Teaching Notes #2
Equilibrium with Complete Markets\textsuperscript{1}

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\textsuperscript{1}These teaching notes draw heavily on Duffie (1996, Chapters 9 and 10) and Karatzas and Shevre (1999, Chapter 3 and 4). They are intended for students of Business 35909 only. Please, do not distribute without my prior consent.
1 Competitive Equilibrium

- We now use the results in TN 1 to determine the competitive equilibrium.

- The notion of equilibrium in this set up is as follows:
  
  1. There are $m$ agents in the economy, each endowed with a stream of consumption good;
  2. The consumption good is immediately perishable, so that it must be consumed immediately;
  3. Agents can trade their endowments, by selling/buying financial securities;
  4. All financial securities are in zero-net supply: For every buyer there must be a seller.

- This is the standard, general equilibrium notion of a pure-exchange economy.

- Notice in particular that there is no production.

1.1 Primitives

- Let $\mathbf{B} = (B^1, \ldots, B^d)$ be a $d$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t\}$ denote the standard filtration of $\mathbf{B}$.

- Let us fix a horizon $T$ and let the consumption space be the set $L$ of adapted processes such that $E \left( \int_0^T c^2_t dt \right) < \infty$.

- Suppose there are $m$ agents, indexed by $i = 1, \ldots, m$.

- Each agent receives an endowment $\{e^i_t\} \in L_+$.
• Each agent has a strictly increasing utility function $U_i : L_+ \rightarrow \mathcal{R}$.

• All agents have a common discount rate $\phi$. We shall assume it constant, but it could also be a function of time.

• Each agent will maximize

$$U (c^i) = E_0 \left[ \int_0^T e^{-\phi t} u_i (c_t^i) \, dt \right]$$

• Notice that we assume here no utility from final wealth, although it could have been inserted without any trouble.

• We shall assume everywhere that condition A in TN1 hold.

1.2 Financial Markets

• We assume complete markets. With no loss of generality, let there be $d$ risky securities with price processes

$$dS_t = I_S \mu_t \, dt + I_S \sigma_t \, dB_t$$

• where $I_S$ is the diagonal matrix with $S_i$ on the $ii$-th element, and $\mu_t$ and $\sigma_t$ are adapted processes in $\mathcal{L}$ and $\mathcal{L}^2$.

• Market completeness is achieved by assuming that $\sigma_t$ is invertible almost everywhere.

• Also, there is a risk-free security, with short rate process $r_t$ and price

$$\beta_t = e^{\int_0^t r_u \, du}$$
• As in the previous notes, define the market price of risk process
\[ \nu_t = \sigma_t^{-1}(\mu_t - r_t1_d) \]

• Assume that the Novikov’s condition
\[ E \left( \exp \left( \frac{1}{2} \int_0^T \nu'_t \nu_t dt \right) \right) < \infty \]

• is satisfied and let
\[ \xi_t = \exp \left( -\int_0^t \nu'_t dB_u - \frac{1}{2} \int_0^t \nu'_u \nu_u du \right) \]

• We recall that from Novikov’s theorem, \( \xi_t \) is a \( P \)-martingale.

• Finally, define the state-price density process
\[ \pi_t = \beta_t^{-1} \xi_t \]  

• We saw already that \( \pi_t \) is such that \( S^\pi = S\pi \) is a martingale.

• As usual, a trading strategy \( (\theta^0, \theta) \) is a vector process in \( \mathcal{H}^2(S) \), that is, a space with sufficient integrability conditions to rule out doubling strategies.

• For convenience, let \( \tilde{\theta} = (\theta^0, \theta), \tilde{S} = (\beta, S) \).

• A trading strategy \( \tilde{\theta} = (\theta^0, \theta) \) finances a consumption process \( c \) given income \( e \) if
\[ \tilde{\theta}_t \cdot \tilde{S}_t = \int_0^t \tilde{\theta}_u \cdot d\tilde{S}_u + \int_0^t (e_u - c_u) du \geq 0 \]  
\[ \tilde{\theta}_T \cdot \tilde{S}_T = 0 \]
• Notice that the wealth $W_t = \tilde{\theta}_t \cdot \tilde{S}_t$ grows also because of the additional endowment that has to be taken into account.

• The last equality means that no obligations are left at horizon.

• Each agent then faces the problem

\[
\sup_{(c, \theta^0, \theta) \in \Lambda_i} U_i(c^i)
\]  

(4)

where $\Lambda_i = \left\{ (c, \theta^0, \theta) \in L^+ \times \mathcal{H}^2(S) \text{ such that } \theta^0, \theta \text{ finances } c^i \text{ given } e^i \right\}$

• A security-spot market equilibrium is a collection of price processes $(\beta, S)$, consumption processes $(c^i)_{i=1}^m$ and trading strategies $(\tilde{\theta}^i)_{i=1}^m$ such that given $(\beta, S)$, each agent solves (4) and markets clear:

\[
\sum_{i=1}^m \theta^i = 0 \text{ and } \sum_{i=1}^m c^i - e^i = 0
\]

• Notice that this is an “endowment” economy, so that the aggregate consumption is just generated by the aggregate endowment.

• In equilibrium, agents trade their own endowments by selling and buying financial securities.
2 The Individual Agent Optimization Problem

• We first look at the optimization problem of one given agent.

• This is assumed to be “small” in the sense of taking the price processes as given and optimize his/her intertemporal utility given the prices.

• As one can guess, this problem is the same as the one we solved for in TN1.

• The only difference is that now our investor is not endowed with an initial wealth $w$ but with an endowment process $e$.

• However, we can use the same technique used earlier to make “static” the budget constraint (2)-(3).

• Once this is accomplished, it is intuitive that the resulting optimal strategy would be similar.

• Under the assumptions above, let $Q$ be the equivalent martingale measure defined by

$$\xi_T = \exp \left( -\int_0^T \mathbf{\nu}_t d\mathbf{B}_u - \frac{1}{2} \int_0^T \mathbf{\nu}_u' \mathbf{\nu}_u du \right)$$

• Let the discounted future endowment be denoted by:

$$w^i = E^Q \left[ \int_0^T \beta_t^{-1} e_t^i dt \right]$$

• Notice that the expectation is under the $Q$—measure.
• We recall that for given initial wealth $w^i$, the static budget constraint that we obtained in TN1 was

$$E^Q \left[ \int_0^T \beta_t^{-1} c_t^i dt \right] \leq w^i$$

• Hence, in analogy with what found in TN1, we have that the dynamic budget constraint (2)-(3) can be equivalently expressed as

$$E^Q \left[ \int_0^T \beta_t^{-1} c_t^i dt \right] \leq E^Q \left[ \int_0^T \beta_t^{-1} e_t^i dt \right]$$

(5)

• The way to prove this is to go through the same steps as in Proposition 4 in TN1 and define $c_t^* = c_t - e_t$ and let $w = 0$. It is immediate that one gets (5).

• Finally, using the same method as in proposition 4 in TN1 one obtains:

• **Corollary 1**: The static budget constraint (5) is equivalent to

$$E \left[ \int_0^T \pi_t c_t^i dt \right] \leq E \left[ \int_0^T \pi_t e_t^i dt \right]$$

(6)

• where $\pi_t$ is the state price density defined in (1).

• Notice that the expectation is under the original probability measure $P$. 
2.1 Optimal Consumption

- From the result of TN1, we then obtain the following result.

- Let $\mathcal{I}_u^i : R \rightarrow R$ be the inverse of the instantaneous marginal utility function $u^i_c$, that is, it is such that for every $x$ we have $\mathcal{I}_u^i (u^i_c (x)) = x$.

- **Proposition 1**: Let the price process $(\beta, S)$ be given and assume that condition A is satisfied for agent $i$. Then, there exists a solution to the individual investor’s problem with

$$c^i_t = \mathcal{I}_u^i \left( \lambda_i e^{\phi t} \pi_t \right)$$

- where $\lambda_i$ solves

$$E \left[ \int_0^T \pi_t \mathcal{I}_u^i \left( \lambda_i e^{\phi t} \pi_t \right) dt \right] = E \left[ \int_0^T \pi_t e^i_t dt \right] \quad (7)$$

- The analogy with the result in section 8.1.1 in TN1 is the following:

  1. The inverse marginal utility function: In section 8.1.1 we had $u(c, t)$, with $t$ included in the utility function. Hence, the relationship is

$$u_c(c, t) = e^{-\phi t} u_c(c)$$

Hence, if $x = u_c(c, t)$, its inverse is

$$\mathcal{I}_u(x, t) = \mathcal{I}_u(e^{\phi t} x)$$

This explains why we have the term "$e^{\phi t}$" inside the inverse utility function.
2. We defined the function

\[ w(\lambda) = E \left[ \int_0^T \pi_t I_u (\lambda, \pi_t, t) \, dt \right] \]

and we imposed \( w(\lambda) = w \). Clearly, equation (7) is the same condition.

### 2.2 Optimal Portfolio Weights

- As we pointed out earlier, with complete markets it is necessary only to find the optimal consumption.
- We can find the optimal strategy that finances consumption as a residual. We recall the method here again.
- From the proofs in TN1, we found a few important relationships that we must recall first.
- For convenience, define the wealth at time \( t \) as

\[ W_t^i = \theta_{0,i}^t \beta_t + \theta_i^t S_t \]

- and the discounted wealth as

\[ \tilde{W}_t^i = \beta_t^{-1} W_t^i = \theta_{0,i}^t \beta_t^{-1} + \theta_i^t \tilde{S}_t = \theta_{0,i}^t + \theta_i^t \tilde{S}_t \]

- where we recall that \( \tilde{S}_t = S_t \beta_t^{-1} \) is a martingale under \( Q \).
- Hence

\[ d\tilde{S}_t = \tilde{I}_s \sigma_t d\tilde{B}_t \]
• where $\hat{B}_t$ is a Brownian motion under $Q$ generated by Girsanov’s theorem through the formula

$$\hat{B}_t = B_t + \int_0^t \nu_u du$$

• From the dynamic budget constraint we also have

$$\hat{W}_t = \theta_0^0 + \theta_t^i \cdot \hat{S}_t$$

$$= \int_0^t \theta_u^i \cdot \hat{dS}_u + \int_0^t \beta_u^{-1} (e_u^i - c_u^i) \, du \geq 0$$ (9)

• so that

$$d\hat{W}_t = \beta_t^{-1} (e_t^i - c_t^i) \, dt + \theta_t^i \hat{dS}_t$$

$$= \beta_t^{-1} (e_t^i - c_t^i) \, dt + \theta_t^i I_S \sigma_t d\hat{B}_t$$ (11)

• By defining $c_{t*}^i = c_t^i - e_t^i$ and setting $w = 0$, propositions 4 and 5 in TN 1 imply that the current (discounted) wealth is just equal to the expected discounted value of future consumption minus endowment under $Q$:

$$\hat{W}_t = E_t^Q \left( \int_t^T \beta_u^{-1} c_{u*}^i \, du \right) = E_t^Q \left( \int_t^T \beta_u^{-1} (c_u^i - e_u^i) \, du \right)$$

• This equality is due to the assumption of complete markets: The optimal consumption stream can be thought of as a security, and $\hat{W}_t$ as its price at time $t$.

• To review how to transform these expectations under $Q$ into expectations under $P$, recall that the measure $Q$ is defined
through the Radon-Nikodym derivative $\frac{dQ}{dP} = \xi_T$ and hence that we can use the property that for any random variable $Z$ such that $E^Q(|Z|) < \infty$ we obtain

$$E^Q(Z|\mathcal{F}_t) = \frac{E(\xi_T Z|\mathcal{F}_t)}{E(\xi_T|\mathcal{F}_t)}$$

- We then have the following chain of equalities

$$\tilde{W}^i_t = E_t^Q \left( \int_t^T \beta_u^{-1} (c_u^i - e_u^i) \, du \right) = \frac{E_t \left( \xi_T \int_t^T \beta_u^{-1} (c_u^i - e_u^i) \, du \right)}{E_t (\xi_T)} = \frac{E_t \left( \int_t^T \xi_T \beta_u^{-1} (c_u^i - e_u^i) \, du \right)}{\xi_t}$$

- For notational convenience, we can define

$$J^i_T = \int_0^T \pi_u (c_u^i - e_u^i) \, du$$

- so that we can rewrite the discounted wealth as

$$\tilde{W}^i_t = \frac{1}{\xi_t} E_t \left( J^i_T - J^i_t \right) = \frac{1}{\xi_t} (M^i_t - J^i_t) \quad (12)$$

- where we defined the $P$–martingale $M_t$ as

$$M^i_t = E_t \left( J^i_T \right) \quad (13)$$
• From the Martingale Representation Theorem, there exists a $d-$valued process $\eta^i \in (L^2)^d$ such that

$$M^i_t = M^i_0 + \int_0^t \eta^i_u dB_u$$

(14)

• where we now set $M^i_0 = 0$.

• Recall now that

$$d\xi_t = -\xi_t \nu'_tdB_t$$

• Hence, from Ito’s Lemma

$$d\mathcal{W}^i_t = -\left(\frac{M^i_t - J^i_t}{\xi^2_t}\right)d\xi_t + \frac{1}{\xi_t} (dM^i_t - dJ^i_t)$$

$$+ \left(\frac{M^i_t - J^i_t}{\xi^3_t}\right)(d\xi_t)^2 - \frac{1}{\xi^2_t} d\eta_t \left( dM^i_t - dJ^i_t \right)$$

$$= \left(\frac{M^i_t - J^i_t}{\xi_t}\right)\nu'_tdB_t + \frac{1}{\xi_t} \left( \eta^i_t dB_t - \pi_t \left( c^i_t - e^i_t \right) dt \right)$$

$$+ \left(\frac{M^i_t - J^i_t}{\xi_t}\right)\nu'_tdt + \frac{1}{\xi_t} \nu'_t \eta^i_t dt$$

$$= \mathcal{W}^i_t \nu'_tdB_t + \frac{1}{\xi_t} \eta^i_t dB_t - \beta^{-1}_t \left( c^i_t - e^i_t \right) dt$$

$$+ \mathcal{W}^i_t \nu'_tdt + \frac{1}{\xi_t} \nu'_t \eta^i_t dt$$

• Hence, using again $B_t = \mathcal{B}_t - \int_0^t \nu_u du$ we obtain

$$d\mathcal{W}^i_t = -\beta^{-1}_t \left( c^i_t - e^i_t \right) dt + \frac{1}{\xi_t} \left( \eta^i_t + \mathcal{W}^i_t \nu'_t \right) dB_t$$

(15)
• Comparing (10) with (15), we then must have

$$\theta^i I_s \sigma_t = \frac{1}{\xi_t} \left( \eta^i_t + \widehat{W}^i_t \nu'_t \right) = \frac{\eta^i_t}{\xi_t} + \widehat{W}^i_t \nu'_t$$

• Multiplying both sides by $\beta_t$ we finally obtain

$$\theta^i I_s \sigma_t = \frac{\eta^i_t}{\pi_t} + W^i_t \nu'_t \tag{16}$$

• where

$$W^i_t = \frac{1}{\pi_t} E_t \left( \int_t^T \pi_u \left( c^i_u - e^i_u \right) du \right) \tag{17}$$

• Using the definition

$$W^i_t = \theta^{0,i}_t \beta_t + \theta^i_t \cdot S_t$$

• we also find the allocation in bonds

$$\theta^{0,i}_t = \beta_t^{-1} \left( W^i_t - \theta^i_t \cdot S_t \right)$$
3 Equilibrium and the Representative Agent

- So far we merely repeated the exercise in TN1. Now, we impose market clearing conditions and obtain the equilibrium results.

- In this section we are going to skip even more of the details.

- Let the aggregate endowment be denoted as
  \[ e = \sum_{i=1}^{m} e^i \]

- From the results in the previous section, we then have the following

- **Corollary 1**: In any equilibrium, we must have
  \[ e_t = \sum_{i=1}^{m} I^i_u \left( \lambda_i e^{\phi t} \pi_t \right) \]  
  \[ (18) \]

  - where \( \lambda_i \) satisfy the system of equations
    \[ E \left[ \int_{0}^{T} \pi_t \left( I^i_u \left( \lambda_i e^{\phi t} \pi_t \right) - e^i_t \right) dt \right] = 0 \]  
    \[ (19) \]

- The converse is also true: If there exists a vector \( \lambda = (\lambda_1, \ldots, \lambda_m) \) such that (18) and (19) are satisfied, then the market is in equilibrium. In either case, the optimal consumption is given by
  \[ c^i_t = I^i_u \left( \lambda_i e^{\phi t} \pi_t \right) \]  
  \[ (20) \]
The only part of the proof yet to determine is that if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that (18) and (19) are satisfied, then the resulting market is in equilibrium.

This is true because

1. If the vector exists, then we know that (20) maximizes utility.
2. Hence, (18) implies that the commodity market is cleared
   \[ \sum_{i=1}^{m} c^i = \sum_{i=1}^{m} e^i \]

3. From the previous proof, recall that portfolio weights were determined by the martingale
   \[ M_t^i = \int_0^t \eta_u^i dB_u = E_t \left[ \int_0^T \pi_u \left( c_u^i - e_u^i \right) du \right] \]
   * From (19) we have $\sum_{i=1}^{n} M_t^i = 0$ which implies $\sum_{i=1}^{n} \eta^i = 0$.
   * Summing over (17) and then (16) we finally find
     \[ \sum_{i=1}^{m} W_t^i = 0, \sum_{i=1}^{m} \theta_t^i = 0, \sum_{i=1}^{m} \theta_t^{0,i} = 0 \]

Equation (18) is key in the construction of a representative agent.

Given the representative agent, we shall characterize the parameters of the process.

Given a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$, let’s define the function
\[ I_u(x, \lambda) = \sum_{i=1}^{m} I_u^i(\lambda_i x) \]
This a continuous, decreasing function for \( x \in (0, \infty) \).

Define by \( \mathcal{U}_c (\cdot ; \lambda) \) the inverse of \( \mathcal{I} (\cdot ; \lambda) \), that is, that function such that for every \( x \), we have
\[
\mathcal{U}_c (\mathcal{I}_u (x, \lambda), \lambda) = x
\]
(21)

This function is strictly decreasing on \((0, \infty)\).

Notice that the monotonicity of the function implies
\[
\mathcal{I}_u (\mathcal{U}_c (c, \lambda), \lambda) = c
\]

Using these definitions, we can rewrite (18) as
\[
et_t = \mathcal{I}_u (e^{\phi t} \pi_t, \lambda)
\]

We can then invert this function to obtain a formula for the state-price density
\[
\pi_t = e^{-\phi t} \mathcal{U}_c (e_t; \lambda)
\]
(22)

The form of this state price density should look familiar.

From corollary 1, we have

**Corollary 2**: Under condition A, a financial market is in equilibrium if and only if its state-price density is given by \( \pi_t \) in (22) where \( \lambda \) satisfies the system of equations
\[
E \left[ \int_0^T e^{-\phi t} \mathcal{U}_c (e_t; \lambda) \left( \mathcal{I}_u i (\lambda; \mathcal{U}_c (e_t; \lambda)) - e_t^i \right) dt \right] = 0
\]
(23)
• In addition, consumption is given by
\[ c_t^i = \mathcal{I}_u^i (\lambda_t \mathcal{U}_c (e_t; \lambda)) \] (24)

• Corollary 2 characterizes an equilibrium, if it exists.

3.1 The Representative Agent

• Define the function
\[ \mathcal{U} (c; \lambda) = \max_{c_1 \geq 0, \ldots, c_m \geq 0: \sum_{i=1}^m c_i = c} \frac{1}{\lambda_i} u^i (c_i) \] (25)

• We then have

• **Proposition 1:** \( \mathcal{U} (c; \lambda) \) is a strictly increasing utility function satisfying condition A. In addition
\[ \frac{d\mathcal{U} (c, \lambda)}{dc} = \mathcal{U}_c (c, \lambda) \]

• as defined in (21).

• The key point in the proof is to define
\[ \hat{c}_t^i = \mathcal{I}_u^i (\lambda_t \mathcal{U}_c (c; \lambda)) \] (26)

• and show that effectively, these \( \hat{c}_t^i \) satisfy (25). This is easy to see. Notice that (26) implies that
\[ u_c^i (\hat{c}_t^i) = \lambda_i \mathcal{U}_c (c; \lambda) \] (27)
• Hence, consider now any other \((c_1, \ldots, c_m)\) such that \(\sum_{i=1}^{m} c_i = c\).

• From the strict concavity of the utility functions we have
\[
\frac{1}{\lambda_i} u^i(c^i) \leq \frac{1}{\lambda_i} u^i(\bar{c}^i) + (c^i - \bar{c}^i) u^i_c(\bar{c}^i) \]
\[
= \frac{1}{\lambda_i} u^i(\bar{c}^i) + \mathcal{U}_c(c; \lambda) \sum_{i=1}^{m} (c^i - \bar{c}^i) \]
\[
= \frac{1}{\lambda_i} u^i(\bar{c}^i) \]

• So, we have defined a representative agent who assign constant weights \(\lambda = (\lambda_1, \ldots, \lambda_m)\) to the various agents 1 to \(m\).

• Given the result in (23) and (26), the weights \(\lambda_1, \ldots, \lambda_m\) are chosen so that the optimal consumption of the representative agent equals the aggregate endowment.

• We finally notice the following homogeneity property of the aggregate (representative) utility function:

• For every constant \(\alpha\), we have
\[
\mathcal{U}(c, \alpha \lambda) = \alpha^{-1} \mathcal{U}(c, \lambda) \]
\[
\mathcal{U}_c(c, \alpha \lambda) = \alpha^{-1} \mathcal{U}_c(c, \lambda) \]

• Since the marginal utility \(\mathcal{U}_c(c, \alpha \lambda)\) determines the state price density, this property is convenient to renormalize the state price density.
• **Theorem 1**: There exists a vector $\lambda$ that satisfies the system (23). In addition, if for all $i$

$$\frac{-cu_{cc}(c)}{u_c(c)} \leq 1$$

this solution is “unique” in the sense that any other solution $\hat{\lambda}$ implies the existence of a constant $\alpha$ such that

$$\alpha U_c(e_t, \lambda) = U_c(e_t, \hat{\lambda})$$

3.2 Characterizing the Equilibrium

• We now have all the ingredients to solve back for the parameters of the price process and the interest rate process.

• First, notice that we can renormalize the state price density so that

$$\tilde{\pi}_0 = U_c(e_0; \lambda) = 1$$

• Define the process

$$\tilde{\xi}_t = U_c(e_t; \lambda)$$

• By Ito’s lemma we have

$$\tilde{\xi}_t = 1 + \int_0^t U_{cc}(e_u; \lambda) \, de_u + \frac{1}{2} \int_0^t U_{ccc}(e_u; \lambda) \, (de_u)^2$$

• If we assume that the aggregate endowment evolves according to the Ito process

$$de_t = e_t \mu_{e,t} \, dt + e_t \sigma_{e,t} \, dB_t$$
• we obtain
\[ \hat{\xi}_t = 1 + \int_0^t \left( U_{cc}(e_u; \lambda) e_u \mu_{e,u} + \frac{1}{2} U_{ccc}(e_u; \lambda) e_u^2 \sigma_{e,u} \sigma'_{e,u} \right) du \]
\[ + \int_0^t U_{cc}(e_u; \lambda) e_u \sigma_{e,u} dB_u \] (28)

• Recall that from (22), we must have that the state price density is
\[ \hat{\pi}_t = e^{-\phi t} \hat{\xi}_t \]

• On the other hand, we also have that given a system of prices \((\beta, S)\), each following an Ito’s process as described at the beginning of the teaching notes, we must have that the state price density is
\[ \pi_t = \beta_t^{-1} \xi_t \]

• where
\[ \xi_t = \exp \left( -\frac{1}{2} \int_0^t \nu_u \nu'_u du - \int_0^t \nu_u dB_u \right) \]

• Notice that since \(\xi_0 = 1\), \(\xi_t\) satisfies the integral equation
\[ \xi_t = 1 - \int_0^t \xi_u \nu'_u dB_u \]

• Define
\[ \tilde{\xi}_t = \pi_t e^{\phi t} = \xi_t e^{\int_0^t (\phi - r_u) du} \]
• We then have
  \[ \tilde{\xi}_t = 1 + \int_0^t (\phi - r_u) \tilde{\xi}_u \, dt - \int_0^t \xi_u \nu'_u \, dB_u \]  
  (29)

• By definition, in equilibrium we must have
  \[ \hat{\pi}_t = \pi_t \]

• or
  \[ \tilde{\xi}_t = e^{\phi t} \hat{\pi}_t = e^{\phi t} \pi_t = \tilde{\xi}_t \]

• Hence, equating the integral equations (28) and (29) we obtain the equalities
  \[ (\phi - r_t) \tilde{\xi}_t = \mathcal{U}_{cc}(e_t; \lambda) e_t \mu_{e,t} + \frac{1}{2} \mathcal{U}_{ccc}(e_t; \lambda) e^2_t \sigma_{e,t} \sigma'_{e,t} \]
  \[ \xi_t \nu'_t = -\mathcal{U}_{cc}(e_t; \lambda) e_t \sigma_{e,t} \]

• Recall that by definition \( \tilde{\xi}_t = \hat{\xi}_t = \mathcal{U}_c(e_t; \lambda) \), obtaining
  \[ r_t = \phi - \frac{\mathcal{U}_{cc}(e_t; \lambda) e_t}{\mathcal{U}_c(e_t; \lambda)} \mu_{e,t} - \frac{1}{2} \frac{\mathcal{U}_{ccc}(e_t; \lambda) e^2_t}{\mathcal{U}_c(e_t; \lambda)} \sigma_{e,t} \sigma'_{e,t} \]
  \[ \nu_t = -\frac{\mathcal{U}_{cc}(e_t; \lambda) e_t}{\mathcal{U}_c(e_t; \lambda)} \sigma'_{e,t} \]

• We can further define the relative risk aversion of the representative agent as
  \[ \gamma(e_t; \lambda) = -\frac{\mathcal{U}_{cc}(e_t; \lambda) e_t}{\mathcal{U}_c(e_t; \lambda)} \]
and the relative prudence coefficient of the representative agent as

\[ q(e_t; \lambda) = -\frac{U_{ccc}(e_t; \lambda)e_t}{U_{cc}(e_t; \lambda)} \]

- we can rewrite the equations as

\[ r_t = \phi + \gamma(e_t; \lambda) \mu_{e,t} - \frac{1}{2} \gamma(e_t; \lambda) q(e_t; \lambda) \sigma_{e,t} \sigma'_{e,t} \]

\[ \nu_t = \gamma(e_t; \lambda) \sigma'_{e,t} \]

- Therefore, we conclude that

1. The risk-free rate increases linearly with the discount rate \( \phi \) and the relative risk aversion coefficient \( \gamma(e_t; \lambda) \), while it decreases with the relative prudence parameter \( q(e_t; \lambda) \) and with the variance of the endowment process \( \sigma_{e,t} \sigma'_{e,t} \);

2. The market price of risk \( \nu_t \) increases linearly with the relative risk aversion coefficient and the variability of the endowment process \( \sigma'_{e,t} \) (recall that \( \nu_t \) is a vector).

- These results immediately imply the following equilibrium concept.
4 The Equity Premium and the Consumption CAPM

- From the definition of \( \nu_t \) we have

\[
\sigma_t \cdot \nu_t = \mu_t - r_t 1_d
\]

- We then obtain

\[
\mu_t - r_t 1_d = \gamma (e_t; \lambda) \sigma_t \sigma_{e,t}'
\]

- That is

\[
E \left[ \frac{dS^i_t}{S^i_t} \right] - r_t dt = \gamma (e_t; \lambda) \times Cov_t \left( \frac{dS^i_t}{S^i_t}, \frac{de_t}{e_t} \right)
\]

(30)

- That is, the expected excess returns of asset \( i \) depends on the relative risk aversion and the covariance between asset \( i \) and the aggregate endowment process.

- There are two puzzles here:

1. Equation (30) must hold for the market as a whole.

   - The correlation between stock returns and consumption growth is anywhere between .12 and -.05 (see Campbell and Cochrane (1999), Table 7)

   - Even with the optimistic assumptions of correlation = .12, volatility of return = .17 and volatility of consumption growth =.02, we have expected excess return = .4% even with \( \gamma = 10 \).
• So, $\gamma = 100$ would be needed here to obtain an equity premium = 4%.

• Note that the risk free rate may actually end up being reasonable if $\gamma$ is high enough: For instance, $\gamma = 100$ produces $r = 0$ if $\phi = .02$, $\mu_c = .02$ and $\sigma_e = .02$.

• The reason is that the precautionary saving motive $-1/2\gamma^* (\gamma + 1) \sigma^2_e$ kicks in.

2. The second puzzle is about the cross-section: The consumption CAPM does not seem to work.

• Using the Fama French size/book-to-market portfolios as test portfolios, a Fama-MacBeth regression of returns on consumption growth yields an insignificant coefficient $= .22$ and cross-sectional $R^2 = 16\%$ (see Lettau and Ludvigson (2001, Table 3).

• Recent papers however point at noise in the consumption data and they show that if one uses more lags and leads to compute consumption growth, the result works out.

• Typical pitfall: Note that even if a Fama-MacBeth coefficient turns out to be positive and significant (and the $R^2$ is high) there is still an issue of economic significance: The $\gamma$ needed to rationalize the result may be too high, as in the case of the aggregate market.

4.1 The CAPM

• From (30) it is possible to find a “beta” relationship.

• Define the $1 \times d$ process $\psi_t$ using the relationship

\[ \psi_t \cdot I_S \cdot \sigma_t = e_t \cdot \sigma_{e,t} \]
• We can consider $\psi_t$ as a self-financing strategy $(\theta^0, \theta)$ by choosing $\theta^i_t = \psi^i_t$ and $\theta^0_t$ to meet the self-financing constraint.

• Let $S^\psi_t$ be the value of the portfolio $S^\psi_t = \psi_t \cdot S_t$. We then have

$$\frac{dS^\psi_t}{S^\psi_t} = \mu^\psi_{,t} dt + \sigma^\psi_{,t} dB_t$$

• where

$$\sigma^\psi_{,t} = \frac{1}{S_t^\psi} \psi_t \cdot I_{S_t} \cdot \sigma_t = \frac{e_t}{S_t^\psi} \sigma_{e,t}$$

• and $\mu^\psi_{,t}$ satisfies the condition (30)

$$\mu^\psi_{,t} - r_t = \gamma(e_t; \lambda) \times \sigma^\psi_{,t} \sigma'_{e,t}$$

(31)

• Hence, we finally have that for all $i = 1, .., n$

$$\mu^i_t - r_t = \gamma(e_t; \lambda) \sigma^i_t \sigma'_{e,t}$$

$$= \gamma(e_t; \lambda) \times \frac{S_t^\psi}{e_t} \times \sigma^i_t \sigma'_{\psi,t}$$

• Since this holds for the portfolio $\psi$ as well, that is

$$\mu^\psi_t - r_t = \gamma(e_t; \lambda) \times \frac{S_t^\psi}{e_t} \times \sigma^\psi_{,t} \sigma'_{\psi,t}$$
• by substituting for the common term $\gamma (e_t; \lambda) \times \frac{S_t^\psi}{e_t}$ we obtain the CAPM “beta” relationship
\[
E_t \left( \frac{dS_t^i}{S_t^i} \right) - r_t dt = \beta_t^i \times \left( E_t \left( \frac{dS_t^\psi}{S_t^\psi} \right) - r_t 1_d \right)
\]

where
\[
\beta_t^i = \frac{\sigma_t^i \cdot \sigma_{\psi,t}'}{\sigma_{\psi,t} \cdot \sigma_{\psi,t}'} = \frac{\text{cov}_t \left( dS_t^i / S_t^i, dS_t^\psi / S_t^\psi \right)}{\text{var}_t \left( dS_t^\psi / S_t^\psi \right)}
\]

• **Note:**
  - The (conditional) CAPM works with respect to that asset that is perfectly correlated with the endowment process (and thus the stochastic discount factor).
  - This need not be the market portfolio. The existence of labor income, for instance, generate a wedge between the stochastic discount factor and the market portfolio: Thus, the CAPM is not supposed to be working (theoretically) with respect to the market portfolio.
  - But even in the case where there is no labor income it is not obvious that the market portfolio is perfectly correlated with the endowment process.
  - We will do more on this later on in the course.
4.2 An Asset Pricing Relationship

- Suppose that agents $i$ sells a financial asset whose payment is its entire endowment (or a fraction of it), that is $e^i_t$.

- What is the fair price of this stream of payments?

- The above results imply that the price at time $t$ of this claim is

$$S^e_t = \frac{1}{\pi_t} E_t \left[ \int_t^T \pi_u e^i_u du \right]$$  \hspace{1cm} (32)

$$= \frac{1}{U_c(e_t)} E_t \left[ \int_t^T e^{-\phi(u-t)} U_c(e_u) e^i_u du \right]$$ \hspace{1cm} (33)

- In fact, if this was not true one could find a trading strategy $(\theta_0, \theta)$ that finances $e^i_u$ and whose value is (32). This in turn generates an arbitrage opportunity.

- Indeed, the value of total endowment process is simply

$$S_t = \frac{1}{U_c(e_t)} E_t \left[ \int_t^T e^{-\phi(u-t)} U_c(e_u) e_u du \right]$$ \hspace{1cm} (34)

- We will use this pricing equation often.
4.3 A simple example: Log Utility

- Suppose all the agents have logarithmic utility
  \[ u^i(c_t) = \log(c_t) \]

- so that the inverse of the marginal utility is given by
  \[ \mathcal{I}_u^i(x) = x^{-1} \]

- In this case, the marginal utility of the representative investor is
  \[ \mathcal{U}_c(c) = \frac{1}{c} \sum_{i=1}^{m} \lambda_i^{-1} \]

- We can renormalize the weights so that \( \sum_{i=1}^{m} \lambda_i^{-1} = e_0 \), which we can assume positive.

- We then have
  \[ \mathcal{U}_c(e_0) = 1 \]

- The vector \( \lambda = (\lambda_1, \ldots, \lambda_m) \) has to satisfy the system of equations (23),
  \[ E \left[ \int_0^T e^{-\phi t} \mathcal{U}_c(e_t; \lambda) \left( \mathcal{I}_u^i (\lambda_i \mathcal{U}_c(e_t; \lambda)) - e_t^i \right) dt \right] = 0 \quad (35) \]

- which becomes
  \[ E \left[ \int_0^T e^{-\phi t} e_0 \left( \left( \frac{e_0}{e_t} \right)^{-1} - e_t^i \right) dt \right] = 0 \quad (36) \]
and in turn
\[ \lambda_i^{-1} = e_0 \frac{E \left[ \int_0^T e^{-\phi t} \frac{e^i}{e_t} dt \right]}{\int_0^T e^{-\phi t} dt} \]  

(37)

Hence
\[ \pi_t = e^{-\phi t} U_c (e_t; \lambda) = e^{-\phi t} \frac{e_0}{e_t} \]

We then have that
\[ c_t^i = I_u \left( \lambda_i e^{\phi t} \pi_t \right) = I_u \left( \lambda_i \frac{e_0}{e_t} \right) = \lambda_i^{-1} \frac{e_t}{e_0} \]

(38)

Finally, we have
\[ \gamma (e_t; \lambda) = -\frac{e_t U_{cc} (e_t, \lambda)}{U_c (e_t, \lambda)} = -\frac{e_t (e_0/e_t^2)}{e_0/e_t} = 1 \]
\[ q (e_t; \lambda) = -\frac{e_t U_{ccc} (e_t, \lambda)}{U_{cc} (e_t, \lambda)} = -\frac{e_t 2e_0/e_t^3}{(-e_0/e_t^2)} = 2 \]

Hence, the condition for the market equilibrium are
\[ r_t = \phi + \mu_{e,t} - \sigma_{e,t} \sigma'_{e,t} \]
\[ \nu_t = \sigma'_{e,t} \]

4.4 An Alternative Formula for the C-CAPM

Before commenting further, notice also that an alternative expression for the (30) can be obtained as follows.
• We know that at an optimum, the following condition holds (see (27))

\[
u_c^j (\hat{c}_t^j) = \lambda_j U_c (e; \lambda)
\]  

(39)

• Hence, rather than using the representative agent utility function, we may think of using the marginal utility of agent \( j \) defined on the optimal consumption path \( \hat{c}_t^j \).

• Equation (39) ensures that the two approaches are identical.

• Define the process

\[
\hat{\xi}_t^j = \frac{1}{\lambda_j} u_c^j (\hat{c}_t^j)
\]

• Define the state-price density of agent \( j \) as

\[
\hat{\pi}_t^j = \beta_t^{-1} \hat{\xi}_t^j
\]

• By going through the same typo of calculation, it is clear that everywhere we can substitute the “representative agent” utility and endowment, with agent \( i \) utility and consumption.

• As a consequence, one then obtains

\[
E \left[ \frac{dS_t^i}{S_t^i} \right] - r_t dt = \gamma^j (\hat{c}_t^j) \times Cov_t \left( \frac{dS_t^i}{S_t^i}, \frac{d\hat{c}_t^j}{\hat{c}_t^j} \right)
\]

\[
= a^j (\hat{c}_t^j) \times Cov_t \left( \frac{dS_t^i}{S_t^i}, d\hat{c}_t^j \right)
\]
where

\[ a^j \left( \hat{c}_t^j \right) = -\frac{u_c^j (\hat{c}_t^j)}{u_t^j (\hat{c}_t^j)} \]

is the coefficient of absolute risk aversion of agent \( j \).

Divide now both sides by \( a^j \left( \hat{c}_t^j \right) \) and sum across \( j = 1, \ldots, m \). Since from the market clearing condition \( \sum_{j=1}^{n} \hat{c}_t^j = e_t \) we must have

\[
\sum_{j=1}^{m} Cov_t \left( \frac{dS_t^i}{S_t^i}, d\hat{c}_t^j \right) = Cov_t \left( \frac{dS_t^i}{S_t^i}, d\hat{c}_t^j \right)
\]

we obtain

\[
E \left[ \frac{dS_t^i}{S_t^i} \right] - r_t dt = \Gamma(c_t) \times Cov_t \left( \frac{dS_t^i}{S_t^i}, de_t \right)
\]

The coefficient

\[
\Gamma(c_t) = \frac{1}{\sum_{j=1}^{m} a^j \left( \hat{c}_t^j \right)}
\]

is the coefficient of absolute risk aversion of the market itself.