Modern Dynamic Asset Pricing Models

Lecture Notes 2.

Equilibrium with Complete Markets\(^1\)

Pietro Veronesi

*The University of Chicago*

*Booth School of Business*

*CEPR, NBER*

\(^1\)These teaching notes draw heavily on Duffie (1996, Chapters 9 and 10) and Karatzas and Shevre (1999, Chapter 3 and 4). They are intended for students of Business 35907 only. Please, do not distribute without my prior consent.
We now use the results in TN 1 to determine the competitive equilibrium.

The notion of equilibrium in this setup is as follows:

1. There are \( m \) agents in the economy, each endowed with a stream of consumption good;
2. The consumption good is immediately perishable, so that it must be consumed immediately;
3. Agents can trade their endowments, by selling/buying financial securities;
4. All financial securities are in zero-net supply: For every buyer there must be a seller.

This is the standard, general equilibrium notion of a pure-exchange economy.

Notice in particular that there is no production.
Primitives

• Let $B = (B^1, ..., B^d)$ be a $d$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t\}$ denote the standard filtration of $B$.

• Let us fix a horizon $T$ and let the consumption space be the set $L$ of adapted processes such that $E \left( \int_0^T c_i^2 dt \right) < \infty$.

• Suppose there are $m$ agents, indexed by $i = 1, .., m$.

• Each agent receives an endowment $\{e^i_t\} \in L_+$

• Each agent has a strictly increasing utility function $U_i : L_+ \rightarrow \mathbb{R}$.

• All agents have a common discount rate $\phi$. We shall assume it constant, but it could also be a function of time.

• Each agent will maximize $U(c^i) = E_0 \left[ \int_0^T e^{-\phi t} u_i(c^i_t) dt \right]$

• Notice that we assume here no utility from final wealth, although it could have been inserted without any trouble.

• We shall assume everywhere that condition A in TN1 hold.
Financial Markets

- We assume complete markets. With no loss of generality, let there be $d$ risky securities with price processes

$$dS_t = I_S \mu_t dt + I_S \sigma_t dB_t$$

where $I_S$ is the diagonal matrix with $S_i$ on the $i$-th element, and $\mu_t$ and $\sigma_t$ are adapted processes in $\mathcal{L}$ and $\mathcal{L}^2$.

- Market completeness is achieved by assuming that $\sigma_t$ is invertible almost everywhere.

- Also, there is a risk-free security, with short rate process $r_t$ and price

$$\beta_t = e^{\int_0^t r_u du}$$

- As in the previous notes, define the market price of risk process

$$\nu_t = \sigma_t^{-1}(\mu_t - r_t 1_d)$$

- Assume that the Novikov’s condition is satisfied

$$E\left(\exp\left(\frac{1}{2} \int_0^T \nu_t' \nu_t dt\right)\right) < \infty$$
• Let

$$\xi_t = \exp \left( -\int_0^t \nu'_u dB_u - \frac{1}{2} \int_0^t \nu'_u \nu_u du \right)$$

• We recall that from Novikov’s theorem, $\xi_t$ is a $P$-martingale.

• Finally, define the state-price density process

$$\pi_t = \beta_t^{-1} \xi_t$$ (1)

• We saw already that $\pi_t$ is such that $S^{\pi} = S\pi$ is a martingale.

• As usual, a trading strategy $(\theta^0, \theta)$ is a vector process in $\mathcal{H}^2 (S)$, that is, a space with sufficient integrability conditions to rule out doubling strategies.

• For convenience, let $\tilde{\theta} = (\theta^0, \theta)$, $\tilde{S} = (\beta, S)$.

• A trading strategy $\tilde{\theta} = (\theta^0, \theta)$ finances a consumption process $c$ given income $e$ if

$$\tilde{\theta}_t \cdot \tilde{S}_t = \int_0^t \tilde{\theta}_u \cdot d\tilde{S}_u + \int_0^t (e_u - c_u) \, du \geq 0$$ (2)

$$\tilde{\theta}_T \cdot \tilde{S}_T = 0$$ (3)
• Notice that the wealth $W_t = \tilde{\theta}_t \cdot \tilde{S}_t$ grows also because of the additional endowment that has to be taken into account.

• The last equality means that no obligations are left at horizon.

• Each agent then faces the problem

$$\sup_{(c, \theta^0, \theta) \in \Lambda_i} U_i \left(c^i\right)$$

where $\Lambda_i = \left\{ (c, \theta^0, \theta) \in L_+ \times H^2(S) \text{ such that } (\theta^0, \theta) \text{ finances } c^i \text{ given } e^i \right\}$

• A security-spot market equilibrium is a collection of price processes $(\beta, S)$, consumption processes $(c^i)_{i=1}^m$ and trading strategies $(\tilde{\theta}^i)_{i=1}^m$ such that given $(\beta, S)$, each agent solves (4) and markets clear:

$$\sum_{i=1}^m \theta^i = 0 \text{ and } \sum_{i=1}^m c^i - e^i = 0$$

• Notice that this is an “endowment” economy, so that the aggregate consumption is just generated by the aggregate endowment.

• In equilibrium, agents trade their own endowments by selling and buying financial securities.
The Individual Agent Optimization Problem

- We first look at the optimization problem of one given agent.
- This is assumed to be “small” in the sense of taking the price processes as given and optimize his/her intertemporal utility given the prices.
- As one can guess, this problem is the same as the one we solved for in TN1.
- The only difference is that now our investor is not endowed with an initial wealth $w$ but with an endowment process $e$.
- However, we can use the same technique used earlier to make “static” the budget constraint (2)-(3).
- Once this is accomplished, it is intuitive that the resulting optimal strategy would be similar.
- Under the assumptions above, let $Q$ be the equivalent martingale measure defined by

$$\xi_T = \exp \left(- \int_0^T \nu'_t dB_u - \frac{1}{2} \int_0^T \nu'_u \nu_u du \right)$$

- Let the discounted future endowment be denoted by:

$$w^i = E^Q \left[ \int_0^T \beta_t^{-1} e^i_t dt \right]$$
• Notice that the expectation is under the $Q$—measure.

• We recall that for given initial wealth $w^i$, the static budget constraint that we obtained in TN1 was

$$E^Q \left[ \int_0^T \beta_t^{-1} c^i_t \, dt \right] \leq w^i$$

• Hence, in analogy with what found in TN1, we have that the dynamic budget constraint (2)-(3) can be equivalently expressed as

$$E^Q \left[ \int_0^T \beta_t^{-1} c^i_t \, dt \right] \leq E^Q \left[ \int_0^T \beta_t^{-1} e^i_t \, dt \right]$$

(5)

• The way to prove this is to go through the same steps as in Proposition 4 in TN1 and define $c^*_t = c_t - e_t$ and let $w = 0$. It is immediate that one gets (5).

• Finally, using the same method as in proposition 4 in TN1 one obtains:

• **Corollary 1**: The static budget constraint (5) is equivalent to

$$E \left[ \int_0^T \pi_t c^i_t \, dt \right] \leq E \left[ \int_0^T \pi_t e^i_t \, dt \right]$$

(6)

— where $\pi_t$ is the state price density defined in (1).

• Notice that the expectation is under the original probability measure $P$. 
Optimal Consumption and Portfolio Weights

- From the result of TN1, we then obtain the following result.
- Let \( I^i_u : R \rightarrow R \) be the inverse of the instantaneous marginal utility function \( u^i_c \), that is, it is such that for every \( x \) we have \( I^i_u (u^i_c (x)) = x \).
- **Proposition 1**: Let the price process \((\beta, S)\) be given and assume that condition A is satisfied for agent \( i \). Then, there exists a solution to the individual investor’s problem with

\[
c^i_t = I^i_u (\lambda_i e^{\phi t} \pi_t)
\]

where \( \lambda_i \) solves

\[
E \left[ \int_0^T \pi_t I^i_u (\lambda_i e^{\phi t} \pi_t) \, dt \right] = E \left[ \int_0^T \pi_t e^{\phi t} dt \right]
\]

(7)

- The relation with the result in TN1 is the following:

1. The inverse marginal utility function: In TN 1 we had \( u(c, t) \), with \( t \) included in the utility function. Hence, the relationship is

\[
u_c (c, t) = e^{-\phi t} u_c (c)
\]

Hence, if \( x = u_c (c, t) \), its inverse is

\[
I_u (x, t) = I_u (e^{\phi t} x)
\]

This explains why we have the term “\( e^{\phi t} \)” inside the inverse utility function.
2. We defined the function

\[ w(\lambda) = E \left[ \int_0^T \pi_t I_u(\lambda \pi_t, t) \, dt \right] \]

and we imposed \( w(\lambda) = w \). Clearly, equation (7) is the same condition.

- We finally find the weights to support the optimal consumption. This is identical to our findings in TN 1, but it is useful to review them here again.

- From the proofs in TN1, we found a few important relationships that we must recall first.

- For convenience, define the wealth at time \( t \) as

\[ W_t^i = \theta_t \beta_t + \theta_t^i S_t \]

- and the discounted wealth as

\[ \hat{W}_t^i = \beta_t^{-1} W_t^i = \theta_t^0 \beta_t^{-1} + \theta_t^i S_t \beta_t^{-1} = \theta_t^0 \beta_t^{-1} + \theta_t^i \hat{S}_t \]

— where we recall that \( \hat{S}_t = S_t \beta_t^{-1} \) is a martingale under \( Q \).
Optimal Consumption and Portfolio Weights

- Hence
  \[ d\hat{S}_t = I_{\hat{S}}^\sigma_t d\hat{B}_t \]

- where \( \hat{B}_t \) is a Brownian motion under \( Q \) generated by Girsanov’s theorem through the formula
  \[ \hat{B}_t = B_t + \int_0^t \nu_u du \]

- From the dynamic budget constraint we also have
  \[ \hat{W}_t^i = \theta_{t_0}^{0,i} + \theta^i_t \cdot \hat{S}_t \]
  \[ = \int_0^t \theta^i_u \cdot d\hat{S}_u + \int_0^t \beta^{-1}_u (e^i_u - c^i_u) \, du \geq 0 \] (8)

  so that
  \[ d\hat{W}_t^i = \beta^{-1}_t (e^i_t - c^i_t) \, dt + \theta^i_t I_{\hat{S}}^\sigma_t d\hat{B}_t \] (10)
By defining \( c^*_i \equiv c^i_t - e^i_t \) and setting \( w = 0 \), propositions 4 and 5 in TN 1 imply that the current (discounted) wealth is just equal to the expected discounted value of future consumption minus endowment under \( Q \):

\[
\hat{W}_t = E_t^Q \left( \int_t^T \beta_u^{-1} c^*_u \, du \right) = E_t^Q \left( \int_t^T \beta_u^{-1} (c^i_u - e^i_u) \, du \right)
\]

This equality is due to the assumption of complete markets: The optimal consumption stream can be thought of as a security, and \( \hat{W}_t \) as its price at time \( t \).

To review how to transform these expectations under \( Q \) into expectations under \( P \), recall that the measure \( Q \) is defined through the Radon-Nikodym derivative \( \frac{dQ}{dP} = \xi_T \) and hence that we can use the property that for any random variable \( Z \) such that \( E^Q(|Z|) < \infty \) we obtain

\[
E^Q(Z|\mathcal{F}_t) = \frac{E(\xi_T Z|\mathcal{F}_t)}{E(\xi_T|\mathcal{F}_t)}
\]
Optimal Consumption and Portfolio Weights

- We then have the following chain of equalities

\[
\tilde{W}_t^i = E_t^Q \left( \int_t^T \beta_u^{-1} (c_u^i - e_u^i) \, du \right) = \frac{E_t \left( \xi_T \int_t^T \beta_u^{-1} (c_u^i - e_u^i) \, du \right)}{E_t (\xi_T)} = \frac{E_t \left( \int_t^T \xi_u \beta_u^{-1} (c_u^i - e_u^i) \, du \right)}{E_t (\xi_T)}
\]

- For notational convenience, we can define

\[
J_T^i = \int_0^T \pi_u (c_u^i - e_u^i) \, du
\]
Optimal Consumption and Portfolio Weights

• We can rewrite the discounted wealth as

\[ \hat{W}^i_t = \frac{1}{\xi_t} E_t \left( J^i_T - J^i_t \right) = \frac{1}{\xi_t} \left( M^i_t - J^i_t \right) \]  
(12)

• where we defined the \( P \)-martingale \( M_t \) as

\[ M^i_t = E_t \left( J^i_T \right) \]  
(13)

• From the Martingale Representation Theorem, there exists a \( d \)-valued process \( \eta^i \in \mathcal{L}^2 \) such that

\[ M^i_t = M^i_0 + \int_0^t \eta^i_u d\mathbf{B}_u \]  
(14)

  where we now set \( M^i_0 = 0 \).

• Recall now that

\[ d\xi_t = -\xi_t \nu_t' dB_t \]
• Hence, from Ito’s Lemma

\[ \hat{W}_t^i = -\frac{(M_t^i - J_t^i)}{\xi_t^2} d\xi_t + \frac{1}{\xi_t} (dM_t^i - dJ_t^i) \]

\[ + \frac{(M_t^i - J_t^i)}{\xi_t^3} (d\xi_t)^2 - \frac{1}{\xi_t^2} d\xi_t (dM_t^i - dJ_t^i) \]

\[ = \frac{(M_t^i - J_t^i)}{\xi_t} \nu_t dB_t + \frac{1}{\xi_t} (\eta_t dB_t - \pi_t (c_t^i - e_t^i) dt) \]

\[ + \frac{(M_t^i - J_t^i)}{\xi_t} \nu_t \nu_t dt + \frac{1}{\xi_t} \nu_t \eta_t^i dt \]

\[ = \hat{W}_t^i \nu_t dB_t + \frac{1}{\xi_t} \eta_t dB_t - \beta_t^{-1} (c_t^i - e_t^i) dt \]

\[ + \hat{W}_t^i \nu_t \nu_t dt + \frac{1}{\xi_t} \nu_t \eta_t^i dt \]

• Hence, using again \( B_t = \hat{B}_t - \int_0^t \nu_u du \) we obtain

\[ d\hat{W}_t^i = -\beta_t^{-1} (c_t^i - e_t^i) dt + \frac{1}{\xi_t} \left( \eta_t^i + \hat{W}_t^i \nu_t' \right) d\hat{B}_t \]
Optimal Consumption and Portfolio Weights

• Comparing (10) with (15), we then must have

\[ \theta_t^i I_S \sigma_t = \frac{1}{\xi_t} \left( \eta_t^i + \hat{W}_t^i \nu'_t \right) = \frac{\eta_t^i}{\xi_t} + \hat{W}_t^i \nu'_t \]

• Multiplying both sides by \( \beta_t \) we finally obtain

\[ \theta_t^i I_S \sigma_t = \frac{\eta_t^i}{\pi_t} + W_t^i \nu'_t \]  

(16)

— where

\[ W_t^i = \frac{1}{\pi_t} E_t \left( \int_t^T \pi_u (c_u^i - e_u^i) du \right) \]  

(17)

• Using the definition

\[ W_t^i = \theta_t^{0,i} \beta_t + \theta_t^i \cdot S_t \]

• we also find the allocation in bonds

\[ \theta_t^{0,i} = \beta_t^{-1} (W_t^i - \theta_t^i \cdot S_t) \]
Equilibrium and the Representative Agent

- So far we merely repeated the exercise in TN1. Now, we impose market clearing conditions and obtain the equilibrium results.
- In this section we are going to skip even more of the details.
- Let the aggregate endowment be denoted as

\[ e = \sum_{i=1}^{m} e^i \]

- From the results in the previous section, we then have the following
- **Corollary 1**: In any equilibrium, we must have

\[ e_t = \sum_{i=1}^{m} \mathcal{I}_u^i (\lambda_i e^{\phi t} \pi_t) \]  

(18)

- where \( \lambda_i \) satisfy the system of equations

\[ E \left[ \int_0^T \pi_t (\mathcal{I}_u^i (\lambda_i e^{\phi t} \pi_t) - e^i_t) \ dt \right] = 0 \]  

(19)
Equilibrium and the Representative Agent

- The converse is also true: If there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that (18) and (19) are satisfied, then the market is in equilibrium. In either case, the optimal consumption is given by

$$c_t^i = \mathcal{I}_{u}^i \left( \lambda_i e^{\phi_t \pi_t} \right)$$

(20)

- The only part of the proof yet to determine is that if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that (18) and (19) are satisfied, then the resulting market is in equilibrium.

- This is true because
  1. If the vector exists, then we know that (20) maximizes utility.
  2. Hence, (18) implies that the commodity market is cleared

$$\sum_{i=1}^{m} c^i = \sum_{i=1}^{m} e^i$$

  3. From the previous proof, recall that portfolio weights were determined by the martingale

$$M_t^i = \int_0^t \eta_u^i dB_u = E_t \left[ \int_0^T \pi_u (c_u^i - e_u^i) \, du \right]$$
From (19) we have \( \sum_{i=1}^{n} M_i^t = 0 \) which implies \( \sum_{i=1}^{n} \eta^i = 0 \).

Summing over (17) and then (16) we finally find

\[
\sum_{i=1}^{m} W_i^t = 0, \quad \sum_{i=1}^{m} \theta^i_t = 0, \quad \sum_{i=1}^{m} \theta_0^t = 0
\]

Equation (18) is key in the construction of a representative agent.

- Given the representative agent, we shall characterize the parameters of the process.

Given a vector \( \lambda = (\lambda_1, \ldots, \lambda_m) \), let’s define the function

\[
I_u(x, \lambda) = \sum_{i=1}^{m} I^i_u(\lambda_i x)
\]

This a continuous, decreasing function for \( x \in (0, \infty) \).

Define by \( U_c(\cdot; \lambda) \) the inverse of \( I(\cdot; \lambda) \), that is, that function such that for every \( x \), we have

\[
U_c(I_u(x, \lambda), \lambda) = x
\]

This function is strictly decreasing on \( (0, \infty) \).

Notice that the monotonicity of the function implies

\[
I_u(U_c(c, \lambda), \lambda) = c
\]
• Using these definitions, we can rewrite (18) as

\[ e_t = \mathcal{I}_u \left( e^{\phi t} \pi_t, \lambda \right) \]

• We can then invert this function to obtain a formula for the state-price density

\[ \pi_t = e^{-\phi t} \mathcal{U}_c (e_t; \lambda) \]  

(22)

• The form of this state price density should look familiar.

• From corollary 1, we have

• **Corollary 2**: Under condition A, a financial market is in equilibrium if and only if its state-price density is given by \( \pi_t \) in (22) where \( \lambda \) satisfies the system of equations

\[ E \left[ \int_0^T e^{-\phi t} \mathcal{U}_c (e_t; \lambda) \left( \mathcal{I}_u^i (\lambda_i \mathcal{U}_c (e_t; \lambda)) - e_t^i \right) dt \right] = 0 \]  

(23)

• In addition, consumption is given by

\[ c_t^i = \mathcal{I}_u^i (\lambda_i \mathcal{U}_c (e_t; \lambda)) \]  

(24)

• Corollary 2 characterizes an equilibrium, if it exists.
The Representative Agent

- Define the function

\[ U(c; \lambda) = \max_{c_1 \geq 0, \ldots, c_m \geq 0: \sum_{i=1}^{m} c_i = c} \sum_{i=1}^{m} \frac{1}{\lambda_i} u^i(c_i) \]  

(25)

- We then have

- Proposition 1: \( U(c; \lambda) \) is a strictly increasing utility function satisfying condition A. In addition

\[ \frac{dU(c, \lambda)}{dc} = U_c(c, \lambda) \]

— as defined in (21).

- The key point in the proof is to define

\[ \tilde{c}_t^i = I_u^i(\lambda_i U_c(c; \lambda)) \]  

(26)

- and show that effectively, these \( \tilde{c}_t^i \) satisfy (25).
• This is easy to see. Notice that (26) implies that

\[ u^i_c\left(\hat{c}_t^i\right) = \lambda_i U_c\left(c; \lambda\right) \]  

(27)

• Hence, consider now any other \((c_1, \ldots, c_m)\) such that \(\sum_{i=1}^{m} c_i = c\).

• From the strict concavity of the utility functions we have

\[
\sum_{i=1}^{m} \frac{1}{\lambda_i} u^i\left(c^i\right) \leq \sum_{i=1}^{m} \frac{1}{\lambda_i} \left(u^i\left(\hat{c}^i\right) + \left(c^i - \hat{c}^i\right) u_c\left(\hat{c}^i\right)\right)
\]

\[
= \sum_{i=1}^{m} \frac{1}{\lambda_i} u^i\left(\hat{c}^i\right) + U_c\left(c; \lambda\right) \sum_{i=1}^{m} \left(c^i - \hat{c}^i\right)
\]

\[
= \sum_{i=1}^{m} \frac{1}{\lambda_i} u^i\left(\hat{c}^i\right)
\]

• So, we have defined a representative agent who assign constant weights \(\lambda = (\lambda_1, \ldots, \lambda_m)\) to the various agents 1 to \(m\).

• Given the result in (23) and (26), the weights \(\lambda_1, \ldots, \lambda_m\) are chosen so that the optimal consumption of the representative agent equals the aggregate endowment.
• We finally notice the following homogeneity property of the aggregate (representative) utility function:

• For every constant $\alpha$, we have

$$U(c, \alpha \lambda) = \alpha^{-1} U(c, \lambda)$$

$$U_c(c, \alpha \lambda) = \alpha^{-1} U_c(c, \lambda)$$

• Since the marginal utility $U_c(c, \alpha \lambda)$ determines the state price density, this property is convenient to renormalize the state price density.

• **Theorem 1**: There exists a vector $\lambda$ that satisfies the system (23). In addition, if for all $i$

$$-cu_{cc}(c) \leq 1$$

this solution is “unique” in the sense that any other solution $\hat{\lambda}$ implies the existence of a constant $\alpha$ such that

$$\alpha U_c(e_t, \lambda) = U_c(e_t, \hat{\lambda})$$
Characterizing the Equilibrium

- We now have all the ingredients to solve back for the parameters of the price process and the interest rate process.

- First, notice that we can renormalize the state price density so that
  \[
  \hat{\pi}_0 = U_c(e_0; \lambda) = 1
  \]

- Define the process
  \[
  \hat{\xi}_t = U_c(e_t; \lambda)
  \]

- By Ito’s lemma we have
  \[
  \hat{\xi}_t = 1 + \int_0^t U_{cc}(e_u; \lambda) \, de_u + \frac{1}{2} \int_0^t U_{ccc}(e_u; \lambda) \, (de_u)^2
  \]

- If we assume that the aggregate endowment evolves according to the Ito process
  \[
  de_t = e_t \mu_{e,t} \, dt + e_t \sigma_{e,t} \, dB_t
  \]
Characterizing the Equilibrium

• we obtain

\[ \hat{\xi}_t = 1 + \int_0^t \left( \mathcal{U}_{cc}(e_u; \lambda) e_u \mu_{e,u} + \frac{1}{2} \mathcal{U}_{ccc}(e_u; \lambda) e_u^2 \sigma_{e,u} \sigma_{e,u}' \right) du \]

\[ + \int_0^t \mathcal{U}_{cc}(e_u; \lambda) e_u \sigma_{e,u} dB_u \]  

(28)

• Recall that from (22), we must have that the state price density is

\[ \hat{\pi}_t = e^{-\phi t} \hat{\xi}_t \]

• On the other hand, we also have that given a system of prices \((\beta, S)\), each following a Ito’s process as described at the beginning of the teaching notes, we must have that the state price density is

\[ \pi_t = \beta_t^{-1} \xi_t \]

• where

\[ \xi_t = \exp \left( -\frac{1}{2} \int_0^t \nu_u \nu_u' du - \int_0^t \nu_u dB_u \right) \]
• Notice that since $\xi_0 = 1$, $\xi_t$ satisfies the integral equation

$$\xi_t = 1 - \int_0^t \xi_u \nu'_u d\mathbb{B}_u$$

• Define

$$\tilde{\xi}_t = \pi_t e^{\phi t} = \xi_t e\int_0^t (\phi - r_u) du$$

• We then have

$$\tilde{\xi}_t = 1 + \int_0^t (\phi - r_u) \tilde{\xi}_u du - \int_0^t \xi_u \nu'_u d\mathbb{B}_u$$

(29)

• By definition, in equilibrium we must have

$$\hat{\pi}_t = \pi_t$$

• or

$$\hat{\xi}_t = e^{\phi t} \pi_t = e^{\phi t} \pi_t = \tilde{\xi}_t$$
Characterizing the Equilibrium

• Hence, equating the integral equations (28) and (29) we obtain the equalities

\[
(\phi - r_t) \xi_t = U_{cc} (e_t; \lambda) e_t \mu_{e,t} + \frac{1}{2} U_{ccc} (e_t; \lambda) e_t^2 \sigma_{e,t} \sigma'_{e,t} \\
\xi_t \nu'_t = -U_{cc} (e_t; \lambda) e_t \sigma_{e,t}
\]

• Recall that by definition \( \tilde{\xi}_t = \hat{\xi}_t = U_c (e_t; \lambda) \), obtaining

\[
r_t = \phi - \frac{U_{cc} (e_t; \lambda) e_t}{U_c (e_t; \lambda)} \mu_{e,t} - \frac{1}{2} \frac{U_{ccc} (e_t; \lambda) e_t^2}{U_c (e_t; \lambda)} \sigma_{e,t} \sigma'_{e,t} \\
\nu_t = -\frac{U_{cc} (e_t; \lambda) e_t}{U_c (e_t; \lambda)} \sigma'_{e,t}
\]

• We can further define the relative risk aversion of the representative agent as

\[
\gamma (e_t; \lambda) = -\frac{U_{cc} (e_t; \lambda) e_t}{U_c (e_t; \lambda)}
\]

• and the relative prudence coefficient of the representative agent as

\[
q (e_t; \lambda) = -\frac{U_{ccc} (e_t; \lambda) e_t}{U_{cc} (e_t; \lambda)}
\]
Characterizing the Equilibrium

- We can rewrite the equations as

\[ r_t = \phi + \gamma(e_t; \lambda) \mu_{e,t} - \frac{1}{2} \gamma(e_t; \lambda) q(e_t; \lambda) \sigma_{e,t} \sigma'_{e,t} \]
\[ \nu_t = \gamma(e_t; \lambda) \sigma'_{e,t} \]

- Therefore, we conclude that

1. The risk-free rate increases linearly with the discount rate \( \phi \) and the relative risk aversion coefficient \( \gamma(e_t; \lambda) \), while it decreases with the relative prudence parameter \( q(e_t; \lambda) \) and with the variance of the endowment process \( \sigma_{e,t} \sigma'_{e,t} \).

2. The market price of risk \( \nu_t \) increases linearly with the relative risk aversion coefficient and the variability of the endowment process \( \sigma'_{e,t} \) (recall that \( \nu_t \) is a vector).

- These results immediately imply the following equilibrium concept.
The Equity Premium and the Consumption CAPM

• From the definition of $\nu_t$ we have

$$\sigma_t \cdot \nu_t = \mu_t - r_t 1_d$$

• We then obtain

$$\mu_t - r_t 1_d = \gamma(e_t; \lambda) \sigma_t \sigma'_e, t$$

• That is

$$E \left[ \frac{dS^i_t}{S^i_t} \right] - r_t dt = \gamma(e_t; \lambda) \times Cov_t \left( \frac{dS^i_t}{S^i_t}, \frac{de_t}{e_t} \right)$$

(30)

• That is, the expected excess returns of asset $i$ depends on the relative risk aversion and the covariance between asset $i$ and the aggregate endowment process.
There are two puzzles here:

1. Equation (30) must hold for the market as a whole.
   - The correlation between stock returns and consumption growth is anywhere between .12 and -.05 (see Campbell and Cochrane (1999), Table 7).
   - Even with the optimistic assumptions of correlation = .12, volatility of return = .17 and volatility of consumption growth = .02, we have expected excess return = .4% even with $\gamma = 10$.
   - So, $\gamma = 100$ would be needed here to obtain an equity premium = 4%.
   - Note that the risk free rate may actually end up being reasonable if $\gamma$ is high enough: For instance, $\gamma = 100$ produces $r = 0$ if $\phi = .02$, $\mu_c = .02$ and $\sigma_e = .02$.
   - The reason is that the precautionary saving motive $-1/2 \gamma (\gamma + 1) \sigma_e^2$ kicks in.

2. The second puzzle is about the cross-section: The consumption CAPM does not seem to work.
   - Using the Fama French size/book-to-market portfolios as test portfolios, a Fama-MacBeth regression of returns on consumption growth yields an insignificant coefficient = .22 and cross-sectional $R^2 = 16\%$ (see Lettau and Ludvigson (2001, Table 3).
   - Recent papers however point at noise in the consumption data and they show that if one uses more lags and leads to compute consumption growth, the result works out.
   - Typical pitfal: Note that even if a Fama-MacBeth coefficient turns out to be positive and significant (and the $R^2$ is high) there is still an issue of economic significance: The $\gamma$ needed to rationalize the result may be too high, as in the case of the aggregate market.
From (30) it is possible to find a "beta" relationship.

Define the $1 \times d$ process $\psi_t$ using the relationship

$$\psi_t \cdot I_S \cdot \sigma_t = e_t \cdot \sigma_{e,t}$$

We can consider $\psi_t$ as a self-financing strategy $(\theta^0, \theta)$ by choosing $\theta^i = \psi^i_t$ and $\theta^0$ to meet the self-financing constraint.

Let $S_t^{\psi}$ be the value of the portfolio $S_t^{\psi} = \psi_t \cdot S_t$. We then have

$$\frac{dS_t^{\psi}}{S_t^{\psi}} = \mu_{\psi,t} dt + \sigma_{\psi,t} dB_t$$

where

$$\sigma_{\psi,t} = \frac{1}{S_t^{\psi}} \psi_t \cdot I_S \cdot \sigma_t = \frac{e_t}{S_t^{\psi}} \sigma_{e,t}$$

and $\mu_{\psi,t}$ satisfies the condition (30)

$$\mu_{\psi,t} - r_t = \gamma (e_t, \lambda) \times \sigma_{\psi,t} \sigma_{e,t}'$$  (31)
• Hence, we finally have that for all $i = 1, \ldots, n$

$$\mu^i_t - r_t = \gamma(e_t; \lambda) \sigma^{i'}_{e,t}$$

$$= \gamma(e_t; \lambda) \times \frac{S^\psi_t}{e_t} \times \sigma^{i'}_{\psi,t}$$

• Since this holds for the portfolio $\psi$ as well, that is

$$\mu^\psi_t - r_t = \gamma(e_t; \lambda) \times \frac{S^\psi_t}{e_t} \times \sigma_{\psi,t}^{\psi'}$$

• by substituting for the common term $\gamma(e_t; \lambda) \times \frac{S^\psi_t}{e_t}$ we obtain the CAPM “beta” relationship

$$E_t \left( \frac{dS^i_t}{S^i_t} \right) - r_t dt = \beta^i_t \times \left( E_t \left( \frac{dS^\psi_t}{S^\psi_t} \right) - r_t 1_d \right)$$

— where

$$\beta^i_t = \frac{\sigma^{i'}_{\psi,t} \cdot \sigma^{\psi'}_{\psi,t}}{\sigma_{\psi,t} \cdot \sigma^{\psi'}_{\psi,t}} = \frac{cov_t \left( dS^i_t / S^i_t, dS^\psi_t / S^\psi_t \right)}{var_t \left( dS^\psi_t / S^\psi_t \right)}$$
• Note:
  – The (conditional) CAPM works with respect to that asset that is perfectly correlated with the endowment process (and thus the stochastic discount factor).
  
  – This need not be the market portfolio. The existence of labor income, for instance, generate a wedge between the stochastic discount factor and the market portfolio: Thus, the CAPM is not supposed to be working (theoretically) with respect to the market portfolio.
  
  – But even in the case where there is no labor income it is not obvious that the market portfolio is perfectly correlated with the endowment process.
  
  – We will do more on this later on in the course.
An Asset Pricing Relationship

• Suppose that agents $i$ sells a financial asset whose payment is its entire endowment (or a fraction of it), that is $e^i_t$.

• What is the fair price of this stream of payments?

• The above results imply that the price at time $t$ of this claim is

$$S^{e^i}_t = \frac{1}{\pi_t} E_t \left[ \int_t^T \pi_u e^i_u du \right]$$  \hspace{1cm} (32)

$$= \frac{1}{U_c(e^i_t)} E_t \left[ \int_t^T e^{-\phi(u-t)} U_c(e_u) e^i_u du \right]$$  \hspace{1cm} (33)

• In fact, if this was not true one could find a trading strategy $(\theta_0, \boldsymbol{\theta})$ that finances $e^i_u$ and whose value is (32). This in turn generates an arbitrage opportunity.

• Indeed, the value of total endowment process is simply

$$S_t = \frac{1}{U_c(e^i_t)} E_t \left[ \int_t^T e^{-\phi(u-t)} U_c(e_u) e_u du \right]$$  \hspace{1cm} (34)

• We will use this pricing equation often.
A simple example: Log Utility

- Suppose all the agents have logarithmic utility
  \[ u^i(c_t) = \log(c_t) \]

- so that the inverse of the marginal utility is given by
  \[ I_u^i(x) = x^{-1} \]

- In this case, the marginal utility of the representative investor is
  \[ U^c(c) = \frac{1}{c} \sum_{i=1}^{m} \lambda_i^{-1} \]

- We can renormalize the weights so that \( \sum_{i=1}^{m} \lambda_i^{-1} = e_0 \), which we can assume positive.

- We then have
  \[ U^c(e_0) = 1 \]

- The vector \( \lambda = (\lambda_1, \ldots, \lambda_m) \) has to satisfy the system of equations \((23)\),
  \[ E \left[ \int_0^T e^{-\phi t} U^c(e_t; \lambda) \left( I_u^i(\lambda U^c(e_t; \lambda) - e_t^i) \right) dt \right] = 0 \] (35)
A simple example: Log Utility

- which becomes

\[
E \left[ \int_0^T e^{-\phi t} e_0 \left( \left( \lambda_i \frac{e_0}{e_t} \right)^{-1} - e_t^i \right) dt \right] = 0 \quad (36)
\]

- and in turn

\[
\lambda_i^{-1} = e_0 \frac{E \left[ \int_0^T e^{-\phi t} e_t^i dt \right]}{\int_0^T e^{-\phi t} dt} \quad (37)
\]

- Hence

\[
\pi_t = e^{-\phi t} U_c(e_t; \lambda) = e^{-\phi t} \frac{e_0}{e_t}
\]

- We then have that

\[
c_t^i = \mathcal{I}_u \left( \lambda_i e^{\phi t} \pi_t \right) = \mathcal{I}_u \left( \lambda_i \frac{e_0}{e_t} \right) = \lambda_i^{-1} \frac{e_t}{e_0} \quad (38)
\]
A simple example: Log Utility

- Finally, we have

$$
\gamma(e_t; \lambda) = - \frac{e_t U_{cc}(e_t, \lambda)}{U_c(e_t, \lambda)} = - \frac{e_t (-e_0/e_t^2)}{e_0/e_t} = 1
$$

$$
q(e_t; \lambda) = - \frac{e_t U_{ccc}(e_t, \lambda)}{U_{cc}(e_t, \lambda)} = - \frac{e_t^2 e_0/e_t^3}{(-e_0/e_t^2)} = 2
$$

- Hence, the condition for the market equilibrium are

$$
r_t = \phi + \mu_{e,t} - \sigma_{e,t} \sigma_{e,t}'
$$

$$
\nu_t = \sigma_{e,t}'
$$
An Alternative Formula for the C-CAPM

• Before commenting further, notice also that an alternative expression for the (30) can be obtained as follows.

• We know that at an optimum, the following condition holds (see (27))

\[ u^j_c \left( \hat{c}^j_t \right) = \lambda_j U_c (e; \mathbf{\lambda}) \]  (39)

• Hence, rather than using the representative agent utility function, we may think of using the marginal utility of agent \( j \) defined on the optimal consumption path \( \hat{c}^j \).

• Equation (39) ensures that the two approaches are identical.

• Define the process

\[ \hat{\xi}_t^j = \frac{1}{\lambda_j} u^j_c \left( \hat{c}^j_t \right) \]

• Define the state-price density of agent \( j \) as

\[ \hat{\pi}_t^j = \beta_t^{-1} \hat{\xi}_t^j \]
An Alternative Formula for the C-CAPM

• By going through the same type of calculation, it is clear that everywhere we can substitute the "representative agent" utility and endowment, with agent $i$ utility and consumption.

• As a consequence, on then obtains

$$E \left[ \frac{dS^i_t}{S^i_t} \right] - r_t dt = \gamma^j \left( \hat{c}_t^j \right) \times Cov_t \left( \frac{dS^i_t}{S^i_t}, \frac{d\hat{c}_t^j}{\hat{c}_t^j} \right)$$

$$= a^j \left( \hat{c}_t^j \right) \times Cov_t \left( \frac{dS^i_t}{S^i_t}, d\hat{c}_t^j \right)$$

• where

$$a^j \left( \hat{c}_t^j \right) = -\frac{u^j_{cc} \left( \hat{c}_t^j \right)}{u^j_c \left( \hat{c}_t^j \right)}$$

• is the coefficient of absolute risk aversion of agent $j$.

• Divide now both sides by $a^j \left( \hat{c}_t^j \right)$ and sum across $j = 1, ..., m$. Since from the market clearing condition $\sum_{j=1}^n \hat{c}_t^j = e_t$ we must have

$$\sum_{j=1}^m Cov_t \left( \frac{dS^i_t}{S^i_t}, \frac{d\hat{c}_t^j}{\hat{c}_t^j} \right) = Cov_t \left( \frac{dS^i_t}{S^i_t}, d\hat{c}_t^j \right)$$
An Alternative Formula for the C-CAPM

- we obtain

\[ E \left[ \frac{dS^i_t}{S^i_t} \right] - r_t dt = \Gamma (c_t) \times Cov_t \left( \frac{dS^i_t}{S^i_t}, de_t \right) \]

- The coefficient

\[ \Gamma (c_t) = \frac{1}{\sum_{j=1}^{m} a^j \left( \hat{c}^j_t \right)^{-1}} \]

- is the coefficient of absolute risk aversion of the market itself.