Modern Dynamic Asset Pricing Models

Lecture Notes 1.

Dynamic Portfolio Allocation Strategies

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1. Review of Merton / Samuelson Portfolio Allocation Problem
   • The Puzzles

2. Strategic Asset Allocation under Predictability of Stock Returns
   • The Problem and its solution
   • Implications for Dynamic Asset Allocation

3. Learning about Average Returns
   • Implications for Dynamic Asset Allocation
   • Comparison with the case of Predictability

4. Strategic Asset Allocation with Model Misspecification
   • The Problem and Its solution
   • The Example of Constant Investment Opportunity Set

5. Conclusion
Review of Merton/Samuelson Portfolio Allocation Problem

- There are $n$ stocks. Stock $i$ return

  \[ dR^i_t = \frac{dS^i_t + D^i_t dt}{S^i_t} \]

- $d\mathbf{R}_t = (dR^1_t, ..., dR^n_t)'$

- Assume:

  \[ d\mathbf{R}_t = \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\mathbf{B}_t \]

- $d\mathbf{B}_t = (dB^1_t, ..., dB^n_t) = \text{vector of independent Brownian motions.}$

- Investor problem:

  \[ J(W_0, 0) = \max_{\{(C_t), (\theta_t)\}} E_0 \left[ \int_0^T u(C_t, t) dt \right] \]

- subject to

  \[ dW_t = \{W_t (\theta'_t(\boldsymbol{\mu} - r \mathbf{1}_n) + r) - C_t\} dt + W_t \theta'_t \sigma d\mathbf{B}_t \]
The Bellman Equation

- Bellman Equation:
  \[ 0 = \sup_{(C_t, \theta)} u(C_t, t) + E\left[dJ(W, t)\right]/dt \]

- with boundary condition \( J(W_T, T) = 0 \)

- Why this form?
  - The discrete time Bellman equation over a small \( \Delta \)
    \[ J(W_{t+\Delta}, t + \Delta) = \max_{C, \theta} \{ u(C, t) \Delta + E[J(W_{t+\Delta}, t + \Delta) | W_t]\} \]
    \[ \implies 0 = \max_{C, \theta} u(c, t) \Delta + E_t [J(W_{t+\Delta}, t + \Delta) - J(W_t, t)] \]

- Note that by Itô's Lemma:
  \[ E[dJ(W, t)]/dt = J_t + J_W E_t[dW]/dt + \frac{1}{2} J_{WW} E_t[dW^2]/dt \]
  \[ = J_t + J_W \{ W_t (\theta'_t(\mu - r) + r) - C_t \} + \frac{1}{2} J_{WW} W_t^2 \theta'_t \sigma \sigma' \theta_t \]
The Optimal Consumption and Portfolio Allocation

• FOC with respect to $C$:

$$u_c (C_t, t) = J_W (W, t)$$

  - Example: Power utility

$$u (C_t, t) = e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} \implies C_t = e^{-\frac{\rho t}{\gamma}} J_W (W, t)^{-\frac{1}{\gamma}}$$

• FOC with respect to $\theta_t$:

$$\theta_t = \frac{1}{RRA(W)} \left( \sigma \sigma' \right)^{-1} (\mu - r1_n)$$

  - where

$$RRA(W) = -\frac{W J_{WW} (W, t)}{J_W (W, t)}$$

• We now solve for $J(W, t)$ in the power utility case.
The Explicit Solution via an Ordinary Differential Equation

1. Conjecture:

\[ J(W, t) = e^{-\rho t} \frac{W^{1-\gamma}}{1-\gamma} F(t)^\gamma \]

2. Compute \( J_t, J_W \) and \( J_{WW} \);

3. Optimal consumption and portfolio holdings:

\[ C_t = WF(t)^{-1}; \quad \text{and} \quad \theta_t = \frac{1}{\gamma} (\sigma \sigma')^{-1} (\mu - r1) \]

4. To find \( F(t) \), substitute \( J_t, J_W \) and \( J_{WW} \) and optimal strategies in Bellman equation

\[
0 = e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} - \rho J + J\gamma \frac{F_t}{F} + W^{-1} (1-\gamma) J(W_t (\theta_t' (\mu - r1_n) + r) - C_t) \\
- \frac{1}{2} \gamma (1-\gamma) W^{-2} JW_t^2 \theta_t' \sigma \sigma' \theta_t
\]
The Explicit Solution via a Ordinary Differential Equation

5. Simplify all that can be simplified, to find the ODE

\[ 0 = 1 - aF(t) + F_t \]

where \( F(T) = 0 \) and

\[ a = \frac{1}{\gamma} \left\{ \rho - (1 - \gamma) r - \frac{1 - \gamma}{2\gamma} (\mu - r1_n)' (\sigma \sigma')^{-1} (\mu - r1_n) \right\} \]

6. The solution is

\[ F(t) = \frac{1}{a} \left( 1 - e^{-a(T-t)} \right) \]

- As \( t \to T \), consume a higher fraction of wealth.

7. The last point is to verify that “Conjecture” is indeed optimal.
The Puzzles

• For $n = 1$

$$\theta_t = (\mu - r) / (\gamma \sigma^2)$$

1. $\theta_t$ is independent of age $t$, and thus of remaining life $T - t$.
   - Against empirical evidence: an inverted U shaped $\theta_t$
   - Against the typical recommendation of portfolio advisors.

2. Too large $\theta$. Using $\mu - r = 7\%$ and $\sigma = 16\%$

<table>
<thead>
<tr>
<th>Risk Aversion $\gamma$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>136%</td>
<td>68%</td>
<td>45%</td>
<td>34%</td>
<td>27%</td>
</tr>
</tbody>
</table>

- Typical household holds between 6 % to 20 % in equity.
- Conditional on participation, $\approx 40\%$ of financial assets.
Strategic Asset Allocation with Time Varying Expected Returns

- $n$ stocks: 
  $$dR_t = \mu_t dt + \sigma dB_t$$

  $- \mu_t = E_t[dR_t]$ is now time varying.

- For convenience (later), denote the expected excess return
  $$\lambda_t = \mu_t - r1_n$$

- Assume a VAR process
  $$d\lambda_t = (A_0 + A_1 \lambda_t) dt + \Sigma dB_t$$

- Note:
  - Assume $dB_t$ is now $n \times m$.
  - E.g. $n = 1$ (1 stock), $m = 2$ (two shocks) with
    $$\sigma = (\sigma_1, 0) \quad \Sigma = (\Sigma_1, \Sigma_2) \quad \Rightarrow \text{Cov} \ (dR, d\lambda) = \sigma \Sigma' = \sigma_1 \Sigma_1$$
The Bellman Equation with Time Varying Expected Returns

- **Investor problem:**
  \[
  J (W_0, \lambda_0, 0) = \max_{\{ (C_t), (\theta_t) \}} \mathbb{E}_0 \left[ \int_0^T u (C_t, t) \, dt \right]
  \]

- subject to
  \[
  dW_t = \left\{ W_t (\theta_t' \lambda + r) - C_t \right\} dt + W_t \theta_t' \sigma dB_t
  \]

- The Bellman equation is
  \[
  0 = \max_{C_t, \theta_t} \frac{e^{-\rho t} C_t^{1-\gamma}}{1-\gamma} + \mathbb{E}_t [dJ_t] / dt
  \]

- with
  \[
  \mathbb{E}_t [dJ_t] / dt = J_t + J_W \mathbb{E}_t [dW_t] + \frac{1}{2} J_{WW} \mathbb{E}_t [dW_t^2]
  \]
  \[
  + J'_{\lambda} \mathbb{E}_t [d\lambda_t] + J_{W\lambda} \mathbb{E}_t [d\lambda_t dW_t] + \frac{1}{2} tr \left( J_{\lambda\lambda} \mathbb{E}_t [d\lambda_d d\lambda_t'] \right)
  \]
Optimal Consumption and Portfolio Allocation

- Substitute expectations in Bellman equation:

\[
0 = \max_{C_t, \theta_t} e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} + J_t + J_W (W_t (\theta_t' \lambda_t + r) - C_t) + \frac{1}{2} J_{WW} W_t^2 \theta_t' \sigma \sigma' \theta_t \\
+ J'_\lambda (A_0 + A_1 \lambda_t) + J_{W\lambda} W_t \Sigma \sigma' \theta_t + \frac{1}{2} tr (J_{\lambda\lambda} \Sigma \Sigma')
\]

- FOC with respect to \( C_t \):

\[
C_t = e^{-\frac{\rho t}{\gamma}} J_W^{-\frac{1}{\gamma}}
\]

  - Same form as before.
  - But recall that \( J_W \) is not different.

- FOC with respect to \( \theta_t \):

\[
\theta_t = \frac{1}{RRA(W_t)} (\sigma \sigma')^{-1} \lambda_t - (\sigma \sigma')^{-1} \sigma \Sigma' \frac{J_{W\lambda}}{J_{WW} W}
\]

  - There is one additional term.
Optimal Portfolio Allocation

- Optimal Portfolio Allocation:
  \[ \theta_t = \theta_t^M + \theta_t^H \]

- Myopic Demand
  \[ \theta_t^M = \frac{1}{RRA(W_t)} (\sigma \sigma')^{-1} \lambda_t \]
  - Same as before.

- Hedging Demand
  \[ \theta_t^H = - (\sigma \sigma')^{-1} \sigma \sum' \frac{J_{W\lambda}}{J_{WW\lambda} W} \]
  - Recall that expected returns \( \lambda_t \) also (obviously) affect intertemporal utility.
  - \( \Longrightarrow \) The asset allocation must “hedge” against the negative impact that the variation in expected returns has on the marginal utility.

- If \( \theta_t^H \) depends on age \((t)\) and is negative, we may “resolve” the two puzzles.
Optimal Portfolio Allocation under Power Utility

- Solving this problem is substantially more complicated.
- Conjecture 1:
  \[ J(W_t, \lambda_t, t) = e^{-\rho t} W_t^{1-\gamma} \frac{1}{1-\gamma} F(\lambda_t, t)^\gamma \]
- Compute \( J_t, J_W, J_{WW}, J_{W\lambda}, J_{\lambda} \) and \( J_{\lambda\lambda} \).
- This yields
  \[ C_t = W_t F^{-1} \quad \text{and} \quad \theta_t = \frac{1}{\gamma} (\sigma \sigma')^{-1} \lambda_t + (\sigma \sigma')^{-1} \sigma \Sigma' \frac{F_{\lambda}}{F} \]
- To solve for \( F(\lambda, t) \), substitute everything into the Bellman equation.
The Bellman Equation and its Solution

\[ 0 = F^{-1} + \left( (1 - \gamma) r - \rho \right) \frac{1}{\gamma} + \frac{F_t}{F} + \frac{1}{2} tr \left( \frac{F \lambda}{F} \Sigma \Sigma' \right) + \frac{(1 - \gamma)}{2\gamma^2} \lambda_t' (\sigma \sigma')^{-1} \lambda_t + \]
\[ \frac{(1 - \gamma)}{\gamma} \frac{F'}{F} \Sigma \sigma' (\sigma \sigma')^{-1} \lambda_t + \frac{F'}{F} (A_0 + A_1 \lambda_t) + \]
\[ \frac{1}{2} (1 - \gamma) \, tr \left( \left( \frac{F \lambda}{F} \frac{F'}{F} \right) \left( \Sigma \sigma' (\sigma \sigma')^{-1} \sigma \Sigma' - \Sigma \Sigma' \right) \right) \]

- This is horrible. There is:
  - A quadratic term in \( \lambda_t \);
  - A linear term in \( \lambda_t \);
  - A quadratic term in \( F \lambda \).

- Yet, by applying recent techniques developed in Fixed Income, an analytical solution exists for the case

\[ \Sigma \sigma' (\sigma \sigma')^{-1} \sigma \Sigma' - \Sigma \Sigma' = 0 \]
Towards an Analytical Solution

• Conjecture 2:

\[ F(\lambda, t; T) = \int_t^T f(\lambda, t; \tau) \, d\tau \]

• with \( f(\lambda, t, t) = 1. \)

• After some algebra, we find the following PDE for \( f(\lambda_t, t; \tau) \):

\[
0 = ((1 - \gamma) r - \rho) \frac{1}{\gamma} f + f_t + \frac{1}{2} tr (f_{\lambda\lambda} \Sigma \Sigma') + \frac{(1 - \gamma)}{2\gamma^2} \lambda_t' (\sigma \sigma')^{-1} \lambda_t f + \\
+ \frac{(1 - \gamma)}{\gamma} f_{\lambda}' \Sigma \sigma' (\sigma \sigma')^{-1} \lambda_t + f_{\lambda}' (A_0 + A_1 \lambda_t)
\]

• Perhaps this does not look any better to most, but it is a very standard PDE in Fixed Income Asset Pricing.

– The solution is an exponential linear-quadratic function of \( \lambda_t \)
An Analytical Solution

- Use method of undetermined coefficients.
- Conjecture 3:
  \[ f(\lambda,t;\tau) = e^{\alpha_0(t;\tau) + \alpha_1(t;\tau)'\lambda_t + \frac{1}{2}\lambda_t'\alpha_2(t;\tau)\lambda_t} \]

1. Take the derivatives \( f_t, f_\lambda \) and \( f_{\lambda\lambda} \)
2. Substitute and pool terms together

- to obtain

\[
0 = \left( (1-\gamma)r - \rho \right) \frac{1}{\gamma} \frac{\partial \alpha_0(t;\tau)}{\partial t} + \alpha_1(t,\tau)'A_0 + \frac{1}{2}tr(\alpha_2(t,\tau)\Sigma\Sigma') + \frac{1}{2}tr\left( \alpha_1(t,\tau)\alpha_1(t,\tau)'\Sigma\Sigma' \right) \\
+ \left( \frac{\partial \alpha_1(t,\tau)}{\partial t} + (1-\gamma)\alpha_1(t,\tau)'\Sigma\sigma\frac{1}{\gamma}(\sigma\sigma')^{-1} + \alpha_1(t,\tau)'A_1 + A_0'\alpha_2(t,\tau) + \alpha_1(t,\tau)\Sigma\Sigma'\alpha_2(t,\tau) \right) \lambda_t \\
+ tr\left( \left( \frac{1}{2} \frac{\partial \alpha_2(t,\tau)}{\partial t} + \frac{1}{2}(1-\gamma)\frac{1}{\gamma^2} (\sigma\sigma')^{-1} \right) + (1-\gamma)\alpha_2(t,\tau)'\Sigma\sigma\frac{1}{\gamma}(\sigma\sigma')^{-1} + \alpha_2(t,\tau)'A_1 + \frac{1}{2}\alpha_2(t,\tau)'\Sigma\Sigma'\alpha_2(t,\tau) \right) \lambda_t \lambda_t' \]

- In order for the right hand side to be zero independently of \( \lambda_t \), the following must hold.
An Analytical Solution

- A system of ODE:

\[
0 = \frac{\partial \alpha_2 (t, \tau)}{\partial t} + (1 - \gamma) \frac{1}{\gamma} \left( \frac{1}{\gamma} + 2 \alpha_2 (t, \tau)' \Sigma \sigma' \right) \left( \sigma' \sigma \right)^{-1} + 2 \alpha_2 (t, \tau)' A_1 + \alpha_2 (t, \tau)' \Sigma \Sigma' \alpha_2 (t, \tau)
\]

\[
0 = \frac{\partial \alpha_1 (t, \tau)'}{\partial t} + (1 - \gamma) \alpha_1 (t, \tau)' \Sigma \sigma' \frac{1}{\gamma} \left( \sigma' \sigma \right)^{-1} + \alpha_1 (t, \tau)' A_1 + A_0 \alpha_2 (t, \tau) + \alpha_1 (t, \tau) \Sigma \Sigma' \alpha_2 (t, \tau)
\]

\[
0 = \frac{\partial \alpha_0 (t; \tau)}{\partial t} + (1 - \gamma) r - \rho \frac{1}{\gamma} + \alpha_1 (t, \tau)' A_0 + \frac{1}{2} \text{tr} (\alpha_2 (t, \tau) \Sigma \Sigma') + \frac{1}{2} \text{tr} (\alpha_1 (t, \tau) \alpha_1 (t, \tau)' \Sigma \Sigma')
\]

- with final conditions \( \alpha_i (\tau, \tau) = 0, \ i = 0, 1, 2. \)

- These ODEs can be easily solved numerically, independently of the dimension.
  - Just start with the final condition at \( \tau \) and move backwards over time (it is three lines of code: one for each ODE).
Application 1: Portfolio Allocation under Predictability

- Let $n = 1$ and $dR_t$ be the return on the aggregate stock market.
- Much of the literature uses the log dividend price ratio as a predictor.
- Let $x_t = \log \left( \frac{D_t}{P_t} \right)$ and let it follow the mean reverting process

$$dx_t = (\eta - \phi x_t) \, dt + \sum x_1 dB^1_t$$
Application 1: Portfolio Allocation under Predictability

- Using $x_t$ a predictor of excess stock returns, we can estimate

$$R_{t,t+dt} = \tilde{\beta}_0 + \tilde{\beta}_1 x_t + \epsilon_{t+dt}$$

Sample: 1947 - 2001. $dt = .25$

<table>
<thead>
<tr>
<th>$\beta_0$ (t-stat)</th>
<th>$\beta_1$ (t-stat)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1898 (3.7042)</td>
<td>0.0531 (3.2424)</td>
<td>3.53%</td>
</tr>
</tbody>
</table>

![Graph showing expected return and realized return over time from 1940 to 2010. The graph includes a dotted line for realized return and a solid line for expected return. The x-axis represents the years from 1940 to 2010, and the y-axis shows the return values ranging from -0.4 to 0.3.]
Application 1: Portfolio Allocation under Predictability

- The annualized expected return $\lambda_t = E_t[R_{t,t+dt}/dt]$ is given by
  \[ \lambda_t = \beta_0 + \beta_1 x_t \]
- with $\beta_i = \ddot{\beta}_i/dt$
- Ito’s Lemma implies
  \[ d\lambda_t = (A_0 + A_1 \lambda_t) dt + \Sigma_1 dB_t^1 \]
  with
  \[ A_0 = \beta_1 \eta + \phi \beta_0; \quad A_1 = -\phi; \quad \Sigma_1 = \beta_1 \Sigma x_1 \]
- The process for stock returns is
  \[ dR_t = (r + \lambda_t) dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2 \]
Application 1: Portfolio Allocation under Predictability

Model:

\[
\begin{align*}
    d\lambda_t &= (A_0 + A_1 \lambda_t) \, dt + \Sigma_1 dB^1_t \\
    dR_t &= (r + \lambda_t) \, dt + \sigma_1 dB^1_t + \sigma_2 dB^2_t
\end{align*}
\]

Sample: 1947 - 2001. \( dt = .25 \)

<table>
<thead>
<tr>
<th></th>
<th>( A_0 )</th>
<th>( A_1 )</th>
<th>( \Sigma_1 )</th>
<th>( \sigma_1 )</th>
<th>( \sigma_2 )</th>
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<tbody>
<tr>
<td></td>
<td>0.0077</td>
<td>-0.1405</td>
<td>-0.0317</td>
<td>0.1183</td>
<td>0.1057</td>
</tr>
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</table>

- **Note 1**: Negative \( \Sigma_1 \) simply means \( Cov(dR, d\lambda) = \Sigma_1 \sigma_1 = -.0038 < 0 \)
  - Positive shocks to dividend yield increase expected returns but are *contemporaneously* negatively correlated with returns.
    * This is intuitive: dividend yield moves mainly because of prices.
    * If \( P_t \downarrow \implies dR_t < 0 \) and \( \log(D/P) \uparrow \)

\[ A \bad\ news \( dR < 0 \) \text{ is not very bad, as it increases expected returns} \]
Application 1: Portfolio Allocation under Predictability

- **Note 2:** The condition for an exact analytical solution is violated:

\[
\Sigma \sigma' (\sigma \sigma')^{-1} \sigma \Sigma - \Sigma \Sigma' = 0 \Rightarrow \frac{(\sigma_1 \Sigma_1)^2}{\sigma_1^2 + \sigma_2^2} = \Sigma_1^2 \Rightarrow \sigma_2^2 = 0
\]

- \(\implies\) Exact formula really holds under the assumption of complete markets.
  - Stock returns span all of the uncertainty.

- Instead, we found \(\sigma_2 > 0\).
  - Part of the problem is the use of quarterly data. At monthly frequency the (negative) correlation between returns and dividend yield is higher.
  - For the sake of argument, I will assume a perfect negative correlation between returns and dividend yield.
    * In what follows I then use \(\sigma_1 = .1612\) and \(\sigma_2 = 0\).
Myopic and Hedging Demand for Various Risk Aversion Parameter

- Myopic Demand
  - $\gamma = 5$
  - $\gamma = 10$
  - $\gamma = 20$

- Hedging Demand
  - $\gamma = 5$
  - $\gamma = 10$
  - $\gamma = 20$
Hedging Demand with Predictable Returns

**Finding 1:** The hedging demand is positive.

- The intuition is simple:
  - If we have a bad shock to returns, we have that $\mu_t$ increases (intuitively, the $D/P$ increases, implying higher expected return).
  - But a higher $\lambda_t$ implies that investor now want to buy more of the stock.
  - Anticipating this correlation, the investor buys more of the stock today, compared to the case where the hedging demand is zero.

- This finding is bad news for the portfolio holding puzzle:
  - We already showed that the agent would hold too much of the stock even with simple myopic demand (no time varying investment opportunity set).
  - * The total demand now of the stock is even higher, deepening the puzzle.
Total Demand with Predictable Returns

![Graph showing total demand with predictable returns for different values of $\gamma$.](image)

- $\gamma = 5$
- $\gamma = 10$
- $\gamma = 20$

Expected Return vs. Total Demand
Finding 2: Hedging demands help to address the life-cycle allocation puzzle.

- As it can be see, the shorter the life expectancy $T$ the lower the share in stocks, especially if current expected return is high.
- In this case, mean reversion kicks in and the investor is wary about the negative consequences of a decrease in expected returns.
Still, because of the hedging demand, an investor with 5 years to live would still substantially exposed to stocks.
• What is the variation over time of the optimal allocation to stock?

• Consider investor with $T = 15$ (constant) and $\gamma = 1, 5, 20$.

• The pattern for $\gamma = 20$ seems more reasonable than $\gamma = 1$ or 5.
1. The predictability of stock returns is still source of heated debate.
   
   – Here we take the strong view that investors take empirical estimates as “true” parameters.
   
   – Much recent literature tried to relax this assumption, and use Bayesian methods in portfolio allocation

   
   * These methodologies are very numerically intensive.
Strategic Asset Allocation: Discussion

2. As shown in Menzly, Santos and Veronesi (JPE, 2004), the dividend yield in which dividends are corrected for stock repurchases is a superior forecaster of future returns that the traditional dividend yield.

- Without repurchases we have

<table>
<thead>
<tr>
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<th>( \hat{\beta}_0 )</th>
<th>t-stat</th>
<th>( \hat{\beta}_1 )</th>
<th>t-stat</th>
<th>( R^2 )</th>
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<tbody>
<tr>
<td>w/ repurchases</td>
<td>0.1233</td>
<td>2.5376</td>
<td>0.0310</td>
<td>2.0815</td>
<td>2.24%</td>
</tr>
<tr>
<td>w/o repurchases</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
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</table>
3. The setting above can be easily extented to multiple assets and multiple predictors.

- Analytical solutions are quite useful in this case.
- Most models of strategic asset allocation do not go over the two or three assets.

- As an illustration, next pictures show the strategic asset allocation for an investor who in addition to a market index, he has access to the returns from mutual funds specialized in 4 strategies:
  1. Value / Small Cap
  2. Value / Large Cap
  3. Growth / Small Cap
  4. Growth / Large Cap
Allocation to 6 Size - BM sorted portfolios and market

- Myopic demand
- Hedging demand
- Total demand
Allocation to 6 Size - BM sorted portfolios and market

Position in Stocks

- Small Growth
- Small Value
- Large Growth
- Large Value
- Mkt

Year:
- 1940
- 1950
- 1960
- 1970
- 1980
- 1990
- 2000
- 2010

Value:
- −1.5
- −1
- −0.5
- 0
- 0.5
- 1
- 1.5
- 2
- 2.5
Application 2: Learning about Average Returns

- Consider the same setting as in the original Merton problem
  \[ dR_t = \mu dt + \sigma dB_t \]

- Differently from Merton, assume that average returns \( \mu \) are not observable.

- Investors observe realized returns \( dR_t \) and infer the value of \( \mu \).

- Since the risk free rate \( r \) is observable, we can equivalently assume that agents infer the value of the average excess return \( \lambda = \mu - r1_n \).

- The following filtering result holds.
A Filtering Result

- Result: Let investors prior distribution at time 0 on $\lambda$ be given by
  \[ \lambda_{t_0} \sim N(\tilde{\lambda}_0, \tilde{q}_0) \]
- Then, the posterior distribution at any time $t$ is given by
  \[ \lambda_t \sim N(\tilde{\lambda}_t, \tilde{q}_t) \]
- where
  \[ d\tilde{\lambda}_t = \Sigma_t d\tilde{B}_t \]
  \[ \Sigma_t = \tilde{q}_t (\sigma')^{-1} \]
  \[ d\tilde{q}_t = -\tilde{q}_t (\sigma \sigma')^{-1} \tilde{q}_t \]
- The innovation process is
  \[ d\tilde{B}_t = \sigma^{-1} [dR_t - E_t (dR_t)] \] (1)
An Informational Equivalent Setting

• We can rewrite the system of returns then as follows

\[ dR_t = (r + \lambda) dt + \sigma dB_t \]
\[ d\lambda_t = \sum_t dB_t \]

• This is very similar to the previous case. Note the following:

1. We are back to complete markets: Conditional on investors’ information, the set of BMs that drive returns \( dR_t \) is the same that drive expected return \( \lambda_t \).
   – The reason is that the information filtration is generated by the return process \( dR_t \).
   – Thus, expected returns will depend on the observation of \( dR_t \) only: if we observe high returns we change our posterior to on expected future returns. That is, expected returns and realized returns become perfectly correlated.
   – \( \Rightarrow \) The asset allocation solution is exact!
2. The only difference from the problem discussed earlier is the fact that the volatility of $\tilde{\lambda}_t$ depends on $t$.
   - However, this volatility declines deterministically.
   - Thus, the methodology developed earlier applies here too, once we are careful to remember that $\tilde{\Sigma}_t$ is a function of time.

3. The volatility $\tilde{\Sigma}_t$ converges to zero as $t \to \infty$
   - This is because we assume $\lambda$ is constant forever. Assuming some time variation in underlying average return will prevent the posterior variance from converging.
   - E.g. for the case $n = 1$,
     \[ q_t = \frac{1}{q_0^{-1} + \sigma^{-2}t} \]
4. Learning has a bite: It has a prediction about the correlation between returns and expected returns.

\[ Cov_t (dR_t, d\lambda_t) = \sigma \Sigma'_t = \sigma (\sigma)^{-1} \tilde{q}_t = \tilde{q}_t \]

- They are positively correlated: A negative innovation in returns decreases expected return.
- The hedging demand will go in the right direction here:

  *Bad news on returns are “twice bad news”. You lost money, and now you expect to gain even less in the future.*

- This is opposite of what we found in our earlier exercise, where we used the “predictability” intuition: negative returns increases the dividend price ratio, which predicts higher returns. That is, realized returns and expected returns were negatively correlated.
An Equivalent Portfolio Problem

- Investor problem:

\[ J(W_0, \hat{\lambda}_0, 0) = \max_{\{(C_t, \theta_t)\}} E_0 \left[ \int_0^T u(C_t, t) \, dt \right] \]

- subject to

\[ dW_t = \{ W_t (\theta_t' \hat{\lambda}_t + r) - C_t \} \, dt + W_t \theta_t' \sigma \, dB_t \]

- At this point, the solution is “almost” the same as before.

  - We need to set \( A_0 = A_1 = 0 \)
  
  - Remember that \( \hat{\Sigma}_t \) depends on time \( t \).

    * The computation is in fact straightforward, as we can simply iterate forward the ODE that defines \( \hat{q}_t \) (Riccati equation)
How Fast Would an Investor Learn?

- First, how fast does “uncertainty” decline?
  - From a prior uncertainty $\sqrt{q_0} = 5\%$, it declines rather slowly.
• The most important effect of learning is that hedging demand this time is negative.

• The intuition, recall, is that bad news are twice bad news here:
  – not only you get a negative return to stock, but now you expected even lower returns for the future.
  – Thus, investors’ optimally reduce their holding of stocks.
  – This mechanism was first observed by Brennan (1998, European Finance Review), but then analyzed by many others.

• The following figures show the hedging demand and total demand for three different value of initial uncertainty

\[ \sqrt{q_0} = 1\%, 3\%, 5\% \]
Strategic Asset Allocation with Learning: The Role of Prior Uncertainty

hedging demand

expected return vs. hedging demand

Prior Unc = 1 %
Prior Unc = 3 %
Prior Unc = 5 %

total demand

expected return vs. total demand

Prior Unc = 1 %
Prior Unc = 3 %
Prior Unc = 5 %
What effect does risk aversion have on hedging demands?

Higher risk aversion decreases (in absolute value) the hedging demand.
Strategic Asset Allocation with Learning: The Role of Risk Aversion

- Why does higher risk aversion decreases (in absolute value) the hedging demand?
  - This is due to the sensitivity of the consumption / wealth ratio $C/W$ to changes in expected returns.
  - As we increase $\gamma$, the myopic demand for stocks decreases.

  $\star \implies$ The consumption to wealth ratio $C/W$ becomes more and more insensitive to variation expected return.

  $\star \implies$ Eventually, changes in expected return have no impact on $C/W$, and thus no need of hedging demand.

  $\star \implies$ The relation between $\gamma$ and hedging demand is non-linear, as hedging demand are close to zero both for $\gamma$ close to 1 and for $\gamma$ large.
Strategic Asset Allocation with Learning: The Life Cycle Implications

• How does learning affect the allocation of investors with different life expectancies?
Strategic Asset Allocation with Learning: The Life Cycle Implications

• Learning does not seem to have a large impact on the asset allocation as a function of time $T$.

• The little that is has goes in the opposite direction:
  – The reason, again, is the EIS.
  – The longer the horizon, the higher the impact of an increase in expected return on future consumption.
  – $\Rightarrow$ larger decrease in $\theta_t$ due to consumption smoothing.
Strategic Asset Allocation with Learning over time

- Consider an investor in 1947 with prior uncertainty \( \sqrt{q_0} = 5\% \).
  - How would his asset allocation change over time?
Strategic Asset Allocation with Learning over time

• Case 1: Assume a declining uncertainty over time
• Case 2: Assume a constant uncertainty (e.g. small probability of jumps)
Strategic Asset Allocation and Expected Returns: Comparison

- Learning about average returns:
  - \(\Rightarrow\) Investor behave like “momentum” traders (or trend chasers)
    * They buy when prices increase.

- Forecasting returns using the dividend yield:
  - \(\Rightarrow\) Investors behave like reversal traders
    * They buy when prices drop
Strategic Asset Allocation and Expected Returns: Comparison

![Graph showing Price/Dividend Ratio from 1940 to 2010 with data points labeled for each decade.](image1)

![Graph showing Trading Strategies: Learning versus Forecasting from 1940 to 2010 with two lines representing Learning ($\gamma = 5$) and Forecasting ($\gamma = 20$).](image2)
Strategic Asset Allocation with Model Misspecification

- What if investors are uncertain about the “model” and would like to take decisions that are “robust” to small misspecification?
  - We now discuss preferences for robustness and their implications for strategic portfolio allocation
  - The framework is the one of Anderson, Hansen, Sargent (ReStud 1999) as well as Maenhout (RFS, 2004)

- Consider (again!) the usual setting, with

\[
d\mathbf{R}_t = (r + \lambda_t) dt + \sigma d\mathbf{B}_t
\]

\[
d\lambda_t = (A_0 + A_1 \lambda_t) dt + \Sigma d\mathbf{B}_t
\]

- Let \( P \) denote the probability measure that is defined by these processes.
- We call this the “reference model”. 
Modeling “Model Misspecification”

• The investor is worried about “small” model misspecification.

• Two questions:
  1. How can we model a model misspecification?
  2. How can we model investor “aversion” to such misspecification?

• We can model “model misspecification” by introducing a set of “plausible” probability measures $Q$ that are “close” to the original one $P$.

• In continuous time, we can “perturb” the reference model and obtain new probability measures $Q$ by replacing $dB_t$ by

$$dB_t = d\hat{B}_t + h_t dt$$

• where $h_t$ is another stochastic process.
The class of misspecified models is then those defined by the
\[ dR_t = (r + \lambda_t) dt + \sigma (d\hat{B}_t + h_t dt) \]

\[ d\lambda_t = (A_0 + A_1 \lambda_t) dt + \Sigma (d\hat{B}_t + h_t dt) \]

for “plausible” \( h_t \) processes.

How can we introduce “preferences” for robustness?
The *multiplier robust control problem* can be formulated as

\[
\sup_{C, \theta} \inf_h \left\{ \widetilde{E} \left[ \int_0^T e^{-\rho t} \left( u(C_t) + \frac{\eta}{2} \mathbf{h}_t \mathbf{h}_t' \right) dt \right] \right\}
\]

subject to the “perturbed” budget equations

\[
dW_t = \left( W_t (\theta'_t \lambda_t + r) - C_t \right) dt + W_t \theta'_t \sigma \left( d\overline{B}_t + \mathbf{h}_t dt \right)
\]

- Here \( \eta \) is a penalty imposed on the discrepancy between \( Q \) and \( P \).
- For given \( \eta \), the “robust” investor
  1. considers the probabilities \( Q \) (each defined by a process \( \mathbf{h}_t \)) that lead to low utility (\( \inf_h \) part)
  2. maximizes utility taking into account these worst case scenarios (\( \max_{C, \theta} \) part)
A high $\eta$ implies a choice of $h_t$ that is close to 0, i.e. a probability $Q$ that is close to $P$, because we are taking the “inf” with respect to $h_t$.

- If $\eta = 0$, we consider all the possible $Q$’s.
- If $\eta = \infty$, we consider only $P$. 
Strategic Asset Allocation with Model Misspecification

- How can we solve this “max min” problem?
- It is convenient to stack all the state variables. Define $Y_t = (W_t, \lambda'_t)'$, so that we have

$$dY_t = \mu_Y(Y_t, \theta_t, C_t) \, dt + \sigma_Y(Y_t, \theta_t, C_t)(dB_t + h_t \, dt)$$

- The following Bellman Isaac condition is the necessary condition for the solution to the max min problem
- There exists a value function $J(Y)$ such that

$$\delta J = \max_{C, \theta} \min_h \left\{ u(C) + \frac{\eta}{2} hh' + (\mu_Y + \sigma_Y h')' J_Y + \frac{1}{2} tr \left( \sigma_Y' J_{YY} \sigma_Y' \right) \right\}$$
Towards a Solution to the Asset Allocation

- Solving for the minimum $h$, one obtains
  \[ h' = -\frac{1}{\eta} \sigma'_Y J_Y \]

- Notice that then
  \[ \frac{\eta}{2} hh' = \frac{1}{2\eta} J'_Y \sigma_Y \sigma'_Y J_Y \]
  \[ \sigma_Y h' = -\frac{1}{\eta} \sigma_Y \sigma'_Y J_Y \]

- Substitute into Bellman Isaac equation to find
  \[ \delta J = \max_{C,\theta} \left\{ u(C) - \frac{1}{2\eta} J'_Y \sigma_Y \sigma'_Y J_Y + \mu'_Y J_Y + \frac{1}{2} \text{tr} (\sigma'_Y J_{YY} \sigma_Y) \right\} \]

- This is similar to earlier problem.
Optimal Consumption and Asset Allocation under Model Misspecification

- The FOC with respect to consumption lead to the usual condition

\[ u_C = J_W \]

- But \( J_W \) is different from before. It will depend on robustness preferences

- Instead, the FOC for optimal portfolio weights imply

\[
\theta_t = -\frac{J_W}{W_t \left( J_{WW} - \frac{1}{\eta} J_W^2 \right)} (\sigma \sigma')^{-1} (\lambda_t) \\
+ \frac{-1}{W_t \left( J_{WW} - \frac{1}{\eta} J_W^2 \right)} (\sigma \sigma')^{-1} \sigma \Sigma' J_W \lambda \\
+ \frac{\frac{1}{\eta} J_W}{W_t \left( J_{WW} - \frac{1}{\eta} J_W^2 \right)} (\sigma \sigma')^{-1} \sigma \Sigma' J_\lambda
\]
Strategic Asset Allocation under Model Misspecification

- The portfolio rule has then three components:

   - Notice that the denominator is adjusted for robustness, implying a lower investment in the stocks (because \( J_W^2 \frac{1}{\eta} > 0 \)).

2. The standard Merton’s hedging demand.

3. An additional hedging demand arising from robustness preferences.
   - If \( \eta \to \infty \), i.e. we consider the class of probability \( Q \) that are closer and closer to the reference \( P \), we have back the usual results.
   - Note in particular that the last term drops out.
An Exact Solution for the Original Merton Problem

- Consider the original setting without time varying expected returns.
  - i.e. $A_0 = 0$, $A_1 = 0$ and $\Sigma = 0$

- In this case, the FOC with respect to $h_t$ yield

  $$h_t = -\frac{1}{\eta} \sigma'_W J_W$$

- and the Bellman Isaac equation is then given by

  $$\delta J = \max_{C, \theta} \left\{ u(C) - \frac{1}{2\eta} J_W^2 \sigma_W \sigma'_W + \mu W J_W + \frac{1}{2} J_{WW} \sigma_W \sigma'_W \right\}$$

- Using $u_c = J_W$ we obtain

  $$\theta_t = \frac{-W_t}{W_t \left( J_{WW} - \frac{1}{\eta} J_W^2 \right)} \left( \sigma \sigma' \right)^{-1} \left( \mu - r 1_d \right)$$
An Exact Solution for the Original Merton Problem

- One complication with the previous problem is that, generically, it is not “scale invariant”
  - It is hard to solve as the solution depends on wealth.

- Maenheut (2004) proposes to scale the penalty parameter $\eta$ by the value function $J$ itself, in a way to make the model again scale independent.

  $$\eta = \eta(J) = \eta^* (1 - \gamma) J(W, t)$$

- The value function is then given by

  $$J(W, t) = \left( \frac{1 - e^{-a(T-t)}}{a} \right)^\gamma \frac{W^{1-\gamma}}{1 - \gamma}$$

- where

  $$a = \frac{1}{\gamma} \left[ \rho - (1 - \gamma) r - \frac{1 - \gamma}{2(\gamma + \eta)} (\mu - r 1_n)' (\sigma \sigma')^{-1} (\mu - r 1_n) \right]$$

  $\rho$ is the correlation between the assets and the market, $r$ is the risk-free rate, $\mu$ is the expected return on the assets, $\sigma$ is the covariance matrix of the assets, $1_n$ is a vector of ones.
The optimal consumption and asset allocation are

\[ C_t = \frac{a}{1 - e^{-a(T-t)}}W_t \]

\[ \theta_t = \frac{1}{\gamma + 1/\eta^*} (\sigma \sigma')^{-1} (\mu - r1_d) \]

- Preferences for robustness clearly go in the right direction to “solve” the asset allocation puzzle
- A lower \( \eta^* \) translates into a higher “aversion” to model misspecification.
- In this case, the allocation to stocks decreases.
- Yet, the allocation is still independent of life expectancy \( T - t \).
  * We need to introduce predictability for that.
How much pessimism is plausible?

- Clearly, by decreasing \( \eta^* \) we can match any empirically observed level of asset holdings.
- However, the question is then what is a “reasonable” level of \( \eta^* \).
- Consider the case \( n = 1 \) (one stock) for simplicity.
  - For each level of \( \eta^* \), there is a given worst case scenario, defined by the FOC
    \[
    h_t = -\frac{1}{\eta} \sigma W J_W = -\frac{1}{(1 + \gamma \eta^*) \sigma} (\mu - r)
    \]
  - where I substitute for \( \sigma_W = W \theta_t \sigma, \ J_W \) and \( \eta = \eta^* (1 - \gamma) J \).
- A robust investor thinks that stock returns are given by
  \[
  dR_t = (\mu + \sigma h_t) dt + \sigma d\bar{B}_t
  \]
How much pessimism is plausible?

- Thus, the equity premium for a robust investor is

$$E_t^h [dR - r] = (\mu + \sigma h_t) - r = (\mu - r) \left( 1 - \frac{1}{1 + \gamma \eta^*} \right)$$

- We can use the “implied” perceived equity premium of the robust investor as a reasonable metric to assess whether $\eta^*$ is too small.

Optimal Portfolio Allocation under Robustness

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Recent Applications of Robust Control

- The approach of robust control theory has found numerous applications in finance in recent times.

1. Liu, Pan and Wang (JF, 2005): uncertainty on rare events to explain options premia, along with the standard result on return equity premium.
2. Routledge and Zin (2004): rare events and market liquidity. \( \Rightarrow \) uncertainty aversion may lead agents not to trade after big market events.
3. Uppal and Wang (JF, 2003): extend the above model to the case of different aversions to uncertainty across assets.
   - For some assets there is less “ambiguity” about the probabilities.
   - Under-diversification: even a limited amount of aversion to uncertainty on some stocks \( \Rightarrow \) over-invest in those with less uncertainty aversion.

Conclusion

• The last decade has seen a boom in research about optimal asset allocation.

• The groundwork set by Samuelson and Merton has found application only recently, as researchers were able to solve long-standing problems
  – The concept of hedging demands date back 30+ years
  – But only recently these hedging demands have been characterized in a quantitative fashion.

• Yet, we are still far from explaining all of the puzzles in a nice, convincing theory.
  – Predictability has the right implication for life cycle, but wrong for asset allocation magnitudes
  – Learning has the right implication for the magnitudes, but wrong for life cycle
  – Preferences for robustness imply unreasonable levels of pessimism.