lambda + \mu) + \gamma r p_\pi h(\pi)^2 + f'(\pi)h(\pi)^2$ , and  $P_0(\pi) = \gamma r p_\pi^2 h(\pi)^2 + 2\gamma\sigma p_\pi h(\pi) + f'(\pi)\frac{\sigma}{r}h(\pi) + f'(\pi)p_\pi h(\pi)^2$ .

As in Lemma B1, the locus  $y_1' = 0$  is the vertical axis, so that to the right of the vertical axis the system moves north and to its left it moves south. The locus of points where  $y_2' = 0$  is given by the straight line  $y_1 = -\frac{P_2}{r}y_2 - \frac{P_0}{r}$ . Again, as  $\pi \rightarrow 0$ , the locus  $y_2' = 0$  tends to  $y_1 = \frac{\mu}{r}y_2$ , while for  $\pi \rightarrow 1$  it tends to  $y_1 = -\frac{\lambda}{r}y_2$ . The phase diagram is in Figure 11. Notice that  $P_0(\pi) > 0$  for all  $\pi$ : in fact, we can write

$$P_0(\pi) = (\gamma r p_\pi^2 + f'(\pi)p_\pi)h(\pi)^2 + \left(2\gamma\sigma p_\pi + f'(\pi)\frac{\sigma}{r}\right)h(\pi).$$

We know that  $h(\pi) > 0$  for all  $\pi$  and from Lemma B1 and Equation (38) that  $f'(0) = x_2^*(0) > \hat{x}_2 > -\gamma r p_\pi > -2\gamma r p_\pi$ . Since  $f'(\pi)$  is increasing, this implies that  $f'(\pi) > \gamma r p_\pi > -2\gamma r p_\pi$  for all  $\pi$  and hence  $P_0(\pi) > 0$ . Therefore the  $y_2' = 0$  locus crosses the vertical axis at a negative point.

As in Lemma B1, we can find a point on  $y_1 = \frac{\mu}{r}y_2$  which does not move when we move from  $\pi = 0$  to  $\pi = \varepsilon$ . Let us compute

$$\begin{aligned} \frac{dy_1}{d\pi} \Big|_{\pi=0} &= -\frac{1}{r} \left[ \left( \frac{dP_2}{d\pi} \Big|_{\pi=0} \right) y_2 + \frac{dP_0}{d\pi} \Big|_{\pi=0} \right] \\ &= -\frac{1}{r} \left[ (\gamma(\bar{\theta} - \underline{\theta}) + \lambda + \mu)y_2 + (\bar{\theta} - \underline{\theta}) \left( 2\gamma p_\pi + x_2^*(0)\frac{1}{r} \right) \right]. \end{aligned}$$

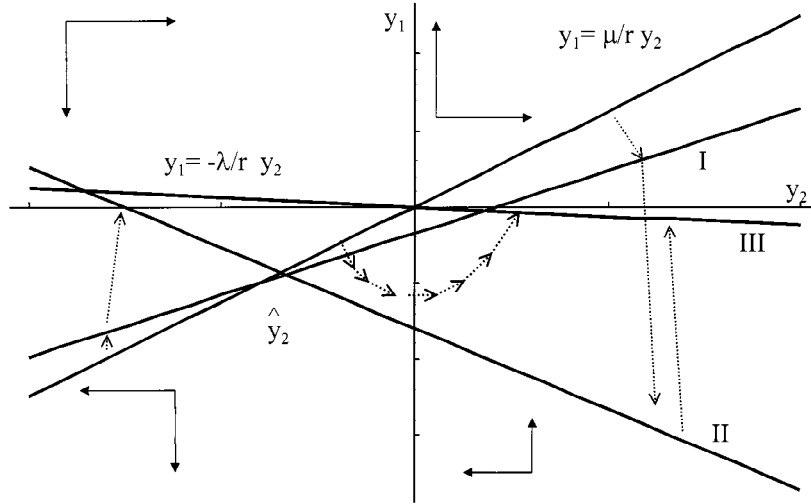


Figure 11  
Phase diagram of Lemma B2

Let

$$\hat{y}_2 = -(\bar{\theta} - \underline{\theta}) \frac{2\gamma p_\pi}{\gamma(\bar{\theta} - \underline{\theta}) + \lambda + \mu} - \frac{x_2(0)^*}{r(\gamma(\bar{\theta} - \underline{\theta}) + \lambda + \mu)}$$

so that

$$\frac{dy_1}{d\pi} \Big|_{\pi=0} \begin{cases} > 0 & \text{if } x_2 < \hat{y}_2 \\ = 0 & \text{if } x_2 = \hat{y}_2 \\ < 0 & \text{if } x_2 > \hat{y}_2. \end{cases}$$

This implies that to the left of  $\hat{y}_2$ , the  $y'_2 = 0$  locus moves upward and to the right of  $\hat{y}_2$  it moves downward. Now, the same argument as in Lemma B1 shows the existence of a solution for  $S(\pi)$  which is bounded, with bounded first derivative. From the phase diagram in Figure 11, we can also see that  $S(\pi)$  is negative, convex, and U-shaped. ■

### Appendix C

*Proof of Corollary 2.* From Equation (15), Equation (3),  $\lambda = \mu$ , and Itô's lemma we obtain

$$d\sigma_{P^*} = p_\pi \frac{(\bar{\theta} - \underline{\theta})}{\sigma} \left[ 4\lambda \left( \frac{1}{2} - \pi \right)^2 - h(\pi)^2 \right] dt + 2p_\pi \frac{\bar{\theta} - \underline{\theta}}{\sigma} \left( \frac{1}{2} - \pi \right) h(\pi) dv.$$

From Equation (15) again, we see that  $\left( \frac{1}{2} - \pi \right) = \pm \frac{1}{2} \sqrt{1 - \frac{4\sigma}{p_\pi \Delta \theta} \left( \sigma_{P^*} - \frac{\sigma}{r} \right)}$ . By substituting for  $\left( \frac{1}{2} - \pi \right)$ ,  $h(\pi)$  we obtain the result. ■

*Proof of Proposition 5.* In Appendix A, we defined  $E_t[dQ] = (D - rP + \alpha_P(\pi))dt$ . By using Equation (26) and the definition of  $g(\pi)$  one obtains

$$\begin{aligned} E_t[dQ]/dt &= (D - rP + \alpha_P(\pi)) \\ &= f'(\pi)\sigma_P(\pi)h(\pi) - r\gamma S'(\pi)\sigma_P(\pi)h(\pi) + \gamma r\sigma_P(\pi)^2 \\ &= \sigma_P(\pi)[f'(\pi)h(\pi) - r\gamma S'(\pi)h(\pi) + \gamma r\sigma_P(\pi)]. \end{aligned}$$

Substituting for  $\sigma_P(\pi) = \frac{\sigma}{r} + (p_\pi + S'(\pi))h(\pi)$  in the brackets yields the result. ■

### References

- Birkoff, G., and G. C. Rota, 1989, *Ordinary Differential Equations*, Wiley, New York.
- Bittlingmayer, G., 1998, "Output, Stock Volatility and Political Uncertainty in a Natural Experiment: Germany, 1880–1940," *Journal of Finance*, 53, 6, 2243–2257.
- Black, F., 1976, "Studies of Stock, Price Volatility Changes," Proceedings of the 1976 meetings of the Business and Economics Statistics Section, American Statistical Association.
- Bollerslev, T., R. Chou, and K. Kroner, 1992, "ARCH Modelling in Finance: A Review of the Theory and Empirical Evidence," *Journal of Econometrics*, 52, 5–59.
- Brennan, M. J., and Y. Xia, 1998, "Stock Price Volatility, Learning and the Equity Premium Puzzle," working paper, UCLA.
- Campbell, J. Y., 1997, "Asset Prices, Consumption and the Business Cycle," forthcoming in J. B. Taylor and M. Woodford (eds.), *The Handbook of Macroeconomics*, North Holland, Amsterdam.
- Campbell, J. Y., and J. H. Cochrane, 1999, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107, 205–251.
- Campbell, J. Y., and L. Hentschel, 1992, "No News is Good News: An Asymmetric Model of Changing Volatility in Stock Returns," *Journal of Financial Economics*, 31, 281–318.
- Campbell, J. Y., and A. S. Kyle, 1993, "Smart Money, Noise Trading and Stock Price Behavior," *Review of Economic Studies*, 60, 1–34.
- Cox, J., J. Ingersoll, and S. Ross, 1985, "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53.
- David, A., 1997, "Fluctuating Confidence in Stock Markets: Implications for Returns and Volatility," *Journal of Financial and Quantitative Analysis*, 32, 427–462.
- Detemple, J. B., 1986 "Asset Pricing in a Production Economy with Incomplete Information," *Journal of Finance*, 41.
- Detemple, J. B., 1991, "Further Results on Asset Pricing with Incomplete Information," *Journal of Economic Dynamics and Control*, 15.
- French, K. R., W. G. Schwert, and R. F. Stambaugh, 1987, "Expected Stock Returns and Volatility," *Journal of Financial Economics*, 19, 3–29.
- Genotte, G., 1986, "Optimal Portfolio Choice Under Incomplete Information," *Journal of Finance* 41.
- Glosten, L. R., R. Jagannathan, and D. E. Runkle, 1993, "On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks," *Journal of Finance*, 58, 1779–1801.
- Grundy, B. D., and Y. Kim, 1995, "Stock Market Volatility in a Heterogeneous Information Economy," working paper, University of Pennsylvania.

*Stock Market Overreaction to Bad News in Good Times*

- Hamilton, J. D., 1989, "A New Approach to the Analysis of Nonstationary Returns and the Business Cycle," *Econometrica*, 57, 357–384.
- Karlin, S., and H. M. Taylor, 1975, *A First Course in Stochastic Processes*, Academic Press, New York.
- Liptser, R. S., A. N. Shiryaev, 1977, *Statistics of Random Processes: I, II*, Springer-Verlag, New York.
- Merton, R. C., 1973, "An Intertemporal Asset Pricing Model," *Econometrica*, 41, 867–888.
- Merton, R. C., 1980, "On Estimating the Expected Return on the Market: an Exploratory Investigation," *Journal of Financial Economics*, 8, 326–361.
- Schwert, G. W., 1989, "Why Does Stock Market Volatility Change over Time?," *Journal of Finance*, 44, 1115–1153.
- Shiller, R. J., 1984, "Stock Price and Social Dynamics," *Brookings Papers in Economic Activity*, 2, 457–498.
- Veronesi, P., 1997, "Stock Market Volatility in Regime Shift Models," PhD. dissertation, Harvard University.
- Veronesi, P., 1999, "How Does Information Quality Affect Stock Returns? Results from a Dynamic Model of Learning," CRSP Working Paper 462, University of Chicago; forthcoming in *Journal of Finance*.
- Wang, J., 1993, "A Model of Intertemporal Asset Prices under Asymmetric Information," *Review of Economic Studies*, 60, 249–282.