Modern Dynamic Asset Pricing Models

Lecture Notes 5.

Heterogeneity

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1These teaching notes draw heavily on the papers quoted in the references or in the Syllabus. They are intended for students of Business 35907 only. Please, do not distribute without my prior consent
This teaching notes discuss three papers that deal with particular types of heterogeneity:

  * Agents have different constraints on stock market investing. In particular, some simply do not.
- Jiang Wang (1993): Asymmetric Information
  * Agents have different information about the fundamentals of the stock. Prices partly reveal this information to uninformed agents.
- Basak and Crotouru (2001): Differences of Opinions
  * Agents have the same information, but different opinions. In other words, they interpret public signals differently.
• The paper by Basak and Cuoco (1998) is particularly interesting because it provides nice insights on the effects of limited stock market participation on risk premia, interest rates and so on.

• The idea is simple: Consider an economy where some agents are constrained and cannot participate to the stock market. This can be justified by a class of agents facing very high transaction costs.

• We can study such a set-up in an environment similar to the one studied in TN1 and 2.

• Consider a standard exchange economy with a riskless and risky asset (as usual, we assume standard requirements to the probability space and the filtration)

• The riskless asset is in zero net supply and its price is given by

$$\beta_t = \exp\left(\int_0^t r_{\tau} d\tau\right)$$

• where $r_{\tau}$ is the (equilibrium) short-term risk-free rate.

• The risky stock is in positive net supply (normalized to 1) and it is characterized by a dividend rate process

$$\delta_t = \delta_0 + \int_0^t \mu_{\delta,\tau} d\tau + \int_0^t \sigma_{\delta,\tau} dB_{\tau}$$

• where $\mu_{\delta}$ and $\sigma_{\delta}$ are progressively measurable processes.
The stock price evolves according to the Ito process

\[ S_t = S_0 + \int_0^t (S_{\tau}\mu_{\tau} - \delta_{\tau}) \, d\tau + \int_0^t S_{\tau}\sigma_{\tau}dB_{\tau} \]

where the processes \( \{\mu_t\} \) and \( \{\sigma_t\} \) have to be determined by equilibrium conditions.

A trading strategy is given by a vector \((\varphi^0, \varphi)\) of amounts invested in the riskless asset and risky asset. They are admissible if they satisfy the usual integrability (and square integrability) conditions so that the wealth process is well defined and doubling strategies are ruled out.

Given consumption plan \( c \in L_+ \), we have the dynamic budget constraint for the wealth process \( W_t = \varphi_t^0 + \varphi_t^1 \) is

\[ dW_t = (\varphi_t^0 r_t + \varphi_t^1 \mu_t - c_t) \, dt + \varphi_t^1 \sigma_t dB_t \]

There are two agents:

- Agent 2 is constrained and has log utility \( u^2(c) = \ln(c) \);
- Agent 1 is unconstrained and has a utility function \( u^1(.) \) satisfying condition A in TN1 (differentiability and Inada Conditions)

Both maximize

\[ U^i(c) = E \left[ \int_0^T e^{-\phi t} u^i(c_t) \, dt \right] \]

subject to the dynamic budget constraint.
• Agent 1 (unconstrained) is endowed with 1 unit of stock and a short position of $b$ units of bond.
• Agent 2 (constrained) is endowed only with $b$ units of bonds.
• Assume that
  \[ b < \frac{1 - e^{-\delta_0}}{\phi} \]
• That is, agent 1 is not too indebted that it will make it impossible to settle his position before $T$.
• An equilibrium for this economy is a pair of price processes $(r, S)$ and a set $(\hat{c}^i; (\hat{\phi}^{0,i}, \hat{\phi}^i))_{i=1}^2$ such that
  1. For each $i$, $(\hat{\phi}^{0,i}, \hat{\phi}^i)$ finances $\hat{c}^i$;
  2. $\hat{c}^1$ maximizes agent 1 utility over the set of admissible trading strategies such that $\phi_0 + \phi_0^1 = S_0 - b$;
  3. $\hat{c}^2$ maximizes agent 2 utility over the set of admissible trading strategies such that $\phi_0^2 = b$ and $\phi_t^2 = 0$;
  4. Markets clear: $\hat{c}^1 + \hat{c}^2 = \delta$, $\hat{\phi}^{0,1} + \hat{\phi}^{0,2} = 0$ and $\hat{\phi}^1 = S$. 
The Benchmark Case: Unrestricted Investors

- We have already seen how to solve for the equilibrium in the unconstrained case.

- Define the aggregate utility function

\[ U(c; \lambda) = \max_{c^1 + c^2 = c} u^1(c^1) + \lambda u^2(c^2) \]

- for some \( \lambda > 0 \). Notice that

1. For simplicity, I use the reciprocal of the “\( \lambda \)”’s in TN2. Recall that there, we wrote \( U(c; \lambda) = \max i \tilde{c} \sum_{i=1}^{n} \frac{1}{\lambda_i} u^i(c^i) \). So, don’t get confused when you check out the notation.

2. We renormalize \( \lambda_1 = 1 \).

- Since in an exchange economy, consuming the aggregate endowment must be optimal for the representative agent, we then have that the state price density is given by

\[ \pi_t = e^{-\phi_t} \frac{U_c(\delta_t; \lambda)}{U_c(\delta_0; \lambda)} \]

(1)

- Recalling that the state-price density obtained by the condition of no arbitrage \( \tilde{\pi}_t \) is given by

\[ \tilde{\pi}_t = \exp \left( -\int_0^t \left( r_s + \frac{1}{2} \nu_s^2 \right) ds - \int_0^t \nu_t dB_t \right) \]

where \( \nu_t = \sigma_t^{-1}(\mu_t - r_t) \)
• We obtain the dynamics

\[ d\tilde{\pi}_t = -r_t\tilde{\pi}_t dt - \tilde{\pi}_t \nu_t dB_t \quad (2) \]

• Equating the dynamics of (1) and (2) we obtain the standard results

\[ r_t = -\frac{D\left(e^{-\phi t}U_c(\delta_t; \lambda)\right)}{U_c(\delta_0; \lambda)} \]

• where \( D\left(e^{-\phi t}U_c(\delta_t; \lambda)\right) \) denotes the drift of the process \( e^{-\phi t}U_c(\delta_t; \lambda); \) and

\[ S_t = E_t \left[ \int_t^T e^{-\phi(\tau-t)} \frac{U_c(\delta_\tau; \lambda)}{U_c(\delta_0; \lambda)} \delta_\tau d\tau \right] \]

• Hence, one obtains

\[ r_t = \phi + a(\delta_t; \lambda) \mu_{\delta,t} - \frac{1}{2} a(\delta_t; \lambda) q(\delta_t; \lambda) \sigma_\delta^2 \quad (3) \]

\[ \mu_t - r_t = a(\delta_t; \lambda) \sigma_{\delta,t} \sigma_t \quad (4) \]

• where

\[ a(\delta_t; \lambda) = -\frac{U_{cc}(\delta_t; \lambda)}{U_c(\delta_0; \lambda)} \]

\[ q(\delta_t; \lambda) = -\frac{U_{ccc}(\delta_t; \lambda)}{U_{cc}(\delta_0; \lambda)} \]
• Finally, from the results in TN2 the optimal consumption plans are given by
\[
\tilde{c}_1^t = I_u^1(\mathcal{U}_c(\delta_t; \lambda)) \\
\tilde{c}_2^t = I_u^2\left(\frac{1}{\lambda}\mathcal{U}_c(\delta_t; \lambda)\right)
\]

• The “λ” must satisfy the budget constraint for agent 2 (because of the renormalization), that is
\[
b = E \left[ \int_0^T e^{-\phi t} \frac{\mathcal{U}_c(\delta_t; \lambda)}{\mathcal{U}_c(\delta_0; \lambda)} \tilde{c}_2^t \, dt \right] = \frac{1 - e^{-\phi T}}{\phi} \frac{\lambda}{\mathcal{U}_c(\delta_0; \lambda)}
\]

• One can also solve for the optimal strategies for agent 1 and 2.

• Notice that the wealth of agent 2 (log utility) is given by
\[
W_2^t = E_t \left[ \int_t^T e^{-\phi(t-\tau)} \frac{\mathcal{U}_c(\delta_\tau; \lambda)}{\mathcal{U}_c(\delta_t; \lambda)} \tilde{c}_2^\tau \, d\tau \right]
\]
\[
= E_t \left[ \int_t^T e^{-\phi(t-\tau)} \mathcal{U}_c(\delta_\tau; \lambda) \frac{\lambda}{\mathcal{U}_c(\delta_t; \lambda) \mathcal{U}_c(\delta_\tau; \lambda)} \, d\tau \right]
\]
\[
= \frac{\lambda}{\mathcal{U}_c(\delta_t; \lambda)} \left[ 1 - e^{-\phi(T-t)} - \frac{\phi}{\phi} \frac{1 - e^{-\phi(T-t)}}{1 - e^{-\phi T}} \right]
\]
\[
= b \frac{\mathcal{U}_c(\delta_0; \lambda)}{\mathcal{U}_c(\delta_t; \lambda)} \left[ 1 - e^{-\phi(T-t)} - \frac{\phi}{\phi} \frac{1 - e^{-\phi(T-t)}}{1 - e^{-\phi T}} \right]
\]
• By Ito’s Lemma, we finally have that the diffusion part of $dW_t^2$, call it $\sigma_{W,t}$ is

$$\sigma_{W,t} = \left( \frac{b}{U_c(\delta_0; \lambda)} \frac{1 - e^{-\phi(T-t)}}{1 - e^{-\phi T}} \right) \frac{-U_{cc}(\delta_t; \lambda)}{U_c(\delta_t; \lambda)} \sigma_{\delta,t}$$

$$= W_t^2 a (\delta_t; \lambda) \sigma_{\delta,t}$$

$$= W_t^2 \sigma_t^{-1} (\mu_t - r_t)$$

• where the last equality stems from (4)

• Since, by the dynamic budget constraint, we must also have

$$\sigma_{W,t} = \varphi_t \sigma_t$$

• we finally obtain

$$\varphi_t^2 = W_t^2 \sigma_t^{-2} (\mu_t - r_t)$$

$$\varphi_t^{0.2} = W_t^2 \left( 1 - \sigma_t^{-2} (\mu_t - r_t) \right)$$

• Clearly, the position for agent 1 can be obtained as complement of these, so that

$$W_t^1 = S_t - W_t^2$$

$$\varphi_t^1 = S_t - \varphi_t^1$$

$$\varphi_t^{0,1} = -\varphi_t^{0,1}$$
Finally, one finds that the equilibrium consumption process evolves according to
\[ d\hat{c}_t^i = \mu_{c,t}^i dt + \sigma_{c,t}^i dB_t \]  

where
\[ \mu_{c,t}^i = \frac{a_t}{a_t^i} \mu_{\delta,t} - \frac{1}{2} \frac{a_t}{a_t^i} q_t \sigma_{\delta,t}^2 + \frac{1}{2} \left( \frac{a_t}{a_t^i} \right)^2 q_t^i \sigma_{\delta,t}^2 \]  
\[ \sigma_{c,t}^i = \frac{a_t}{a_t^i} \sigma_{\delta} \]  

and where
\[ a_t = - \frac{U_{cc} (\delta_t; \lambda)}{U_c (\delta_t; \lambda)}; \quad a_t^i = - \frac{u_{cc}^i (\hat{c}_t^i)}{u_c^i (\hat{c}_t^i)}; \quad \text{and} \quad q_t^i = - \frac{u_{ccc}^i (\hat{c}_t^i)}{u_{cc}^i (\hat{c}_t^i)} \]
The Restricted Case

• Obtaining “nice” formulas in this case is more challenging.
• First of all, since agent 2 is restricted from participating in the market, the “first best,” Pareto efficient allocation cannot be achieved.
• In this case, one cannot even construct a representative agent with fixed weights, as we did in TN2.
• However, we can construct a representative agent with *stochastic* weights.
• This approach has been first put forward by Cuoco and He (1994) in the context of incomplete markets.
• More specifically, consider a representative agent with stochastic weights

\[
U(c, \lambda) = E \left[ \int_0^T e^{-\phi t} U(c_t, \lambda_t) \, dt \right]
\]

• where

\[
U(c_t, \lambda_t) = \max_{c_t^1 + c_t^2 = c_t} u^1(c_t^1) + \lambda_t u^2(c_t^2)
\]

(8)

• and \(\lambda_t\) is a stochastic weight yet to be determined.
• As before, consuming the whole endowment must be optimal for the representative agent and hence
\[
\pi_t = e^{-\phi_t} \frac{U_c(\delta_t, \lambda_t)}{U_c(\delta_0, \lambda_0)}
\] (9)

• must still identify the state-price density for the whole economy, which evolves in equilibrium as
\[
d\pi_t = -r_t \pi_t dt - \pi_t \nu_t dB_t
\] (10)

• where \( \nu_t \) is the market price of risk.

• From the same type of argument as in TN2 (i.e. from the convexity of \( u^1 \) and \( u^2 \)) we have that the optimal consumption \( \hat{c}_t^1 \) and \( \hat{c}_t^2 \) that solve (8) when \( c_t = \delta_t \) and \( \lambda_t \) is given must still be
\[
\hat{c}_t^1 = \mathcal{I}^1_u (U_c(\delta_t, \lambda_t))
\]
\[
\hat{c}_t^2 = \mathcal{I}^2_u \left( \frac{1}{\lambda_t} U_c(\delta_t, \lambda_t) \right)
\]

• This implies
\[
u^1_c(\hat{c}_t^1) = U_c(\delta_t, \lambda_t) = \lambda_t \nu^2_c(\hat{c}_t^2)
\]

• which yields
\[
\lambda_t = \frac{\nu^1_c(\hat{c}_t^1)}{\nu^2_c(\hat{c}_t^2)}
\] (11)
• Now, agent 1 faces dynamically complete markets (he can trade in both securities) and hence we also know that

\[ \hat{c}_t^1 = \mathcal{I}_u^1 \left( \psi_1 e^{\phi t} \pi_t \right) \]  

(12)

• for some \( \psi_1 > 0 \), which in turn implies

\[ u^1_c (\hat{c}_t^1) = \psi_1 e^{\phi t} \pi_t \]  

(13)

• Agent 2 instead is barred from investing in the market (too high transaction/information costs? Too poor?).

• Hence, agent 2 faces a dynamically incomplete market. Since he/she has log utility, however, we can still solve for his/her optimal consumption.

• In fact, since she is barred from investing in stocks, effectively this investor faces a “restricted” state-price density

• This state-price density reflects the only investment opportunity that is available to the restricted agent, namely, the bond.

• That is, the state-price density of this agent is simply given by

\[ \pi_t^2 = \beta_t^{-1} \]

• Hence, his/her optimal consumption will be

\[ \hat{c}_t^2 = \mathcal{I}_u^2 \left( \psi_2 e^{\phi t} \beta_t^{-1} \right) \]  

(14)
• for some $\psi_2 > 0$, which in turn implies

$$u^2_c(\hat{c}^2_t) = \psi_2 e^{\phi t} \beta_t^{-1}$$

• In other words, because the two agents face different investment opportunity sets, they also face different state-price densities.

• Substitute everything into equation (11) to find

$$\lambda_t = \frac{u^1_c(\hat{c}^1_t)}{u^2_c(\hat{c}^2_t)} = \frac{\psi_1 \pi_t \beta_t}{\psi_2}$$

• Apply Ito’s Lemma to find

$$d\lambda_t = \frac{\psi_1 \beta_t d\pi_t}{\psi_2} + \frac{\psi_1}{\psi_2} \pi_t d\beta_t$$

$$= -\frac{\psi_1}{\psi_2} \beta_t r_t \pi_t dt - \frac{\psi_1}{\psi_2} \beta_t \pi_t \nu_t dB_t + \frac{\psi_1}{\psi_2} \pi_t r_t \beta_t dt$$

$$= -\lambda_t \nu_t dB_t$$

• We must also have

$$\pi_t = e^{-\phi t} U_c(\delta_t, \lambda_t)$$

$$\frac{\lambda_t}{U_c(\delta_0, \lambda_0)}$$

(15)
• which yields (we write $\mathcal{U}(\delta_t, \lambda_t) = \mathcal{U}(t)$ for simplicity)

$$d\pi_t = \frac{\mathcal{D}(e^{-\phi t}\mathcal{U}_c(t))}{\mathcal{U}_c(0)} + \frac{e^{-\phi t}}{\mathcal{U}_c(0)} (\mathcal{U}_{cc}(t) \sigma_\delta - \mathcal{U}_{c\lambda}(t) \lambda_t \nu_t) dB_t$$

$$= \frac{\mathcal{D}(e^{-\phi t}\mathcal{U}_c(t))}{e^{-\phi t}\mathcal{U}_c(0)} \pi_t + \frac{\pi_t}{\mathcal{U}_c(t)} (\mathcal{U}_{cc}(t) \sigma_\delta - \mathcal{U}_{c\lambda}(t) \lambda_t \nu_t) dB_t$$

• where $\mathcal{D}(e^{-\phi t}\mathcal{U}_c(\delta_t, \lambda_t))$ denotes the drift of the process $e^{-\phi t}\mathcal{U}_c(\delta_t, \lambda_t)$.

• Hence, confronting with (10) we find

$$r_t = -\frac{\mathcal{D}(e^{-\phi t}\mathcal{U}_c(\delta_t, \lambda_t))}{e^{-\phi t}\mathcal{U}_c(\delta_t, \lambda_t)}$$

$$\nu_t = \frac{\mathcal{U}_{cc}(t) \sigma_\delta - \mathcal{U}_{c\lambda}(t) \lambda_t \nu_t}{\mathcal{U}_c(t)}$$

• From the last relationship, we can solve for the equilibrium market price of risk $\nu_t$

$$\nu_t = -\frac{\mathcal{U}_{cc}(t)}{\mathcal{U}_c(t) - \mathcal{U}_{c\lambda}(t) \lambda_t} \sigma_\delta$$

• It is possible to show that (see Basak and Cuoco (1998, Lemma 2))

$$\frac{\mathcal{U}_{cc}(t)}{\mathcal{U}_c(t) - \mathcal{U}_{c\lambda}(t) \lambda_t} = \frac{u_{1c}^1(\mathcal{I}_{c\lambda}^1(\mathcal{U}_c(\delta_t, \lambda_t)))}{\mathcal{U}_c(\delta_t, \lambda_t)}$$
• We can finally rewrite the process for the stochastic weight $\lambda$ as
\[
d\lambda_t = \lambda_t \frac{u_{cc}^1 (\mathcal{I}_{u} (\mathcal{U}_c (\delta_t, \lambda_t)))}{\mathcal{U}_c (\delta_t, \lambda_t)} \sigma \delta dB_t
\]  
(19)

• These are necessary conditions.

• One has still to prove (1) that the equilibrium exists; and (2) that given the interest rate $r_t$ in (16), the market price of risk in (18) and the process for $\lambda_t$ in (19), the optimal consumption plans are given by (12) and (14).

• This is done in the paper, who report the following Theorem:

• **Theorem:** Suppose that there exists a strictly positive solution $\lambda$ to the SDE (19), with $\lambda_0$ being the solution of

\[
b = \frac{1 - e^{-\phi T}}{\phi} U_{\lambda} (\delta_0, \lambda_0)
\]

• and that

\[
E \left[ \int_0^T e^{-\phi t} U_c (\delta_t, \lambda_t) \delta_t dt \right] < \infty
\]

• Then, there exists an equilibrium where

1. The interest rate is given by:
\[
r_t = - \frac{D (e^{-\phi t} U_c (\delta_t, \lambda_t))}{e^{-\phi t} U_c (\delta_t, \lambda_t)}
\]
2. The stock price is (if $\lambda_t$ is in fact a martingale):

$$S_t = E_t \left[ \int_t^T e^{-\phi(\tau-t)} \frac{U_c(\delta_{\tau}, \lambda_{\tau})}{U_c(\delta_t, \lambda_t)} \delta_{\tau} d\tau \right]$$

3. The bond price is:

$$\beta_t = e^{\phi t} \frac{\lambda_t U_c(\delta_0, \lambda_0)}{\lambda_0 U_c(\delta_t, \lambda_t)}$$

4. The optimal consumptions are

$$\hat{c}_t^1 = I_u^1 (U_c(\delta_t, \lambda_t))$$

$$\hat{c}_t^2 = I_u^2 \left( \frac{1}{\lambda_t} U_c(\delta_t, \lambda_t) \right)$$

5. The optimal trading strategies are

$$\varphi_{t,1}^{0,1} = -b \frac{e^{-\phi t} - e^{-\phi T}}{1 - e^{-\phi t}} \beta_t$$

$$\varphi_{t,1}^{1,1} = S_t$$

$$\varphi_{t,2}^{0,2} = b \frac{e^{-\phi t} - e^{-\phi T}}{1 - e^{-\phi t}} \beta_t$$

$$\varphi_{t,2}^{1,2} = 0$$

• Proof. See Basak and Cuoco (1998)
• The key point of this representation is the weighting process \( \lambda_t \). Recall that this represents the ratio of the marginal utilities of consumption of the two agents, or equivalently, the ratio of the two state-price densities.

• As this ratio moves stochastically over time, we are giving more or less weight to the marginal utility of one or the other agent within the “representative agent” representation of the economy.

• This in turn affects stock returns.

• **Corollary:** The equilibrium interest rate and risk premium have representations

\[
\begin{align*}
\frac{rt}{\mu_t - r_t} &= \phi + a_t \mu_{\delta,t} - \frac{1}{2} a_t q_1^1 \sigma_{\delta,t}^2 \\
\mu_t - r_t &= a_t \sigma_{\delta,t} \sigma_t
\end{align*}
\]

(20) (21)

• In addition, the optimal consumption policies satisfy

\[
d\hat{c}_t^i = \mu_{c,t}^i dt + \sigma_{c,t}^i dB_t
\]

(22)

• where

\[
\begin{align*}
\mu_{c,t}^1 &= \frac{a_t}{a_t^1} \mu_{\delta,t} - \frac{1}{2} \frac{a_t}{a_t^1} q_1^1 \sigma_{\delta,t}^2 + \frac{1}{2} q_1^1 \sigma_{\delta,t}^2 \\
\mu_{c,t}^2 &= \frac{a_t}{a_t^2} \mu_{\delta,t} - \frac{1}{2} \frac{a_t}{a_t^2} q_1^1 \sigma_{\delta,t}^2 \\
\sigma_{c,t}^1 &= \sigma_{\delta} \\
\sigma_{c,t}^2 &= 0
\end{align*}
\]

(23) (24) (25) (26)
Remarks:

1. From (20), the interest rate has a very similar form as in the unrestricted case. However, the interest rate is negatively related to the volatility of endowment through the prudence parameter of the unrestricted agent only. This stems from (25) and (26), that show that only the unrestricted agent has a “risky” consumption path.

2. The equity premium only depends on the risk aversion of the unrestricted agent.

3. Compared to the unrestricted economy, the volatility of consumption of agent 1 is increased while the one of agent 2 is decreased. This is because the restricted agent is now the only one that has to bear the risk of stemming from the a stochastic dividend.
Calibration

- Does it matter for stock returns? Can we solve “puzzles”? Do we create new ones?
- Basak and Cuoco (1998) calibrate the model by assuming the standard CRRA utility function
  \[ u^1(c) = \frac{c^{1-\gamma}}{1-\gamma} \]

- First, they notice that as the fraction \( x \) of nonstockholders increases
  1. The equilibrium interest rate decreases (very quickly), obtaining a plausible explanation for low interest rates;
  2. The equity market price of risk increases, as the stockholders must hold more and more of the aggregate risk.
- Mankiew and Zeldes (1991) estimate \( x = .68 \). Hence, the authors can match a real interest rate of 1.3% and the historical market price of risk of 0.37 with a coefficient of risk aversion \( \gamma = 3.3 \) which implies a representative agent with risk aversion equal to 1.3.
Figure 1  
**Behavior of the equilibrium interest rate**

The graph plots the equilibrium interest rate implied by the model as a function of the nonstockholders' share of aggregate consumption, assuming that stockholders display CRRA preferences with a relative risk aversion coefficient γ equal to 1, 4, 7, and 10. The parameters of the aggregate consumption process are chosen to match the estimates reported by Mehra and Prescott (1985). The dotted line corresponds to Mehra and Prescott's estimate of the mean real interest rate.
Figure 2
Behavior of the equilibrium market price of risk
The graph plots the equilibrium market price of risk implied by the model as a function of the nonstockholders' share of aggregate consumption, assuming that stockholders display CRRA preferences with a relative risk aversion coefficient $\gamma$ equal to 1, 4, 7, and 10. The parameters of the aggregate consumption process are chosen to match the estimates reported by Mehra and Prescott (1985). The dotted line corresponds to Mehra and Prescott's estimate of the market price of risk.
We start by studying the paper of Jiang Wang (1993).

The economy is different from the one studied so far, but it has been used widely in the literature (see e.g. Campbell and Kyle (1993), Veronesi (1999)).

Consider a probability space \((\Omega, P, \mathcal{F})\) on which a \(3 \times 1\) dimensional Brownian motion \(\{B_t\}\) is defined.

**Risky Asset:** There is a risky asset generating a flow of output (dividend) at the rate \(D_t\), where

\[
dD_t = (\Pi_t - kD_t) dt + b_D dB_t
\]

\(\Pi_t\) is a state-variable following an O-U process

\[
d\Pi_t = a_\Pi (\Pi - \Pi_t) dt + b_\Pi dB_t
\]

Assume \(k \geq 0, a_\Pi > 0, \Pi\) are constants and \(b_D\) and \(b_\Pi\) are \(1 \times 3\) vectors of constants.

Notice that for \(k > 0\) and \(a_\Pi > 0\), the dividend process is then stationary. When \(k = 0\) the dividend process becomes non-stationary (but the growth rate of dividends is).
- **Noise**: The total amount of the risky asset is $1 + \Theta_t$, where evolves according to the O-U process

  $$d\Theta_t = -a\Theta_t dt + b\Theta dB_t$$

- The noise process $\Theta_t$ is necessary to make sure that the equilibrium price does not reveal all the information to the less informed agent (to be defined below).

- **Riskless Asset**: There is a riskless asset, returning a *constant* interest rate $r$. This is to be thought of as a risk-free storage technology. All investors have access to the storage technology at no costs.

- For later reference, we denote the price of the risky asset $P_t$. We will find its equilibrium value below.

- **Investors**: There are two types of investors:

  1. *Informed Investors*: There is a fraction $(1 - w)$ of informed investors, who have full information on the economy. Specifically, they observe the dividends, the growth rate of dividends and the price of the stock. Hence, their information set is

     $$\mathcal{F}_{t}^{i} = \{D_{\tau}, P_{\tau}, \Pi_{\tau} : \tau \leq t\}$$

  2. *Uninformed Investors*: There is a fraction $w$ of uninformed investors, who have only partial information about the economy. They are assumed to observe only the dividends and the price of the stock, but not the current growth rate of dividends. Their information set is

     $$\mathcal{F}_{t}^{u} = \{D_{\tau}, P_{\tau} : \tau \leq t\}$$
• **Remark**: Informed agents do not observe directly the level of noise trading $\Theta_t$. However, they will be able to figure it out from the equilibrium price function. Hence, equivalently, we will impose throughout that

$$G^i_t = \{D_\tau, P_\tau, \Pi_\tau, \Theta_\tau : \tau \leq t\}$$

• **Preferences**: Agents are identical and endowed with the CARA instantaneous utility of consumption

$$u(c_t, t) = -\exp(-\rho t - \gamma c_t)$$

where $\rho$ is the time-impatience parameter and $\gamma$ the coefficient of absolute risk aversion. We set $\gamma = 1$ for notational simplicity (results can be extended easily).

• **Independent Shocks**: It is assumed that

$$b_D = (\sigma_D, 0, 0)$$
$$b_\Pi = (0, \sigma_\Pi, 0)$$
$$b_\Theta = (0, 0, \sigma_\Theta)$$

• The solution of the general case is feasible, but more messy!
The Benchmark Case: No Asymmetric Information

- To check what is the effect of asymmetric information on asset prices, it is important to first discuss the equilibrium prices when there is no asymmetric information.
- In this case we could have all informed \((w = 0)\) or all uninformed \((w = 1)\).
- We study the simpler case where all agents are informed and discuss the “uninformed” case later.

**Proposition 1**: Under perfect information, the equilibrium price of the economy is

\[
P_t^B = P_t^{B*} + (p_0^B + p_\Theta^B \Theta_t)
\]  

(27)

where

\[
P_t^{B*} = E_t \left[ \int_t^\infty e^{-r(s-t)} D_s ds \right] = \phi + p_D^B D_t + p_\Pi^B \Pi_t
\]  

(28)

- is the risk-neutral price, and where

\[
\phi = \frac{a\Pi p_\Pi^B \Pi}{r}, \quad p_D^B = \frac{1}{r + k},
\]  

(29)

\[
p_0^B = - \left[ (p_D^B)^2 \sigma_D^2 + (p_\Pi^B)^2 \sigma_\Pi^2 \right] < 0
\]  

(30)

\[
p_\Theta^B < 0
\]  

(31)
This proposition shows that the price of the asset $P_t^B$ is given by the risk-neutral price $P_t^{B*}$, that is, the price that would occur if agents were all risk neutral, plus a (likely) negative term (from (30) and (31)).

Hence, it has the form of a discount on the price.

The form of the risk-neutral price $P_t^{B*}$ denotes that the price function is linear in dividends $D_t$ and the growth rate of dividends $\Pi_t$.

This is clearly due to the linearity of the processes for dividends and dividend growth together with the CARA utility function.

**Expected Excess Returns:** It is convenient to work in is dollar terms, rather than percent terms.

Let $dQ_t^B$ be the excess return to one share of the asset (financed at the risk free rate)

$$dQ_t^B = dP_t^B + D_t dt - rP_t^B dt$$

From Ito’s lemma, it is simple to show that

$$E_t [dQ_t^B] = [-rp_0^{B*} - (r + a_{\Theta}) p_{\Theta}^{B*} \Theta_t] dt$$

Hence, expected returns are time varying as the level of noise trading changes over time.

The intuition is that as $\Theta_t$ increases, investors are exposed to a higher amount of aggregate shocks (recall that the total supply of the asset is $1 + \Theta_t$).

Hence, a higher risk premium should be given in order to make investors buy the stock.
• Finally, again by Ito’s lemma one can see that

\[
\sigma_Q^2 = (p^*_D)^2 \sigma_D^2 + (p^*_\Pi)^2 \sigma_{\Pi}^2 + (p^*_\Theta)^2 \sigma_{\Theta}^2
\]

• Hence, the variance of dollar returns is constant.

• This implies by itself that the variance (volatility) of percentage returns \((= \sigma_Q/P_t)\) increases when \(P_t\) decreases, generating the typical increase in volatility during downmarkets.

• The econometric properties of this model with symmetric information but with noise traders have been studied in Campbell and Kyle (1993).
Equilibrium with Asymmetric Information

• To solve for an equilibrium we proceed as follows:
  1. Conjecture a price function;
  2. Solve the intertemporal maximization problem of the two agents, conditioning on the conjectured price function;
  3. Check that market clearing conditions and optimality conditions lead to a solution that has the same form as the conjectured price function.

• The price function will depend on the state variables of the problem.
• What are the state variables?
• Given the assumed CARA utility function, wealth will not enter as a state variable.
• Certainly, \(D_t, \Pi_t\) and \(\Theta_t\) are state variables.
• However, uninformed agents do not observe neither \(\Pi_t\) nor \(\Theta_t\). They will be able to estimate them using some filtering algorithm and the information from equilibrium prices.
• Hence, we must include also the “induced” state variables, \(\hat{\Pi}_t = E_t [\Pi_t | \mathcal{F}_t^u]\) and \(\hat{\Theta}_t = E_t [\Theta_t | \mathcal{F}_t^u]\).
• The following proposition is the (first) main result of Wang (1993)
**Proposition 2:** There exists a stationary rational expectations equilibrium with

\[
P_t = (\phi + p_0) + p_B^D D_t + p_{\Pi} \Pi_t + p_\Theta \Theta_t + p_\Delta \hat{\Pi}_t
\]

\[
= P_t^{B*} + (p_0 + p_\Theta \Theta_t) + p_\Delta \Delta_t
\]

- where \( p_{\Pi} = p_B^{\Pi} - p_\Delta \), \( P_t^{B*} \) is given by (28) and \( \Delta_t = \hat{\Pi}_t - \Pi_t \).

**Remarks:**
1. \( \Delta_t \) is the estimation error of investors;
2. The equilibrium price reveals \( \Theta_t \) to the informed investors, as claimed above;
3. The equilibrium price reveals to the uninformed investors the following sum of \( \Pi_t \) and \( \Theta_t \):

\[
\Lambda_t = p_{\Pi} \Pi_t + p_\Theta \Theta_t
\]

- Hence, since we can write

\[
P_t = (\phi + p_0) + p^D D_t + p_\Delta \hat{\Pi}_t + \Lambda_t
\]

- we have that \( \Lambda_t \) is the variable that captures the information content of prices.
- In other words, observing \( D_t \) and \( P_t \) is equivalent to observing \( D_t \) and \( \Lambda_t \).

4. The estimate of the state variable \( \Pi_t \) of the uninformed investor, \( \hat{\Pi}_t \), enters into the price function. We would expect also \( \hat{\Theta}_t \) to enter, but this is not necessary, because one can show

\[
p_{\Pi} \Pi_t + p_\Theta \Theta_t = p_{\Pi} \hat{\Pi}_t + p_\Theta \hat{\Theta}_t
\]

- or

\[
p_{\Pi} (\Pi_t - \hat{\Pi}_t) = -p_\Theta (\Theta_t - \hat{\Theta}_t)
\]
The Filtering Problem of the Uninformed Investors

- Given the conjectured form of the price function, we can now apply the previous result about optimal filtering and obtain the solution of the uninformed investors’ estimates $\hat{\Pi}_t$ and $\hat{\Theta}_t$.
- Within the set-up of TN 3, investors receive two signals, $s_t = (D_t, \Lambda_t)'$ and must estimate the state variables $z_t = (\Pi_t, \Theta_t)'$.
- Notice that $\Lambda_t$ follows the process

$$d\Lambda_t = [a_{\Pi}p_{\Pi}(\Pi_t - \Pi_t) - a_{\Theta}p_{\Theta}\Theta_t] \, dt + b_{\Lambda} dB_t$$

- where $b_{\Lambda} = (0, p_{\Pi}\sigma_{\Pi}, p_{\Theta}\sigma_{\Theta})$.
- Hence, we can write the process for the signals

$$\begin{pmatrix} dD_t \\ d\Lambda_t \end{pmatrix} = \begin{pmatrix} \Pi_t - kD_t \\ a_{\Pi}p_{\Pi}(\Pi_t - \Pi_t) - a_{\Theta}p_{\Theta}\Theta_t \end{pmatrix} dt + \begin{pmatrix} b_D \\ b_{\Lambda} \end{pmatrix} dB_t$$

- All the processes are linear, we then obtain the result:
**Proposition 3:** \( \hat{\Pi}_t \) and \( \hat{\Theta}_t \) satisfy the SDEs

\[
\begin{pmatrix}
d\hat{\Pi}_t \\
d\hat{\Theta}_t
\end{pmatrix} = \begin{pmatrix}
a_{\Pi} (\Pi - \hat{\Pi}_t) \\
-a_{\Theta} \hat{\Theta}
\end{pmatrix} dt + \begin{pmatrix}
h_{\Pi D} & h_{\Pi \Lambda} \\
h_{\Theta D} & h_{\Theta \Lambda}
\end{pmatrix} (b_s b'_s)^{-\frac{1}{2}} d\tilde{B}_t
\]

where

\[
d\tilde{B}_t = \left( b_s b'_s \right)^{-\frac{1}{2}} \begin{pmatrix}
dD_t - (\hat{\Pi}_t - kD_t) dt \\
da_{\Pi} p_{\Pi} \left( \Pi - \hat{\Pi}_t \right) - a_{\Theta} p_{\Theta} \hat{\Theta}_t dt
\end{pmatrix}
\]

where \( h_{ij} \) are constants and \( \tilde{B}_t \) is a standard BM with respect to \( F_t^u \).

**Remarks:**

1. Here Wang (1993) only looks at the stationary equilibrium, hence the variance covariance matrix \( H_t = E_t \left[ (z_t - \hat{z}_t)(z_t - \hat{z}_t)' \right] \), which evolves according to the Riccati equation, already reached the steady state. Hence, \( H \) is assumed to be constant.

2. Recall that we started with the assumption that

\[
E \left[ d\Pi_t dD_t \right] = E \left[ d\Theta_t dD_t \right] = 0
\]

3. Instead, from the above result, we obtain

\[
E \left[ d\hat{\Pi}_t dD_t \right] > 0 \quad (32)
\]

\[
E \left[ d\hat{\Theta}_t dD_t \right] > 0 \quad (33)
\]
— The intuition for this is rather simple: Investors use dividends to estimate $\Pi_t$, hence positive innovations in dividends increase the estimated value of the dividend growth, generating (32).

— Similarly, when investors receive a good news in dividends, given a value $\Lambda_t = p_\Pi \hat{\Pi}_t + p_\Theta \hat{\Theta}_t$, they must increase the estimate of $\hat{\Theta}_t$ to offset the increase in $\hat{\Pi}_t$ (recall that $p_\Theta < 0$).

— These induced correlations due to learning will be important to understand the effects of asset prices.

3. Finally, it is possible to show that the estimation error $\Delta_t = \hat{\Pi}_t - \Pi_t$ also follows a O-U process

\[ d\Delta_t = -a_\Delta \Delta dt + b_\Delta dB_t \]

• for two constant $a_\Delta$ and $b_\Delta$.

• The last result also shows that one can rewrite

\[ d\hat{\Pi}_t = d\Pi_t + d\Delta_t \]
Return Processes

- At this point we can consider what are the return processes perceived by the investors.

- From the price function

\[ P_t = (\phi + p_0) + p_D^B D_t + p_\Pi \Pi_t + p_\Theta \Theta_t + p_\Delta \hat{\Pi}_t \]

- we obtain

\[ dP_t = p_D^B dD_t + p_\Pi d\Pi_t + p_\Theta d\Theta_t + p_\Delta d\hat{\Pi}_t \]

\[ = \left\{ p_D^B (\Pi_t - kD_t) + a_\Pi p_D^B (\Pi - \Pi_t) - a_\Theta p_\Theta \Theta_t - a_\Delta p_\Delta \Delta_t \right\} dt + b_P dB_t \]

- Hence, we can obtain an expression for the excess return process

\[ dQ_t = (D_t - rP_t) dt + dP_t \]

- **Proposition 4**: The excess (dollar) return process is given by

\[ dQ_t = (e_0 + e_\Theta \Theta_t + e_\Delta \Delta_t) dt + b_P dB_t \]

- where \( e_0 = -rp_0, \ e_\Theta = -(r + a_\Theta) p_\Theta \) and \( e_\Delta = -(r + a_\Delta) p_\Delta \).
• Clearly, asymmetric information implies a different perception of expected returns.

• For the *informed* agent, we have

\[ E \left[ dQ_t | \mathcal{F}^i_t \right] / dt = e_0 + e_\Theta \Theta_t + e_\Delta \Delta_t \]

• We already commented on why \( \Theta_t \) should enter the expected return for informed agents.

• Now we see that also the estimation error \( \Delta_t \) enters in the expected return process.

• The intuition is clear: When \( \Delta_t > 0 \), uninformed investors are overestimating the expected growth rate of dividends, an error that likely will be corrected in the future as more information arrives.

• Hence, informed agents forecast a price drop, leading to lower expected returns (as \( e_\Delta < 0 \)).

• For the *uninformed* investor, we have

\[ E \left[ dQ_t | \mathcal{F}^u_t \right] / dt = e_0 + e_\Theta \widehat{\Theta}_t \]

• For them, the only source of risk is the (perceived) aggregate risk in the economy, which is summarized by \( \widehat{\Theta}_t \).
We have to solve two intertemporal maximizations.

The key is that under both investors information sets, the price $P_t$ is linear in variables that follow Gaussian processes.

Hence, both investors are going to have linear demand functions with respect to prices.

This is what we need to close the loop, solve out for prices and show that it is indeed linear.

Let $X_t^i$ be the number of shares bought by agent $i$ at time $t$. Then:

**Informed Agent:**

$$\max_{X^i, c^i} E \left[ - \int_t^\infty e^{-\rho s - c^i_s} ds \left| \mathcal{F}_t^i \right. \right]$$

subject to

$$dW_t^i = (rW_t^i - c_t^i) \, dt + X_t^i dQ_t$$

Letting $J^i(W_t, \Theta, \Delta, t)$ denote the value function, we have that the Hamilton - Bellman - Jacobi equation is

$$0 = \max_{X^i, c^i} \left\{ -e^{-\rho s - c^i_s} + \mathcal{D}J^i \right\}$$

with some transversality condition.
• **Result:** The value function $J^i(W^i, \Theta, \Delta, t)$ is

$$J^i(W^i, \Theta, \Delta, t) = -e^{-\rho t - r W^i - V^i(\Theta, \Delta)}$$

• where $V^i(\Theta, \Delta) = \frac{1}{2} \Psi^i \Psi^i$ is a quadratic function, with $\Psi^i = (1, \Theta, \Delta)$ and $\Psi$ is a $3 \times 3$ matrix of constants.

• In addition, demand for stocks is

$$X_t^i = f^i_0 + f^i_\Theta \Theta_t + f^i_\Delta \Delta_t$$

• The optimization is more difficult for the uninformed agent:

• **Uninformed Agent:**

$$\max_{X^u_t, c^u_t} E \left[ -\int_t^\infty e^{-\rho s - c^u_s} ds | F^u_t \right]$$

• subject to

$$dW^u_t = (r W^u_t - c^u_t) dt + X^u_t dQ_t$$

• This maximization is complicated by the fact that the agent has only partial information on the underlying economy.

• Luckily, the information structure implied by $\mathcal{F}^u_t = \{D_{\tau}, \Lambda_{\tau} : \tau \leq t \}$ turns out to be the same as the information structure generated by $\{\tilde{B}_t \}$. 
This implies that if we can restate the whole problem for the uninformed in terms of these two innovations, we can use an approach similar to the previous one.

Luckily, this is the case. From its definition we have

\[ d\tilde{B}_t = (b_s b'_s)^{-\frac{1}{2}} \left( dD_t - \left( \hat{\Pi}_t - kD_t \right) dt, \right) \]

\[ d\Lambda_t - \left( a_{\Pi} p_{\Pi} \left( \Pi - \hat{\Pi}_t \right) - a_{\Theta} p_{\Theta} \hat{\Theta}_t \right) dt, \right) \]

which we can invert to write

\[ \left( \begin{array}{c} dD_t \\ d\Lambda_t \end{array} \right) = \left( \begin{array}{c} \left( \hat{\Pi}_t - kD_t \right) \\ a_{\Pi} p_{\Pi} \left( \Pi - \hat{\Pi}_t \right) - a_{\Theta} p_{\Theta} \hat{\Theta}_t \end{array} \right) dt + (b_s b'_s)^{\frac{1}{2}} d\tilde{B}_t \]

In this case, we can use the definition of the price function as

\[ P_t = (\phi + p_0) + p_D^B D_t + p_{\Pi} \Pi_t + p_{\Theta} \Theta_t + p_{\Delta} \hat{\Pi}_t \]

\[ = (\phi + p_0) + p_D^B D_t + p_{\Delta} \hat{\Pi}_t + \Lambda_t \]

Hence, we can compute the expected returns in terms of the innovation \( d\tilde{B}_t \) rather than \( dB_t \).

As we have seen, it turns out that we have

\[ E [dQ_t | \mathcal{F}^u_t] / dt = e_0 + e_{\Theta} \hat{\Theta}_t \]
• Hence, $\hat{\Theta}_t$ is the only relevant state variable for the uninformed investors, because it determines the expected returns. (Recall that the volatility is constant).

• Letting $J^u \left( W^u, \hat{\Theta}, t \right)$ denote the value function, we have that the Hamilton - Bellman - Jacobi equation is

$$0 = \max_{X^u, c^u} \left\{ -e^{-\rho s - c^u} + DJ^u \right\}$$

• with some transversality condition.

• **Result:** The value function $J^u \left( W^u, \hat{\Theta}, t \right)$ is

$$J^u \left( W^u, \hat{\Theta}, t \right) = -e^{-\rho t - rW^u - V^u(\hat{\Theta})}$$

• where $V^u \left( \hat{\Theta} \right) = \frac{1}{2} \Psi^u \Psi^u$ is a quadratic function, with $\Psi^u = \left( 1, \hat{\Theta} \right)$ and $v^i$ is a $2 \times 2$ matrix of constants.

• In addition, demand for stocks is

$$X^u_t = f^u_0 + f^u_\Theta \hat{\Theta}_t$$
Market Clearing

- Finally, we must have that the asset market clears:

$$1 + \Theta_t = (1 - w) X^i_t + w X^u_t$$

- That is

$$1 + \Theta_t = (1 - w) \left( f^i_0 + f^i_\Theta \Theta^i_t + f^i_\Delta \Delta^i_t \right) + w \left( f^u_0 + f^u_\Theta \hat{\Theta}^i_t \right)$$

- or

$$1 + \Theta_t = (1 - w) \left( f^i_0 + f^i_\Theta \Theta^i_t - f^i_\Delta \left( \Pi_t - \hat{\Pi}_t \right) \right)$$

$$+ w \left( f^u_0 + f^u_\Theta \frac{\Pi}{\hat{\Pi}} \left( \Pi_t - \hat{\Pi}_t \right) + f^u_\Theta \Theta^i_t \right)$$

- Hence, we finally obtain the following system of equations

$$\begin{align*}
(1 - w) f^i_0 + w f^u_0 &= 1 \\
(1 - w) f^i_\Theta + w f^u_\Theta &= 1 \\
-(1 - w) f^i_\Delta + w f^u_\Theta \frac{\Pi}{\hat{\Pi}} &= 0
\end{align*}$$

- If one can show that a solution to this system of equation exists, then the loop is closed.
We can now investigate what are the effects of asymmetric information on asset prices.

### Stock Prices

1. Compared to the benchmark, we have

\[ P_t - P_t^B = p_\Delta \Delta + \left( (p_0 + p_\Theta \Theta_t) - (p_0^B + p_\Theta^B \Theta_t) \right) \]

   - The first piece, \( p_\Delta \Delta \), is driven by uninformed investors’ estimation error.
   - If they are optimistic, the price is going to be higher compared to the benchmark case, while if they are pessimistic, the price is going to be lower.
   - The second piece is the difference in the discounts required. This is a negative shift, showing that the presence of uninformed investors decrease the price of the stock.

2. Prices are “history” dependent: Uninformed investors base their estimates of dividend growth partially on the information revealed by prices themselves.
   - Suppose uninformed investors observe a sequence of good shocks to prices. They can believe partly that this is due to \( \Theta_t \) decreasing and partly that is due to \( \Pi_t \) increasing.
   - Hence, they increase \( \hat{\Pi}_t \), increase the demand for stocks and drive up the price further.
Price volatility

1. Compared with the benchmark case, increase in the number of uninformed agents tend to increase the volatility of stock prices.
   - The reason is that there is an extra kick from imperfect information. Under perfect information, an increase of dividends increase the price “1 to 1.”
   - If a fraction of agents are uninformed, an increase in price also increase their estimate \( \hat{\Pi}_t \), giving an extra kick to the current price (because they now believe that dividends are going to be even higher in the future). This increases volatility.

2. However, it turns out that the relationship is not monotonic. When \( \sigma_{\Theta} \) is high, an increase in the number of uninformed may decrease the volatility of stock prices.
   - The reason is subtle: Consider the case where everyone is uninformed. Then, since \( \Pi_t \) drops out of the price function, investors can perfectly observe \( \Theta_t \). Hence, prices are not used to filter out future information about dividend growth.
   - Now, suppose that some informed traders are introduced in the market. On the one hand, they stabilize the prices, because there is more aggregate information in the market.
   - On the other, now the uninformed must partly use prices to filter out what is the current \( \Pi_t \). Since increases in prices must be correlated with increases in \( \hat{\Pi}_t \), the volatility of prices can actually increase.

3. The above discussion implies that it is not necessarily true that more informed agents in the market tend to stabilize the prices. They may actually de-stabilize it!
• Risk Premium

1. The risk premium is given by $E_t[dQ_t/P_t]$.
   - Increasing the number of the uninformed agents affect the risk premium in two ways:
     - The first, through the numerator: Now $E[dQ]$ may increase (for the informed agent).
     - The second is through the denominator: We just saw that the price $P_t$ decreases (at least in the long run).
   - Hence, we can expect that the introduction of asymmetric information increases the risk premium that informed agents require on stocks.

• Autocorrelation

1. Wang (1993) shows that the model is consistent with either positive or negative autocorrelation of stock returns, even in the case of symmetric information.
   - Intuitively, this stems from the mean reversion of $\Theta$ that induces a time varying equity premium
   - However, Wang (1993) shows that the presence of asymmetric information increases the autocorrelation of stock returns, because the feedback effect through the prices.
• **Optimal Investment Strategies:**
  
  — How do the two classes of agents trade?
  
  — This can be obtained by the two demand functions

  \[
  X^i_t = f^i_0 + f^i_\Theta \Theta_t + f^i_\Delta \Delta_t \\
  X^u_t = f^u_0 + f^u_\Theta \hat{\Theta}_t
  \]

  — It is possible to show that depending on parameter values

  \[
  E_t [dX^u_t dP_t] > 0
  \]

  — That is, increases in the price of stock increases the demand for assets of the uninformed trader.
  
  — That is, uninformed traders would act as “trend chaser,” in the attempt to extract information from the more informed agents.
  
  — Given that market clearing must hold, the above discussion would imply that informed agents would tend to act as “contrarians,” by buying losers and selling winners.
  
  — The idea is the informed traders trade on their superior information, knowing the type of mistake that the uninformed agents are making.
  
  — Since they expect the mistake to “revert” back, they act in opposition.
  
  — Notice that this does not imply that there is any arbitrage: All these investments are risky and hence risk aversion prevents them from taking infinite positions.
Differences in Opinions

• A totally different model is one that goes under the heading of “differences in opinion.”
• Under the asymmetric information model studied above the idea is that there are a group of agents who have more information than others.
• Hence, the “others” try to use equilibrium market price to partially infer the information of the informed agents.
• The differences of opinion (or beliefs) literature focuses on a different issue.
• The basic assumption is that everybody have the same information, but they either use different models to interpret it or have different prior beliefs at the “origin” able to generate persistent differences in beliefs.
• For example, consider again the case where there is a single state variable and a single signal

\[ dz_t = [a z_0 + a z_z z_t] dt + b_z dB_{1,t} \]
\[ ds_t = z_t dt + b_s dB_{2,t} \]

• with \( dB_{1,t} dB_{2,t} = 0. \)
• From the result in TN 3, if \( q^* \) is the stationary mean square error \( q^* = E \left[ (z_t - \hat{z}_t)^2 \right] \), we have that the posterior mean \( \hat{z}_t \) evolves according to the SDE

\[
d\hat{z}_t = \left[ a_{z0} + a_{zz}\hat{z}_t \right] dt + [q^* a_{zz}] b_s^{-1} d\tilde{B}_2,t
\]

• where

\[
d\tilde{B}_{2,t} = b_s^{-1} (ds_t - E[ds_t|\mathcal{F}_t]) = b_s^{-1} (ds_t - \hat{z}_t dt)
\]

• Now, consider two agents who start at time zero with a different mean \( \hat{z}_0^1 \neq \hat{z}_0^2 \).
• The filtering result then implies

1. Each agent has its own process for \( \hat{z}_t^i \)

\[
d\hat{z}_t^i = \left[ a_{z0} + a_{zz}\hat{z}_t^i \right] dt + [q a_{zz}] b_s^{-1} d\tilde{B}_{2,t}^i
\]

2. Since the innovation \( d\tilde{B}_{2,t} \) depend also on the prior \( \hat{z}_t^i \),

\[
d\tilde{B}_{2,t}^i = b_s^{-1} (ds_t - \hat{z}_t^i dt)
\]

— also the Brownian motions are individual specific;
3. Inverting (34) we obtain

\[ ds_t = \hat{z}_t^i dt + b_s d\tilde{B}_2^i t \]

\[ \text{Hence, even though each agent observes the same signal, the process for it is specific.} \]

4. However, we must have the following consistency requirement

\[ \hat{z}_t^1 dt + b_s d\tilde{B}_2^1 t = \hat{z}_t^2 dt + b_s d\tilde{B}_2^2 t \]

\[ \text{Hence, we can always rewrite} \]

\[ d\tilde{B}_2^2 t = d\tilde{B}_2^1 t + \frac{\hat{z}_t^1 - \hat{z}_t^2}{b_s} \]

- We now use the results as above to study the implications of differences of opinion for stock returns.
- In addition, we will find interesting equilibrium "mispricings" under portfolio constraints.
Differences in Beliefs on Dividend Growth

• We follow the recent Basak and Croitoru (2000), even though differences in beliefs was not the focus of their paper (see below).

• Consider a standard economy where a dividend process is given by

\[ d\delta_t = \delta_t \mu_{\delta,t} dt + \delta_t \sigma_{\delta,t} dB_t \]

• The mean growth \( \mu_{\delta,t} \) and volatility \( \sigma_{\delta,t} \) are assumed to satisfy standard regularity conditions.

• There are two (classes of) agents. They both observe \( \delta_t \) but ignore \( \mu_{\delta,t} \). Hence, they have a common information set \( F_t = \{ \delta_\tau : \tau \leq t \} \).

• They use the information from \( \delta_t \) to deduce the value of \( \mu_{\delta,t} \).

• Rather than specifying the process for \( \mu_{\delta,t} \) and obtain “filtering results” etc. as we have done above, it suffices for us to assume that agents have a different probability measure \( P^i \) on the measurable space \( (\Omega, \mathcal{F}) \) that disagree on some sigma-field \( \mathcal{H} \).

• Hence, we can define

\[ \mu^i_{\delta,t} = E^i_t [\mu_{\delta,t}|F_t] \]

• where \( E^i_t [\cdot] \) denotes expectation with respect to the probability \( P^i \)
• The innovation process $B^i_t$ induced by agent $i$, given the Brownian filtration $\{\mathcal{F}_t\}$ is then

$$dB^i_t = \frac{1}{\sigma_{\delta,t}} \left[ \frac{d\delta_t}{\delta_t} - \mu^{i}_{\delta,t} dt \right] = dB_t + \frac{\mu^1_{\delta,t} - \mu^2_{\delta,t}}{\sigma_{\delta,t}}$$

• Hence, we can also write

$$dB^2_t = dB^1_t + \mu_t$$

• where

$$\mu_t = \frac{\mu^1_{\delta,t} - \mu^2_{\delta,t}}{\sigma_{\delta,t}}$$

• Clearly, $\mu_t$ parametrizes agents’s disagreement.

• Remark: Notice that each agent knows (or can deduce) the other agent forecast. They have common information although heterogeneous priors.

• The agents agree to disagree! Under this condition, the standard Milgrom-Stokey no-trade theorem does not apply. (It applies to the common prior case).
Securities Market

• Let there be a stock in constant net supply of 1, paying the continuous dividend rate \( \delta_t \).

• Its price \( S_t \) has dynamics

\[
dS_t + \delta_t dt = S_t \left( \mu_{S,t} dt + \sigma_{S,t} dB_t \right)
\]

• We can immediately rewrite this dynamics under the two different (perceived) Brownian motions

\[
dS_t + \delta_t dt = S_t \left( \mu^i_{S,t} dt + \sigma_{S,t} dB^i_t \right)
\]

• That is, we must have the consistency requirement

\[
\begin{align*}
\mu^1_{S,t} dt + \sigma_{S,t} dB^1_t &= \mu^2_{S,t} dt + \sigma_{S,t} dB^2_t \\
&= \mu^2_{S,t} dt + \sigma_{S,t} dB^1_t + \sigma_{S,t} \bar{\mu}_t dt \\
&= \left( \mu^2_{S,t} + \sigma_{S,t} \bar{\mu}_t \right) + \sigma_{S,t} dB^1_t
\end{align*}
\]

• Hence

\[
\frac{\mu^1_{S,t} - \mu^2_{S,t}}{\sigma_{S,t}} = \bar{\mu}_t = \frac{\mu^1_{\delta,t} - \mu^2_{\delta,t}}{\sigma_{\delta,t}}
\]

• The relative disagreement in dividends must be equal to the relative disagreement in stock returns!

• Let there be a riskless bond in zero net supply

\[
d\beta_t = \beta_t r_t dt
\]
Logarithmic Preferences

• Consider the case where investors have logarithmic preferences and no time discount.

• That is, they maximize

\[ E^i \left[ \int_0^T \log (c_t) \, dt \right] \]

• The time-preference parameter could be added without changing any of the conclusions. It is just convenient to have it equal to zero.

• We should notice that each investor \( i \) faces a different (personal) state price density \( \pi_t^i \) because of the different beliefs.

• That is, define the market price of risk for agent \( i \) as

\[ \nu_t^i = \sigma_{S,t}^{-1} \left( \mu_{S,t}^i - r_t \right) \]

• Notice that we also have

\[ \overline{\mu}_t = \nu_t^1 - \nu_t^2 \]

• Agent \( i \) state price density is

\[ \pi_t^i = \exp \left( - \int_0^t \left( r_t + \frac{1}{2} (\nu_t^i)^2 \right) \, dt \right) - \int_0^t \nu_t^i dB_t^i \]
Then, standard results imply that optimal consumption and allocation $\varphi_t^i$ (fraction of wealth) are

$$c_t^i = \frac{1}{y_t^i \pi_t^i} = \frac{W_t^i}{T - t} \quad \text{and} \quad \varphi_t^i = \frac{\mu_t^i - r_t}{\sigma_{S,t}^2}$$

Notice in particular that

$$\varphi_t^1 = \frac{\mu_t^1 - r_t}{\sigma_{S,t}^2} = \frac{1}{\sigma_{S,t}} \nu_t^1 = \frac{1}{\sigma_{S,t}} \nu_t^2 + \frac{\bar{\mu}_t}{\sigma_{S,t}}$$

$$= \varphi_t^2 + \frac{\bar{\mu}_t}{\sigma_{S,t}}$$

Differences of opinions generates trading in the securities market. As commented, the no-trade theorem does not apply.

On passing, notice that the no-trade theorem did not apply in the asymmetric model of Wang (1993) either: In that case, agents also did not have common prior, due to the presence of noise trading.
• How do we construct an equilibrium?
• We have already encountered a problem where agents have different state price densities in the work of Basak and Cuoco (1998) (limited market participation).

Recalling the procedure we adopted in that case, we construct a representative agent with utility function

\[
U(c_t, \lambda_t) = \max_{c_1^1 + c_2^2 = c_t} \log (c_1^1) + \lambda_t \log (c_2^2)
\]

\[
= \log \left( \frac{c_t}{1 + \lambda_t} \right) + \lambda_t \log \left( \frac{\lambda t c_t}{1 + \lambda_t} \right)
\]

• Define the state price density

\[
\hat{\pi}_t = \frac{U_c(\delta_t, \lambda_t)}{U_c(\delta_0, \lambda_0)}
\]

• We saw in TN2 that the optimal allocation is

\[
\hat{c}_1^1 = \mathcal{I}_u^{1}(U_c(\delta_t, \lambda_t)) = \frac{1}{U_c(\delta_t, \lambda_t)}
\]

\[
\hat{c}_2^1 = \mathcal{I}_u^{2}\left(\frac{1}{\lambda_t} U_c(\delta_t, \lambda_t)\right) = \frac{\lambda_t}{U_c(\delta_t, \lambda_t)}
\]
Hence, we must have
\[ \lambda_t = \frac{c_t^2}{c_t^1} = \frac{y_1\pi_t^1}{y_2\pi_t^2} \]

- By definition, we have
\[ \frac{\pi_t^1}{\pi_t^2} = \exp \left( - \int_0^t \left( \frac{1}{2} \nu_t^1 - \frac{1}{2} \nu_t^2 \right)^2 - \nu_t^1 dB_t^1 + \nu_t^2 dB_t^2 \right) \]
\[ = \exp \left( - \int_0^t \left( \frac{1}{2} \nu_t^1 - \frac{1}{2} \nu_t^2 - \nu_t^2 \mu_t dt \right) \right) - \int_0^t \left( \nu_t^1 - \nu_t^2 \right) dB_t^1 \]

where we used the fact that \( dB_t^2 = dB_t^1 + \mu_t dt \).

- Hence, recalling that \( \mu_t = \nu_t^1 - \nu_t^2 \), we have
\[ \frac{1}{2} \left( \nu_t^1 \right)^2 - \frac{1}{2} \left( \nu_t^2 \right)^2 - \nu_t^2 \mu_t = \frac{1}{2} \left( \nu_t^1 - \nu_t^2 \right)^2 \]

which yields
\[ \lambda_t = \frac{y_1\pi_t^1}{y_2\pi_t^2} = \exp \left( - \int_0^t \frac{1}{2} \mu_t^2 dt - \int_0^t \mu_t dB_t^1 \right) \]

- Hence, the process for \( \mu_t \) determines the process for the stochastic weight \( \lambda_t \)
\[ \frac{d\lambda_t}{\lambda_t} = - \mu_t dB_t^1 \]
By using standard arguments developed in previous teaching notes, we obtain the following results:

1. The optimal consumption for the two agents are
   \[ c_1^t = \frac{\delta_t}{1 + \lambda_t} \text{ and } c_2^t = \frac{\lambda_t \delta_t}{1 + \lambda_t} \]

2. The stock price is given by
   \[ S_t = (T - t) \delta_t \]

3. The market price of risk is
   \[ \nu_1^t = \sigma_{\delta,t} + \frac{\lambda_t}{1 + \lambda_t} \bar{\mu}_t \]
   \[ \nu_2^t = \sigma_{\delta,t} - \frac{1}{1 + \lambda_t} \bar{\mu}_t \]

4. The equilibrium interest rate is
   \[ r_t = \frac{1}{1 + \lambda_t} \mu_1^{\delta,t} + \frac{\lambda_t}{1 + \lambda_t} \mu_2^{\delta,t} - \sigma_{\delta}^2 \]

- It also turns out that
  \[ \frac{W_1^t}{W_1^t + W_2^t} = \frac{1}{1 + \lambda_t} \]

- Hence, the risk-free rate is given by a wealth-weighted average of the drift rates of the dividend process, according to the two investors beliefs.
• Notice that the stock price is proportional to the dividend, exactly as in the case of common beliefs. This is an artifact of the log-utility case. More general preferences would give a role for differences in beliefs for the stock market price as well.

• The market price of risk of the two agents differ, depending on the disagreement. If agent 1 is more optimistic, then his sharpe ratio increases, while the one of agent 2 decreases (quite obviously).
Portfolio Constraints and “Mispricing”

• Basak and Croitoru (2000) assume that a redundant security (derivative) is introduced in the market.

• Let its price be $P_t$. Suppose it pays no dividends, so that

\[ dP_t = P_t \left( \mu_{P,t} dt + \sigma_{P,t} dB_t \right) = P_t \left( \mu_{P,t}^i dt + \sigma_{P,t} dB_t^i \right) \]

• Because of the same argument as before, we have

\[ \mu_{P,t}^1 - \mu_{P,t}^2 = \sigma_{P,t} \bar{\mu}_t \]

• Clearly, there are two securities and one BM, hence we know that the payoff of $P_t$ can be perfectly replicated, unless we introduce some frictions to prevent market completeness.

• Let $\varphi^i = (\varphi_B^i, \varphi_S^i, \varphi_P^i)$ be a trading strategy in terms of percentage of wealth $W_t^i$ invested in each security (they must add up to 1).

• Basak and Croitoru (2000) then impose a short sale constraint on the stock and an upper bound to the fraction of wealth that can be invested in the derivative security (no unlimited borrowing)

\[ \varphi_S^i \geq 0 \text{ and } \varphi_P^i \leq \gamma \]
• It may happen then that in equilibrium the sharpe ratio of the stock and the derivative security differ.

• Let us denote

\[ \Delta_{P,S,t}^i = \frac{\mu_{P,t}^i - r_t}{\sigma_{P,t}} - \frac{\mu_{S,t}^i - r_t}{\sigma_{S,t}} \]

• We say that \( P \) is favorable if \( \Delta_{P,S,t}^i > 0 \).

• The consistency of security prices yields immediately the following

• Result: Agents agree on the misspricing:

\[ \Delta_{P,S,t}^1 = \Delta_{P,S,t}^2 = \Delta_{P,S,t} = \frac{\mu_{P,t} - r_t}{\sigma_{P,t}} - \frac{\mu_{S,t} - r_t}{\sigma_{S,t}} \]
It is convenient to solve the allocation problem in two steps.

In the first step, define a composite risky security by

$$\Phi_t^i = \varphi_{S,t}^i + \frac{\sigma_{P,t}^i}{\sigma_{S,t}^i} \varphi_{P,t}^i$$

Since the volatility of wealth is $W_t^i \Phi_t^i \sigma_{S,t}^i$, the composite risky security $\Phi_t^i$ can be interpreted as agent $i$’s “composite” risk exposure, defined as his wealth volatility per unit of stock volatility.

Lemma: Let $\Phi_t^i$ be given. Then

1. If there is no mispricing, all pairs such that

$$\varphi_{S,t}^i = \Phi_t^i - \frac{\sigma_{P,t}^i}{\sigma_{S,t}^i} \varphi_{P,t}^i$$

— is an optimal allocation.

2. If $P$ is favorable, $i$’s optimal risk allocation between $S$ and $P$ is

$$\varphi_{S,t}^i = \max \left( \Phi_t^i - \frac{\sigma_{P,t}^i}{\sigma_{S,t}^i} \gamma, 0 \right)$$

$$\varphi_{P,t}^i = \gamma - \frac{\sigma_{S,t}^i}{\sigma_{P,t}^i} \max \left( -\left( \Phi_t^i - \frac{\sigma_{P,t}^i}{\sigma_{S,t}^i} \gamma \right), 0 \right)$$

When $\Phi_t^i$ is given, it is simple to compute the optimal allocation: Check which security has the highest Sharpe ratio and invest as much as possible there.
Allocation in risky and riskless asset

- The problem is to choose between $\Phi_t^i$ and consumption.
- The problem is that there are borrowing and short sales constraints.
- TN 6 explain the methodology in detail. But, in words the idea is to generate a space of fictitious economies, indexed by some process $v_t$, such as to transform the model with constraints in a set of un-constrained maximization problems.
- Suppose that out of the class of possible fictitious economies, you found the process $v_t^{i*}$ that satisfies the short sale constraints above.
- Then, an application of the results in TN6 yields that the optimal consumption is

$$c_t^i = \frac{1}{y^i \pi_t^i} \text{ and } \Phi_t^i = \frac{\nu_t^i}{\sigma_{S,t}}$$

- where $\nu_t^i$ and $\pi_t^i$ are the market price of risk and the state price density obtained by using $v_t^{i*}$, that is

$$\nu_t^i = \frac{\mu_{S,t}^i - r_t}{\sigma_{S,t}} + \frac{v_t^{i*}}{\sigma_{S,t}}$$

- and

$$\pi_t^i = \exp \left( - \int_0^t \left( r_t^i + \frac{1}{2} (\nu_t^i)^2 \right) dt - \int_0^t \nu_t^i dB_t^i \right)$$

- and where $r_t^i$ also is the perturbed interest rate obtained by adding the first component in $v_t^{i*}$. 
Equilibrium

- It is possible to check that the only situations where the allocations are consistent with the equilibrium and the consistency of security returns are either of the following

1. No misspricing:
   \[ \Delta_{P,S} = 0 \]

2. \( P \) favorable with
   \[
   \frac{\mu^i_{P,t} - r_t}{\sigma_{P,t}} > \frac{\mu^i_{S,t} - r_t}{\sigma_{S,t}} \geq \gamma \sigma_{P,t} \geq \frac{\mu^j_{P,t} - r_t}{\sigma_{P,t}} > \frac{\mu^j_{S,t} - r_t}{\sigma_{S,t}} \]  
   \( (35) \)

3. The converse.
   - That is, for both agents \( P \) is favorable, but for agent \( i \) investing in the risky asset rather than in bonds is strictly better than for \( j \), as the Sharpe ratio is strictly higher.
   - But this implies that the “mispricing” is going to remain in equilibrium: Investor \( i \) would like to invest even more in the composite risky security, but it cannot because it cannot borrow unlimitedly.
   - Also agent \( j \) would like to sell the stock, but he instead hit the short sale constraints.
   - By applying a method similar to the one we wrote above for the no-frictions case, one can find the following results:
• Proposition:

1. If

$$-\frac{1 + \lambda_t}{\lambda_t} (\lambda_t \bar{\gamma} \sigma_{P,t} + \sigma_{\delta,t}) \leq \bar{\mu}_t \leq \frac{1 + \lambda_t}{\lambda_t} (\lambda_t \bar{\gamma} \sigma_{P,t} + \sigma_{\delta,t})$$

- Then, there is no misspricing and we are back in the case with no frictions.

2. If

$$\bar{\mu}_t > \frac{1 + \lambda_t}{\lambda_t} (\lambda_t \bar{\gamma} \sigma_{P,t} + \sigma_{\delta,t})$$

- Then (35) holds with $i = 1$ and $j = 2$. In this case, the equilibrium misspricing is

$$\Delta_{P,S,t} = \bar{\mu}_t - \frac{1 + \lambda_t}{\lambda_t} (\bar{\gamma} \sigma_{P,t} + \lambda_t \sigma_{\delta,t}) > 0$$

3. If

$$\bar{\mu}_t < -\frac{1 + \lambda_t}{\lambda_t} (\lambda_t \bar{\gamma} \sigma_{P,t} + \sigma_{\delta,t})$$

- Then (35) holds with $i = 2$ and $j = 1$. In this case, the equilibrium misspricing is

$$\Delta_{P,S,t} = -\bar{\mu}_t - \frac{1 + \lambda_t}{\lambda_t} (\lambda_t \bar{\gamma} \sigma_{P,t} + \sigma_{\delta,t}) > 0$$

• The bottom line is that portfolio constraints generate equilibrium “mispricing” when agents have motives to trade.

• In equilibrium, the two agents have different views of the world and would like to take advantage of their own beliefs.
Since they cannot take any position, the prices of otherwise redundant securities go out of line.

For example, consider again

\[
\frac{\mu_{P,t}^i - r_t}{\sigma_{P,t}} > \frac{\mu_{S,t}^i - r_t}{\sigma_{S,t}} \geq \bar{\gamma}\sigma_{P,t} \geq \frac{\mu_{P,t}^j - r_t}{\sigma_{P,t}} > \frac{\mu_{S,t}^j - r_t}{\sigma_{S,t}}
\]

Both prefer \( P \) to \( S \), but \( i \) is more optimistic than \( j \) on both securities with respect to the bond.

Hence, \( i \) wants to load on the “risky” security while \( j \) want to sell short the “risky” security.

Therefore, \( i \) will be driven to his upper bound \( (\varphi_P = \bar{\gamma}) \) in \( P \), and it would still want more of the stock \( S \).

Similarly, agent \( j \) will be driven to the lowest bound for \( S \) \( (\varphi_S = 0) \) and still wants to sell short \( P \).

These desires are incompatible with market clearing!

Since \( P \) is in zero net supply and it is the only security that can be sold short, the total dollar amount of \( j \)'s short sale is limited by \( i \)'s upper constraint on \( P \) (that is \( \varphi_P^j W_j \geq -\bar{\gamma} W_i \)).

Hence, prices must adjust to make \( P \) more favorable (and hence short selling it more costly) with respect to \( S \).

The equilibrium mispricing follows!

Notice that homogenous beliefs kill the “mispricing” because in this case we have \( \bar{\mu}_t = 0 \), which yields the no-friction case (even if there are frictions).