Teaching Notes #4  
(Addendum)  
Portfolio Selection with recursive preferences and time varying expected return\textsuperscript{1}  

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- Consider again the examples in TN 1 and TN 3, Addendum.  
- Consider the case with only one asset, with stock dynamics  
\begin{align*}  
dS_t/S_t &= \mu_t dt + \sigma_S dB_t  
\end{align*}  
and  
\begin{align*}  
d\mu_t &= (A_0 + A_1 \mu_t) dt + \sigma_{\mu,t} dB_t  
\end{align*}  

- We are in complete markets and \( \lambda_t = \mu_t - r \). Of course  
\begin{align*}  
d\lambda_t &= (\hat{A}_0 + A_1 \lambda_t) dt + \sigma_{\lambda,t} dB_t  
\end{align*}  
where  
\begin{align*}  
\hat{A}_0 = (A_0 + A_1 r) \quad \text{and} \quad \sigma_{\lambda,t} = \sigma_{\mu,t}  
\end{align*}  

\textsuperscript{1}This teaching notes partly draw on Campbell, Chacko, Rodriguez and Viceira "Strategic Asset Allocation in a Continuous Time VAR model", JEDC, 2004.
• We may also think of this setting as one where the true expected return $\bar{\mu}$ is constant but investors learn about it. In this case $\sigma_{\mu,t}$ decreases to zero according to a specific deterministic path (see TN 3).

• We consider $\lambda_t$ our state variable

• For a normalized aggregator, the Bellman equation turns out to be

$$\sup_{(c, \vartheta)} \mathcal{D}^{(c, \vartheta)} J (\lambda, w, t) + \overline{f} (c, J (\lambda, w, t)) = 0 \quad (1)$$

• where

$$\mathcal{D}^{(c, \vartheta)} J (\lambda, w, t) = J_t + J_\lambda (A_0 + A_1 \lambda_t) + J_w (w \vartheta \lambda_t + wr - c) + \frac{1}{2} \left( \sigma_{\lambda,t}^2 J_{\lambda \lambda} + 2 w \vartheta J_{w \lambda} \sigma_{\lambda,t} \sigma_S + \sigma_S^2 w^2 \vartheta^2 J_{ww} \right)$$

• The FOC with respect to consumption and portfolio holdings are

$$J_w = \overline{f}_c$$

$$\vartheta = - \frac{J_w w \lambda_t + w J_{w \lambda} \sigma \sigma_S}{w^2 \sigma_S^2 J_{ww}}$$

• For KP, we have

$$\overline{f} (c, J) = \frac{\phi \ c^\rho - (\alpha J)^{\frac{\rho}{\alpha}}} {\rho \ (\alpha J)^{\left(\frac{\rho}{\alpha} - 1\right)}} = \frac{\phi \ J^\rho}{\rho \ J^{\left(\frac{\rho}{\alpha} - 1\right)}} = \frac{\phi \ J^{\rho}}{\rho \ J^{\left(\frac{\rho}{\alpha} - 1\right)}} - 1 \quad (2)$$
• Recall that we have the following interpretation of parameters

\[
\rho = 1 - \frac{1}{\psi} \\
\alpha = 1 - \gamma
\]

where \( \gamma \) is a risk aversion parameter, and \( \psi \) is the elasticity of intertemporal substitution.

• For these teaching notes, I consider the special case where \( \psi = 1 \), that is \( \rho = 0 \). This corresponds to the log utility in the power utility (\( \alpha = \rho \)).

• The reason is that in this case, we obtain a closed form formula for both consumption and portfolio allocation rules.

• Campbell and Viceira (2002), and Campbell et al. (2004) contain the methodology and results to obtain approximate solutions for the case \( \psi \neq 1 \).

• In the case \( \rho = 0 \) the aggregator is

\[
f(c, J) = \phi \alpha J \left[ \log(c) - \frac{1}{\alpha} \log(\alpha J) \right]
\]

• Conjecture

\[
J = \psi(\lambda, t) \frac{w^\alpha}{\alpha}
\]

• Then

\[
J_w = \psi w^{\alpha - 1} = \alpha Jw^{-1} \\
J_{ww} = \psi (\alpha - 1) w^{\alpha - 2} = (\alpha - 1) \alpha Jw^{-2} \\
J_t = \psi_t \frac{w^\alpha}{\alpha} = \frac{\psi_t}{\psi} J
\]
\[
J_\lambda = \psi_\lambda \frac{w^\alpha}{\alpha} = \frac{\psi_\lambda}{\psi} J \\
J_{\lambda\lambda} = \psi_{\lambda\lambda} \frac{w^\alpha}{\alpha} = \frac{\psi_{\lambda\lambda}}{\psi} J \\
J_{\lambda w} = \psi_\lambda w^{\alpha-1} = \frac{\psi_\lambda}{\psi} \alpha J w^{-1}
\]

- From FOC with respect to consumption

\[
f_c = \phi \alpha J c^{-1} = J_w
\]

- we obtain

\[
c = \phi \frac{\alpha J}{J_w} = \phi w
\]

- That is, a unit elasticity of intertemporal substitution implies a constant consumption/wealth ratio.

  - An increase in expected return has two effects: it increases future expected consumption, thereby increasing the desire to consume more today to smooth out consumption.
  
  - However, it also increases the desire to invest more money in assets, and hence to consume less today.
  
  - When EIS = 1, the two effects exactly balance each other.

- In addition, the FOC with respect to \( \vartheta \) become

\[
\vartheta_t = \frac{\lambda_t}{\sigma_S^2 (1 - \alpha)} + \frac{\psi_\lambda/\psi}{(1 - \alpha) \sigma_S} \sigma_{\lambda,t}
\]

- We have the usual two terms: Myopic demand plus hedging demand.
• Substitute everything in the bellman equation, and we find

\[ 0 = \phi \alpha J \left[ \log (\phi) - \frac{1}{\alpha} \log (\psi (\lambda, t)) \right] \]

\[ + \frac{\psi_t}{\psi} J + \frac{\psi_\lambda}{\psi} J \left( \ddot{A}_0 + A_1 \lambda_t \right) \]

\[ + \alpha J \left( \dot{\theta}_t \lambda_t + r - \phi \right) \]

\[ + \frac{1}{2} \left( \sigma_{\lambda,t}^2 \frac{\psi_{\lambda\lambda}}{\psi} J + 2 \dot{\theta} \frac{\psi_\lambda}{\psi} \right) \alpha J \sigma_{\lambda,t} \sigma_S + \sigma_S^2 \dot{\theta}^2 (\alpha - 1) \alpha J \]

• After some algebra we find the PDE:

\[ 0 = \phi \alpha \log (\phi) - \phi \log (\psi) + \frac{\psi_t}{\psi} + \frac{\psi_\lambda}{\psi} \left( \ddot{A}_0 + A_1 \lambda_t \right) \]

\[ + \frac{1}{2} \alpha \frac{\lambda_t^2}{\sigma_S^2 (1 - \alpha)} + \alpha r - \alpha \phi + \frac{1}{2} \sigma_{\lambda,t}^2 \frac{\psi_{\lambda\lambda}}{\psi} \]

\[ + \frac{1}{2} \frac{(\psi_\lambda)^2}{\sigma_{\lambda,t} \alpha} \frac{\sigma_{\lambda,t}^2 \alpha}{(1 - \alpha)} \]

• Conjecture that the solution to the PDE is given by

\[ \psi (\lambda, t) = e^{a_0 (t; T) + a_1 (t; T) \lambda_t + a_2 (t; T) \lambda_t^2} \]

• This implies

\[ \psi_t = \left( a'_0 + a'_1 \lambda_t + a'_2 \lambda_t^2 \right) \psi \]

\[ \psi_\lambda = (a_1 + 2a_2 \lambda_t) \psi \]

\[ \psi_{\lambda\lambda} = \left( 2a_2 + (a_1 + 2a_2 \lambda_t)^2 \right) \psi \]

• Substitute to find

\[ 0 = \phi \alpha \log (\phi) - \phi \left( a_0 + a_1 \lambda_t + a_2 \lambda_t^2 \right) \]
\[
+ \left( a_0' + a_1' \lambda_t + a_2' \lambda_t^2 \right) + (a_1 + 2a_2 \lambda_t) \left( \tilde{A}_0 + A_1 \lambda_t \right) \\
+ \frac{1}{2} \alpha \sigma_{\lambda, t}^2 \left( \frac{\lambda_t^2}{\sigma^2_s (1 - \alpha)} \right) + \alpha r - \alpha \phi + \frac{1}{2} \sigma_{\lambda, t}^2 (2a_2 + (a_1 + 2a_2 \lambda_t)^2) \\
+ \frac{\lambda_t \alpha \sigma_{\lambda, t}}{\sigma_s (1 - \alpha)} (a_1 + 2a_2 \lambda_t) + \\
+ \frac{1}{2} \frac{\sigma_{\lambda, t}^2 \alpha}{(1 - \alpha)} (a_1 + 2a_2 \lambda_t)^2
\]

- Collect terms

\[
0 = \left( a_1' - \phi a_1 + a_1 A_1 + 2a_2 \tilde{A}_0 + +\sigma_{\lambda, t}^2 a_1 a_2 + \frac{\alpha \sigma_{\lambda, t}}{\sigma_s (1 - \alpha)} a_1 a_1 \sigma_{\lambda, t}^2 \alpha \right) \lambda_t \\
+ \left( a_2' - \phi a_2 + 2a_2 A_1 + \frac{1}{2} \frac{\alpha}{\sigma^2_s (1 - \alpha)} + 2\sigma_{\lambda, t}^2 a_2^2 + \frac{\alpha \sigma_{\lambda, t}}{\sigma_s (1 - \alpha)} 2a_2 + 2 \frac{\sigma_{\lambda, t}^2 \alpha}{(1 - \alpha)} a_2^2 \right) \lambda_t^2 \\
+ a_0' + \phi \alpha \log(\phi) - \phi a_0 + a_1 \tilde{A}_0 + \alpha r - \alpha \phi + \sigma_{\lambda, t}^2 a_2 + \frac{1}{2} \sigma_{\lambda, t}^2 a_2^2 + \frac{1}{2} \frac{\sigma_{\lambda, t}^2 \alpha}{(1 - \alpha)} a_2^2
\]

- and obtain the recursive system

\[
a_2' - \phi a_2 + 2a_2 A_1 + \frac{1}{2} \alpha \sigma_{\lambda, t}^2 a_2^2 + \frac{\alpha \sigma_{\lambda, t}}{\sigma_s (1 - \alpha)} 2a_2 + 2 \frac{\sigma_{\lambda, t}^2 \alpha}{(1 - \alpha)} a_2^2 = 0 \\
a_1' - \phi a_1 + a_1 A_1 + 2a_2 \tilde{A}_0 + +\sigma_{\lambda, t}^2 a_1 a_2 + \frac{\alpha \sigma_{\lambda, t}}{\sigma_s (1 - \alpha)} a_1 a_1 \sigma_{\lambda, t}^2 \alpha = 0 \\
a_0' + \phi \alpha \log(\phi) - \phi a_0 + a_1 \tilde{A}_0 + \alpha r - \alpha \phi + \sigma_{\lambda, t}^2 a_2 + \frac{1}{2} \sigma_{\lambda, t}^2 a_2^2 + \frac{1}{2} \frac{\sigma_{\lambda, t}^2 \alpha}{(1 - \alpha)} a_2^2 = 0
\]

- **Calibration**

- We use the learning model studied discussed in TN 3 Addendum.

- Recall that the parameters are as follows:
Parameter Choice

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of time preference $\phi$</td>
<td>0.0624</td>
</tr>
<tr>
<td>Risk free rate $r$</td>
<td>0.0168</td>
</tr>
<tr>
<td>Volatility of stock prices $\sigma_S$</td>
<td>0.1510</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>3</td>
</tr>
<tr>
<td>$T$</td>
<td>25</td>
</tr>
<tr>
<td>Uncertainty $\sqrt{q_0}$</td>
<td>2%</td>
</tr>
</tbody>
</table>

- The only difference from TN 3 are the assumptions for the base case parameters. Here, we assume $\gamma = 3$ instead of 5, $T = 25$ instead of 50, and uncertainty $\sqrt{q_0} = 2\%$ instead of 3%.

- The reason is that the effects are all stronger with EZ preferences and EIS = 1, and so we do not need extreme values to obtain reasonable effects.

- Figure 1 plots the hedging demand and total demand for various $\gamma$, plotted against expected returns.
Figure 1: Portfolio Allocation for various $\gamma$

- Hedging demand – power utility
- Total demand – power utility
- Hedging demand – EZ preferences
- Total demand – EZ preferences

Graphs show the allocation of portfolio for different values of $\gamma$: $\gamma = 2$, $\gamma = 3$, $\gamma = 4$. The x-axis represents the expected return, while the y-axis shows the demand for hedging or total demand.
• Figure 2 plots the hedging demand and total demand for various uncertainty levels $\sigma_0 = \sqrt{q_t}$, plotted against expected returns.
• For given parameter $\gamma$, the hedging demand effect for EZ preferences is stronger than for power utility.

• Recall the intuition for the existence of hedging demand with learning:

  – Bad news about stock returns (a negative shock) is double bad news, as it also implies that expected future returns are low.

  – The investor, forecasting this correlation between returns and expected return, decreases today the demand of stock compared to the myopic case.

  – When EIS $\neq 1$, however, part of the bad news is amortized through consumption smoothing: if the investor receive a bad news, he/she will adapt not only the asset allocation, but also consumption.

  – Instead, EIS $= 1$ implies that $C/W = \phi = \text{constant}$, and thus the agent cannot (i.e. is not willing to) amortize bad shocks to changes in consumption.

  – It then decrease substantially the demand for stocks to avoid to be over-exposed to return shocks.