1 The Portfolio Problem

- Consider the following portfolio problem with time-varying expected returns.

- Let $\beta, S$ be processes of the form

\[
  d\beta_t = r_t \beta_t dt \quad \text{with } \beta_0 > 0 \\
  dS^i_t = \mu^i_t S^i_t dt + S^i_t \sigma^i_t dB_t \quad \text{with } S^i_0 > 0
\]

- where $r_t, \mu_t = (\mu^1_t, ..., \mu^d_t)$ and $\sigma^i_t$ are bounded, adapted processes.
• For concreteness, assume that $\mu_t = (\mu^1_t, \ldots, \mu^d_t)$ follow a continuous time, VAR process

$$d\mu_t = (A_0 + A_1 \mu_t) \, dt + \Sigma dB_t$$

• Notice that the set of Brownian motions moving $\mu_t$ is the same as the ones moving $S_t$.

• Instead, assume $r_t = r$ is constant and that $\sigma^i_t = \sigma$ is also constant, for every $i$.

• Assume again that $\{\sigma\}$ is invertible.

1.1 The dynamics of the market price of risk

• Define by $\lambda_t = \mu_t - r 1_d$ the excess return process.

• Let the $d \times 1$ market price of risk process

$$\nu_t = \sigma^{-1} \lambda_t$$

• Clearly, also $\nu_t$ follows a VAR process

$$d\nu_t = \sigma^{-1} d\mu_t$$

$$= (\sigma^{-1} A_0 + \sigma^{-1} A_1 \mu_t) \, dt + \sigma^{-1} \Sigma dB_t$$

$$= (\tilde{A}_0 + \tilde{A}_1 \nu_t) \, dt + \tilde{\Sigma} dB_t$$

• with $\tilde{A}_0 = \sigma^{-1} A$, $\tilde{A}_1 = \sigma^{-1} A_1 \sigma$ and $\tilde{\Sigma} = \sigma^{-1} \Sigma$. 
• **Result:** Given an initial condition $\nu_0 = \tilde{\nu}$, then for $\tau > 0$

$$\nu_\tau \sim \mathcal{N} (\boldsymbol{\alpha} (\nu_0, \tau), S (\tau))$$

where

$$\boldsymbol{\alpha} (\nu_0, \tau) = \Psi (\tau) \nu_0 + \zeta (\tau)$$

$$S (\tau) = \int_0^\tau \Psi (\tau - s) \bar{\Sigma} \bar{\Sigma}' \Psi (\tau - s)' ds$$

$$\zeta (\tau) = \int_0^\tau \Psi (\tau - s) \bar{A}_0 ds$$

• and $\Psi (\tau)$ solves the system of differential equation

$$\frac{d\Psi (t)}{dt} = \bar{A}_1 \Psi (t)$$

with initial condition $\Psi (0) = I$.

• If $\bar{B}$ has distinct and real eigenvalues, then the solution is

$$\Psi (\tau) = U \exp (\Lambda \cdot \tau) U^{-1}$$

where, $\Lambda$ is the diagonal matrix with $\bar{A}_1$ eigenvalues on the principal diagonal, $U$ is the matrix of the associated eigenvectors, and $\exp (\Lambda \cdot T)$ is the diagonal matrix with $e^{\lambda_i T}$ in its $ii$–th position.

• Clearly, we need $\lambda_i \leq 0$ to ensure that the solution does not explode.

• In this case, we have that the Novikov’s condition is satisfied:

$$E \left[ \exp \left( \frac{1}{2} \int_0^T \nu'_t \nu_t dt \right) \right] < \infty$$
• Thus,
\[
\xi_t = \exp \left( -\frac{1}{2} \int_0^t \nu_u' \nu_u du - \int_0^t \nu_u' dB_u \right)
\]
defines a P-martingale.

1.2 The Optimal Consumption Plan

• Define the state-price deflator
\[
\pi_t = e^{-rt} \xi_t = \exp \left( - \left( \int_0^t r + \frac{1}{2} \nu_u' \nu_u du \right) - \int_0^t \nu_u' dB_u \right)
\] (3)

• We then have that the optimal consumption is given by
\[
C_t^* = \mathcal{I}_u (\lambda \pi_t, t)
\] (4)

• where \( \mathcal{I}_u \) is the inverse of the utility functions.

• In addition, by defining again
\[
\hat{w} (\lambda) = E \left( \int_0^T \pi_t \mathcal{I}_u (\lambda \pi_t, t) dt \right)
\] (5)

• the solution to \( \lambda^* \) is given by the equality \( \hat{w} (\lambda) = w \).

• Assume for instance a power utility
\[
u (C, t) = e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma}
\]
• this implies 

\[ u_c = e^{-\rho t} C_t^{-\gamma} \]

• and thus 

\[ I_u(x, t) = e^{-\frac{\rho t}{\gamma}} x^{-\frac{1}{\gamma}} \]

• Hence 

\[ C_t^* = e^{-\frac{\rho t}{\gamma}} (\lambda \pi_t)^{-\frac{1}{\gamma}} \]

\[ = e^{-\frac{\rho t}{\gamma}} \left( \lambda \exp \left( - \int_0^t r + \frac{1}{2} \nu'_u \nu_u du - \int_0^t \nu'_u dB_u \right) \right)^{-\frac{1}{\gamma}} \]

\[ = \lambda^{-\frac{1}{\gamma}} \exp \left( \int_0^t \frac{1}{\gamma} (r - \rho) + \frac{1}{2\gamma} \nu'_u \nu_u du + \frac{1}{\gamma} \int_0^t \nu'_u dB_u \right) \]

• where \( \lambda \) is a constant determined by the budget constraint.

• The process for log consumption \( c_t = \log (C_t) \) is then given by

\[ c_t = -\frac{1}{\gamma} \log (\lambda) + \left( \int_0^t \frac{1}{\gamma} (r - \rho) + \frac{1}{2\gamma} \nu'_u \nu_u du + \frac{1}{\gamma} \int_0^t \nu'_u dB_u \right) \]

• implying

\[ dc_t = \left( \frac{1}{\gamma} (r - \rho) + \frac{1}{2\gamma} \nu'_t \nu_t \right) dt + \frac{1}{\gamma} \nu'_t dB_t \]
1.3 The optimal portfolio weights

- Consider now all the processes under $Q$: Define the new Brownian motion

$$d\hat{B}_t = dB_t + \nu_t dt$$

- so that, under $Q$, we have

$$dc_t = \left( \frac{1}{\gamma} (r - \rho) - \frac{1}{2\gamma} \nu_t' \nu_t \right) dt + \frac{1}{\gamma} \nu_t' d\hat{B}_t \quad (6)$$

$$d\nu_t = (\tilde{A}_0 + \tilde{A}_1 \nu_t) dt + \tilde{\Sigma} d\hat{B}_t$$

- with $\tilde{A}_1 = \tilde{A}_1 - \tilde{\Sigma}$.

- We now find the portfolio weights that support $C_t^*$.

- Recall the steps (caveat: we can work under $Q$ or under $P$. Last time we did it under $P$. Now we do it under $Q$):

- First, define the $Q$-martingale

$$M_t = E^Q \left[ \int_0^T \beta_u^{-1} C_u du \right] \quad (7)$$

- We know that $M_0 = w =$ wealth at time 0.

- From the martingale representation theorem, we know that there exists $\tilde{\eta}_t$ such that

$$dM_t = \tilde{\eta}_t d\hat{B}_t$$
- The (discounted) wealth is given by
  \[ \hat{W}_t = \beta_t^{-1} W_t = E^Q_t \left[ \int_t^T \beta_u^{-1} C_u du \right] = M_t - J_t \]  
  (8)

- where
  \[ J_t = \int_0^t \beta_u^{-1} C_u du \]

- Thus, the process for the discounted wealth is
  \[ d\hat{W}_t = -\beta_t^{-1} C_t dt + \tilde{\eta}_t d\hat{B}_t \]  
  (9)

- We also have that the wealth is always equal to the total amount invested in stocks and bonds, which must satisfy the budget constraint
  \[ \hat{W}_t = \theta_0^t + \theta_t \hat{S}_t = \int_0^t \theta_u d\hat{S}_t - \int_0^t \beta_u^{-1} C_u du \]

- where \( \hat{S}_t \) is a martingale under Q
  \[ d\hat{S}_t = \hat{S}_\sigma d\hat{B}_t \]

- Thus, the process for the discounted wealth under Q is
  \[ d\tilde{W}_t = -\beta_t^{-1} C_t + \theta_t I_{\hat{S}} \sigma d\tilde{B}_t \]

- which, comparing with (9), yields immediately
  \[ \theta_t I_{\hat{S}} \sigma = \tilde{\eta}_t \]
2 How do we get $\bar{\eta}_t$ practically?

2.1 The (discounted) wealth process

• Notice that we can rewrite (8) as

$$\tilde{W}_t = E_t^Q \left[ \int_t^T \beta_u^{-1} C_u du \right]$$

$$= \beta_t^{-1} C_t E_t^Q \left[ \int_t^T \frac{\beta_u^{-1} C_u}{\beta_t^{-1} C_t} du \right]$$

$$= \beta_t^{-1} C_t E_t^Q \left[ \int_t^T \frac{\beta_u^{-1} e^{c_u-c_t}}{\beta_t^{-1}} du \right]$$

• From the process for optimal consumption (6), we see that the conditional expectation $E_t^Q \left[ \int_t^T \beta_u^{-1} e^{c_u-c_t} du \right]$ depends only on $\nu_t$.

• In other words, we can define the function

$$F (\nu_t, t; T) = E_t^Q \left[ \int_t^T \frac{\beta_u^{-1} e^{c_u-c_t}}{\beta_t^{-1}} du \right]$$

(10)

• and therefore the process

$$\tilde{W}_t = \beta_t^{-1} C_t F (\nu_t, t; T)$$

• Thus, using Ito’s lemma, the diffusion part of the discounted wealth process $d\tilde{W}_t$ must be given by

$$\tilde{\sigma}_W' = \tilde{W}_t \left( \frac{1}{\gamma} \nu_t' + \frac{1}{F} \sum_{i=1}^n \frac{\partial F}{\partial \nu^i} \tilde{\sigma}^i \right)$$
• with \( \tilde{\sigma}^i = [\tilde{\Sigma}]_i \), the \( i \)-th row of \( \tilde{\Sigma} \).

• In other words, from (9) we must have \( \tilde{\eta}_t = \tilde{\sigma}'_W \).

• This yields

\[
\theta_t I_S \sigma = W_t \left( \frac{1}{\gamma} \nu'_t + \frac{1}{F} \sum_{i=1}^{n} \frac{\partial F}{\partial \nu^i} \tilde{\sigma}^i \right)
\]

• or

\[
\theta_t \tilde{I}_S = W_t \left( \frac{1}{\gamma} \nu'_t + \frac{1}{F} \sum_{i=1}^{n} \frac{\partial F}{\partial \nu^i} \tilde{\sigma}^i \right) \sigma^{-1}
\]

2.2 Myopic and Hedging Demand

• In terms of fraction of wealth, and recalling \( \nu_t = \sigma^{-1}(\mu_t - r1_n) \)

\[
\nu'_t = \frac{\theta_t \tilde{I}_S}{W_t} = \frac{\theta_t I_S}{W_t} = \left( \mu_t - r1_n \right)' \left( \sigma \sigma' \right)^{-1} + \sum_{i=1}^{n} \frac{1}{F} \frac{\partial F}{\partial \nu^i} \tilde{\sigma}^i \sigma^{-1}
\]

\[
(11)
\]

• The first term on the RHS is the usual “myopic term”: Higher excess return increase the portfolio holding, while higher risk and higher risk aversion decreases it.

• The second term is the hedging demand component. Notice first we can write

\[
\tilde{\sigma}^i \sigma^{-1} = \left( \tilde{\sigma}^i \sigma \right) \left( \sigma \sigma' \right)^{-1}
\]
• In addition, it turns out (see later) that

\[
\frac{1}{F} \frac{\partial F}{\partial \nu^i} = - \frac{J_{W\nu^i}}{WJ_{WW}}
\]

• where \( J(W, t; T) = E_t \left[ e^{-\rho(\tau - t)C_{1-\gamma}} \right] \) is the indirect utility function.

• Thus, the hedging demand is given by

\[
\text{Hedging Demand} = - \sum_{i=1}^{n} \frac{J_{W\nu^i}}{WJ_{WW}} \left( \bar{\sigma}^i \sigma \right) \left( \sigma \sigma' \right)^{-1}
\]

• I give an intuition of this term below in the context of a specific example.

2.3 How do we compute \( F(\nu_t, t; T) \) in (10)?

• From contingent claim valuation: Consider a security that pays out \( C_t \) over time as dividend.

• Under \( Q \), its value is

\[
V(C_t, \nu_t, t; T) = E_t^Q \left[ \int_t^T e^{-r(\tau - t)}C_\tau d\tau \right] = C_t F(\nu_t, t; T)
\]

• The total expected return (under \( Q \)) on this security must equal the risk free rate, so that

\[
E_t^Q [dV] + Cdt = Vr dt
\]
• From Ito's Lemma
\[ E_t^Q [dV] = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial C}C_t \mu_C (\nu_t) + \sum_{i=1}^{d} \frac{\partial V}{\partial \nu_i} \mu_i (\nu_t) \]
\[ + \sum_{i=1}^{d} \frac{\partial^2 V}{\partial C \partial \nu^i} C_t \frac{\nu_t}{\gamma} \sigma_i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 V}{\partial \nu^i \partial \nu^j} \tilde{\sigma}_i \tilde{\sigma}_j \]

where
\[ \mu_C (\nu_t) = \frac{1}{\gamma} (r - \rho) + \frac{\nu_t}{2 \gamma} \left( \frac{1}{\gamma} - 1 \right) \] (13)
\[ \mu_i (\nu_t) = \tilde{A}_{0,i} + \tilde{A}_{1,i} \nu_t \] (14)

• Finally, since
\[ \frac{\partial V}{\partial t} = C \frac{\partial F}{\partial t}; \frac{\partial V}{\partial C} = F \frac{\partial F}{\partial \nu^i} \]
\[ \frac{\partial^2 V}{\partial C \partial \nu^i} = \frac{\partial F}{\partial \nu^i} \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} C \]

• substituting everything into (12), we find
\[ F r = 1 + \frac{\partial F}{\partial t} + F \mu_C (\nu_t) + \sum_{i=1}^{d} \frac{\partial F}{\partial \nu^i} \mu_i (\nu_t) + \sum_{i=1}^{d} \frac{\partial F}{\partial \nu^i} \frac{\nu_t}{\gamma} \]
\[ + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} \tilde{\sigma}_i \tilde{\sigma}_j \]

• Or
\[ F (r - \mu_C (\nu_t)) = 1 + \frac{\partial F}{\partial t} + \sum_{i=1}^{d} \frac{\partial F}{\partial \nu^i} \left( \tilde{A}_{0,i} + \left( \tilde{A}_{1,i} + \frac{\tilde{\sigma}_i}{\gamma} \right) \nu_t \right) \]
\[ + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} \tilde{\sigma}_i \tilde{\sigma}_j \] (15)
• with final condition

\[ F(\nu_T, T; T) = 0. \]

• If a solution is not available, the PDE can often be computed numerically.

• From \( F \), one can then obtain \( \eta_t \) and, in addition, also consumption. In fact, recall that

\[ \bar{W}_t = \beta_t^{-1} C_t F(\nu_t, t; T) \]

which gives

\[ C_t = W_t F(\nu_t, t; T)^{-1} \]

2.4 The Hedging Demand Again

• We can finally show that

\[ \frac{1}{\partial F} \partial \nu^i = - \frac{J_{W\nu^i}}{W \cdot J_{WW}} \]  \hspace{1cm} (16) \]

• From the Bellman equation (see TN 1), we have that the first order conditions with respect to consumption \( C \) is

\[ J_W = u_c(C_t^*) \]
• where \( C^*_t = C(W_t, \nu_t) = W_t F(\nu_t, t; T)^{-1} \) is the optimal policy function.

• Differentiating both sides with respect to \( W \) and, say, \( \nu^i \) yields the equalities

\[
J_{WW} = u_{cc} \frac{\partial C^*_t}{\partial W} = F(\nu_t, t; T)^{-1}
\]

\[
J_{W\nu^i} = u_{cc} \frac{\partial C^*_t}{\partial \nu^i} = -WF(\nu_t, t; T)^{-2} \frac{\partial F}{\partial \nu^i}
\]

• It is immediate to verify that the RHS and LHS of (16) coincide.
3 The case $\gamma = 1$

- Notice that if $\gamma = 1$, then $F(\nu_t, t; T) = F(t; T)$ is the solution to the PDE.

- In fact, in this case, we obtain

$$F(t; T)\rho = 1 + F'(t; T)$$

which yields

$$F(t; T) = He^{\rho t} + \frac{1}{\rho}$$

- The final condition

$$F(T; T) = He^{\rho T} + \frac{1}{\rho} = 0$$

yields

$$H = \frac{e^{-\rho T}}{\rho}$$

- Thus

$$F(t; T) = \frac{1}{\rho} \left(1 - e^{-\rho(T-t)}\right)$$

- This implies that the consumption to wealth ratio is deterministic and given by

$$C_t = \frac{W_t\rho}{(1 - e^{-\rho(T-t)})}$$
• In addition, the optimal portfolio choice is
\[
\vartheta_t' = \frac{\left(\mu_t - r\mathbf{1}_n\right)'}{\gamma} (\sigma \sigma')^{-1}
\]

4 The Solution in the Univariate Case for $\gamma > 1$

• Consider the univariate case ($d = 1$).

• In this case, the portfolio holding of the market is given by
\[
\vartheta_t = \frac{\mu_t - r}{\gamma \sigma^2} + \left(\frac{1}{F} \frac{\partial F}{\partial \nu}\right) \tilde{\sigma} \sigma
\]

• where $F$ has to satisfy the PDE (15) becomes
\[
F(r - \mu_C(\nu_t)) = 1 + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \nu} \left( \tilde{A}_0 + \left( \tilde{A}_1 + \frac{\tilde{\sigma}}{\gamma} \right) \nu_t \right) + \frac{1}{2} \frac{\partial^2 F}{\partial \nu^2} \tilde{\sigma}^2
\]

• Unfortunately, it does not have a closed form solution.

• Wachter (2002) however finds a nice way out: Rather than considering a claim to the whole sequence of consumption plan $\{C_\tau\}_t^T$, that is, the security
\[
V(C, \nu, t; T) = EQ \left[ \int_0^T e^{-r(\tau - t)} C_\tau d\tau \right]
\]
she first considers a set of claims, each to the exact consumption “coupon” $C_\tau$ paid at that particular $\tau$, for $\tau \in [0, T]$. 
• By no arbitrage, the value of the security paying the process \( \{C_\tau\} \) will be the sum (i.e. the integral) of all these individual claims.

• Let the value of each claim to the “coupon” \( C_{\tau a u} \) be given by

\[
v(C, \nu, t; \tau) = E^Q \left[ e^{-r(\tau-t)}C_\tau \right].
\]

• The homogeneity discussed earlier entails that \( v(C, \nu, t; \tau) = Cf(\nu, t; \tau) \) for some \( f(.) \).

• Under \( Q \) also this claim must earn the risk free rate

\[
E^Q [dv] = rv dt
\]

• Ito’s Lemma then gives

\[
rv = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial C} C_t \mu_C(\nu_t) + \frac{\partial v}{\partial \nu} (\tilde{A}_0 + \tilde{A}_1 \nu_t) + \frac{1}{2} \frac{\partial^2 v}{\partial \nu^2} \tilde{\sigma}^2
\]

\[
+ \frac{\partial^2 v}{\partial C \partial \nu} C_{\nu_t} \tilde{\sigma} \frac{\nu_t}{\gamma}
\]

• with final condition \( v(C_\tau, \nu_\tau, \tau; \tau) = C_\tau \)

• Recall also that under \( Q \)

\[
\mu_C(\nu_t) = \frac{1}{\gamma} (r - \rho) + \frac{\nu_t^2}{2\gamma} \left( \frac{1}{\gamma} - 1 \right)
\]
• As before, we can substitute the following quantities

\[
\frac{\partial v}{\partial t} = C \frac{\partial f}{\partial t}; \quad \frac{\partial v}{\partial C} = f; \quad \frac{\partial v}{\partial \nu} = C \frac{\partial f}{\partial \nu}
\]

\[
\frac{\partial^2 v}{\partial \nu^2} = C \frac{\partial^2 f}{\partial \nu^2}; \quad \frac{\partial^2 v}{\partial C \partial \nu} = \frac{\partial f}{\partial \nu}
\]

• Substituting also \( v = Cf, \mu_C(\nu) \) and deleting \( C \) on both sides yields

\[
r f = \frac{\partial f}{\partial t} + f \left( \frac{1}{\gamma} (r - \rho) + \frac{\nu^2}{2\gamma} \left( \frac{1}{\gamma} - 1 \right) \right) + \frac{\partial f}{\partial \nu} (\tilde{A}_0 + \tilde{A}_1 \nu_t) + \frac{1}{2} \frac{\partial^2 f}{\partial \nu^2} \tilde{\sigma}^2 + \frac{\partial f}{\partial \nu} \frac{\nu_t}{\gamma} \tilde{\sigma}
\]

• with final condition \( f(\nu_{\tau}, \tau; \tau) = 1 \)

• Fortunately, a “solution” to this PDE instead exists.

• **How can we find it constructively?**

• Use the **method of undetermined coefficients**

  – This methodology is extensively used to obtain the prices of Fixed Income Securities.

• Conjecture (this comes with experience)

\[
f(\nu, t; \tau) = \exp \left( a_0(t; \tau) + a_1(t; \tau) \nu + a_2(t; \tau) \nu^2 \right)
\]

• Then we obtain

\[
\frac{\partial f}{\partial t} = \left( a'_0 + a'_1 \nu_t + a'_2 \nu_t^2 \right) f
\]
\[ \frac{\partial f}{\partial \nu} = (a_1 + 2a_2 \nu_t) f \]
\[ \frac{\partial^2 f}{\partial \nu^2} = (2a_2 + (a_1 + 2a_2 \nu_t)^2) f \]

- Substitute and delete \( f \) on both sides, to find

\[
r = a_0' + a_1' \nu + a_2' \nu^2 + \left( \frac{1}{\gamma} (r - \rho) + \frac{\nu_t^2}{2\gamma} \left( \frac{1}{\gamma} - 1 \right) \right) + (a_1 + 2a_2 \nu) \left( \bar{A}_0 + \bar{A}_1 \nu + \frac{\nu_t}{\gamma} \right) + \frac{1}{2} (2a_2 + (a_1 + 2a_2 \nu)^2) \bar{\sigma}^2
\]

- Finally, bunch up together terms in \( \nu, \nu^2 \) etc.

- One obtains:

\[
0 = a_0' - r + \left( \frac{1}{\gamma} (r - \rho) \right) + a_1 \bar{A}_0 + \frac{1}{2} \left( 2a_2 + a_1^2 \right) \bar{\sigma}^2 + \left( a_1' + a_1 \left( \bar{A}_1 + \frac{\bar{\sigma}}{\gamma} \right) + 2a_2 \bar{A}_0 + 2a_1 a_2 \bar{\sigma}^2 \right) \nu_t + \left( a_2' + \frac{1}{2\gamma} \left( \frac{1}{\gamma} - 1 \right) + 2a_2 \left( \bar{A}_1 + \frac{\bar{\sigma}}{\gamma} \right) + 2a_1^2 \bar{\sigma}^2 \right) \nu_t^2
\]

- This equation is satisfied if the following system of ODEs is satisfied

\[
a_2' + 2a_2 \left( \bar{A}_1 + \frac{\bar{\sigma}}{\gamma} \right) + 2a_2^2 \bar{\sigma}^2 + \frac{1}{2\gamma} \left( \frac{1}{\gamma} - 1 \right) = 0
\]
\[
a_1' + a_1 \left( \bar{A}_1 + \frac{\bar{\sigma}}{\gamma} \right) + 2a_2 \bar{A}_0 + 2a_1 a_2 \bar{\sigma}^2 = 0
\]
\[
a_0' - r + \frac{1}{\gamma} (r - \rho) + a_1 \bar{A}_0 + \frac{1}{2} \left( 2a_2 + a_1^2 \right) \bar{\sigma}^2 = 0
\]
• Note that the system can be easily solved recursively: Solve the first equation (for $a_2$), then plug in the solution into the second equation (for $a_1$) and then finally, obtain the solution of the last equation (for $a_0$).

• It is possible to find exact closed formulas for $a_2$ and $a_1$. However, this is as easy to obtain numerically.

• ODEs are infinitely simpler than PDE, as you can solve then backward: Start with the final condition $a_0(\tau) = a_1(\tau) = a_2(\tau) = 0$ and then simply move backwards. Below are the details.

• Once we have the solution for $a_0(t; \tau)$, $a_1(t; \tau)$ and $a_2(t; \tau)$ for every $\tau \in [0, T]$, the function $F(\nu, t; T)$ can be obtained easily.

• In fact, recall that

$$V(C, \nu, t; T) = CF(\nu, t; T)$$
$$= E^Q_t \left[ \int_t^T e^{-r(\tau-t)} C\tau d\tau \right]$$
$$= \int_t^T E^Q_t \left[ e^{-r(\tau-t)} C\tau \right] d\tau$$
$$= \int_t^T v(C, \nu, t; \tau) d\tau$$
$$= C \int_t^T f(\nu, t; \tau) d\tau$$
• This implies that simply
\[
F(\nu, t; T) = \int_t^T f(\nu, t; \tau) d\tau
\]
\[
= \int_t^T \exp \left( a_0(t; \tau) + a_1(t; \tau)\nu_t + a_2(t; \tau)\nu_t^2 \right) d\tau
\]
• The portfolio holdings require the computation of
\[
\frac{\partial F}{\partial \nu} = \int_t^T (a_1(t; \tau) + a_2(t; \tau)\nu_t) \exp \left( a_0(t; \tau) + a_1(t; \tau)\nu_t + a_2(t; \tau)\nu_t^2 \right) d\tau
\]
• To conclude, we have the following portfolio rule
\[
\vartheta_t = \text{Miopic Demand} + \text{Hedging Demand}
\]
with
\[
\text{Myopic Demand} = \frac{\mu_t - r}{\gamma \sigma^2}
\]
\[
\text{Hedging Demand} = \left( \frac{1}{F} \frac{\partial F}{\partial \nu} \right) \frac{\tilde{\sigma}}{\sigma}
\]
\[
= \frac{\tilde{\sigma} \int_t^T (a_1(t; \tau) + a_2(t; \tau)\nu_t) f(\nu_t, t; \tau) d\tau}{\sigma \int_t^T f(\nu_t, t; \tau) d\tau}
\]
• Also note that the optimal consumption was given by \( C_t = W_t F(\nu, t; T)^{-1} \) implying that the \( C/W \) ratio is given by
\[
\frac{C}{W} = \frac{1}{\int_t^T f(\nu_t, t; \tau) d\tau}
4.1 A Calibration

- An obvious application of the setting above is the one where returns $\mu_t$ are predictable from the dividend price ratio.

- We can think then of $\mu_t$ to be just

$$
\mu_t = \alpha + \beta \log \left( \frac{D_t}{P_t} \right)
$$

where $\alpha$ and $\beta$ are the regression coefficients of some sort of predictive regression.

- Note that if $\log(D_t/P_t)$ follows a mean reverting process, so does $\mu_t$ and so we are back to the case discussed in this section.

- Also, there is a natural negative correlation between returns and $D/P$ ratio: a negative return implies that $P_t$ decreased. Since dividends do not move much, this implies that $D/P$ went up.

- How do we impose a negative correlation in the model? Just assume that $\bar{\sigma} < 0$

- The following parameters have been used by many, including Barberis (JF 2000), and Wachter (JFQA 2002)

<table>
<thead>
<tr>
<th>Parameter Choice</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of time preference $\rho$</td>
<td>0.0624</td>
</tr>
<tr>
<td>Risk free rate $r$</td>
<td>0.0168</td>
</tr>
<tr>
<td>Volatility of stock prices $\sigma$</td>
<td>0.1510</td>
</tr>
<tr>
<td>Volatility of $\nu_t$</td>
<td>-0.0655</td>
</tr>
<tr>
<td>Mean Reversion $A_1$</td>
<td>-0.0226</td>
</tr>
<tr>
<td>Drift $A_0$</td>
<td>0.0062</td>
</tr>
</tbody>
</table>
The following figures show the C/W ratio, Hedging Demand and Total Demand as a function of expected return $\mu_t$.

**Figure 1: C/W ratio**

- Note that the C/W ratio initially declines with $\gamma$, as intuition would have it: Higher risk aversion implies a higher desire to save and thus consume less.

- However, as $\gamma$ increase, the C/W increases again as the investor elasticity of intertemporal substitution (EIS = $1/\gamma$) decreases.

- In the limit, as $\gamma$ increases, the investor’s desire to smooth out consumption is so large that changes in $\mu_t$ have no impact on C/W.

- In this case, the average C/W across possible $\mu_t$ must be the same as the one under different $\gamma$’s, implying a higher C/W for low $\mu_t$ and lower for high $\mu_t$.

- Later on we will do recursive utility, and see the implications of EIS and risk aversion independently.
• The hedging demand is positive. The intuition is simple:
   - If we have a bad shock to returns, we have that $\mu_t$ increases (intuitively, the $D/P$ increases, implying higher expected return).
   - But a higher $\mu_t$ implies that investor now want to buy more of the stock.
   - Anticipating this correlation, the investor buys more of the stock today, compared to the case where the hedging demand is zero.

• This finding is bad news for the portfolio holding puzzle: We already showed that the agent would hold too much of the stock even with simple myopic demand (no time varying investment opportunity set).

• The total demand now of the stock is even higher, deepening the puzzle.
• We will see other channels that would decrease the holding of stocks later on.
• The predictability, however, helps a little to generate asset holding that depend on life cycle

• Using the same parameters, with $\gamma = 5$ but for three different maturities $T$ we obtain the following.

Figure 4: Total Demand

• As it can be see, the shorter the time to “death” the lower the share in stocks, especially if current expected return is high.

• In this case, mean reversion kicks in and the investor is wary about the negative consequences of a decrease in expected returns.